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Periodic pattern formation in the coupled chemotaxis-(Navier–)Stokes system with mixed nonhomogeneous boundary conditions

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We consider the coupled chemotaxis-fluid model for periodic pattern formation on two- and three-dimensional domains with mixed nonhomogeneous boundary value conditions, and prove the existence of nontrivial time periodic solutions. It is worth noticing that this system admits more than one periodic solution. In fact, it is not difficult to verify that $(0, c, 0, 0)$ is a time periodic solution. Our purpose is to obtain a time periodic solution with nonconstant bacterial density.

Keywords: chemotaxis-fluid system; mixed boundary; time periodic pattern; classical solution

2010 *Mathematics subject classification:* 92C17; 92C15; 35M10

1. Introduction

Pattern can be widely observed in the nature, for example, the patterns on butterflies, the stripes on zebras. However, the processes that produce them are unknown. The pattern generation mechanisms have always attracted people's attention. In 1952, Turing [24] proposed a novel idea, for the following diffusion system,

$$\begin{cases} u_t = D_u \Delta u + f(u, v), \\ v_t = D_v \Delta v + g(u, v), \end{cases}$$

in the absence of diffusion ($D_u = D_v = 0$), the solutions tend to a linearly stable uniform steady state, while under certain conditions, these systems are capable of generating spatially inhomogeneous patterns if $D_u \neq D_v$, and this phenomenon is known as Turing (diffusion-driven) instability. Diffusion is usually considered a stabilizing process, and Turing's discovery broke this knowledge. After that, this phenomenon has been widely studied, and it has been shown that reaction-diffusion mechanisms can generate spatially inhomogeneous steady states [19]. Besides the reaction-diffusion mechanisms, the time delay, the prey-taxis, cell-chemotaxis, etc., also have been proposed as the possible causes of pattern formation [13, 16, 19, 26, 32, 34].

In this paper, we consider the time periodic pattern formation for the following chemotaxis-fluid model

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \chi \nabla \cdot (n \nabla c) + \mu n(a(x,t) - n), \text{ in } Q, \\ c_t + u \cdot \nabla c = \Delta c - cn, \text{ in } Q, \\ u_t + \tau u \cdot \nabla u = \Delta u - \nabla \pi + n \nabla \varphi, \text{ in } Q, \\ \nabla \cdot u = 0, \text{ in } Q, \end{cases} \tag{1.1}$$

where $Q = \Omega \times \mathbb{R}^+$, Ω is a bounded domain in $\mathbb{R}^N$ with $\partial\Omega \in C^{2+\alpha}$. This model describes the motion of oxygen-driven bacteria living in a water drop containing oxygen. n, c denote the bacterial density, the oxygen concentration respectively, $J = n\nabla c$ is the chemotactic flux, a is a nonconstant time periodic function with period T, $\mu > 0$ is a parameter, $\mu n(a(x,t) - n)$ reflects the proliferation and death of bacteria in a logistic law, $-cn$ is the consumption term of oxygen, u, π are the fluid velocity and the associated pressure with $\tau = 0$ or 1, $\nabla\varphi$ is the gravitational potential.

The chemotaxis-fluid model was initially introduced by Tuval et al. [**25**] in 2005, which describes the motion of oxygen-driven swimming bacteria in incompressible fluid, that is, the bacillus subtilis suspending in a drop of water will move towards higher concentration of oxygen, this model can be written as follows

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (\chi(c) n \nabla c), \\ c_t + u \cdot \nabla c = \Delta c - k(c)n, \\ u_t + \tau u \cdot \nabla u = \Delta u - \nabla \pi + n \nabla \varphi, \\ \nabla \cdot u = 0. \end{cases} \tag{1.2}$$

However, the viscous force plays a leading role in slow viscous flows (low Reynolds number), and the inertial force is far less than the viscous force. Thus, for which, the Navier–Stokes equations can be approximated using Stokes equations by ignoring the inertia force $u \cdot \nabla u$ [**11**]. The study of chemotaxis-Navier–Stokes system or chemotaxis-Stokes system has attracted much attention in the past decade. For example, for Cauchy problem of this system, a global weak solution in dimension $2(\tau = 1)$ was established in [**18**], and for the initial and boundary value problem with zero-flux boundary condition for n, c, and no-slip boundary for u, a unique global classical solution in the two-dimensional space with $\tau = 1$, and a global weak solution in the three-dimensional space with $\tau = 0$ were obtained respectively, and the authors further proved that the weak solution will become smooth eventually and converge to the semi-trivial steady $(\overline{u}_0, 0, 0)$ [**27**, **28**, **30**], which implies that there is no pattern formation for this system. While if the proliferation and death of bacteria is considered, then the system (1.2) becomes (1.1). For which, the global existence of classical solutions in dimension 2 can be obtained using the same methods as [**27**]. While in dimension 3, a global weak solution was obtained by Lankeit [**14**] for $a \equiv 1$. It is also shown that the weak solution will become smooth after some time and finally converge to a semi-trivial steady state. So, there is no pattern formation for this system. However, the existence of global classical solution for small μ with any large initial datum remains open in dimension 3, which was obtained only for large μ or small initial value [**15**].

On the other hand, if the mobility of bacteria is characterized by the porous medium diffusion, that is

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n^m - \chi \nabla \cdot (n \nabla c), \\ c_t + u \cdot \nabla c = \Delta c - cn, \\ u_t = \Delta u - \nabla \pi + n \nabla \varphi, \\ \nabla \cdot u = 0, \end{cases} \tag{1.3}$$

a global 'very' weak solution ($n \ln n \in L^1$) is obtained for $m = \frac{4}{3}$ by Liu and Lorz [**18**] in dimension 3. Subsequently, Duan and Xiang perfected this result, and established the global existence of this kind of solution for any $m \geqslant 1$ [**4**]. However, this kind of weak solution may be unbounded, and it is impossible to identify the singularity of the solution. Hence, many mathematicians began to turn their attention to searching for a bounded global weak solution. In 2012, Tao and Winkler [**22**] established the existence of global bounded weak solution in the two-dimensional framework for any $m > 1$. While, the study for the 3-D case is much more difficult. In 2010, Di Francesco [**3**] obtained the existence of a global bounded weak solution for $m \in ((7 + \sqrt{217})/12, 2]$; a locally bounded global weak solution was then obtained for $m \in (\frac{8}{7}, +\infty)$ in 2013 [**23**]; the uniform boundedness of solutions was subsequently supplemented for $m \in (\frac{7}{6}, +\infty)$ [**29**]; further extension was made by Winkler for $m > \frac{9}{8}$ in [**31**] to a convex domain; recently, we also [**10**] improved the results to the case $m > \frac{11}{4} - \sqrt{3}$ (approximating to $\frac{56}{55}$). However, if a logistic term reflecting the cell proliferation is added to this model, Jin [**7**] established the existence of global bounded weak solutions for any $m > 1$ to the fluid-free case in dimension 3. Recently, a non-homogeneous boundary value problem is considered in [**1**], the global existence of classical solution in dimension 2 and the global existence of weak solution in dimension 3 were obtained respectively.

From above results, we see that if a is a positive constant, for the homogeneous boundary value problem, there is no pattern formation, since all these solutions converge to the semi-trivial steady state. Thus, in the present paper, we assume that a is a time periodic function with period T. We consider the non-homogeneous mixed boundary value problem of the model (1.1). For u, we still consider the no-slip boundary condition, namely, no fluid motion takes place on the surface of the water drop,

$$u|_{\partial\Omega} = 0.$$

We also assume that there is no bacteria flux through the fluid–air interface, that is

$$\left. \frac{\partial n}{\partial \nu} - \chi n \frac{\partial c}{\partial \nu} \right|_{\partial\Omega} = 0.$$

For oxygen, we absorb the ideas of [**1**, **25**], that is, if the water drop is surrounded by air, oxygen exchange will take place on the boundary of Ω, that is, the solved oxygen in the water drop may leave, and the free oxygen in the air may diffuse into the drop. The behaviour of the oxygen exchange can be described by Raoult's law, which connects the rate of incoming oxygen to the partial vapour pressure of the oxygen in the surroundings. We assume that the vapour pressure of the free oxygen is given, and thereby, the incoming rate of oxygen is known. The leaving rate of the oxygen molecules is proportional to the total number of molecules on the surface. Therefore, we have the following Robin boundary condition

$$\frac{\partial c}{\partial \nu}\Big|_{\partial\Omega} = -a_1(x,t)c(x,t) + a_2(x,t),$$

where $a_1 \in C^\infty(\partial\Omega) \times [0,+\infty))$, $a_2 \in C^\infty(\partial\Omega \times [0,+\infty))$, $a_1 > 0$ is the leaving rate of the oxygen molecules, $a_2 \geqslant 0$ with $a_2 \not\equiv 0$ is the incoming oxygen and depends on the known vapour pressure of the free oxygen. By [**1**, **17**], there exist g_1, g_2 with

$$g_1 \in C_T^\infty(\overline{\Omega} \times [0,+\infty)), g_2 \in C_T^\infty(\overline{\Omega} \times [0,+\infty)) \tag{1.4}$$

such that

$$\begin{aligned} &\frac{\partial g_1(x,t)}{\partial \nu} = -a_1(x,t) < 0, \quad g_2(x,t) = \frac{a_2}{a_1} \geqslant 0, \\ &\text{with } \frac{\partial g_2(x,t)}{\partial \nu} = 0, \quad (x,t) \in \partial\Omega \times [0,+\infty). \end{aligned} \tag{1.5}$$

Thus, we have the following mixed boundary conditions

$$\begin{aligned} &\frac{\partial n}{\partial \nu} - \chi n \frac{\partial c}{\partial \nu} = 0, \quad \frac{\partial c}{\partial \nu} = \frac{\partial g_1(x,t)}{\partial \nu}(c(x,t) - g_2(x,t)), \\ &u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,+\infty). \end{aligned} \tag{1.6}$$

In the present paper, we study the time periodic patterns formation, we aimed to show the existence of nontrivial time periodic solutions of (1.1) and (1.6). One will see that the time periodic solution of the problem (1.1) and (1.6) is not unique, and there are more than one periodic solution. Firstly it is easy to show that $(n, c, u, \pi) = (0, c, 0, 0)$ is the time periodic solution of (1.1) and (1.6), where $c \geqslant 0$ is the unique time periodic solution of (1.7). In fact, we see that when $n = 0$, $u = 0$, then (1.1) and (1.6) are equivalent to

$$\begin{cases} c_t = \Delta c, \quad \text{in } Q, \\ \frac{\partial c}{\partial \nu}\big|_{\partial\Omega} = \frac{\partial g_1(x,t)}{\partial \nu}(c(x,t) - g_2(x,t)). \end{cases} \tag{1.7}$$

It is easy to verify that $\|g_2\|_{L^\infty}$, 0 are the upper and lower periodic solutions of (1.7) respectively, then by an iterative process, see for example [**33**], the problem (1.7) admits a time periodic solution c with

$$0 \leqslant c \leqslant \|g_2\|_{L^\infty}.$$

Similar to the proof of lemma 3.2, we also have the uniqueness of periodic solutions for (1.7). However, in any case, the main purpose of the present paper is to find a time periodic solution with nontrivial n.

We give the main theorems of this paper as follows.

THEOREM 1.1. *Assume $N = 2, \tau = 1$, (1.4) and (1.5) hold, $a, \nabla\varphi \in L_T^\infty(Q), a(x,t)$ has a positive lower bound, that is there exists a positive constant ρ such that*

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$a(x,t) > \rho$. Then the problem (1.1) and (1.6) admits a bounded strong time periodic solution (n, c, u, π) with $n, c \geqslant 0, n$ is not a constant state, and

$$\begin{aligned}
&n \in L_T^\infty(Q) \cap L_T^\infty(\mathbb{R}^+, H^1(\Omega)) \cap W_2^{2,1}(Q_T),\\
&\tilde{c} \in L_T^\infty(\mathbb{R}^+, H^2(\Omega) \cap W^{1,\infty}(\Omega)) \cap W_2^{3,1}(Q_T), \quad \tilde{c}_t \in L_T^\infty(\mathbb{R}^+, L^2(\Omega)),\\
&u \in L_T^\infty(\mathbb{R}^+, H_\sigma^1(\Omega)) \cap W_2^{2,1}(Q_T),\\
&\pi \in L_T^2(\mathbb{R}^+, H^1(\Omega)).
\end{aligned}$$

In particular, if $a, \nabla\phi \in C_T^{\alpha,\alpha/2}(\overline{\Omega} \times \mathbb{R}^+)$, then we also have that (n, c, u, π) is a classical solution with

$$n, c, u \in C_T^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times \mathbb{R}^+), \quad \pi \in C_T^{1+\alpha,\alpha/2}(\overline{\Omega} \times \mathbb{R}^+).$$

In dimension 3, we consider the chemotaxis-Stokes system, and have the following result.

