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THE FOCUSING NONLINEAR SCHRÖDINGER EQUATION ON THE QUARTER PLANE WITH TIME-PERIODIC BOUNDARY CONDITION: A RIEMANN–HILBERT APPROACH

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Abstract We consider the focusing nonlinear Schrödinger equation on the quarter plane. Initial data vanish at infinity while boundary data are time-periodic. The main goal of this paper is to introduce a Riemann–Hilbert problem whose solution gives the solution of our initial–boundary-value problem. This is a preliminary step to obtain uniform long-time asymptotics for the solution of this equation using both the stationary phase and the Deift–Zhou methods.

Keywords: nonlinear Schrödinger equation; inverse scattering; Riemann–Hilbert problem; initial–boundary-value problem; time-periodic boundary condition

AMS 2000 *Mathematics subject classification:* Primary 35Q55
Secondary 37K15; 35Q15

1. Introduction

Since the discovery of the inverse (spectral) transform method to solve initial and boundary value (IBV) problems on the whole line with vanishing conditions for the Korteweg–de Vries (KdV), nonlinear Schrödinger (NLS), sine-Gordon (sG) and other integrable equations several attempts have been made to extend this method to more difficult IBV problems: Dirichlet, Neumann, Robin conditions are prescribed on the half-line or on the interval. The main difficulty associated to this type of IBV problems for nonlinear as well as linear PDEs is the presence of unknown boundary values [**15**]. Such a problem is described by constructing a relation among all boundary values in terms of the given ones. This relation is called 'global relation' and in the nonlinear case this relation takes the form of a set of nonlinear and non-local equations involving the unknown boundary values (see [**5**–**9**]). This paper is devoted to a complete and rigorous study of an IBV problem related to integrable focusing nonlinear Schrödinger equation in a quarter plane $x > 0$, $t > 0$ with time periodic boundary data, and initial condition vanishing at infinity. To make this program we use a Riemann–Hilbert approach proposed in [**14**], where was developed a new transform method to solve IBV problems for linear and nonlinear integrable equations based on the fact that these type of equations has a remarkable property: they possess a Lax pair. The resulting spectral analysis allows the solution to be represented in a simple form. The initial and boundary conditions must satisfy a certain global constraint relation for the IBV problem to be well posed (see also the review [**15**]) for this method. One of the important advantages of this method is that we obtain the solution in a very convenient form to study its long-time asymptotics. Using a steepest descent method [**10**–**12**] for oscillatory Riemann–Hilbert problems the long-time asymptotics of several IBV problems were studied in [**16**–**18**, **20**], under the assumption that the boundary values at $x = 0$ *vanish* for $t \to +\infty$. It is necessary to emphasize that an IBV problem posed on the quarter plane $x > 0$, $t > 0$ differs from one posed on a semi-strip $x > 0$, $0 < t < T < \infty$, because the corresponding Lax operator related to the t-equation has a spectrum of an essentially different nature in both cases. To the best of our knowledge such a type of IBV problems has not been considered elsewhere, anyway in the framework of RH problem. We provide such an implementation for the simplest periodic boundary data. Fortunately, this simple case contains all the novel ingredients necessary for the corresponding RH problem with general periodic boundary data.

The Dirichlet IBV problem

We consider the following initial–boundary-value problem for the 'focusing nonlinear Schrödinger' equation:

$$\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0, \quad \text{with } x, t \in \mathbb{R}_+, \tag{1.1 a}$$

$$q(x, 0) = q_0(x), \tag{1.1 b}$$

$$q(0, t) = g_0(t) = a\mathrm{e}^{2\mathrm{i}\omega t + \mathrm{i}\alpha}, \tag{1.1 c}$$

$$q_0(0) = g_0(0) = a\mathrm{e}^{\mathrm{i}\alpha}, \tag{1.1 d}$$

where $q_0(x)$ vanishes for $x \to +\infty$, $a > 0$, α and ω are real numbers. Let us even *assume* $q_0(x) \in \mathcal{S}(\mathbb{R}_+)$ where $\mathcal{S}(\mathbb{R}_+)$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}_+$:

$$\mathcal{S}(\mathbb{R}_+) = \{u(x) \in \mathcal{C}^\infty(\mathbb{R}_+) \mid x^n u^{(m)}(x) \in L^\infty(\mathbb{R}_+) \text{ for any } n, m \geqslant 0\}.$$

The Lax pair

The focusing nonlinear Schrödinger equation admits [**21**] a Lax pair consisting of the following two eigenvalue equations [**1**, **13**]. For the study of the initial–boundary-value problem (1.1) we shall use simultaneous spectral analysis of the eigenvalue problems for the linear x-equation,

$$\Phi_x + \mathrm{i}k\sigma_3\Phi = Q(x, t)\Phi, \tag{1.2 a}$$

$$Q(x, t) := \begin{pmatrix} 0 & q(x, t) \\ -\bar{q}(x, t) & 0 \end{pmatrix} \tag{1.2 b}$$

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with $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and for the linear t-equation,

$$\Phi_t + 2\mathrm{i}k^2\sigma_3\Phi = \tilde{Q}(x,t;k)\Phi, \tag{1.3 a}$$

$$\tilde{Q}(x,t;k) := 2kQ(x,t) - \mathrm{i}(Q^2(x,t) + Q_x(x,t))\sigma_3, \tag{1.3 b}$$

where $\Phi(x,t;k)$ is a 2×2 matrix-valued function, $k \in \mathbb{C}$. This is the well-known Zakharov–Shabat [**21**] or Ablowitz–Kaup–Newel–Segur [**1**] system of linear equations, which are compatible if and only if $q(x,t)$ solves the nonlinear Schrödinger equation (1.1 a).

Floquet solution

The focusing NLS equation (1.1 a) admits the exact solution

$$q_{\mathrm{p}}(x,t) = a\mathrm{e}^{2\mathrm{i}bx+2\mathrm{i}\omega t+\mathrm{i}\alpha},$$
$$\omega := a^2 - 2b^2, \quad a > 0.$$

Let $\tilde{Q}_{\mathrm{p}}(t;k) = \tilde{Q}_{\mathrm{p}}(0,t;k)$ where $\tilde{Q}_{\mathrm{p}}(x,t;k)$ is defined like $\tilde{Q}(x,t;k)$ but starting from $q_{\mathrm{p}}(x,t)$ instead of $q(x,t)$, i.e.

$$\tilde{Q}_{\mathrm{p}}(t;k) := 2kQ_{\mathrm{p}}(t) - \mathrm{i}(Q_{\mathrm{p}}^2(t) + (Q_{\mathrm{p}})_x(t))\sigma_3,$$

with

$$Q_{\mathrm{p}}(t) := Q_{\mathrm{p}}(0,t) = \begin{pmatrix} 0 & a\mathrm{e}^{2\mathrm{i}\omega t+\mathrm{i}\alpha} \\ -a\mathrm{e}^{-2\mathrm{i}\omega t-\mathrm{i}\alpha} & 0 \end{pmatrix},$$

$$(Q_{\mathrm{p}})_x(t) := (Q_{\mathrm{p}})_x(0,t) = \begin{pmatrix} 0 & 2\mathrm{i}ab\mathrm{e}^{2\mathrm{i}\omega t+\mathrm{i}\alpha} \\ 2\mathrm{i}ab\mathrm{e}^{-2\mathrm{i}\omega t-\mathrm{i}\alpha} & 0 \end{pmatrix},$$

$$Q_{\mathrm{p}}(x,t) := \begin{pmatrix} 0 & q_{\mathrm{p}}(x,t) \\ -\bar{q}_{\mathrm{p}}(x,t) & 0 \end{pmatrix}.$$

Consider now the t-part (1.3 a) of the Lax pair associated with $\tilde{Q}_{\mathrm{p}}(t)$, i.e.

$$\Psi_t(t;k) + 2\mathrm{i}k^2\sigma_3\Psi(t;k) = \tilde{Q}_{\mathrm{p}}(t;k)\Psi(t;k), \quad t > 0,\ k \in \mathbb{C}, \tag{1.4}$$

where $\Psi(t;k)$ is 2×2 matrix-valued. A particular (Floquet) solution of (1.4) is given by

$$\Psi(t;k) = \mathcal{E}(t;k)\mathrm{e}^{\mathrm{i}(\omega-\Omega(k))\sigma_3 t}, \tag{1.5 a}$$

$$\mathcal{E}(t;k) = \mathrm{e}^{\mathrm{i}\omega\hat{\sigma}_3 t}E(k) := \mathrm{e}^{\mathrm{i}\omega\sigma_3 t}E(k)\mathrm{e}^{-\mathrm{i}\omega\sigma_3 t}, \tag{1.5 b}$$

$$E(k) = \begin{pmatrix} \sqrt{\dfrac{X(k)+k+b}{2X(k)}} & \mathrm{i}\mathrm{e}^{\mathrm{i}\alpha}\sqrt{\dfrac{X(k)-k-b}{2X(k)}} \\ \mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha}\sqrt{\dfrac{X(k)-k-b}{2X(k)}} & \sqrt{\dfrac{X(k)+k+b}{2X(k)}} \end{pmatrix}, \tag{1.5 c}$$

$$\Omega(k) = 2(k-b)X(k), \tag{1.5 d}$$

$$X(k) = \sqrt{(k+b)^2 + a^2}, \tag{1.5 e}$$

$$\omega := a^2 - 2b^2. \tag{1.5 f}$$

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We fix the branches of the square roots by their asymptotics, for $k \to \infty$:

$$X(k) = \sqrt{(k+b)^2 + a^2} = k + b + O(k^{-1}), \tag{1.6 a}$$

$$\sqrt{2X(k)} = \sqrt{2k} + O(1), \tag{1.6 b}$$

$$\sqrt{k + b \pm X(k)} = \begin{cases} \sqrt{2k} + O(1), \\ O\left(\dfrac{1}{\sqrt{k}}\right), \end{cases} \tag{1.6 c}$$

where $\sqrt{2k}$ is positive when $\arg k = 0$.

Assumptions. Throughout the paper except in § 3 we assume that the Dirichlet IBV problem (1.1) has a global solution $q(x,t)$, sufficiently smooth and with sufficient decay for $x \to +\infty$. We also assume that the Neumann boundary values take the form

$$q_x(0,t) := g_1(t) = 2\mathrm{i}ab\mathrm{e}^{2\mathrm{i}\omega t} + v_1(t), \tag{1.7 a}$$

$$2b^2 = a^2 - \omega > 0 \tag{1.7 b}$$

with $v_1(t) \in \mathcal{S}(\mathbb{R}_+)$.

Actually one can consider the Neumann IBV problem:

$$\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0, \quad \text{with } x, t \in \mathbb{R}_+, \tag{1.8 a}$$

$$q(x,0) = q_0(x), \tag{1.8 b}$$

$$q_x(0,t) = g_1(t) = 2\mathrm{i}ab\mathrm{e}^{2\mathrm{i}\omega t + \mathrm{i}\alpha}, \tag{1.8 c}$$

$$q_{0x}(0) = g_1(0) = 2\mathrm{i}ab\mathrm{e}^{\mathrm{i}\alpha}, \tag{1.8 d}$$

instead of the Dirichlet problem (1.1). In this case we have to suppose that

$$q(0,t) = g_0(t) = a\mathrm{e}^{2\mathrm{i}\omega t + \mathrm{i}\alpha} + v_0(t), \tag{1.9}$$

with $v_0(t) \in \mathcal{S}(\mathbb{R}_+)$.

A justification of these assumptions which can be done by an asymptotic investigation of the basic Riemann–Hilbert problem formulated in § 2.5 when $x = 0$ and t tends to infinity as well as the asymptotic analysis of the whole IBV problem for large time is now in progress with A. R. Its.

In this paper we restrict our attention to new RH problems connected with the Dirichlet IBV problem (1.1) or with the Neumann IBV problem (1.8).

Notation. If μ is a 2×2 matrix we denote its columns by $[\mu]_1$ and $[\mu]_2$.

2. Direct scattering problem

We analyse the direct scattering problem associated to a given solution of the considered initial–boundary-value problem. Let us *assume* that this solution $q(x,t)$ is $\mathcal{C}^\infty$, continuous with all its derivatives up to the boundary $\{x = 0\} \cup \{t = 0\}$ of the quarter xt-plane and $q(x,t) \in \mathcal{S}(\mathbb{R}_+)$ in x for any fixed $t \in \mathbb{R}_+$.

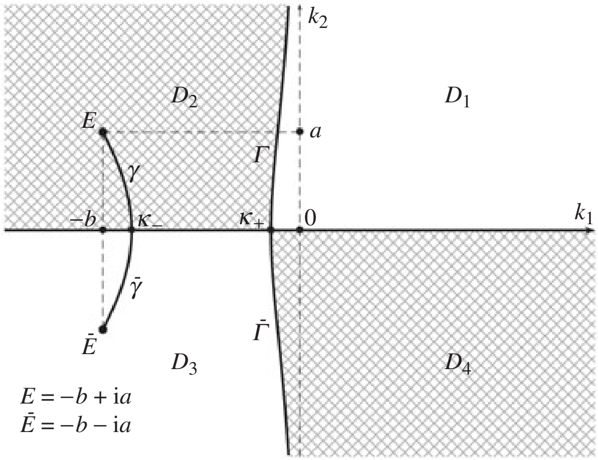

Figure 1. The domains D_j for the case $\omega < -3a^2$, $a > 0$, $b > 0$.

2.1. Domains of boundedness

We consider

$$\Sigma := \{k \in \mathbb{C} \mid \operatorname{Im} \Omega(k) = 0\}.$$

If $k_1 = \operatorname{Re} k$ and $k_2 = \operatorname{Im} k$, the equation $\operatorname{Im} \Omega(k) = 0$ implies

$$k_2 = 0$$

or

$$k_1 k_2^2 = (k_1 - b)(k_1^2 + bk_1 + \tfrac{1}{2}a^2) = (k_1 - b)(k_1 - \kappa_-)(k_1 - \kappa_+) \quad \text{with } |k_1| \leqslant |b|.$$

In what follows we *suppose* $b > 0$. The case $b < 0$ is similar. There are two cases.

Case $\omega \leqslant -3a^2$

If $b^2 \geqslant 2a^2$, i.e. $\omega \leqslant -3a^2$, $\kappa_\pm$ are real and

$$\kappa_\pm = -\tfrac{1}{2}b \pm \sqrt{\tfrac{1}{4}b^2 - \tfrac{1}{2}a^2},$$

with $-b < \kappa_- \leqslant -b/2 \leqslant \kappa_+ < 0$ (Figures 1 and 2).