THEOREM 1.2. *Assume $N = 3, \tau = 0$, (1.4) and (1.5) hold, $a, \nabla\varphi \in L_T^\infty(Q), a(x,t)$ has a positive lower bound, that is there exists a positive constant ρ such that $a(x,t) > \rho$. Then when μ/χ^2 is appropriately large, the problem (1.1) and (1.6) admits a bounded strong time periodic solution (n, c, u, π) with $n, c \geqslant 0, n$ is not a constant state, and*

$$\begin{aligned}
&n \in L_T^\infty(Q) \cap L_T^\infty(\mathbb{R}^+, H^1(\Omega)) \cap W_2^{2,1}(Q_T),\\
&\tilde{c} \in L_T^\infty(\mathbb{R}^+, H^2(\Omega) \cap W^{1,\infty}(\Omega)) \cap W_2^{3,1}(Q_T), \quad \tilde{c}_t \in L_T^\infty(\mathbb{R}^+, L^2(\Omega)),\\
&u \in L_T^\infty(\mathbb{R}^+, H_\sigma^1(\Omega)) \cap W_2^{2,1}(Q_T),\\
&\pi \in L_T^2(\mathbb{R}^+, H^1(\Omega)).
\end{aligned}$$

In particular, if $a, \nabla\phi \in C_T^{\alpha,\alpha/2}(\overline{\Omega} \times \mathbb{R}^+)$, then we also have that (n, c, u, π) is a classical solution with

$$n, c, u \in C_T^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times \mathbb{R}^+), \quad \pi \in C_T^{1+\alpha,\alpha/2}(\overline{\Omega} \times \mathbb{R}^+).$$

2. Preliminaries

We first give some notations, which will be used throughout this paper.

Notations: $\|\cdot\|_{L^p} := \|\cdot\|_{L^p(\Omega)}$; $f \in L_T^p(\mathbb{R}^+; \mathcal{X}) \Longleftrightarrow f$ is a time periodic function with period T, and $f \in L^p((0,T); \mathcal{X})$; $f \in C_T^{\alpha,\beta}(\Omega \times R^+) \Longleftrightarrow f$ is a time periodic function with period T and $f \in C^{\alpha,\beta}(\Omega \times R^+)$; $C_{0,\sigma}^\infty(\Omega)$ denotes the set of all $C^\infty(\Omega)$-real functions $\phi = (\phi_1, \cdots, \phi_n)$ with compact support in Ω, such that $\mathrm{div}\phi = 0$. The closure of $C_{0,\sigma}^\infty(\Omega)$ with respect to norm L^r is denoted by $L_\sigma^r(\Omega)$. By [**6**], each $u \in L^r$ has a unique decomposition

$$u = v + \nabla p, \quad v \in L_\sigma^r, \nabla p \in G^r,$$

with $G^r = \{\nabla p; \nabla p \in L^r; p \in L_{loc}^r\}$, and the projection $P : L^r(\Omega) \to L_\sigma^r(\Omega)$ is called Helmholtz projection. Let $A\omega := -P\Delta\omega$, then A generates a bounded analytic semigroup $\{e^{-tA}\}_{t\geqslant 0}$ on L_σ^r, and the time periodic solution u of (1.1) can be expressed

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as

$$u = \int_{-\infty}^{t} e^{-(t-s)A} P(-u \cdot \nabla u + n(s) \nabla \varphi(s)) \, \mathrm{d}s. \tag{2.1}$$

For more details, please refer to [**5**, **12**].

Next, we show the following two lemmas, which will be used throughout this paper.

Lemma 2.1. *Let $T > 0, a > 0, \sigma \geqslant 0$, and suppose that $f : \mathbb{R}^+ \to [0, \infty)$ is absolutely continuous, f, h are time periodic functions with period T, and f satisfies*

$$f(t) - f(t_0) + a \int_{t_0}^{t} f^{1+\sigma}(s) \, \mathrm{d}s \leqslant \int_{t_0}^{t} h(s) \, \mathrm{d}s, \text{ for any } 0 \leqslant t_0 < t, \tag{2.2}$$

where $0 \leqslant f, h \in L^1_T(\mathbb{R}^+)$ and

$$\int_0^T h(s) \, \mathrm{d}s \leqslant \beta.$$

Then we have

$$\sup_{t \in (0,T)} f(t) + a \int_0^T f(t) \, \mathrm{d}t \leqslant \left(\frac{\beta}{aT} \right)^{1/(1+\sigma)} + 2\beta. \tag{2.3}$$

Proof. Taking $t_0 = 0, t = T$ in (2.2), we obtain

$$\int_0^T f^{1+\sigma}(t) \, \mathrm{d}t \leqslant \frac{\beta}{a}.$$

Using mean value theorem of integrals, there exists $t^* \in (0, T)$ such that

$$f(t^*) \leqslant \left(\frac{\beta}{aT} \right)^{1/(1+\sigma)}.$$

Then by (2.2), we obtain

$$\sup_{t \in (0,T)} f(t) = \sup_{t \in (t^*, t^*+T)} f(t) \leqslant f(t^*) + \beta \leqslant \left(\frac{\beta}{aT} \right)^{1/(1+\sigma)} + \beta.$$

Then (2.3) is proved. $\square$

LEMMA 2.2. *Let $T > 0, a > 0, \sigma > 0$, and suppose that $f : \mathbb{R}^+ \to [0, \infty)$ is absolutely continuous, f, g, h are time periodic functions with period T, and satisfies*

$$f(t) - f(t_0) + a\int_{t_0}^{t} f^{1+\sigma}(s)\,\mathrm{d}s \leqslant \int_{t_0}^{t} g(s)f(s)\,\mathrm{d}s + \int_{t_0}^{t} h(s)\,\mathrm{d}s \text{ for any } 0 \leqslant t_0 < t, \tag{2.4}$$

where $g(t), h(t) \geqslant 0$ with $g, h \in L^1_T(\mathbb{R}^+)$ and

$$\int_0^T g(s)\,\mathrm{d}s \leqslant \alpha, \quad \int_0^T h(s)\,\mathrm{d}s \leqslant \beta.$$

Then we have

$$\sup_{t\in(0,T)} f(t) + a\int_0^T f^{1+\sigma}(t)\,\mathrm{d}t \leqslant C, \tag{2.5}$$

where C is a constant depending only on a, α, β, T. While, if $a = 0$ in (2.4), and

$$\int_0^T f(s)\,\mathrm{d}s \leqslant \gamma.$$

Then we also have

$$\sup_{t\in(0,T)} f(t) \leqslant C, \tag{2.6}$$

where C is a constant depending only on γ, α, β, T.

Proof. Taking $t_0 = 0$, $t = T$ in (2.4), we obtain

$$a\int_0^T f^{1+\sigma}(t)\,\mathrm{d}t \leqslant \int_0^T h(t)\,\mathrm{d}t + \int_0^T g(t)f(t)\,\mathrm{d}t \leqslant \beta + \alpha \sup_{t\in(0,T)} f(t). \tag{2.7}$$

Using mean value theorem of integrals, there exists $t^* \in (0, T)$ such that

$$f(t^*) \leqslant \left(\frac{\beta}{aT} + \frac{\alpha}{aT}\sup_{t\in(0,T)} f(t)\right)^{1/(1+\sigma)}.$$

By (2.4), we see that for any $t \in [t^*, t^* + T)$

$$f(t) \leqslant f(t^*) + \int_{t^*}^{t} g(s)f(s)\,\mathrm{d}s + \beta.$$

By Gronwall's inequality, for any $t \in [t^*, t^* + T)$,

$$\begin{aligned} f(t) &\leqslant (f(t^*) + \beta)\int_{t^*}^{t} g(s)\,\mathrm{d}s \leqslant \left(\frac{\beta}{aT} + \frac{\alpha}{aT}\sup_{t\in(0,T)} f(t)\right)^{1/(1+\sigma)} e^{\alpha} + \beta e^{\alpha} \\ &\leqslant \left(\frac{\beta}{aT}\right)^{1/(1+\sigma)} e^{\alpha} + \left(\frac{\alpha}{aT}\sup_{t\in(0,T)} f(t)\right)^{1/(1+\sigma)} e^{\alpha} + \beta e^{\alpha} \\ &\leqslant \frac{1}{2}\sup_{t\in(0,T)} f(t) + C. \end{aligned}$$

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By the periodicity of f, it implies that

$$\sup_{t\in(0,T)} f(t) \leqslant 2C,$$

combining with (2.7), we obtain (2.5). By

$$\int_0^T f(s)\,\mathrm{d}s \leqslant \gamma,$$

there exists $t_0 \in (0,T)$ such that

$$f(t_0) \leqslant \frac{\gamma}{T}.$$

Then similar to the proof above, we obtain (2.6). □

By [**8**, **20**], we also have the following lemma.

LEMMA 2.3. *Assume that $f \in L^p_T(\mathbb{R}^+; L^p(\Omega))$. Then the following problem*

$$\begin{cases} u_t - \Delta u + u = f(x,t), \\ \frac{\partial u}{\partial \mathbf{n}}\big|_{\partial\Omega} = 0 \end{cases} \tag{2.8}$$

admits a unique strong time periodic solution $u \in W^{2,1}_p(Q_T)$, and

$$\int_0^T \|u\|^p_{W^{2,1}_p}\,\mathrm{d}s \leqslant C \int_0^T \|f\|^p_{L^p}\,\mathrm{d}s, \tag{2.9}$$

where C is a positive constant.

In fact, by [**20**], we see that if f is Höder continuous, then (2.8) admits a classical time periodic solution. We take $f_k \in C^{\alpha,\alpha/2}(\overline{\Omega} \times R^+)$ such that $f_k \to f$ in $L^p(Q_T)$, and the corresponding solution is denoted by u_k. By [**8**],

$$\int_0^T \|u_k\|^p_{W^{2,1}_p}\,\mathrm{d}s \leqslant C \int_0^T \|f_k\|^p_{L^p}\,\mathrm{d}s \leqslant \tilde{C}.$$

Letting $k \to \infty$ (take a subsequence if necessary), then

$$u_k \rightharpoonup u, \quad \text{in } \; W^{2,1}_p(Q_T),$$

and $u \in W^{2,1}_p(Q_T)$ is a strong time periodic solution of (2.8), thus the existence is proved. And (2.9) is a direct result of [**8**]. The uniqueness is easy to be proved by (2.9). In fact, let u_1, u_2 be two time periodic solutions of (2.8), then $u_1 - u_2$ is a time periodic solution of (2.8) with $f = 0$. Using (2.9),

$$\int_0^T \|u_1 - u_2\|^p_{W^{2,1}_p}\,\mathrm{d}s \leqslant 0,$$

it implies that $u_1 = u_2$ a.e. in Q_T.

3. Existence and Regularity of Time Periodic Solutions for the Linearized Problem in Dimension 2 and 3

Note that the boundary condition of oxygen concentration c is inhomogeneous, and the standard Neumann heat semigroup argument can not be applied directly. So, we first make a transformation. Let

$$\tilde{c} = e^{-g_1}(c - g_2).$$

Then we have

$$\left.\frac{\partial \tilde{c}}{\partial \nu}\right|_{\partial\Omega} = -e^{-g_1}\left.\frac{\partial g_2}{\partial \nu}\right|_{\partial\Omega} = 0$$

since $\partial g_2/\partial\nu|_{\partial\Omega} = 0$. And the problem (1.1) and (1.6) is transformed into

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \chi\nabla\cdot(e^{g_1}n\nabla\tilde{c} + e^{g_1}n\tilde{c}\nabla g_1 + n\nabla g_2) + \mu n(a(x,t) - n), \text{ in } Q,\\ \tilde{c}_t - \Delta\tilde{c} + (u - 2\nabla g_1)\nabla\tilde{c} = (|\nabla g_1|^2 + \Delta g_1 - n - u\nabla g_1 - g_{1t})\tilde{c}\\ \quad + e^{-g_1}(\Delta g_2 - u\nabla g_2 - n g_2 - g_{2t}), \text{ in } Q,\\ u_t + \tau u\cdot\nabla u = \Delta u - \nabla\pi + n\nabla\varphi, \text{ in } Q,\\ \nabla\cdot u = 0, \text{ in } Q,\\ \left.\frac{\partial n}{\partial\nu} - \chi e^{g_1}n\tilde{c}\frac{\partial g_1}{\partial\nu}\right|_{\partial\Omega} = \frac{\partial\tilde{c}}{\partial\nu}\Big|_{\partial\Omega} = 0, \quad u|_{\partial\Omega} = 0. \end{cases} \tag{3.1}$$

Consider the linearized problem

$$\begin{cases} u_t - \Delta u + \nabla\pi + \tau\hat{u}\cdot\nabla u = \eta\hat{n}\nabla\varphi, \text{ in } Q,\\ \nabla\cdot u = 0, \text{ in } Q,\\ u|_{\partial\Omega} = 0, \end{cases} \tag{3.2}$$

where $\eta\in[0,1]$ is a constant. By [**8**], we have

LEMMA 3.1. *Assume that* $\hat{u}\in L^4_T(\mathbb{R}^+, L^4_\sigma(\Omega)), \hat{n}\in L^2_T(\mathbb{R}^+, L^2(\Omega))$. *Then when* $N = 2$ *with* $\tau = 1$, *or* $N = 3$ *with* $\tau = 0$, (3.2) *admits a unique strong time periodic solution* u *with* $u\in L^\infty((0,T), H^1_\sigma(\Omega))\cap L^2((0,T), H^2_\sigma(\Omega))$, *and* $u_t\in L^2((0,T), L^2_\sigma(\Omega))$.