In this case, Σ consists of the real axis $\mathbb{R}$, the finite arc $\gamma \cup \bar{\gamma}$ whose endpoints are the branch points $E = -b + \mathrm{i}a$ and $\bar{E} = -b - \mathrm{i}a$, and the contour $\Gamma \cup \bar{\Gamma}$:

$$\Sigma = \mathbb{R} \cup \gamma \cup \bar{\gamma} \cup \Gamma \cup \bar{\Gamma}.$$

The D_j, $j = 1, 2, 3, 4$, are the following domains:

$$\left.\begin{aligned} D_1 &:= \{k \in \mathbb{C} \mid \operatorname{Im} k > 0,\ \operatorname{Im} \Omega(k) > 0\}, \\ D_2 &:= \{k \in \mathbb{C} \mid \operatorname{Im} k > 0,\ \operatorname{Im} \Omega(k) < 0\}, \\ D_3 &:= \{k \in \mathbb{C} \mid \operatorname{Im} k < 0,\ \operatorname{Im} \Omega(k) > 0\}, \\ D_4 &:= \{k \in \mathbb{C} \mid \operatorname{Im} k < 0,\ \operatorname{Im} \Omega(k) < 0\}; \end{aligned}\right\} \tag{2.1}$$

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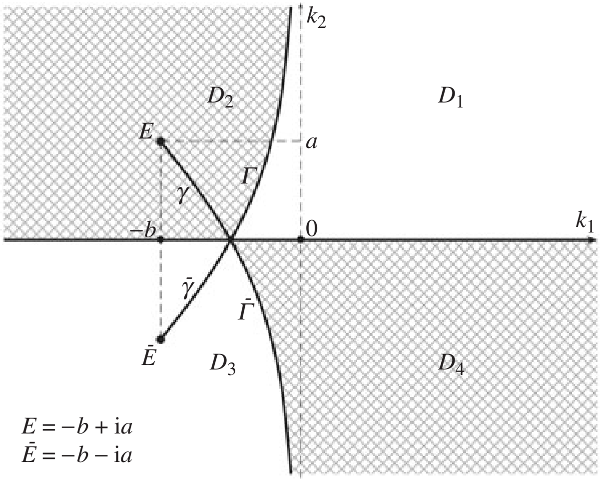

Figure 2. The domains D_j for the case $\omega = -3a^2$, $a > 0$, $b > 0$.

and we also define

$$D_+ := D_1 \cup D_3 = \{k \in \mathbb{C} \mid \operatorname{Im} \Omega(k) > 0\},$$
$$D_- := D_2 \cup D_4 = \{k \in \mathbb{C} \mid \operatorname{Im} \Omega(k) < 0\}.$$

So we obtain a partition of the complex k-plane $\mathbb{C}$:

$$D_1 \cup D_2 \cup D_3 \cup D_4 \cup \Sigma = \mathbb{C}.$$

Case $\omega > -3a^2$

If $b^2 < 2a^2$, i.e. $\omega > -3a^2$, $\kappa_\pm$ are complex conjugate. In this case, in order to define $D_1, \dots, D_4$ it is necessary to consider the Riemann surface $\mathfrak{X}$ of genus zero defined by the algebraic function

$$z = \Omega(k) := 2(k - b)X(k)$$

with

$$X(k) = \sqrt{(k + b)^2 + a^2}.$$

It is a two-sheeted Riemann surface obtained by gluing two copies $\mathbb{C}^{\text{upper}}$ and $\mathbb{C}^{\text{lower}}$ of the complex plane cut along the contours Γ and $\bar{\Gamma}$ where $\operatorname{Im} \Omega(k) = 0$ (see Figures 3 and 4).

Let $\Gamma_{12} = \Gamma_1^+ \wedge \Gamma_2^-$ be obtained by gluing the paths Γ_1^+ and Γ_2^-, and similarly, $\Gamma_{21} = \Gamma_1^- \wedge \Gamma_2^+$. Let also $\bar{\Gamma}_{12} = \bar{\Gamma}_1^+ \wedge \bar{\Gamma}_2^-$ and $\bar{\Gamma}_{21} = \bar{\Gamma}_1^- \wedge \bar{\Gamma}_2^+$. Let $\infty_1 = \infty_1^+ \wedge \infty_2^-$, $\infty_2 = \infty_1^- \wedge \infty_2^+$, $E = E_1 \wedge E_2$ and $\bar{E} = \bar{E}_1 \wedge \bar{E}_2$.

The contour $\Sigma \subset \mathfrak{X}$ is as follows:

$$\Sigma = \mathbb{R}^{\text{upper}} \cup \mathbb{R}^{\text{lower}} \cup \Gamma_{12} \cup \Gamma_{21} \cup \bar{\Gamma}_{12} \cup \bar{\Gamma}_{21}.$$

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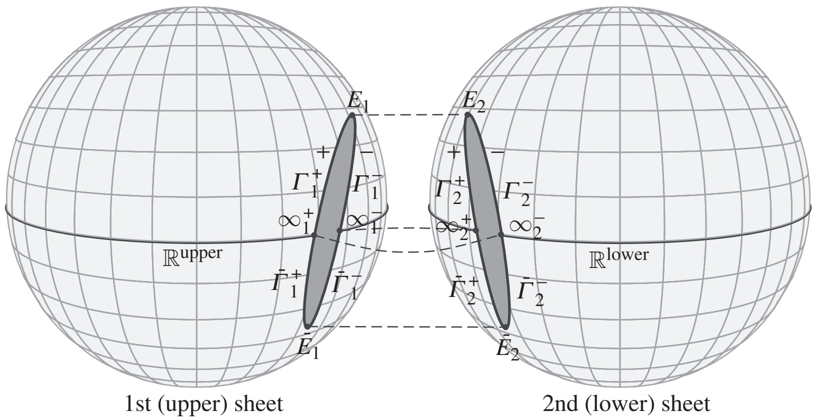

Figure 3. Two copies of the Riemann sphere with cuts along $\Gamma \cup \bar{\Gamma}$ from E to $\bar{E}$.

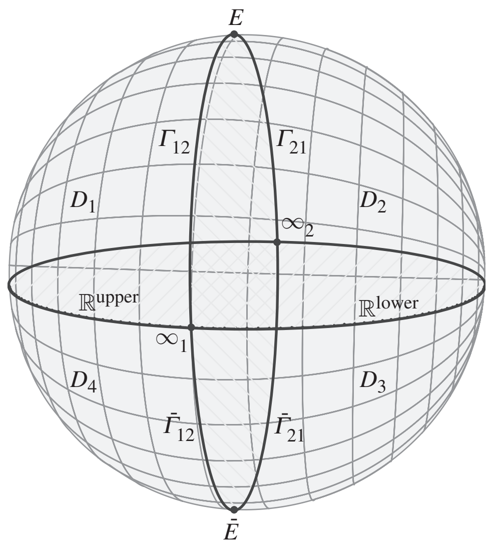

Figure 4. The domains D_j on the Riemann surface $\mathfrak{X}$ for the case $\omega > -3a^2$.

The D_j, $j = 1, 2, 3, 4$, are the following domains:

$$\left.\begin{aligned} D_1 &= \{k \in \mathfrak{X} \mid \operatorname{Im} k > 0,\ \operatorname{Im} \Omega(k) > 0\}, \\ D_2 &= \{k \in \mathfrak{X} \mid \operatorname{Im} k > 0,\ \operatorname{Im} \Omega(k) < 0\}, \\ D_3 &= \{k \in \mathfrak{X} \mid \operatorname{Im} k < 0,\ \operatorname{Im} \Omega(k) > 0\}, \\ D_4 &= \{k \in \mathfrak{X} \mid \operatorname{Im} k < 0,\ \operatorname{Im} \Omega(k) < 0\}. \end{aligned}\right\} \tag{2.2}$$

The domains $D_1 = \mathbb{C}_+^{\text{upper}} \setminus (\Gamma_{12} \cup \Gamma_{21})$ and $D_4 = \mathbb{C}_-^{\text{upper}} \setminus (\bar{\Gamma}_{12} \cup \bar{\Gamma}_{21})$ are on the upper sheet of the Riemann surface $\mathfrak{X}$, while $D_2 = \mathbb{C}_+^{\text{lower}} \setminus (\Gamma_{12} \cup \Gamma_{21})$ and $D_3 = \mathbb{C}_-^{\text{lower}} \setminus (\bar{\Gamma}_{12} \cup \bar{\Gamma}_{21})$ are on the lower sheet (see Figures 4 and 5).

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So we obtain a partition of the Riemann surface $\mathcal{X}$:

$$D_1 \cup D_2 \cup D_3 \cup D_4 \cup \Sigma = \mathcal{X}.$$

We define

$$\begin{aligned} D_+ &= D_1 \cup D_3 = \{k \in \mathcal{X} \mid \operatorname{Im} \Omega(k) > 0\}, \\ D_- &= D_2 \cup D_4 = \{k \in \mathcal{X} \mid \operatorname{Im} \Omega(k) < 0\}. \end{aligned}$$

2.2. Eigenfunctions

We *assume* that there exists a unique global solution $q(x,t)$ satisfying (1.1) and (1.7) and we consider the associated functions $Q(x,t)$ and $\tilde{Q}(x,t;k)$ defined by (1.2 *b*) and (1.3 *b*), respectively.

Define the 2×2 matrix-valued functions $\{\mu_j(x,t;k)\}_{j=1}^3$ for $0 < x < \infty$ and $0 < t < \infty$, as the solutions of the following Volterra integral equations:

$$\mu_3(x,t;k) = I - \int_x^\infty \mathrm{e}^{\mathrm{i}k(\xi - x)\hat{\sigma}_3}(Q\mu_3)(\xi,t;k)\,\mathrm{d}\xi, \tag{2.3 a}$$

$$\mu_2(x,t;k) = I + \mathrm{e}^{-\mathrm{i}kx\hat{\sigma}_3}\int_0^t \mathrm{e}^{-2\mathrm{i}k^2(t-\tau)\hat{\sigma}_3}(\tilde{Q}\mu_2)(0,\tau;k)\,\mathrm{d}\tau + \int_0^x \mathrm{e}^{-\mathrm{i}k(x-\xi)\hat{\sigma}_3}(Q\mu_2)(\xi,t;k)\,\mathrm{d}\xi, \tag{2.3 b}$$

$$\begin{aligned} \mu_1(x,t;k) &= \mathrm{e}^{-\mathrm{i}kx\hat{\sigma}_3 + \mathrm{i}\omega t\hat{\sigma}_3}E(k) \\ &\quad + \mathrm{e}^{-\mathrm{i}kx\hat{\sigma}_3}\mathcal{E}(t;k)\int_\infty^t \mathrm{e}^{\mathrm{i}[\omega - \Omega(k)](t-\tau)\hat{\sigma}_3}\mathcal{E}^{-1}(\tau;k)\tilde{Q}_0(\tau;k)\mu_1(0,\tau;k)\,\mathrm{d}\tau \\ &\quad + \int_0^x \mathrm{e}^{-\mathrm{i}k(x-\xi)\hat{\sigma}_3}(Q\mu_1)(\xi,t;k)\,\mathrm{d}\xi, \end{aligned} \tag{2.3 c}$$

where $E(k)$, $\mathcal{E}(t;k)$, and $\Omega(k)$ are defined by (1.5 *c*), (1.5 *b*), and (1.5 *d*), respectively, and

$$\tilde{Q}_0(t;k) := \tilde{Q}(0,t;k) - \tilde{Q}_{\mathrm{p}}(t;k).$$

Proposition 2.1. *The 2×2 matrices $\{\mu_j(x,t;k)\}_{j=1}^3$ have the following properties.*

(i) *For $j = 1,2,3$,*

$$\det \mu_j(x,t;k) \equiv 1. \tag{2.4}$$

(ii) *The functions $\{\Phi_j\}_{j=1}^3$ defined by*

$$\Phi_1(x,t;k) := \mu_1(x,t;k)\mathrm{e}^{-\mathrm{i}kx\sigma_3 + \mathrm{i}[\omega - \Omega(k)]t\sigma_3}, \tag{2.5}$$

$$\Phi_j(x,t;k) := \mu_j(x,t;k)\mathrm{e}^{-\mathrm{i}kx\sigma_3 - 2\mathrm{i}k^2 t\sigma_3}, \quad j = 2,3, \tag{2.6}$$

satisfy the Lax pair (1.2), (1.3).

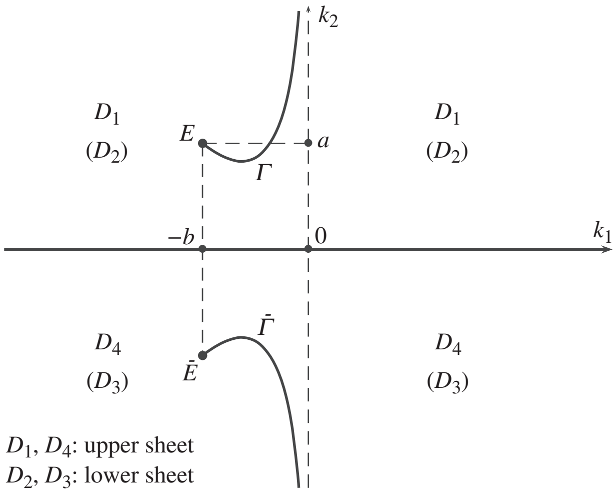

Figure 5. The domains D_j for the case $\omega < -3a^2$, $a > 0$, $b > 0$.

(iii) *For* $j = 1, 2, 3$,

$$\mu_j(x,t;k) = I + O\left(\frac{1}{k}\right), \quad \operatorname{Im} k = 0, \ k \to \infty. \tag{2.7}$$

(iv) *Near* $k = -b \pm \mathrm{i}a$, *the matrix* $\mu_1(x,t;k)$ *exhibits inverse fourth-root singularities like those the matrix* $E(k)$ *has.*

(v) *The matrix* $\mu_1(x,t;k)$ *has different boundary values along a cut connecting the two points* $k = -b \pm \mathrm{i}a$, *which are the branch points of the function* $X(k)$.

(vi) *The matrix* $\mu_2(x,t;k)$ *is entire in* $k \in \mathbb{C}$. *Furthermore,*

$$\mu_1 = \begin{pmatrix} \mu_1^{(2)} & \mu_1^{(3)} \end{pmatrix}, \qquad \mu_2 = \begin{pmatrix} \mu_2^{(1)} & \mu_2^{(4)} \end{pmatrix}, \qquad \mu_3 = \begin{pmatrix} \mu_3^{(34)} & \mu_3^{(12)} \end{pmatrix}, \tag{2.8}$$

where $\mu_1^{(2)}$ *means that the first column vector* $[\mu_1(x,t;k)]_1$ *is bounded and analytic in* D_2; $\mu_1^{(3)}$ *means that the second column vector* $[\mu_1(x,t;k)]_2$ *is bounded and analytic in* D_3; $\mu_3^{(12)}$ *means that* $[\mu_3(x,t;k)]_2$ *is bounded and analytic in* $D_1 \cup D_2$; *etc.*

Proof. The matrix-valued function $\Psi(t;k)$ defined in (1.5) is bounded in $t \in \mathbb{R}_+$ for $k \in \Sigma$. The vector functions $[\Psi(t;k)]_1$ and $[\Psi(t;k)]_2$ are bounded in $t \in \mathbb{R}_+$ for $k \in \Sigma$. They are $\mathcal{C}^\infty$ for $t \in \mathbb{R}_+$ and $k \in \Sigma \setminus (E \cup \bar{E})$. For $k \in \mathbb{C} \setminus \Sigma$ the matrix-valued function $\Psi(t;k)$ is unbounded for $t \in \mathbb{R}_+$. However, its first column $[\Psi(t;k)]_1$ has exponential decay in $D_- = D_2 \cup D_4$ for $t \to +\infty$, while the second column $[\Psi(t;k)]_2$ has exponential decay in $D_+ = D_1 \cup D_3$:

$$[\Psi(t;k)]_{1,2} = O(\mathrm{e}^{\pm 2 \operatorname{Im} \Omega(k) t}), \quad t \to +\infty.$$

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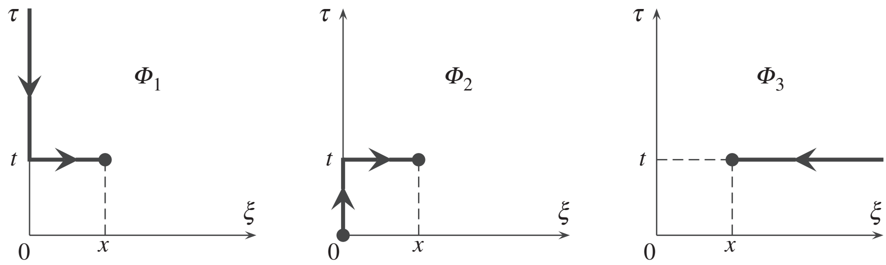

Figure 6. Paths of integration for the construction of Φ_1, Φ_2 and Φ_3.