For the above solution u, we consider the following linear problem for any $\eta\in[0,1]$.

$$\begin{cases} c_t - \Delta c + u\cdot\nabla c + (1-\eta)c = -\hat{n}_+ c, \text{ in } Q,\\ \frac{\partial c}{\partial\nu}\Big|_{\partial\Omega} = \eta\frac{\partial g_1(x,t)}{\partial\nu}(c(x,t) - g_2(x,t)). \end{cases} \tag{3.3}$$

Letting

$$\tilde{c} = e^{-\eta g_1}(c - g_2).$$

Then (3.3) is equivalent to

$$\begin{cases} \tilde{c}_t - \Delta\tilde{c} + (u - 2\eta\nabla g_1)\cdot\nabla\tilde{c} + (1 - \eta + \hat{n}_+)\tilde{c} = \eta(\eta|\nabla g_1|^2 + \Delta g_1 - u\nabla g_1 - g_{1t})\tilde{c}\\ \quad + e^{-\eta g_1}(\Delta g_2 - u\nabla g_2 - \hat{n}_+ g_2 - g_{2t} - g_2), \text{ in } Q,\\ \frac{\partial\tilde{c}}{\partial\nu}\Big|_{\partial\Omega} = 0. \end{cases} \tag{3.4}$$

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For (3.4), we have

Lemma 3.2. *Assume $\hat{n} \in L^2_T(\mathbb{R}^+, H^1(\Omega))$. Let u be the time periodic solution of the problem (3.2). Then (3.4) (or (3.3)) admits a unique strong time periodic solution $\tilde{c}$ with $c \geqslant 0, \tilde{c} \in L^\infty_T(\mathbb{R}^+ \times \Omega) \cap L^\infty_T(\mathbb{R}^+, H^2(\Omega)) \cap L^2_T(\mathbb{R}^+, H^3(\Omega))$, and $\tilde{c}_t \in L^\infty_T(\mathbb{R}^+, L^2(\Omega))$.*

Proof. It is easy to see that $\|g_2\|_{L^\infty}$ and 0 are the upper and lower periodic solutions of (3.3) respectively, then by an iterative process, see for example [**33**], the problem (3.3) admits a time periodic solution c with

$$0 \leqslant c \leqslant \|g_2\|_{L^\infty}, \tag{3.5}$$

which implies that the solution $\tilde{c}$ of (3.4) is bounded.

In what follows, to obtain further regularity estimates, for simplicity, we may assume that the solution $\tilde{c}$ is sufficiently smooth, otherwise, we can approximate u, $\hat{n}_+$ with a sequence of sufficiently smooth functions $u_k, \hat{n}_k$ such that the corresponding solutions $\tilde{c}_k$ are sufficiently smooth, and the following energy estimates can be obtained through an approximate process.

Multiplying the first equation of (3.4) by $\tilde{c}$, then integrating it over $\Omega \times (0, T)$, and noticing that $\tilde{c}$ is periodic and bounded, then

$$\int_0^T \int_\Omega |\nabla \tilde{c}|^2 \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_\Omega (1 - \eta + \hat{n}_+)|\tilde{c}|^2 \,\mathrm{d}x \,\mathrm{d}t \leqslant C \left(1 + \int_0^T \int_\Omega |u|^2 \,\mathrm{d}x \,\mathrm{d}t\right).$$

By lemma 3.1, it gives

$$\int_0^T \int_\Omega |\nabla \tilde{c}|^2 \,\mathrm{d}x \,\mathrm{d}t \leqslant C. \tag{3.6}$$

Multiplying the first equation of (3.4) by $\Delta \tilde{c}$, then integrating it over $\Omega \times (t_0, t)$ for any $0 \leqslant t_0 < t$, using lemma 3.1, we obtain

$$\begin{aligned}
&\frac{1}{2} \int_\Omega |\nabla \tilde{c}(x,t)|^2 \,\mathrm{d}x - \frac{1}{2} \int_\Omega |\nabla \tilde{c}(x,t_0)|^2 \,\mathrm{d}x + \int_{t_0}^t \int_\Omega |\Delta \tilde{c}|^2 \,\mathrm{d}x \,\mathrm{d}s \\
&\quad \leqslant \int_{t_0}^t \|u\|_{L^4} \|\nabla \tilde{c}\|_{L^4} \|\Delta \tilde{c}\|_{L^2} \,\mathrm{d}s \\
&\qquad + \frac{1}{4} \int_{t_0}^t \int_\Omega |\Delta \tilde{c}|^2 \,\mathrm{d}x \,\mathrm{d}s + C \int_{t_0}^t \int_\Omega (1 + |\hat{n}_+|^2 + |u|^2) \,\mathrm{d}x \,\mathrm{d}s \\
&\quad \leqslant C \int_{t_0}^t \|u\|_{H^1} (1 + \|\Delta \tilde{c}\|_{L^2}^{1/2}) \|\Delta \tilde{c}\|_{L^2} \,\mathrm{d}s \\
&\qquad + \frac{1}{4} \int_{t_0}^t \int_\Omega |\Delta \tilde{c}|^2 \,\mathrm{d}x + C \int_{t_0}^t \int_\Omega (1 + |\hat{n}_+|^2 + |u|^2) \,\mathrm{d}x \,\mathrm{d}s \\
&\quad \leqslant \frac{1}{2} \int_{t_0}^t \int_\Omega |\Delta \tilde{c}|^2 \,\mathrm{d}x \,\mathrm{d}s + C \int_{t_0}^t \int_\Omega (1 + |\hat{n}_+|^2) \,\mathrm{d}x \,\mathrm{d}s.
\end{aligned}$$

By (3.6) and lemma 2.2, we arrive at

$$\sup_{t\in(0,T)}\int_\Omega |\nabla\tilde{c}|^2\,\mathrm{d}x+\int_0^T\int_\Omega |\Delta\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant C. \tag{3.7}$$

Similarly, it is also easy to get

$$\int_0^T\int_\Omega |\tilde{c}_t|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant C.$$

Applying ∇ to the first equation of (3.3), multiplying the resulting equation with $-\nabla\Delta\tilde{c}$, then integrating it over $\Omega\times(t_0,t)$ for any $0\leqslant t_0<t$, and using lemma 3.1, (3.7), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\Delta\tilde{c}(x,t)|^2\,\mathrm{d}x-\frac{1}{2}\int_\Omega |\Delta\tilde{c}(x,t_0)|^2\,\mathrm{d}x\\
&\quad+\int_{t_0}^t\int_\Omega |\nabla\Delta\tilde{c}|^2\,\mathrm{d}x+(1-\eta)\int_\Omega |\Delta\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\leqslant\int_{t_0}^t\int_\Omega \nabla((u-2\eta\nabla g_1)\cdot\nabla\tilde{c})\nabla\Delta\tilde{c}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad-\eta\int_{t_0}^t\int_\Omega \nabla((\eta|\nabla g_1|^2+\Delta g_1-u\nabla g_1-g_{1t})\tilde{c})\nabla\Delta\tilde{c}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\int_{t_0}^t\int_\Omega \nabla(\hat{n}_+\tilde{c})\nabla\Delta\tilde{c}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad-\int_{t_0}^t\int_\Omega \nabla\left(e^{-\eta g_1}(\Delta g_2-u\nabla g_2-\hat{n}_+g_2-g_{2t}-g_2)\right)\nabla\Delta\tilde{c}\,\mathrm{d}x\,\mathrm{d}s\\
&\leqslant\int_{t_0}^t\big((C+\|u\|_{L^4})\|\nabla^2\tilde{c}\|_{L^4}\|\nabla\Delta\tilde{c}\|_{L^2}\\
&\quad+(C+\|\nabla u\|_{L^4})\|\nabla\tilde{c}\|_{L^4}\|\nabla\Delta\tilde{c}\|_{L^2}+\|\nabla\tilde{c}\|_{L^4}\|\hat{n}\|_{L^4}\|\nabla\Delta\tilde{c}\|_{L^2}\big)\,\mathrm{d}s\\
&\quad+\int_{t_0}^t(\|\tilde{c}\|_{L^\infty}\|\nabla\hat{n}\|_{L^2}\|\nabla\Delta\tilde{c}\|_{L^2}\\
&\quad+C(1+\|u\|_{H^1}+\|\hat{n}_+\|_{H^1}+\|u\|_{H^1}\|\nabla\tilde{c}\|_{L^4}+\|\tilde{c}\|_{H^1})\|\nabla\Delta\tilde{c}\|_{L^2})\,\mathrm{d}s\\
&\leqslant C\int_{t_0}^t(\|\tilde{c}\|_{L^\infty}\|\nabla\Delta\tilde{c}\|_{L^2}+\|\tilde{c}\|_{L^\infty}^{1-\alpha}\|\nabla\Delta\tilde{c}\|_{L^2}^{1+\alpha})\,\mathrm{d}s\\
&\quad+\int_{t_0}^t((C+\|\nabla u\|_{H^1})\|\nabla\tilde{c}\|_{H^1}+\|\nabla\tilde{c}\|_{H^1}\|\hat{n}\|_{H^1})\|\nabla\Delta\tilde{c}\|_{L^2}\,\mathrm{d}s\\
&\quad+\int_{t_0}^t(\|\tilde{c}\|_{L^\infty}\|\nabla\hat{n}\|_{L^2}\|\nabla\Delta\tilde{c}\|_{L^2}\\
&\quad+C(1+\|u\|_{H^1}+\|\hat{n}_+\|_{H^1}+\|u\|_{H^1}\|\nabla\tilde{c}\|_{L^4}+\|\tilde{c}\|_{H^1})\|\nabla\Delta\tilde{c}\|_{L^2})\,\mathrm{d}s
\end{aligned}$$

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$$\leqslant \frac{1}{4}\int_{t_0}^{t}\int_{\Omega}|\nabla\Delta\tilde{c}|^2\,\mathrm{d}x\,\mathrm{d}s + C\int_{t_0}^{t}(1+\|\tilde{c}\|_{H^2}^2+\|\hat{n}\|_{H^1}^2+\|u\|_{H^2}^2)\,\mathrm{d}s$$
$$+ C\int_{t_0}^{t}\|\Delta\tilde{c}\|_{L^2}^2(\|\hat{n}\|_{H^1}^2+\|u\|_{H^2}^2)\,\mathrm{d}s,$$

where $\alpha = (8-N)/(12-2N)$. By (3.7), lemmas 3.1 and 2.2, we further have

$$\sup_{t\in(0,T)}\int_{\Omega}|\Delta\tilde{c}|^2\,\mathrm{d}x + \int_0^T\|\tilde{c}\|_{H^3}^2\,\mathrm{d}t \leqslant C. \tag{3.8}$$

By the equation (3.4) and the inequality (3.8), it also gives

$$\sup_{t\in(0,T)}\int_{\Omega}|\tilde{c}_t|^2\,\mathrm{d}x \leqslant C. \tag{3.9}$$

Next, we show the uniqueness. Let c_1, c_2 be the two time periodic solutions of the problem (3.3), and denote $c = c_1 - c_2$, then

$$\begin{cases} c_t - \Delta c + u\cdot\nabla c + (1-\eta)c = -\hat{n}_+ c, \text{ in } Q, \\ \frac{\partial c}{\partial\nu}\big|_{\partial\Omega} = \eta\frac{\partial g_1(x,t)}{\partial\nu}c(x,t). \end{cases} \tag{3.10}$$

Multiplying the first equation of (3.10) by c, and integrating it over $\Omega\times(0,T)$, we obtain

$$-\eta\int_0^T\int_{\partial\Omega}\frac{\partial g_1(x,t)}{\partial\nu}|c|^2\,\mathrm{d}s\,\mathrm{d}t + \int_0^T\int_{\Omega}|\nabla c|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\int_{\Omega}(1-\eta+\hat{n}_+)|c|^2\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Noticing that $-\partial g_1(x,t)/\partial\nu|_{\partial\Omega} = a_1 > 0$, then when $0\leqslant\eta<1$, $c=0$ in $\Omega\times(0,T)$; while if $\eta=1$, then the above inequality implies that $c|_{\partial\Omega}=0$, then by poincaré inequality, we also have

$$\int_0^T\int_{\Omega}|c|^2\,\mathrm{d}x\,\mathrm{d}t = 0,$$

and the uniqueness is proved. □

For the above obtained solutions u, $\tilde{c}$, we consider the following linear parabolic problem.

$$\begin{cases} n_t + u\cdot\nabla n + An = \Delta n - \eta\chi\nabla\cdot(e^{g_1}n\nabla\tilde{c} + e^{g_1}n\tilde{c}\nabla g_1 + n\nabla g_2) \\ \qquad\qquad + \eta(\mu a(x,t)+A)\hat{n}_+ - \mu\hat{n}_+ n, \text{ in } Q, \\ \frac{\partial n}{\partial\nu} - \eta\chi e^{g_1}n\tilde{c}\frac{\partial g_1}{\partial\nu}\Big|_{\partial\Omega} = 0, \end{cases} \tag{3.11}$$

where $\eta\in[0,1]$, A is a sufficiently large positive constant that ensures the uniqueness of time periodic solutions. Similar to the problem (3.4), the existence of time periodic solutions can be easily obtained, and we give the regularity estimates.

LEMMA 3.3. *Assume* $a(x,t) \in L_T^\infty(Q), \hat{n} \in L_T^\infty(\mathbb{R}^+, L^2(\Omega)) \cap L_T^2(\mathbb{R}^+, H^1(\Omega)), u, \tilde{c}$ *are the time periodic solutions of the problem* (3.2) *and* (3.4), *respectively, then for sufficiently large* $A > 0$, *the problem* (3.11) *admits a unique strong time periodic solution* n *with* $n \geqslant 0, n \in L_T^\infty(\mathbb{R}^+, H^1(\Omega)) \cap W_2^{2.1}(Q_T)$.

Proof. The existence of time periodic solutions can be easily obtained by a fixed point method. That is, define a Poincaré map from $n(x,0)$ to $n(x,T)$, the time-periodic solution is then identified as a fixed point of this Poincaré map. We omit the proof, and only give the regularity estimate and the proof of uniqueness.

Next, we show that $n \geqslant 0$. Let us examine the set $J(t) = \{x \in \Omega; n(x,t) < 0\}$, we assume that $J(t)$ is a differentiable submanifold. Noticing that $n = 0$ and $\partial n/\partial \nu \geqslant 0$ on $\partial\{J(t)\} \setminus \partial\Omega$; $\partial n/\partial \nu = \eta\chi e^{g_1} n\tilde{c}(\partial g_1/\partial \nu)$ on $\partial J(t) \cap \partial\Omega$, then by a direct integration on $J(t) \times (0,T)$ gives

$$\begin{aligned} 0 \geqslant & -\int_0^T \int_{\partial\{J(t)\}} \left(\frac{\partial n}{\partial \nu} - \eta\chi e^{g_1} n\tilde{c}\frac{\partial g_1}{\partial \nu}\right) \mathrm{d}s\,\mathrm{d}t + \int_0^T \int_{J(t)} (A + \mu\hat{n}_+) n\,\mathrm{d}x\,\mathrm{d}t \\ & = \eta(\mu a + A) \int_0^T \int_{J(t)} \hat{n}_+\,\mathrm{d}x\,\mathrm{d}t \geqslant 0. \end{aligned}$$

It implies that

$$\int_0^T \int_{J(t)} n\,\mathrm{d}x\,\mathrm{d}t = 0,$$

namely $n \geqslant 0$. While if $J(t)$ is not a regular submanifold, we can construct a sufficiently smooth approximating sequence $\{u_k, \tilde{c}_k, \hat{n}_k\}$ of $(u, \tilde{c}, \hat{n}_+)$ such that the corresponding approximating solutions n_k satisfying that $n_k(\cdot,t)$ are continuously differentiable. Thus, the sets $J_k(t)$ are measurable and $\partial J_k(t)$ are differentiable submanifolds. Then the above result can be obtained by letting $k \to \infty$.

Similarly, in what follows, we still assume that the solution n is sufficiently smooth. Otherwise, the following estimates can be obtained by an approximating process.

By a direct integration over $\Omega \times (t_0, t)$ with $t_0 < t < t_0 + T$ for (3.11), it is easy to obtain

$$\int_\Omega n(x,t)\,\mathrm{d}x - \int_\Omega n(x,t_0)\,\mathrm{d}x + A\int_{t_0}^t \int_\Omega n\,\mathrm{d}x\,\mathrm{d}s + \mu\int_{t_0}^t \int_\Omega \hat{n}_+ n\,\mathrm{d}x\,\mathrm{d}s \leqslant C,$$

which implies

$$\sup_{t\in\mathbb{R}^+} \int_\Omega n\,\mathrm{d}x + \int_0^T \int_\Omega n\,\mathrm{d}x\,\mathrm{d}t \leqslant C. \tag{3.12}$$

Testing the first equation of (3.11) by $n\chi_{[t_0,t]}$ for any $t_0 < t < t_0 + T$, where $\chi_{[t_0,t]}$ is the characteristic function of the segment $[t_0, t]$, and using lemma 3.2, we see that

$$\frac{1}{2}\int_\Omega n^2(x,t)\,\mathrm{d}x - \frac{1}{2}\int_\Omega n^2(x,t_0)\,\mathrm{d}x + \int_{t_0}^t\int_\Omega |\nabla n|^2\,\mathrm{d}x\,\mathrm{d}s + \int_{t_0}^t\int_\Omega (\mu\hat{n}_+ + A)n^2\,\mathrm{d}x\,\mathrm{d}s$$
$$= \eta\chi\int_{t_0}^t\int_\Omega (e^{g_1}\nabla\tilde{c} + e^{g_1}\tilde{c}\nabla g_1 + \nabla g_2)n\nabla n\,\mathrm{d}x\,\mathrm{d}s + \eta\int_{t_0}^t\int_\Omega (\mu a + A)\hat{n}_+ n\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant C\int_{t_0}^t \|\nabla\tilde{c}\|_{L^4}\|n\|_{L^4}\|\nabla n\|_{L^2}\,\mathrm{d}s + C\int_{t_0}^t \|n\|_{L^2}\|\nabla n\|_{L^2}\,\mathrm{d}s + C\int_{t_0}^t \|\hat{n}\|_{L^2}\|n\|_{L^2}\,\mathrm{d}s$$
$$\leqslant \frac{1}{4}\int_{t_0}^t \|\nabla n\|_{L^2}^2\,\mathrm{d}s + \frac{A}{2}\int_{t_0}^t \|n\|_{L^2}^2\,\mathrm{d}s + C_1\int_{t_0}^t \|n\|_{L^4}^2\,\mathrm{d}s + C_2\int_{t_0}^t \|\hat{n}\|_{L^2}^2\,\mathrm{d}s. \tag{3.13}$$

By Gagliardo–Nirenberg interpolation inequality, we have

$$C_1\|n\|_{L^4}^2 \leqslant C_3\|\nabla n\|_{L^2}^{2\alpha}\|\|n\|_{L^1}^{2(1-\alpha)} + C_4\|n\|_{L^1}^2 \leqslant \frac{1}{4}\|\nabla n\|_{L^2}^2 + C_5,$$

where $\alpha = 3N/2(2+N) < 2$. Substituting this inequality into (3.13) gives

$$\frac{1}{2}\int_\Omega n^2(x,t)\,\mathrm{d}x - \frac{1}{2}\int_\Omega n^2(x,t_0)\,\mathrm{d}x + \frac{1}{2}\int_{t_0}^t\int_\Omega |\nabla n|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \frac{A}{2}\int_{t_0}^t\int_\Omega n^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C + C_2\int_{t_0}^t \|\hat{n}\|_{L^2}^2\,\mathrm{d}s. \tag{3.14}$$

Using lemma 2.1, we get

$$\sup_{t\in\mathbb{R}^+}\int_\Omega n^2\,\mathrm{d}x + \int_0^T\int_\Omega (|\nabla n|^2 + n^2)\,\mathrm{d}x\,\mathrm{d}t \leqslant C. \tag{3.15}$$

Multiplying the first equation of (3.11) by $-e^{-\eta\chi\tilde{c}e^{g_1}}\Delta(ne^{-\eta\chi\tilde{c}e^{g_1}})$, then integrating it over $\Omega\times(t_0,t)$, noticing that $\partial(ne^{-\eta\chi\tilde{c}e^{g_1}})/\partial\nu|_{\partial\Omega} = 0$, using lemmas 3.1 and 3.2 and (3.15), we arrive at

$$\frac{1}{2}\int_\Omega \left|\nabla\left(n(x,t)e^{-\eta\chi\tilde{c}(x,t)e^{g_1(x,t)}}\right)\right|^2\mathrm{d}x - \frac{1}{2}\int_\Omega \left|\nabla\left(n(x,t_0)e^{-\eta\chi\tilde{c}(x,t_0)e^{g_1(x,t_0)}}\right)\right|^2\mathrm{d}x$$
$$+ \int_{t_0}^t\int_\Omega \left|\Delta\left(ne^{-\eta\chi\tilde{c}e^{g_1}}\right)\right|^2\mathrm{d}x\,\mathrm{d}s + A\int_{t_0}^t\int_\Omega \left|\nabla\left(ne^{-\eta\chi\tilde{c}e^{g_1}}\right)\right|^2\mathrm{d}x\,\mathrm{d}s$$
$$= \eta\chi\int_{t_0}^t\int_\Omega e^{g_1}n(\tilde{c}_t + \tilde{c}g_{1t})e^{-\eta\chi\tilde{c}e^{g_1}}\Delta\left(ne^{-\eta\chi\tilde{c}e^{g_1}}\right)\mathrm{d}x\,\mathrm{d}s$$
$$+ \eta\chi\int_{t_0}^t\int_\Omega e^{-\eta\chi\tilde{c}e^{g_1}}\nabla$$

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$$
\begin{aligned}
&\cdot (n\nabla g_2)\Delta\left(ne^{-\eta\chi\tilde{c}e^{g_1}}\right)\,\mathrm{d}x\,\mathrm{d}s + \int_{t_0}^{t}\int_\Omega \Delta\left(ne^{-\eta\chi\tilde{c}e^{g_1}}\right)u\nabla(ne^{-\eta\chi\tilde{c}e^{g_1}})\,\mathrm{d}x\,\mathrm{d}s\\
&+\eta\chi\int_{t_0}^{t}\int_\Omega \Delta\left(ne^{-\eta\chi\tilde{c}e^{g_1}}\right)une^{-\eta\chi\tilde{c}e^{g_1}}(e^{g_1}\nabla\tilde{c}+\tilde{c}\nabla e^{g_1})\,\mathrm{d}x\,\mathrm{d}s\\
&-\eta\chi\int_{t_0}^{t}\int_\Omega (e^{g_1}\nabla\tilde{c}+e^{g_1}\tilde{c}\nabla g_1)\,\Delta\left(ne^{-\eta\chi\tilde{c}e^{g_1}}\right)\nabla\left(ne^{-\eta\chi\tilde{c}e^{g_1}}\right)\,\mathrm{d}x\,\mathrm{d}s\\
&-\eta\int_{t_0}^{t}\int_\Omega (\mu a+A)\hat{n}_+e^{-\eta\chi\tilde{c}e^{g_1}}\Delta\left(ne^{-\eta\chi\tilde{c}e^{g_1}}\right)\,\mathrm{d}x\,\mathrm{d}s\\
&+\mu\int_{t_0}^{t}\int_\Omega \hat{n}_+ne^{-\eta\chi\tilde{c}e^{g_1}}\Delta\left(ne^{-\eta\chi\tilde{c}e^{g_1}}\right)\,\mathrm{d}x\,\mathrm{d}s\\
\leqslant{}& \chi\int_{t_0}^{t}\|\Delta\left(ne^{-\chi\tilde{c}e^{g_1}}\right)\|_{L^2}\|ne^{-\chi\tilde{c}e^{g_1}}\|_{L^\infty}\|e^{g_1}(\tilde{c}_t+\tilde{c}g_{1t})\|_{L^2}\,\mathrm{d}s\\
&+C\int_{t_0}^{t}\|\Delta\left(ne^{-\chi\tilde{c}e^{g_1}}\right)\|_{L^2}(\|ne^{-\chi\tilde{c}e^{g_1}}\|_{L^2}+\|\nabla n\|_{L^2})\,\mathrm{d}s\\
&+C\int_{t_0}^{t}\|\Delta\left(ne^{-\chi\tilde{c}e^{g_1}}\right)\|_{L^2}\|\nabla\left(ne^{-\chi\tilde{c}e^{g_1}}\right)\|_{L^4}\|u\|_{L^4}\,\mathrm{d}s\\
&+C\int_{t_0}^{t}\|\Delta\left(ne^{-\chi\tilde{c}e^{g_1}}\right)\|_{L^2}\|ne^{-\chi\tilde{c}e^{g_1}}\|_{L^\infty}\|u\|_{L^4}\|\tilde{c}\|_{W^{1,4}}\,\mathrm{d}s\\
&+C\int_{t_0}^{t}\|\Delta\left(ne^{-\chi\tilde{c}e^{g_1}}\right)\|_{L^2}\Big(\|\nabla\left(ne^{-\chi\tilde{c}e^{g_1}}\right)\|_{L^4}\|\tilde{c}\|_{W^{1,4}}\\
&+\|\hat{n}\|_{L^2}+\|ne^{-\chi\tilde{c}e^{g_1}}\|_{L^\infty}\|\hat{n}\|_{L^2}\Big)\,\mathrm{d}s\\
\leqslant{}& C\int_{t_0}^{t}\|\Delta\left(ne^{-\chi\tilde{c}e^{g_1}}\right)\|_{L^2}^{1+\frac{N}{4}}\|ne^{-\chi\tilde{c}e^{g_1}}\|_{L^2}^{1-\frac{N}{4}}\,\mathrm{d}s\\
&+C\int_{t_0}^{t}\|\Delta\left(ne^{-\chi\tilde{c}e^{g_1}}\right)\|_{L^2}(\|ne^{-\chi\tilde{c}e^{g_1}}\|_{L^2}+\|\nabla n\|_{L^2})\,\mathrm{d}s\\
&+C\int_{t_0}^{t}\|\Delta\left(ne^{-\chi\tilde{c}e^{g_1}}\right)\|_{L^2}^{3/2+N/8}\|ne^{-\chi\tilde{c}e^{g_1}}\|_{L^2}^{(4-N)/8}\,\mathrm{d}s\\
&+C\int_{t_0}^{t}\|\Delta\left(ne^{-\chi\tilde{c}e^{g_1}}\right)\|_{L^2}\,\mathrm{d}s\\
\leqslant{}& \frac{1}{2}\int_{t_0}^{t}\|\Delta\left(ne^{-\chi\tilde{c}e^{g_1}}\right)\|_{L^2}^2\,\mathrm{d}s + C\int_{t_0}^{t}\|\nabla n\|_{L^2}^2\,\mathrm{d}s + C.
\end{aligned}
$$