But, both grow exponentially when $k \in D_+ = D_1 \cup D_3$ and $k \in D_- = D_2 \cup D_4$, respectively. The matrix elements of the function $\Psi(t;k)$ has inverse fourth-root singularities at $k = -b + \mathrm{i}a$ and $k = -b - \mathrm{i}a$ where $X(k)$ vanishes.

We introduce appropriate solutions $\{\Phi_j(x,t;k)\}_{j=1}^3$ of the Lax pair which are normalized at the points $(0,\infty)$, $(0,0)$, (∞,t), respectively (Figure 6).

We construct these solutions as follows: the general solution of (1.2) is

$$\mathrm{e}^{\mathrm{i}kx\sigma_3}\Phi(x,t;k) = \mathrm{e}^{\mathrm{i}kx_0\sigma_3}\Phi(x_0,t;k) + \int_{x_0}^{x} \mathrm{e}^{\mathrm{i}k\xi\sigma_3}(Q\Phi)(\xi,t;k)\,\mathrm{d}\xi. \tag{2.9}$$

Since equations (1.2) and (1.3) are compatible a necessary and sufficient condition for $\Phi(x,t;k)$ to satisfy equation (1.3) is that $\Phi(x_0,t;k)$ satisfies (1.3) evaluated at $x = x_0$.

For $\Phi_3(x_0,t;k)$, we choose $x_0 = \infty$. Thus, since $\Phi(\infty,t;k)$ satisfies (1.3) with $\tilde{Q} = 0$, it follows that $\Phi(\infty,t;k) = \exp[-2\mathrm{i}k^2t\sigma_3]C_3(k)$. Letting $C_3(k) = I$ we find

$$\Phi_3(x,t;k) = \mathrm{e}^{-\mathrm{i}k(x+2kt)\sigma_3} - \int_x^{\infty} \mathrm{e}^{\mathrm{i}k(\xi-x)\sigma_3}(Q\Phi_3)(\xi,t;k)\,\mathrm{d}\xi. \tag{2.10}$$

Then the definition (2.5) and (2.10) yield (2.3 a).

For $\Phi_1(x,t;k)$ and $\Phi_2(x,t;k)$, since $x_0 = 0$, it follows that these functions satisfy the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_j(0,t;k) + 2\mathrm{i}k^2\sigma_3\Phi_j(0,t;k) = (\tilde{Q}_{\mathrm{p}}(t;k) + \tilde{Q}_0(t;k))\Phi_j(0,t;k), \quad j = 1,2.$$

For $\Phi_2(x,t;k)$ this equation yields

$$\Phi_2(x,t;k) = \mathrm{e}^{-\mathrm{i}k(x+2kt)\sigma_3}\left[C_2(k) + \int_0^t \mathrm{e}^{2\mathrm{i}k^2\tau\sigma_3}(\tilde{Q}\Phi_2)(0,\tau;k)\,\mathrm{d}\tau\right] + \int_0^x \mathrm{e}^{\mathrm{i}k(\xi-x)\sigma_3}(Q\Phi_2)(\xi,t;k)\,\mathrm{d}\xi. \tag{2.11}$$

Using the fact that $\Psi(t;k)$ satisfies (1.4) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Psi^{-1}(t;k)\Phi_1(0,t;k)) = \Psi^{-1}(t;k)\tilde{Q}_0(t;k)\Phi_1(0,t;k).$$

Solving this equation (with initial point $t = \infty$) and substituting the expressions for $\Phi_1(0,t;k)$ into (2.9) with $x_0 = 0$ we find

$$\Phi_1(x,t;k) = \mathrm{e}^{-\mathrm{i}kx\sigma_3}\Psi(t;k)\left[C_1(k) + \int_\infty^t \Psi^{-1}(\tau;k)\tilde{Q}_0(\tau;k)\Phi_1(0,\tau;k)\,\mathrm{d}\tau\right] + \int_0^x \mathrm{e}^{\mathrm{i}k(\xi-x)\sigma_3}(Q\Phi_1)(\xi,t;k)\,\mathrm{d}\xi. \quad (2.12)$$

We choose $C_1(k) \equiv C_2(k) \equiv I$. Substituting these expressions into (2.11) and (2.12), and using the definitions (2.6) we find equations (2.3 *b*) and (2.3 *c*). It is easy to verify that properties (2.4) and (2.7) hold.

In order to determinate the analytic properties (2.8) of $\{\mu_j(x,t;k)\}_{j=1}^3$ with respect to k we look at the explicit k-dependence of equations (2.3 *a*) and (2.3 *b*). Recalling that

$$\mathrm{e}^{\mathrm{i}k(\xi-x)\hat{\sigma}_3}A = \begin{pmatrix} A_{11} & A_{12}\mathrm{e}^{2\mathrm{i}k(\xi-x)} \\ A_{21}\mathrm{e}^{-2\mathrm{i}k(\xi-x)} & A_{22} \end{pmatrix}$$

it follows that the second column of μ_3 involves $\mathrm{e}^{2\mathrm{i}k(\xi-x)}$ which is bounded, since $\xi - x \geqslant 0$, and analytic in k for $\operatorname{Im} k > 0$. Similarly, the second column of μ_2 involves $\mathrm{e}^{-2\mathrm{i}kx}$, $\mathrm{e}^{2\mathrm{i}k(\xi-x)}$, and $\mathrm{e}^{-2\mathrm{i}\Omega(k)(t-\tau)}$. Since $x > 0$, $x - \xi \geqslant 0$, and $t - \tau \geqslant 0$ the corresponding terms are bounded and analytic in k for $\operatorname{Im} k < 0$ and $\operatorname{Im} \Omega(k) < 0$. Similar properties are valid for the matrices $\{\Phi_j(x,t;k)\}_{j=1}^3$. Details about eigenfunctions can be found in [**2**–**4**]. □

2.3. Relations among eigenfunctions

Let $\{\Phi_j\}_{j=1}^3$ be the 2×2 matrix-valued functions defined in Proposition 2.1. Then in their domains of definition the functions $\{\Phi_j(x,t;k)\}_{j=1}^3$ satisfy both equations of the Lax pair, and their determinants (2.4) do not vanish. Hence they are linearly dependent and satisfy the following dependence relations:

$$\Phi_3(x,t;k) = \Phi_2(x,t;k)s(k), \quad k \in \mathbb{R}, \quad (2.13)$$

$$\Phi_1(x,t;k) = \Phi_2(x,t;k)S(k), \quad k \in \Sigma, \quad (2.14)$$

where $s(k)$ and $S(k)$ are defined by

$$s(k) := \Phi_3(0,0;k) =: \begin{pmatrix} \bar{a}(\bar{k}) & b(k) \\ -\bar{b}(\bar{k}) & a(k) \end{pmatrix}, \quad (2.15)$$

$$S(k) := \Phi_1(0,0;k) =: \begin{pmatrix} \bar{A}(\bar{k}) & B(k) \\ -\bar{B}(\bar{k}) & A(k) \end{pmatrix}. \quad (2.16)$$

Furthermore, the scattering relations (2.13) and (2.14) yield

$$\Phi_1(x,t;k) = \Phi_3(x,t;k)T(k), \quad (2.17)$$

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where

$$T(k) := s^{-1}(k)S(k). \tag{2.18}$$

We denote by $\{T_{ij}(k)\}_{i,j=1}^{2}$ the entries of the 2×2 matrix $T(k)$. Then (2.15), (2.16) imply:

$$T_{11}(k) = \bar{T}_{22}(\bar{k}) = a(k)\bar{A}(\bar{k}) + b(k)\bar{B}(\bar{k}), \tag{2.19}$$

$$T_{12}(k) = -\bar{T}_{21}(\bar{k}) = a(k)B(k) - b(k)A(k). \tag{2.20}$$

We define

$$c(k) := \frac{T_{21}(k)}{T_{11}(k)} - \frac{\bar{b}(\bar{k})}{a(k)} = -\frac{\bar{B}(\bar{k})}{a(k)T_{11}(k)}, \tag{2.21}$$

which is analytic and bounded in $k \in D_2$ and is $O(k^{-1})$ as $k \to \infty$. This follows from the definition of $c(k)$ and from the corresponding properties of the functions $a(k)$, $b(k)$, $A(k)$, $B(k)$ described in § 3 below.

Notation. Let ϕ be a *scalar* function defined in $V \setminus \Sigma$ where V is a neighbourhood of an oriented contour Σ. We denote by $\phi^+(k)$ and $\phi^-(k)$ the limiting values of $\phi(z)$ as z approaches $k \in \Sigma$ from the positive and negative sides of Σ, respectively.

Then, we introduce the jump of $c(k)$ over γ:

$$f(k) := c^-(k) - c^+(k) = \frac{-\mathrm{i}e^{-\mathrm{i}\alpha}}{T_{11}^-(k)T_{11}^+(k)}. \tag{2.22}$$

Proposition 2.2. *The following relations are valid:*

$$\frac{[\Phi_1(x,t;k)]_1}{T_{11}(k)} - \frac{[\Phi_2(x,t;k)]_1}{a(k)} = c(k)[\Phi_3(x,t;k)]_2, \quad \text{for } k \in \Gamma, \tag{2.23 a}$$

$$\frac{[\Phi_1(x,t;k)]_2}{T_{22}(k)} - \frac{[\Phi_2(x,t;k)]_2}{\bar{a}(\bar{k})} = -\bar{c}(\bar{k})[\Phi_3(x,t;k)]_1, \quad \text{for } k \in \bar{\Gamma}, \tag{2.23 b}$$

$$\frac{[\Phi_1^+(x,t;k)]_1}{T_{11}^+(k)} - \frac{[\Phi_1^-(x,t;k)]_1}{T_{11}^-(k)} = -f(k)[\Phi_3(x,t;k)]_2, \quad \text{for } k \in \gamma, \tag{2.23 c}$$

$$\frac{[\Phi_1^+(x,t;k)]_2}{T_{22}^+(k)} - \frac{[\Phi_1^-(x,t;k)]_2}{T_{22}^-(k)} = \bar{f}(\bar{k})[\Phi_3(x,t;k)]_1, \quad \text{for } k \in \bar{\gamma}. \tag{2.23 d}$$

Proof. Using (2.17) it is easy to find that for the first and second columns we obtain

$$\left.\begin{aligned} \frac{[\Phi_1(x,t;k)]_1}{T_{11}(k)} &= [\Phi_3(x,t;k)]_1 + \rho(k)[\Phi_3(x,t;k)]_2, \\ \frac{[\Phi_1(x,t;k)]_2}{T_{22}(k)} &= [\Phi_3(x,t;k)]_2 - \bar{\rho}(\bar{k})[\Phi_3(x,t;k)]_1, \end{aligned}\right\} \tag{2.24}$$

where

$$\left.\begin{aligned} \rho(k) &:= \frac{T_{21}(k)}{T_{11}(k)} = \frac{\bar{b}(\bar{k})\bar{A}(\bar{k}) - \bar{a}(\bar{k})\bar{B}(\bar{k})}{a(k)\bar{A}(\bar{k}) + b(k)\bar{B}(\bar{k})}, \\ \bar{\rho}(\bar{k}) &= -\frac{T_{12}(k)}{T_{22}(k)} = \frac{b(k)A(k) - a(k)B(k)}{\bar{a}(\bar{k})A(k) + \bar{b}(\bar{k})B(k)}. \end{aligned}\right\} \tag{2.25}$$

In order to prove the jump relation across γ, we note that

$$\frac{[\Phi_1^+(x,t;k)]_1}{T_{11}^+(k)} - \frac{[\Phi_1^-(x,t;k)]_1}{T_{11}^-(k)} = \frac{1}{T_{11}^-T_{11}^+}(\det([\Phi_1^-]_1,[\Phi_3]_2)[\Phi_1^+]_1 - \det([\Phi_1^+]_1,[\Phi_3]_2)[\Phi_1^-]_1)$$
$$= \frac{\det([\Phi_1^-]_1,[\Phi_1^+]_1)[\Phi_3]_2}{T_{11}^-T_{11}^+},$$

where we used

$$T_{11}(k) = \det([\Phi_1]_1,[\Phi_3]_2).$$

Taking into account that

$$\det([\Phi_1^-]_1,[\Phi_1^+]_1) = \det\begin{pmatrix} \bar{A}^-(\bar{k}) & \bar{A}^+(\bar{k}) \\ -\bar{B}^-(\bar{k}) & -\bar{B}^+(\bar{k}) \end{pmatrix} = \det([\Psi^-]_1,[\Psi^+]_1) = \mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha},$$

we obtain the jump condition (2.23 c). The jump condition (2.23 d) across $\bar{\gamma}$ can be derived in a similar manner.

The scattering relation (2.13) gives

$$\left.\begin{aligned} \frac{[\Phi_2(x,t;k)]_1}{a(k)} &= [\Phi_3(x,t;k)]_1 + r(k)[\Phi_3(x,t;k)]_2, \\ \frac{[\Phi_2(x,t;k)]_2}{\bar{a}(\bar{k})} &= [\Phi_3(x,t;k)]_2 - \bar{r}(k)[\Phi_3(x,t;k)]_1, \end{aligned}\right\} \tag{2.26}$$

where

$$r(k) := \frac{\bar{b}(\bar{k})}{a(k)}, \qquad \bar{r}(\bar{k}) = \frac{b(k)}{\bar{a}(\bar{k})}. \tag{2.27}$$

From (2.26) and (2.24) one can easily find the relations (2.23 a) and (2.23 b) with

$$c(k) = \rho(k) - r(k) = -\frac{\bar{B}(\bar{k})}{a(k)[b(k)\bar{B}(\bar{k}) + a(k)\bar{A}(\bar{k})]}.$$

□

Remark. The matrix $s(k) = \Phi_3(x,0;k)|_{x=0}$ (2.15) is uniquely defined in terms of $\varphi(x;k)$ which is a solution of the Volterra integral equation

$$\varphi(x;k) = I - \int_x^\infty \mathrm{e}^{\mathrm{i}k(\xi-x)\sigma_3} Q_0(\xi)\varphi(\xi;k)\,\mathrm{d}\xi.$$

Similarly, the matrix $S(k)$ which is defined by (2.16) can be also defined in terms of

$$\Phi(t;k) = \Psi^{-1}(t;k)\Phi_2(0,t;k),$$

which is a solution of the Volterra integral equation

$$\Phi(t;k) = E^{-1}(k) + \int_0^t \Psi^{-1}(\tau;k)\tilde{Q}_0(\tau;k)\Psi(\tau;k)\Phi(\tau;k)\,\mathrm{d}\tau.$$