Taking advantage of (3.15) and lemma 2.1 gives

$$
\sup_{t\in\mathbb{R}^+}\left\|\nabla\left(ne^{-\chi\tilde{c}e^{g_1}}\right)\right\|_{L^2}^2+\int_0^T\left\|\Delta\left(ne^{-\chi\tilde{c}e^{g_1}}\right)\right\|_{L^2}^2\leqslant C. \tag{3.16}
$$

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Combining with lemmas 3.1 and 3.2, it also implies that

$$\sup_{t\in\mathbb{R}^+} \|\nabla n\|_{L^2}^2 + \int_0^T \|\Delta n\|_{L^2}^2 \leqslant C. \tag{3.17}$$

By this inequality and recalling the equation (3.11), we further have

$$\int_0^T \int_\Omega |n_t|^2 \,\mathrm{d}x \,\mathrm{d}t \leqslant C.$$

Next, we show the uniqueness. Let n_1, n_2 be the two periodic solutions of (3.11), denote $n = n_1 - n_2$. Testing the corresponding equation with n, we have

$$\begin{aligned}
&\int_0^T \int_\Omega |\nabla n|^2 \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_\Omega (\mu\hat{n}_+ + A)n^2 \,\mathrm{d}x \,\mathrm{d}t \\
&= \eta\chi \int_0^T \int_\Omega (e^{g_1}\nabla\tilde{c} + e^{g_1}\tilde{c}\nabla g_1 + \nabla g_2) n\nabla n \,\mathrm{d}x \,\mathrm{d}t \\
&\leqslant \chi \int_0^T \|e^{g_1}\|_{L^\infty} \|\nabla\tilde{c}\|_{L^4} \|n\|_{L^4} \|\nabla n\|_{L^2} \,\mathrm{d}t \\
&\quad + \chi \int_0^T \|e^{g_1}\tilde{c}\nabla g_1 + \nabla g_2\|_{L^\infty} \|n\|_{L^2} \|\nabla n\|_{L^2} \,\mathrm{d}t \\
&\leqslant C_1\chi \int_0^T \|e^{g_1}\|_{L^\infty} \|\nabla\tilde{c}\|_{L^4} \|n\|_{L^2}^{1-N/4} \|\nabla n\|_{L^2}^{1+N/4} \,\mathrm{d}t \\
&\quad + C_2\chi \int_0^T \|e^{g_1}\|_{L^\infty} \|\nabla\tilde{c}\|_{L^4} \|n\|_{L^2} \|\nabla n\|_{L^2} \,\mathrm{d}t \\
&\quad + \chi \int_0^T \|e^{g_1}\tilde{c}\nabla g_1 + \nabla g_2\|_{L^\infty} \|n\|_{L^2} \|\nabla n\|_{L^2} \,\mathrm{d}t \\
&\leqslant \frac{1}{2} \int_0^T \|\nabla n\|_{L^2}^2 \,\mathrm{d}t + M \int_0^T \|n\|_{L^2}^2 \,\mathrm{d}t,
\end{aligned}$$

which implies that

$$\frac{1}{2} \int_0^T \int_\Omega |\nabla n|^2 \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_\Omega (\mu\hat{n}_+ + A - M)n^2 \,\mathrm{d}x \,\mathrm{d}t \leqslant 0. \tag{3.18}$$

When $A > M$, we have

$$\int_0^T \int_\Omega (|\nabla n|^2 + n^2) \,\mathrm{d}x \,\mathrm{d}t = 0,$$

which implies the uniqueness, and this lemma is proved. □

4. Existence of Time Periodic Solutions in Dimension 2

In the following two sections, for simplicity, we let C, C_i, $\tilde{C}$ denote some different positive constants, if there is no special explanation, which depend at most on Ω, T, χ, g_1, g_2, μ, $\nabla\varphi$.

In this section, we show the existence of nontrivial time periodic solutions in dimension 2. We define an operator $\mathcal{F}$ as follows

$$\mathcal{F} : \mathcal{G} \times [0,1] \to \mathcal{G},$$
$$\mathcal{F}(\hat{u}, \hat{n}, \eta) = (u, n),$$

where

$$\mathcal{G} = \{(u,n); u \in L_T^4(\mathbb{R}^+; L_\sigma^4(\Omega)), n \in L_T^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L_T^2(\mathbb{R}^+; H^1(\Omega))\}$$

endowed with the norm

$$\|(u,n)\|_{\mathcal{G}} = \left(\int_0^T \|u(\cdot,t)\|_{L^4}^4\right)^{1/4} + \left(\int_0^T \|n(\cdot,t)\|_{H^1}^2 \,\mathrm{d}t\right)^{1/2} + \sup_t \|n(\cdot,t)\|_{L^2},$$

u, n are the time periodic solutions of (3.2) and (3.11) respectively, and $\tilde{c}$ is defined by (3.4). By lemmas 3.1 and 3.3, we have $(u,n) \in \mathcal{D}$, where

$$\mathcal{D} = \{(u,n) \in W_2^{2,1}(Q_T); u \in L_T^\infty(\mathbb{R}^+; H_\sigma^1(\Omega)), n \in L_T^\infty(\mathbb{R}^+; H^1(\Omega))\}.$$

By Aubin–Lions lemma [**2**, **21**],

$$\mathcal{D} \hookrightarrow \mathcal{G},$$

and the embedding is compact, it is also easy to show that the operator $\mathcal{F}$ is continuous. Thus, we have

Lemma 4.1. *The operator $\mathcal{F}$ is completely continuous.*

We see that solving problem (3.1) is equivalent to solving the equation

$$U - \mathcal{F}(U, 1) = 0; \quad U = (u,n) \in \mathcal{G}.$$

Next, we show energy estimates to the solutions (u, n, η) of the problem $(u,n) - \mathcal{F}(n, u, \eta) = 0$.

Lemma 4.2. *Let (u, n, η) be a time periodic solution of $(u,n) - \mathcal{F}(n,u,\eta) = 0$. Then there exists a constant C such that*

$$\sup_t \|n(\cdot,t)\|_{L^1} + \int_0^T \|n(\cdot,t)\|_{L^2}^2 \,\mathrm{d}t \leqslant C, \tag{4.1}$$

$$\sup_t \|u(\cdot,t)\|_{H^1}^2 + \int_0^T (\|u(\cdot,t)\|_{H^2}^2 + \|u_t(\cdot,t)\|_{L^2}^2) \,\mathrm{d}t \leqslant C. \tag{4.2}$$

Proof. Noticing that $n \geqslant 0$ by lemma 3.3, replacing $\hat{n}_+$ by n in (3.11), and by a direct integration, we obtain (4.1). Replacing $\hat{u}, \hat{n}$ with u, n in (3.2), multiplying

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this equation by u, then integrating it over $\Omega \times (t_0, t)$ for any $t_0 < t$, and using Poincaré inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |u(x,t)|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega |u(x,t_0)|^2\,\mathrm{d}x + \int_{t_0}^t\int_\Omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad\leqslant \eta\int_{t_0}^t \|\nabla\varphi\|_{L^\infty}\|n\|_{L^2}\|u\|_{L^2}\,\mathrm{d}s\\
&\quad\leqslant C\int_{t_0}^t \|n\|_{L^2}\|\nabla u\|_{L^2}\,\mathrm{d}s \leqslant \frac{1}{2}\int_{t_0}^t \|\nabla u\|_{L^2}^2\,\mathrm{d}s + C\int_{t_0}^t \|n\|_{L^2}^2\,\mathrm{d}s.
\end{aligned}$$

Using lemma 2.1, we obtain

$$\sup_t \int_\Omega |u|^2\,\mathrm{d}x + \int_0^T\int_\Omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \tilde{C}\int_0^T \|n\|_{L^2}^2\,\mathrm{d}t. \tag{4.3}$$

By Gagliardo–Nirenberg interpolation inequality, we see that

$$\|u\|_{L^4}^4 \leqslant C_1\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2 + C_2\|u\|_{L^2}^4,$$

then by (4.3), we get

$$\int_0^T\int_\Omega |u|^4\,\mathrm{d}x\,\mathrm{d}t \leqslant C.$$

Recalling lemma 3.1, then we have (4.2). □

LEMMA 4.3. *Let (u, n, η) be a time periodic solution of $(u, n) - \mathcal{F}(n, u, \eta) = 0$, and $\tilde{c}$ is the solution of (3.4). Then there exists a constant C such that*

$$\sup_t(\|\tilde{c}(\cdot,t)\|_{H^2}^2 + \|\tilde{c}_t(\cdot,t)\|_{L^2}^2) + \int_0^T \|\tilde{c}(\cdot,t)\|_{H^3}^2\,\mathrm{d}t \leqslant C, \tag{4.4}$$

$$\sup_t \|n(\cdot,t)\|_{H^1}^2 + \int_0^T (\|n(\cdot,t)\|_{H^2}^2 + \|n_t(\cdot,t)\|_{L^2}^2)\,\mathrm{d}t \leqslant C. \tag{4.5}$$

Proof. By the proof of lemma 3.2, we see that when $n \in L^2(Q_T)$, we have

$$\sup_{t\in(0,T)} \|\tilde{c}\|_{L^\infty} + \|\tilde{c}\|_{H^1}^2 + \int_0^T\int_\Omega (|\tilde{c}_t|^2 + |\Delta\tilde{c}|^2)\,\mathrm{d}x\,\mathrm{d}t \leqslant C. \tag{4.6}$$

By Gagliardo–Nirenberg interpolation inequality, we see that

$$\|\nabla\tilde{c}\|_{L^4}^4 \leqslant C_1\|\nabla\tilde{c}\|_{L^2}^2\|\Delta\tilde{c}\|_{L^2}^2 + C_2\|\nabla\tilde{c}\|_{L^2}^4,$$

which implies that

$$\int_0^T \|\nabla\tilde{c}(\cdot,t)\|_{L^4}^4\,\mathrm{d}t \leqslant C. \tag{4.7}$$

Multiplying the first equation of (3.11) by n, and integrating it over $\Omega \times (t_0, t)$, then using (4.6) and Gagliardo–Nirenberg interpolation inequality, we see that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega n^2(x,t)\,\mathrm{d}x - \frac{1}{2}\int_\Omega n^2(x,t_0)\,\mathrm{d}x + \int_{t_0}^t\int_\Omega |\nabla n|^2\,\mathrm{d}x + \mu\int_{t_0}^t\int_\Omega n^3\,\mathrm{d}x\,\mathrm{d}s\\
&\quad\leqslant \eta\chi\int_{t_0}^t\int_\Omega (e^{g_1}\nabla\tilde{c} + e^{g_1}\tilde{c}\nabla g_1 + \nabla g_2)n\nabla n\,\mathrm{d}x\,\mathrm{d}s + \eta\mu\int_{t_0}^t\int_\Omega an^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad\leqslant C\int_{t_0}^t \|\nabla\tilde{c}\|_{L^4}\|n\|_{L^4}\|\nabla n\|_{L^2}\,\mathrm{d}s + C\int_{t_0}^t\left(\|n\|_{L^2}\|\nabla n\|_{L^2} + \|n\|_{L^2}^2\right)\mathrm{d}s\\
&\quad\leqslant C_1\int_{t_0}^t (\|\nabla\tilde{c}\|_{L^4}\|n\|_{L^2}^{1/2}\|\nabla n\|_{L^2}^{3/2}\\
&\qquad + \|\nabla\tilde{c}\|_{L^4}\|n\|_{L^2}\|\nabla n\|_{L^2} + \|n\|_{L^2}\|\nabla n\|_{L^2} + \|n\|_{L^2}^2)\,\mathrm{d}s\\
&\quad\leqslant \frac{1}{2}\int_{t_0}^t\|\nabla n\|_{L^2}^2\,\mathrm{d}s + C_2\int_{t_0}^t\|\nabla\tilde{c}\|_{L^4}^4\|n\|_{L^2}^2\,\mathrm{d}s\\
&\qquad + C_3\int_{t_0}^t\|\nabla\tilde{c}\|_{L^4}^2\|n\|_{L^2}^2\,\mathrm{d}s + \frac{\mu}{2}\int_{t_0}^t\|n\|_{L^3}^3\,\mathrm{d}s + C_4.
\end{aligned}\tag{4.8}$$

Taking advantage of (4.7) and lemma 2.2, we obtain

$$\sup_t\int_\Omega n^2\,\mathrm{d}x + \int_0^T\int_\Omega(|\nabla n|^2 + n^3)\,\mathrm{d}x\,\mathrm{d}t \leqslant C. \tag{4.9}$$

Using (4.9), combining with lemmas 3.2 and 3.3, we obtain (4.4) and (4.5). □

LEMMA 4.4. *Let (u, n, η) be the time periodic solution of $(u, n) - \mathcal{F}(n, u, \eta) = 0$, and $\tilde{c}$ is the solution of (3.4). Then*

$$\sup_t \|\tilde{c}(\cdot,t)\|_{W^{1,\infty}} \leqslant C, \tag{4.10}$$

$$\sup_t \|n(\cdot,t)\|_{L^\infty} \leqslant C\sup_t\|n(\cdot,t)\|_{L^2}. \tag{4.11}$$