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Thus $S(k)$ can be defined by two different formulae:

$$S^{-1}(k) = \Phi(\infty;k) = E^{-1}(k) + \int_0^{\infty} \Psi^{-1}(\tau;k)\tilde{Q}_0(\tau;k)\Psi(\tau;k)\Phi(\tau;k)\,\mathrm{d}\tau,$$
$$S(k) = \Phi_1(0,0;k) = E(k) - \int_0^{\infty} \Psi^{-1}(\tau;k)\tilde{Q}_0(\tau;k)\Phi_1(0,\tau;k)\,\mathrm{d}\tau.$$

2.4. The global relation

By evaluating (2.17) at $x=0$, $t=t_0 \gg 1$ and using the definition of $T(k) = s^{-1}(k)S(k)$ we obtain

$$S^{-1}(k)s(k) = [\Psi^{-1}(t_0;k) + o(1)]\Phi_3(0,t_0;k),$$

or, since $\Psi^{-1}(t_0;k) = \mathrm{e}^{\mathrm{i}\Omega(k)t_0\sigma_3}E^{-1}(k)\mathrm{e}^{-\mathrm{i}\omega t_0\sigma_3}$,

$$S^{-1}(k)s(k) = \mathrm{e}^{\mathrm{i}\Omega(k)t_0\sigma_3}[E^{-1}(k)\mathrm{e}^{-\mathrm{i}\omega t_0\sigma_3} + o(1)]\mu_3(0,t_0;k)\mathrm{e}^{-2\mathrm{i}k^2 t_0\sigma_3}.$$

By putting

$$\begin{aligned} C(k,t_0) &= E^{-1}(k)\mathrm{e}^{-\mathrm{i}\omega t_0\sigma_3}\mu_3(0,t_0;k) \\ &= E^{-1}(k)\mathrm{e}^{-\mathrm{i}\omega t_0\sigma_3}\left[I - \int_0^{\infty} \mathrm{e}^{\mathrm{i}k\xi\hat{\sigma}_3}(Q\mu_3)(\xi,t_0;k)\,\mathrm{d}\xi\right], \end{aligned}$$

we finally find

$$S^{-1}(k)s(k) = \mathrm{e}^{\mathrm{i}\Omega(k)t_0\sigma_3}[C(k,t_0) + o(1)]\mathrm{e}^{-2\mathrm{i}k^2 t_0\sigma_3}. \tag{2.28}$$

By using (2.15) and (2.16) the $(2.28)_{12}$ component of the relation (2.28) yields

$$b(k)A(k) - a(k)B(k) = [c_E(k,t_0) + o(1)]\mathrm{e}^{\mathrm{i}(\Omega(k)+2k^2)t_0}$$

with the function c_E, defined by

$$c_E(k,t_0) = -\int_0^{\infty} q(\xi,t_0)[\mu_3]_{12}(\xi,t_0;k)\mathrm{e}^{2\mathrm{i}k\xi}\,\mathrm{d}\xi,$$

which is analytic and bounded in $\mathbb{C}_+$ and it is $O(1/k)$ for $k\to\infty$. The left-hand side is analytic in $k\in D_1$. The right-hand side is also analytic in $k\in D_1$ and tends to zero for $t_0\to+\infty$ ($\operatorname{Im} k^2 > 0$). Thus the 'global relation' becomes

$$b(k)A(k) - a(k)B(k) \equiv 0 \quad \text{for } k\in D_1. \tag{2.29}$$

The global relation yields that $T_{12}(k)\equiv 0$ for $k\in D_1$ and $T_{21}(k)\equiv 0$ for $k\in D_4$. In particular, the global relation means that $\rho(k) = c(k) + r(k) \equiv 0$ for $k\in[\kappa_+,+\infty)$. Moreover, it can be shown [**3**, **4**] that $\rho(k)$ and all its derivatives have jumps at $k=\kappa_-$:

$$\left.\frac{\mathrm{d}^l}{\mathrm{d}k^l}\rho(k)\right|_{k=\kappa_- -0} - \left.\frac{\mathrm{d}^l}{\mathrm{d}k^l}\rho(k)\right|_{k=\kappa_- +0} = \left.\frac{\mathrm{d}^l}{\mathrm{d}k^l}f(k)\right|_{k=\kappa_-}, \quad l = 1,2,\dots. \tag{2.30}$$

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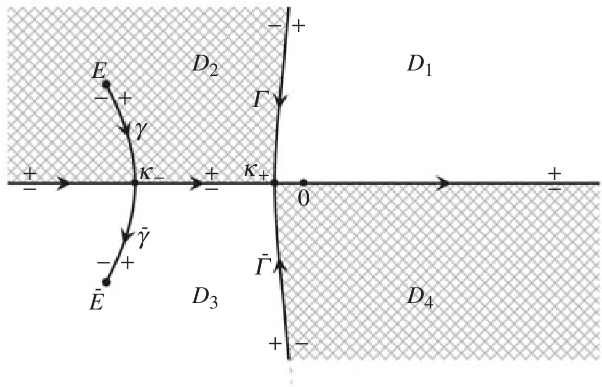

Figure 7. The oriented contour Σ for the case $\omega < -3a^2$, $a > 0$, $b > 0$.

In view of the global relation and the determinant relations

$$|a(k)|^2 + |b(k)|^2 \equiv 1, \quad k \in \mathbb{R},$$
$$A(k)\bar{A}(\bar{k}) + B(k)\bar{B}(\bar{k}) \equiv 1, \quad k \in \Sigma,$$

we find

$$T_{11}(k) = T_{22}^{-1}(k) = \frac{a(k)}{A(k)}, \quad k \in (\kappa_+, \infty). \tag{2.31}$$

Therefore, $T_{11}(k)$ and $T_{22}(k)$ have analytic continuations to the domain D_1. Similarly

$$T_{11}(k) = T_{22}^{-1}(k) = \frac{\bar{A}(\bar{k})}{\bar{a}(\bar{k})} \tag{2.32}$$

for $k \in D_4$. As a consequence it follows that

$$|a(k)| = |A(k)|, \qquad k \in (\kappa_+, \infty), \tag{2.33}$$

$$\left.\begin{aligned} \frac{[\Phi_1(x,t;k)]_1}{\bar{A}(\bar{k})} &= \frac{[\Phi_3(x,t;k)]_1}{\bar{a}(\bar{k})}, \qquad k \in D_4, \\ \frac{[\Phi_1(x,t;k)]_2}{A(k)} &= \frac{[\Phi_3(x,t;k)]_2}{a(k)}, \qquad k \in D_1. \end{aligned}\right\} \tag{2.34}$$

2.5. The basic Riemann–Hilbert problem

We will show that the relations (2.13)–(2.20) among Φ_1, Φ_2, Φ_3 can be rewritten in the form of a Riemann–Hilbert problem RH_{xt}:

$$M_-(x,t;k) = M_+(x,t;k)J(x,t;k), \quad k \in \Sigma, \tag{2.35}$$

where $M_+(x,t;k)$ and $M_-(x,t;k)$ denote the limiting values of $M(x,t;z)$ as $z \to k$ from the left and right sides of Σ, respectively (see Figures 7–9).

The contour Σ

There are two cases: $\omega \leqslant -3a^2$ and $\omega > -3a^2$.