Proof. Recalling (3.4), we see that

$$\tilde{c}_t - \Delta\tilde{c} + \tilde{c} = F(n,\tilde{c},u),$$

where

$$F(n,\tilde{c},u) = -(u - 2\eta\nabla g_1)\cdot\nabla\tilde{c} + (\eta - n)\tilde{c} + \eta(\eta|\nabla g_1|^2 + \Delta g_1 - u\nabla g_1 - g_{1t})\tilde{c}.$$

Noticing that the time periodic solution of (3.4) can be expressed as follows

$$\tilde{c} = \int_{-\infty}^t e^{(s-t)}e^{-(s-t)\Delta}F\,\mathrm{d}s, \tag{4.12}$$

where $\{e^{t\Delta}\}_{t\geqslant 0}$ is the Neumann heat semigroup in Ω, for more properties of Neumann heat semigroup, please refer to [**27**]. By lemmas 4.2 and 4.3, clearly we

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have

$$\sup_t \|F\|_{L^3} \leqslant C.$$

Then

$$\begin{aligned}\|\nabla\tilde{c}\|_{L^\infty} &\leqslant \int_{-\infty}^{t} e^{(s-t)}\|e^{-(s-t)\Delta}F\|_{L^\infty}\,\mathrm{d}s\\ &\leqslant C_1\int_{-\infty}^{t} e^{-(t-s)}(t-s)^{-1/3-1/2}\|F\|_{L^3}\,\mathrm{d}s\\ &\leqslant C_2\int_{-\infty}^{t} e^{-(t-s)}(t-s)^{-5/6}\,\mathrm{d}s\\ &= C_2\int_{0}^{\infty} e^{-s}s^{-5/6}\,\mathrm{d}s \leqslant \tilde{C}.\end{aligned}$$

Replacing $\hat{n}$ with n in (3.11), multiplying it by rn^{r-1} with $r \geqslant 2$, then integrating it over $\Omega\times(t_0,t)$, and using (4.10), we see that

$$\begin{aligned}&\int_\Omega n^r(x,t)\,\mathrm{d}x - \int_\Omega n^r(x,t_0)\,\mathrm{d}x + r(r-1)\int_{t_0}^{t}\int_\Omega n^{r-2}|\nabla n|^2\,\mathrm{d}x\,\mathrm{d}s\\ &\quad+\int_{t_0}^{t}\int_\Omega(\mu r n^{r+1}+n^r)\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant \eta\chi r(r-1)\int_{t_0}^{t}\int_\Omega(e^{g_1}\nabla\tilde{c}+e^{g_1}\tilde{c}\nabla g_1+\nabla g_2)n^{r-1}\nabla n\,\mathrm{d}x\,\mathrm{d}s\\ &\quad+\int_{t_0}^{t}\int_\Omega(\eta r\mu a+1)n^r\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant \frac{r(r-1)}{4}\int_{t_0}^{t}\int_\Omega n^{r-2}|\nabla n|^2\,\mathrm{d}x\,\mathrm{d}s + Cr^2\int_{t_0}^{t}\int_\Omega n^r\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}\tag{4.13}$$

Noticing that

$$\|n\|_{L^r}^r = \|n^{r/2}\|_{L^2}^2 \leqslant C_1\|\nabla n^{r/2}\|_{L^2}\|n^{r/2}\|_{L^1} + C_2\|n^{r/2}\|_{L^1}^2,$$

then

$$Cr^2\int_{t_0}^{t}\int_\Omega n^r\,\mathrm{d}x\,\mathrm{d}s \leqslant \frac{r(r-1)}{4}\int_{t_0}^{t}\int_\Omega n^{r-2}|\nabla n|^2\,\mathrm{d}x\,\mathrm{d}s + Cr^4\int_{t_0}^{t}\left(\int_\Omega n^{r/2}\,\mathrm{d}x\right)^2\mathrm{d}s.$$

Substituting it into (4.13) yields

$$\int_\Omega n^r(x,t)\,\mathrm{d}x - \int_\Omega n^r(x,t_0)\,\mathrm{d}x + \int_{t_0}^{t}\int_\Omega n^r\,\mathrm{d}x\,\mathrm{d}s \leqslant Cr^4\int_{t_0}^{t}\left(\int_\Omega n^{r/2}\,\mathrm{d}x\right)^2\mathrm{d}s. \tag{4.14}$$

Noticing that n is periodic, by a direct calculation, we obtain

$$\sup_t \|n\|_{L^r}^r \leqslant Cr^4\sup_t \|n\|_{L^{r/2}}^r.$$

Let $r_j = 2^j$, $M_j = \sup_t \|n\|_{L^{r_j}}$. Then

$$M_j \leqslant C^{\sum_{k=2}^{j}(1/2^k)} 2^{\sum_{k=2}^{j}(4k/2^k)} M_1.$$

Letting $j \to \infty$, and (4.11) is obtained. □

Next, let us consider the following problem

$$\begin{cases} n_t + u \cdot \nabla n + An = \Delta n - \chi \nabla \cdot (e^{g_1} n \nabla \tilde{c} + e^{g_1} n \tilde{c} \nabla g_1 + n \nabla g_2) \\ \qquad +(\mu a(x,t) + A)\hat{n}_+ - \mu \hat{n}_+ n + \gamma, \text{ in } Q, \\ \frac{\partial n}{\partial \nu} - \chi e^{g_1} n \tilde{c} \frac{\partial g_1}{\partial \nu}\Big|_{\partial\Omega} = 0. \end{cases} \tag{4.15}$$

Similar to the discussion in lemma 3.3, for given $\hat{u} \in L_T^4(\mathbb{R}^+, L_\sigma^4(\Omega))$, $\hat{n} \in L_T^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L_T^2(\mathbb{R}^+; H^1(\Omega))$, $\gamma \in [0,1]$, $\gamma \in [0,1]$, the problem (3.2), (3.4) and (4.15) admits a unique nonnegative time periodic solution $(u, n) \in \mathcal{D}$. Define

$$\mathcal{T} : \mathcal{G} \times [0,1] \to \mathcal{G},$$
$$\mathcal{T}(\hat{u}, \hat{n}, \gamma) = (u, n).$$

Clearly, the operator $\mathcal{T}$ is completely continuous.

LEMMA 4.5. *Assume that $a(x,t) > \rho > 0$. Then there exists a sufficiently small constant $\sigma > 0$ such that the problem $I - \mathcal{T}(\cdot, \gamma) = 0$ admits no solution $(u, n) \in \mathcal{G}$ with*

$$0 < \|(u,n)\|_{\mathcal{G}} \leqslant \sigma.$$

In particular, there is no solution such that $\|(u,n)\|_{\mathcal{G}} \leqslant \sigma$ for any $\gamma \in (0,1]$, and there is only zero solution with $\|(u,n)\|_{\mathcal{G}} \leqslant \sigma$ for $\gamma = 0$.

Proof. Suppose the contrary, there exists a solution (u, n) such that

$$0 < \|(u,n)\|_{\mathcal{G}} \leqslant \sigma,$$

which implies

$$\sup_t \|n(\cdot,t)\|_{L^2} \leqslant \sigma. \tag{4.16}$$

By lemmas 4.2–4.4, we also have $u \in L_T^\infty(\mathbb{R}^+; H_\sigma^1(\Omega)) \cap W_2^{2,1}(Q_T)$, $\nabla \tilde{c} \in W^{1,\infty}(Q_T)$, $n \in L_T^\infty(\mathbb{R}^+; H^1(\Omega)) \cap W_2^{2,1}(Q_T)$.

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Replacing $\hat{n}$ by n in (4.15), multiplying it by rn^{r-1} with $r \geqslant 2$, then integrating it over $\Omega \times (t_0, t)$ for any $t_0 < t$, and using (4.10), we see that

$$\begin{aligned}
&\int_\Omega n^r(x,t)\,\mathrm{d}x - \int_\Omega n^r(x,t_0)\,\mathrm{d}x + r(r-1)\int_{t_0}^t\int_\Omega n^{r-2}|\nabla n|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_{t_0}^t\int_\Omega (\mu r n^{r+1} + n^r)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad \leqslant \eta\chi r(r-1)\int_{t_0}^t\int_\Omega (e^{g_1}\nabla\tilde{c} + e^{g_1}\tilde{c}\nabla g_1 + \nabla g_2)n^{r-1}\nabla n\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_{t_0}^t\int_\Omega ((\eta r\mu a + 1)n^r + \gamma r n^{r-1})\,\mathrm{d}x\,\mathrm{d}s \\
&\quad \leqslant \frac{r(r-1)}{4}\int_{t_0}^t\int_\Omega n^{r-2}|\nabla n|^2\,\mathrm{d}x\,\mathrm{d}s + Cr^2\int_{t_0}^t\int_\Omega n^r\,\mathrm{d}x\,\mathrm{d}s + \gamma r\int_{t_0}^t\int_\Omega n^{r-1}\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Using Gagliardo–Nirenberg interpolation inequality, we see that

$$Cr^2\|n\|_{L^r}^r = Cr^2\|n^{r/2}\|_{L^2}^2 \leqslant C_1r^2\|\nabla n^{r/2}\|_{L^2}\|n^{r/2}\|_{L^1} + C_2r^2\|n^{r/2}\|_{L^1}^2$$

and

$$\|n\|_{L^{r-1}}^{r-1} = \|n^{r/2}\|_{L^{2(r-1)/r}}^{2(r-1)/r} \leqslant C_3\|\nabla n^{r/2}\|_{L^2}^{(r-2)/r}\|n^{r/2}\|_{L^1} + C_4\|n\|_{L^{r/2}}^{r-1},$$

then we have

$$\begin{aligned}
&\int_\Omega n^r(x,t)\,\mathrm{d}x - \int_\Omega n^r(x,t_0)\,\mathrm{d}x + \frac{3(r-1)}{r}\int_{t_0}^t\int_\Omega |\nabla n^{r/2}|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_{t_0}^t\int_\Omega (\mu r n^{r+1} + n^r)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad \leqslant \frac{1}{2}\int_{t_0}^t \|\nabla n^{r/2}\|_{L^2}^2\,\mathrm{d}s + Cr^4\int_{t_0}^t \|n\|_{L^{r/2}}^r\,\mathrm{d}s \\
&\quad + Cr^2\int_{t_0}^t \|n\|_{L^{r/2}}^{r^2/(r+2)}\,\mathrm{d}s + Cr\int_{t_0}^t \|n\|_{L^{r/2}}^{r-1}\,\mathrm{d}s \\
&\quad \leqslant \frac{1}{2}\int_{t_0}^t \|\nabla n^{r/2}\|_{L^2}^2\,\mathrm{d}s + \tilde{C}r^4\int_{t_0}^t \|n\|_{L^{r/2}}^r\,\mathrm{d}s + \tilde{C}r^2\int_{t_0}^t \|n\|_{L^{r/2}}^{r-2}\,\mathrm{d}s.
\end{aligned}$$

Noticing that $3(r-1)/r \geqslant 3/2$, the above inequality implies

$$\begin{aligned}
&\int_\Omega n^r(x,t)\,\mathrm{d}x - \int_\Omega n^r(x,t_0)\,\mathrm{d}x + \int_{t_0}^t\int_\Omega n^r\,\mathrm{d}x \\
&\quad \leqslant \tilde{C}r^4\int_{t_0}^t \|n\|_{L^{r/2}}^r\,\mathrm{d}s + \tilde{C}r^2\int_{t_0}^t \|n\|_{L^{r/2}}^{r-2}\,\mathrm{d}s.
\end{aligned}$$

By a direct calculation, we obtain

$$\sup_t \|n\|_{L^r}^r \leqslant \hat{C}r^4\sup_t \|n\|_{L^{r/2}}^r + \hat{C}r^2\|n\|_{L^{r/2}}^{r-2}. \tag{4.17}$$

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Let $r_j = 2^j$, $M_j = \max\{1, \sup_t \|n\|_{L^{r_j}}\}$. Using (4.16) and (4.17), we see that for every j, M_j is bounded, and we further have

$$\begin{aligned} M_j &\leqslant \left(\hat{C} r_j^4 M_{j-1}^{r_j} + \hat{C} r_j^2 M_{j-1}^{r_j-2}\right)^{1/r_j} \leqslant (2\hat{C})^{1/2^j} 2^{4j/2^j} M_{j-1} \\ &\leqslant (2\hat{C})^{\sum_{k=2}^{j}(1/2^k)} 2^{\sum_{k=2}^{j}(4k/2^k)} M_1. \end{aligned}$$

Noticing that $\sum_{k=2}^{\infty}(1/2^k)$ and $\sum_{k=2}^{\infty} 4k/2^k$ converge. Letting $j \to \infty$, then we finally have

$$\sup_t \|n\|_{L^\infty} \leqslant C. \tag{4.18}$$

Substituting (4.18) into (4.17), we also have

$$\sup_t \|n\|_{L^r}^r \leqslant C_1 r^4 \sup_t \|n\|_{L^{r/2}}^{r-2}$$

with $C_1 > 1$. Let $r_j = 2^j$, $\tilde{M}_j = \sup_t \|n\|_{L^{r_j}}$, then

$$\tilde{M}_j \leqslant C_1^{1/2^j} 2^{4j/2^j} \tilde{M}_{j-1}^{1-1/2^{j-1}} \leqslant C_1^{\sum_{k=2}^{j}(1/2^k)} 2^{\sum_{k=2}^{j}(4k/2^k)} \tilde{M}_1^{\prod_{k=2}^{j}\left(1-1/2^{k-1}\right)}.$$

Next, we show that

$$S = \prod_{k=2}^{\infty}\left(1 - \frac{1}{2^{k-1}}\right) > 0.$$

Noticing that $\ln(1/S) = \sum_{k=2}^{\infty} \ln(1 + 1/(2^{k-1} - 1))$ converges, which implies that $0 < S < 1$. Letting $j \to \infty$.

$$\sup_t \|n\|_{L^\infty} \leqslant C \sup_t \|n\|_{L^2}^S. \tag{4.19}$$

Combining with (4.16), we obtain that when $\|(u, n)\|_{\mathcal{G}} \leqslant \sigma$,

$$\sup_t \|n\|_{L^\infty} \leqslant C \sup_t \|n\|_{L^2}^S \leqslant \tilde{C}\sigma^S := \tilde{\sigma}. \tag{4.20}$$

Replacing $\hat{n}$ by n in (4.15), taking σ sufficiently small such that $\tilde{\sigma} < \rho$, and by a direct integration over $\Omega \times (0, T)$, we see that

$$\begin{aligned} \int_\Omega n(x, T)\,\mathrm{d}x - \int_\Omega n(x, 0)\,\mathrm{d}x &= \int_0^T \int_\Omega \mu n(a(x, t) - n)\,\mathrm{d}x\,\mathrm{d}t + \gamma|\Omega|T \\ &\geqslant \int_0^T \int_\Omega \mu n(\rho - \tilde{\sigma})\,\mathrm{d}x\,\mathrm{d}t + \gamma|\Omega|T \geqslant \gamma|\Omega|T. \end{aligned} \tag{4.21}$$

which implies that if $\|(u, n)\|_{\mathcal{G}} \leqslant \sigma$ for σ sufficiently small, then there is no periodic solution for any $\gamma \in (0, 1]$, and there is only zero solution for $\gamma = 0$. □

Next, we show the existence of nontrivial time periodic solutions.

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Proof of theorem 1.1. In what follows, we apply the topological degree theory. Let $\hat{B}_R$ be the ball of radius R centred at the origin in $\mathcal{G}$. By lemmas 4.2 and 4.3, there exists a sufficiently large $R > 0$ such that if $(u, n) - \mathcal{F}(u, n, \eta) = 0$, then

$$\|(u, n)\|_{\mathcal{G}} < R.$$

That is

$$(I - \mathcal{F}(\cdot, \eta))(\partial \hat{B}_R) \neq 0, \quad \eta \in [0, 1].$$

Recalling (3.2) and (3.11), it is easy to see that when $\eta = 0$, $(u, n) = (0, 0)$. That is $\mathcal{F}(\cdot, 0) \equiv 0$. Then by the homotopy invariance of topological degree, we have

$$\mathbf{deg}(I - \mathcal{F}(\cdot, 1), \hat{B}_R, 0) = \mathbf{deg}(I - \mathcal{F}(\cdot, 0), \hat{B}_R, 0) = \mathbf{deg}(I, \hat{B}_R, 0) = 1. \tag{4.22}$$

By lemma 4.5, for σ appropriately small, we also have

$$\mathbf{deg}(I - \mathcal{F}(\cdot, 1), \hat{B}_\sigma, 0) = \mathbf{deg}(I - \mathcal{T}(\cdot, 0), \hat{B}_\sigma, 0) = \mathbf{deg}(I - \mathcal{T}(\cdot, 1), \hat{B}_\sigma, 0) = 0. \tag{4.23}$$

Combining (4.22) and (4.23), we see that

$$\begin{aligned} \mathbf{deg}(I - \mathcal{F}(\cdot, 1), \hat{B}_R \setminus \hat{B}_\sigma, 0) = \mathbf{deg}(I - \mathcal{F}(\cdot, 1), \hat{B}_R, 0) \\ - \mathbf{deg}(I - \mathcal{F}(\cdot, 1), \hat{B}_\sigma, 0) = 1. \end{aligned} \tag{4.24}$$

It implies that the problem (3.1) admits a time periodic solution $(n, \tilde{c}, u, \pi)$ with $0 \leqslant n \neq 0$, $u \neq 0$. By lemmas 4.2–4.4, we also have

$$\begin{aligned} &n \in L^\infty_T(Q) \cap L^\infty_T(\mathbb{R}^+, H^1(\Omega)) \cap W^{2,1}_2(Q_T), \\ &\tilde{c} \in L^\infty_T(\mathbb{R}^+, H^2(\Omega) \cap W^{1,\infty}(\Omega)) \cap W^{3,1}_2(Q_T), \\ &u \in L^\infty_T(\mathbb{R}^+, H^1_\sigma(\Omega)) \cap W^{2,1}_2(Q_T), \end{aligned}$$

which implies that $(n, \tilde{c}, u)$ is a strong time periodic solution of (3.1).

Recalling (2.1), there exists $\lambda > 0$, such that

$$\begin{aligned} \|u\|_{L^\infty} = &\int_{-\infty}^t \|e^{-(t-s)A} P(n(s)\nabla\varphi(s))\|_{L^\infty} \,\mathrm{d}s \\ &+ \int_{-\infty}^t \|e^{-(t-s)A} P(-\nabla \cdot (u(s) \otimes u(s)))\|_{L^\infty} \,\mathrm{d}s \\ \leqslant &\int_{-\infty}^t e^{-\lambda(t-s)}(t-s)^{-1/2} \|n(s)\nabla\varphi\|_{L^2} \,\mathrm{d}s \\ &+ \int_{-\infty}^t e^{-\lambda(t-s)}(t-s)^{-1/2-1/4} \|u(s) \otimes u(s)\|_{L^4} \,\mathrm{d}s \\ \leqslant &\sup_t \|n\|_{L^2} \|\nabla\varphi\|_{L^\infty} \int_{-\infty}^t e^{-\lambda(t-s)}(t-s)^{-1/2} \,\mathrm{d}s \\ &+ \sup_t \|u\|^2_{L^8} \int_{-\infty}^t e^{-\lambda(t-s)}(t-s)^{-3/4} \,\mathrm{d}s \\ \leqslant &C. \end{aligned}$$

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For any $p > 2$, we have

$$
\begin{aligned}
\|A^{1/2}u\|_{L^p} &\leqslant \int_{-\infty}^{t} \|A^{1/2}e^{-(t-s)A}P(n(s)\nabla\varphi(s))\|_{L^p}\,\mathrm{d}s \\
&\quad + \int_{-\infty}^{t} \|A^{1/2}e^{-(t-s)A}P(-u(s)\cdot\nabla u(s))\|_{L^p}\,\mathrm{d}s \\
&\leqslant \int_{-\infty}^{t} e^{-\lambda(t-s)}(t-s)^{-1/2}\|n(s)\nabla\varphi\|_{L^p}\,\mathrm{d}s \\
&\quad + \int_{-\infty}^{t} e^{-\lambda(t-s)}(t-s)^{-1+1/p}\|u(s)\cdot\nabla u(s)\|_{L^2}\,\mathrm{d}s \\
&\leqslant \int_{-\infty}^{t} e^{-\lambda(t-s)}(t-s)^{-1/2}\|n(s)\nabla\varphi\|_{L^p}\,\mathrm{d}s \\
&\quad + \int_{-\infty}^{t} e^{-\lambda(t-s)}(t-s)^{-1+1/p}\|u(s)\|_{L^\infty}\|\nabla u(s)\|_{L^2}\,\mathrm{d}s \\
&\leqslant C\int_{0}^{+\infty} e^{-\lambda s}s^{-1/2}\,\mathrm{d}s + C\int_{0}^{+\infty} e^{-\lambda s}s^{-1+1/p}\,\mathrm{d}s \\
&\leqslant \tilde{C}.
\end{aligned}
\tag{4.25}
$$

Combining with (4.2), and we have $u \in C^{\beta,\beta/2}(\overline{\Omega}\times\mathbb{R}^+)$ for some $\beta \in (0,\alpha]$, see for example [**9**]. To use the Neumann heat semigroup theory for the homogeneous Neumann boundary problem, we let $\tilde{n} = ne^{-\chi\tilde{c}e^{g_1}}$, then $\partial\tilde{n}/\partial\nu|_{\partial\Omega} = 0$, and we have

$$
\tilde{n}_t - \Delta\tilde{n} + \tilde{n} = \mathbf{F_1}\cdot\nabla\tilde{n} + F_2, \tag{4.26}
$$

where

$$
\begin{aligned}
&\mathbf{F_1} = \nabla(\tilde{c}e^{g_1}) - u - \chi\nabla g_2, \\
&F_2 = -\chi\tilde{n}u\cdot\nabla(\tilde{c}e^{g_1}) - \chi\tilde{n}e^{g_1}(\tilde{c}_t + \tilde{c}g_{1t}) - \chi\tilde{n}\Delta g_2 - \chi^2\tilde{n}\nabla g_2\nabla(\tilde{c}e^{g_1}).
\end{aligned}
$$

Thus for any $p > 2$, we have

$$
\begin{aligned}
\|\nabla\tilde{n}\|_{L^p} &\leqslant \int_{-\infty}^{t} e^{-(t-s)}\|\nabla e^{-(t-s)\Delta}(\mathbf{F_1}\cdot\nabla\tilde{n} + F_2)\|_{L^p}\,\mathrm{d}s \\
&\leqslant C\int_{-\infty}^{t} e^{-(t-s)}(t-s)^{-1+1/p}\|(\mathbf{F_1}\cdot\nabla\tilde{n} + F_2\|_{L^2}\,\mathrm{d}s \\
&\leqslant C\int_{-\infty}^{t} e^{-(t-s)}(t-s)^{-1+1/p}(\|(\mathbf{F_1}\|_{L^\infty}\|\nabla\tilde{n}\|_{L^2} + \|F_2\|_{L^2})\,\mathrm{d}s \\
&\leqslant C\int_{0}^{+\infty} e^{-s}s^{-1+1/p}\,\mathrm{d}s \leqslant \hat{C}.
\end{aligned}
\tag{4.27}
$$

which implies that

$$
\sup_t \|\nabla\tilde{n}(\cdot,t)\|_{L^p} \leqslant C, \text{ for any } p > 2.
$$

Similar to the proof of [9], we have $n \in C^{\beta,\beta/2}(\overline{\Omega} \times \mathbb{R}^+)$. Recalling (3.1), by the standard parabolic regularity theory, we successively obtain $\tilde{c} \in C^{2+\beta,1+\beta/2}(\overline{\Omega} \times \mathbb{R}^+)$, $u \in C^{2+\beta,1+\beta/2}(\overline{\Omega} \times \mathbb{R}^+)$, $n \in C^{2+\beta,1+\beta/2}(\overline{\Omega} \times \mathbb{R}^+)$. Using these results, and going back to the equations (3.1), we further have $n, \tilde{c}, u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times \mathbb{R}^+)$, and theorem 1.1 is proved. □

5. Existence of Time Periodic Solutions in Dimension 3

In this section, we show the existence nontrivial time periodic solutions in dimension 3 for $\tau = 0$. We define an operator $\mathcal{F}$ as follows.

$$\begin{aligned} \mathcal{F} : \mathcal{E} \times [0,1] &\to \mathcal{E}, \\ \mathcal{F}(\hat{n}, \eta) &= n, \end{aligned}$$

where

$$\mathcal{E} = \{n; n \in L^\infty_T(\mathbb{R}^+; L^2(\Omega)) \cap L^2_T(\mathbb{R}^+; H^1(\Omega))\},$$

n is the time periodic solution of (3.11), and u, $\tilde{c}$ are defined by (3.2) and (3.4) respectively. By lemmas 3.1 and 3.3, we have $n \in \tilde{\mathcal{E}}$, where

$$\tilde{\mathcal{E}} = \{n \in L^\infty_T(\mathbb{R}^+; H^1(\Omega)) \cap W^{2,1}_2(Q_T)\}.$$

Then by Aubin–Lions lemma,

$$\tilde{\mathcal{E}} \hookrightarrow \mathcal{E}.$$

and the embedding is compact, it is also easy to show that the operator $\mathcal{F}$ is continuous. Thus, we have

Lemma 5.1. *The operator $\mathcal{F}$ is completely continuous.*

We see that solving problem (3.1) is equivalent to solving the equation

$$n - \mathcal{F}(n, 1) = 0, \quad n \in \mathcal{E}.$$

Next, we show the energy estimates for the solution (n, η) of the problem $n - \mathcal{F}(n, \eta) = 0$.

Lemma 5.2. *Let (n, η) be the time periodic solution of $n - \mathcal{F}(n, \eta) = 0$. Then*

$$\int_0^T \|n(\cdot,t)\|^2_{L^2} \,\mathrm{d}t \leqslant C, \tag{5.1}$$

$$\sup_t \|u(\cdot,t)\|^2_{H^1} + \int_0^T (\|u(\cdot,t)\|^2_{H^2} + \|u_t(\cdot,t)\|^2_{L^2}) \,\mathrm{d}t \leqslant C, \tag{5.2}$$

where the two constants C are independent of μ, χ.

Proof. Replacing $\hat{n}$ with n in (3.11), and testing it by 1, we see that

$$\mu\int_0^T\int_\Omega n^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + A\int_0^T\int_\Omega n(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \eta\mu\int_0^T\int_\Omega a(x,t)n(x,t)\,\mathrm{d}x\,\mathrm{d}t + \eta A\int_0^T\int_\Omega n(x,t)\,\mathrm{d}x\,\mathrm{d}t,$$

which implies that

$$\int_0^T\int_\Omega n^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant \eta\int_0^T\int_\Omega a(x,t)n(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{2}\int_0^T\int_\Omega n^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + C,$$

then we have (5.1). (5.2) is a direct result of lemma 3.1 and (5.1). □

LEMMA 5.3. *Let (n,η) be the time periodic solution of $n-\mathcal{F}(n,\eta)=0,\tilde{c}$ be the solution of (3.4). Then when μ/χ^2 is appropriately large, we have*

$$\sup_{t\in(0,T)}(\|\tilde{c}\|_{L^\infty} + \|\tilde{c}\|_{H^2}^2 + \|\tilde{c}_t\|_{L^2}^2) + \int_0^T \|\tilde{c}(\cdot,t)\|_{H^3}^2\,\mathrm{d}t \leqslant C, \tag{5.3}$$

$$\sup_t \|n(\cdot,t)\|_{H^1}^2 + \int_0^T(\|n(\cdot,t)\|_{H^2}^2 + \|n_t(\cdot,t)\|_{L^2}^2)\,\mathrm{d}t \leqslant C, \tag{5.4}$$

where C are positive constants.

Proof. By the proof of lemma 3.2, we see that when $n\in L^2(Q_T)$, we have

$$\sup_{t\in(0,T)}(\|\tilde{c}\|_{L^\infty} + \|\tilde{c}\|_{H^1}^2) + \int_0^T\int_\Omega(|\tilde{c}_t|^2 + |\Delta\tilde{c}|^2)\,\mathrm{d}x\,\mathrm{d}t \leqslant C, \tag{5.5}$$

where C is independent of μ, χ. Recalling (5.1), there exists $t_0\in[0,T]$ such that

$$\int_\Omega n^2(x,t_0)\,\mathrm{d}x = \frac{1}{T}\int_0^T\int_\Omega n^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant C. \tag{5.6}$$

Multiplying (3.11) by n, and integrating it over $\Omega\times(t_0,t)$ for any $t\in(t_0,t_0+T)$, we obtain

$$\frac{1}{2}\int_\Omega n^2(x,t)\,\mathrm{d}x - \frac{1}{2}\int_\Omega n^2(x,t_0)\,\mathrm{d}x + \int_{t_0}^t\int_\Omega|\nabla n|^2\,\mathrm{d}x\,\mathrm{d}s + \int_{t_0}^t\int_\Omega(\mu n^3 + An^2)\,\mathrm{d}x\,\mathrm{d}s$$
$$= \eta\chi\int_{t_0}^t\int_\Omega(e^{g_1}\nabla\tilde{c} + e^{g_1}\tilde{c}\nabla g_1 + \nabla g_2)n\nabla n\,\mathrm{d}x\,\mathrm{d}s + \eta\int_{t_0}^t\int_\Omega(\mu a + A)n^2\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant \frac{1}{2}\int_{t_0}^t\int_\Omega|\nabla n|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{\chi^2}{2}\int_{t_0}^t\int_\Omega(e^{g_1}\nabla\tilde{c} + e^{g_1}\tilde{c}\nabla g_1 + \nabla g_2)^2 n^2\,\mathrm{d}x\,\mathrm{d}s$$

$$
\begin{aligned}
&+\eta\int_{t_0}^{t}\int_\Omega(\mu a+A)n^2\,\mathrm{d}x\,\mathrm{d}s\\
&\leqslant \frac{1}{2}\int_{t_0}^{t}\int_\Omega|\nabla n|^2\,\mathrm{d}x\,\mathrm{d}s+\frac{\chi^2}{2}\int_{t_0}^{t}\int_\Omega e^{2g_1}|\nabla\tilde{c}|^2n^2\,\mathrm{d}x\,\mathrm{d}s+\frac{\mu}{4}\int_{t_0}^{t}\int_\Omega n^3\,\mathrm{d}x\,\mathrm{d}s\\
&+\eta A\int_{t_0}^{t}\int_\Omega n^2\,\mathrm{d}x\,\mathrm{d}s+C\\
&\leqslant \frac{1}{2}\int_{t_0}^{t}\int_\Omega|\nabla n|^2\,\mathrm{d}x\,\mathrm{d}s+C\frac{\chi^6}{\mu^2}\int_{t_0}^{t}\int_\Omega|\nabla\tilde{c}|^6\,\mathrm{d}x\,\mathrm{d}s+\frac{\mu}{2}\int_{t_0}^{t}\int_\Omega n^3\,\mathrm{d}x\,\mathrm{d}s\\
&+\eta A\int_{t_0}^{t}\int_\Omega n^2\,\mathrm{d}x\,\mathrm{d}s+C\\
&\leqslant \frac{1}{2}\int_{t_0}^{t}\int_\Omega|\nabla n|^2\,\mathrm{d}x\,\mathrm{d}s+C_1\frac{\chi^6}{\mu^2}\|\tilde{c}\|_{L^\infty}^3\int_{t_0}^{t}\int_\Omega|\Delta\tilde{c}|^3\,\mathrm{d}x\,\mathrm{d}s+\frac{\mu}{2}\int_{t_0}^{t}\int_\Omega n^3\,\mathrm{d}x\,\mathrm{d}s\\
&+\eta A\int_{t_0}^{t}\int_\Omega n^2\,\mathrm{d}x\,\mathrm{d}s+C_2,
\end{aligned}
$$

where C_1 is independent of μ,χ, C_2 depends on μ,χ. Taking advantage of (5.5), we obtain

$$
\begin{aligned}
&\sup_t\int_\Omega n^2\,\mathrm{d}x+\int_0^T\int_\Omega|\nabla n|^2\,\mathrm{d}x\,\mathrm{d}t+\mu\int_0^T\int_\Omega n^3\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leqslant C_3\frac{\chi^6}{\mu^2}\int_0^T\int_\Omega|\Delta\tilde{c}|^3\,\mathrm{d}x+C_4,
\end{aligned}\tag{5.7}
$$

where C_3 is independent of μ,χ, C_4 depends on μ,χ. Next, by (3.4) and lemma 2.3, we also have

$$
\begin{aligned}
\int_0^T\int_\Omega|\Delta\tilde{c}|^3\,\mathrm{d}x &\leqslant C\int_0^T\int_\Omega|(u-2\eta\nabla g_1)\cdot\nabla\tilde{c}|^3\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\int_\Omega|(\eta|\nabla g_1|^2+\Delta g_1-u\nabla g_1-g_{1t})\tilde{c}|^3\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\int_\Omega|e^{-\eta g_1}(\Delta g_2-u\nabla g_2-ng_2-g_{2t}-g_2)|^3\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\int_\Omega(|n\tilde{c}|^3+|\tilde{c}|^3)\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant C\int_0^T\int_\Omega n^3\,\mathrm{d}x\,\mathrm{d}t+C\int_0^T\int_\Omega|u|^3\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+C\int_0^T(\|u\|_{L^6}^3\|\nabla\tilde{c}\|_{L^6}^3+\|\nabla\tilde{c}\|_{L^3}^3)\,\mathrm{d}t+C,
\end{aligned}\tag{5.8}
$$

where these constants C are independent of μ, χ. Noting that $\tilde{c}$ is bounded, then in dimension 3, we have

$$\|u\|_{L^6} \leqslant C\|u\|_{H^1}, \quad \|\nabla\tilde{c}\|_{L^3}^3 \leqslant C\|\nabla\tilde{c}\|_{L^2}^{3/2}\|\Delta\tilde{c}\|_{L^2}^{3/2} + C, \quad \|\nabla\tilde{c}\|_{L^6}^3 \leqslant C\|\tilde{c}\|_{L^\infty}^{3/2}\|\Delta\tilde{c}\|_{L^3}^{3/2} + C,$$

then by (5.2) and (5.5), we obtain

$$\int_0^T\int_\Omega |\Delta\tilde{c}|^3\,\mathrm{d}x \leqslant C_5\int_0^T\int_\Omega n^3\,\mathrm{d}x\,\mathrm{d}t + C_6, \tag{5.9}$$

where C_5, C_6 are independent of μ, χ. Combining with (5.7), we obtain

$$\sup_t\int_\Omega n^2\,\mathrm{d}x + \int_0^T\int_\Omega |\nabla n|^2\,\mathrm{d}x\,\mathrm{d}t + \mu\int_0^T\int_\Omega n^3\,\mathrm{d}x\,\mathrm{d}t \leqslant C_7\frac{\chi^6}{\mu^2}\int_0^T\int_\Omega n^3\,\mathrm{d}x + C_8, \tag{5.10}$$

where C_7 is independent of μ, χ, C_8 depends on μ, χ. Taking χ^2/μ appropriately small such that $C_7(\chi^6/\mu^3) \leqslant 1/2$, and we have

$$\sup_t\int_\Omega n^2\,\mathrm{d}x + \int_0^T\int_\Omega |\nabla n|^2\,\mathrm{d}x\,\mathrm{d}t + \mu\int_0^T\int_\Omega n^3\,\mathrm{d}x\,\mathrm{d}t \leqslant C_9, \tag{5.11}$$

where C_9 depends on μ, χ. Taking advantage of (5.11), and recalling lemmas 3.2 and 3.3, we obtain (5.3)–(5.4). □

Similar to lemma 4.4, we also have

Lemma 5.4. *Let (u, n, η) be the time periodic solution of $n - \mathcal{F}(n, \eta) = 0$, $\tilde{c}$ be the solution of (3.4). Then when μ/χ^2 is appropriately large, we have*

$$\sup_t\|\tilde{c}(\cdot,t)\|_{W^{1,\infty}} \leqslant C, \tag{5.12}$$

$$\sup_t\|n(\cdot,t)\|_{L^\infty} \leqslant C\sup_t\|n(\cdot,t)\|_{L^2}^S, \tag{5.13}$$

where $S \in (0,1)$ is a constant.

Proof. Recalling (4.12), and using (5.2)–(5.4), we see that

$$\begin{aligned}\|\nabla\tilde{c}\|_{L^\infty} &\leqslant \int_{-\infty}^t e^{(s-t)}\|e^{-(s-t)\Delta}F\|_{L^\infty}\,\mathrm{d}s\\ &\leqslant \int_{-\infty}^t e^{-(t-s)}(t-s)^{-3/8-1/2}\|F\|_{L^4}\,\mathrm{d}s\end{aligned}$$

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$$
\begin{aligned}
&\leqslant C \int_{-\infty}^{t} e^{-(t-s)}(t-s)^{-7/8} \\
&\qquad \times (\|u\|_{L^6}\|\nabla\tilde{c}\|_{L^{12}} + \|\nabla\tilde{c}\|_{L^4} + (1+\|u\|_{L^6}+\|n\|_{L^6})\|\tilde{c}\|_{L^\infty})\,\mathrm{d}s \\
&\leqslant C \sup_t(\|\nabla\tilde{c}(\cdot,t)\|_{L^{12}}+1)\int_0^\infty e^{-s}s^{-7/8}\,\mathrm{d}s \\
&\leqslant C \sup_t(\|\nabla\tilde{c}(\cdot,t)\|_{L^6}^{1/2}\|\nabla\tilde{c}(\cdot,t)\|_{L^\infty}^{1/2}+1)\int_0^\infty e^{-s}s^{-7/8}\,\mathrm{d}s \\
&\leqslant \tilde{C}\left(1+\sup_t\|\nabla\tilde{c}(\cdot,t)\|_{L^\infty}^{1/2}\right),
\end{aligned}
$$

which implies (5.12). Then similar to the proof of (4.11) and (5.13) is obtained. □

Define

$$
\mathcal{T}:\mathcal{E}\times[0,1]\to\mathcal{E},\quad \mathcal{T}(\hat{n},\gamma)=n,
$$

where n is the solution of (4.15). Similarly, for $\hat{n}\in\mathcal{E}$, we also have $n\in\tilde{\mathcal{E}}$. Thus, $\mathcal{T}$ is completely continuous. Similar to lemma 4.5, we also have

Lemma 5.5. *Assume that $a(x,t)>\rho>0$. Then there exists a sufficiently small constant $\sigma>0$ such that the problem $I-\mathcal{T}(\cdot,\gamma)=0$ admits no solution $n\in\mathcal{G}$ with*

$$
0<\|n\|_{\mathcal{G}}\leqslant\sigma.
$$

In particular, there is no solution with $\|n\|_{\mathcal{G}}\leqslant\sigma$ for any $\gamma\in(0,1]$, and there is only zero solution with $\|n\|_{\mathcal{G}}\leqslant\sigma$ for $\gamma=0$.

Then completely similar to the proof of theorems 1.1 and 1.2 can be proved.

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