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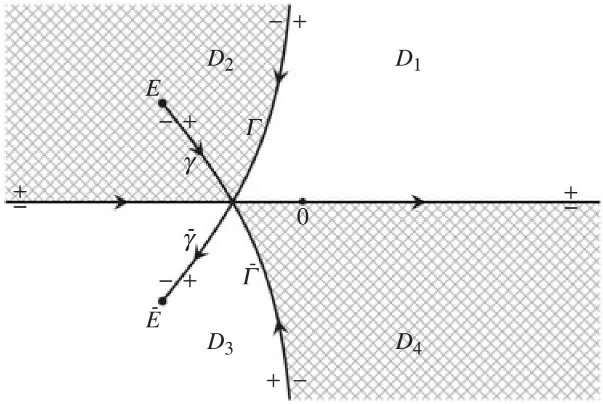

Figure 8. The oriented contour Σ for the case $\omega = -3a^2$, $a > 0$, $b > 0$.

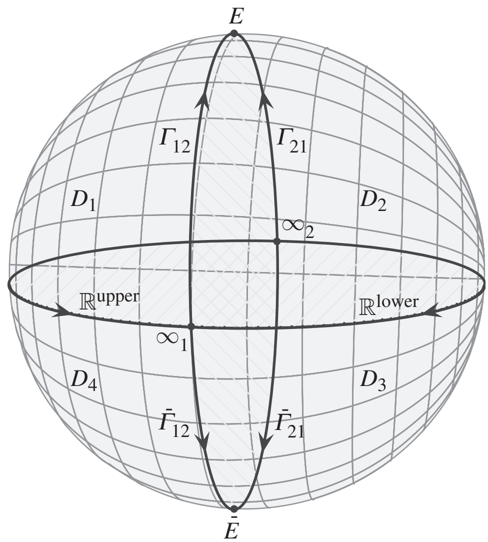

Figure 9. The oriented contour Σ for the case $\omega > -3a^2$, $a > 0$, $b > 0$.

Case $\omega \leqslant -3a^2$

The Riemann–Hilbert problem RH_{xt} is defined on the complex k-plane $\mathbb{C}$ with the oriented contour (see Figures 7 and 8)

$$\Sigma = \mathbb{R} \cup \gamma \cup \bar{\gamma} \cup \Gamma \cup \bar{\Gamma}.$$

Case $\omega > -3a^2$

The Riemann–Hilbert problem RH_{xt} is defined on the two-sheeted Riemann surface $\mathcal{X}$ of genus zero with the oriented contour (see Figure 9)

$$\Sigma = \mathbb{R}^{\text{upper}} \cup \mathbb{R}^{\text{lower}} \cup \Gamma_{12} \cup \Gamma_{21} \cup \bar{\Gamma}_{12} \cup \bar{\Gamma}_{21}.$$

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The jump matrix $J(x,t;k)$

Case $\omega \leqslant -3a^2$

We recall that $\Sigma = \mathbb{R} \cup \Gamma \cup \bar{\Gamma} \cup \gamma \cup \bar{\gamma}$ (see Figures 7 and 8). The jump matrix $J(x,t;k)$ is given by six different expressions:

$$J(x,t;k) = \begin{cases} \begin{pmatrix} 1 & -\bar{\rho}(k)\mathrm{e}^{-2\mathrm{i}(kx+(\Omega(k)-\omega)t)} \\ -\rho(k)\mathrm{e}^{2\mathrm{i}(kx+(\Omega(k)-\omega)t)} & 1+|\rho(k)|^2 \end{pmatrix}, & k \in (-\infty, \kappa_+), \\ \begin{pmatrix} 1 & -\bar{r}(k)\mathrm{e}^{-2\mathrm{i}(kx+(\Omega(k)-\omega)t)} \\ -r(k)\mathrm{e}^{2\mathrm{i}(kx+(\Omega(k)-\omega)t)} & 1+|r(k)|^2 \end{pmatrix}, & k \in (\kappa_+, \infty), \\ \begin{pmatrix} 1 & 0 \\ c(k)\mathrm{e}^{2\mathrm{i}(kx+(\Omega(k)-\omega)t)} & 1 \end{pmatrix}, & k \in \Gamma, \\ \begin{pmatrix} 1 & \bar{c}(\bar{k})\mathrm{e}^{-2\mathrm{i}(kx+(\Omega(k)-\omega)t)} \\ 0 & 1 \end{pmatrix}, & k \in \bar{\Gamma}, \\ \begin{pmatrix} \mathrm{e}^{-2\mathrm{i}\Omega^+(k)t} & 0 \\ -f(k)\mathrm{e}^{2\mathrm{i}(kx-\omega t)} & \mathrm{e}^{2\mathrm{i}\Omega^+(k)t} \end{pmatrix}, & k \in \gamma, \\ \begin{pmatrix} \mathrm{e}^{-2\mathrm{i}\Omega^+(k)t} & -\bar{f}(\bar{k})\mathrm{e}^{-2\mathrm{i}(kx-\omega t)} \\ 0 & \mathrm{e}^{2\mathrm{i}\Omega^+(k)t} \end{pmatrix}, & k \in \bar{\gamma}. \end{cases} \tag{2.36}$$

Case $\omega > -3a^2$

We recall that

$$\Sigma = \mathbb{R}^{\mathrm{upper}} \cup \mathbb{R}^{\mathrm{lower}} \cup \Gamma_{12} \cup \bar{\Gamma}_{12} \cup \Gamma_{21} \cup \bar{\Gamma}_{21}$$

(see Figure 9). The jump matrix $J(x,t;k)$ is also given by six different expressions:

$$J(x,t;k) = \begin{cases} \begin{pmatrix} 1 & -\bar{r}(k)\mathrm{e}^{-2\mathrm{i}(kx+(\Omega(k)-\omega)t)} \\ -r(k)\mathrm{e}^{2\mathrm{i}(kx+(\Omega(k)-\omega)t)} & 1+|r(k)|^2 \end{pmatrix}, & k \in \mathbb{R}^{\mathrm{upper}}, \\ \begin{pmatrix} 1 & -\bar{\rho}(k)\mathrm{e}^{-2\mathrm{i}(kx+(\Omega(k)-\omega)t)} \\ -\rho(k)\mathrm{e}^{2\mathrm{i}(kx+(\Omega(k)-\omega)t)} & 1+|\rho(k)|^2 \end{pmatrix}, & k \in \mathbb{R}^{\mathrm{lower}}, \\ \begin{pmatrix} 1 & 0 \\ c^+(k)\mathrm{e}^{2\mathrm{i}(kx+(\Omega(k)-\omega)t)} & 1 \end{pmatrix}, & k \in \Gamma_{12}, \\ \begin{pmatrix} 1 & 0 \\ -c^-(k)\mathrm{e}^{2\mathrm{i}(kx+(\Omega(k)-\omega)t)} & 1 \end{pmatrix}, & k \in \Gamma_{21}, \\ \begin{pmatrix} 1 & \bar{c}^+(\bar{k})\mathrm{e}^{-2\mathrm{i}(kx+(\Omega(k)-\omega)t)} \\ 0 & 1 \end{pmatrix}, & k \in \bar{\Gamma}_{12}, \\ \begin{pmatrix} 1 & -\bar{c}^-(\bar{k})\mathrm{e}^{-2\mathrm{i}(kx+(\Omega(k)-\omega)t)} \\ 0 & 1 \end{pmatrix}, & k \in \bar{\Gamma}_{21}. \end{cases} \tag{2.37}$$

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Here $c^+(k)$ and $c^-(k)$ are boundary values at k of the function $c(z)$ which is analytic in the domain D_2.

Residue conditions

In the presence of eigenvalues the following residue conditions hold:

$$\left.\begin{aligned}
\operatorname{res}_{k=k_j}[M(x,t;k)]_1 &= \mathrm{i}m_j^1 \mathrm{e}^{2\mathrm{i}(k_jx+(\Omega(k_j)-\omega)t)}[M(x,t;k_j)]_2, && k_j\in D_1,\\
\operatorname{res}_{k=z_j}[M(x,t;k)]_1 &= \mathrm{i}m_j^2 \mathrm{e}^{2\mathrm{i}(z_jx+(\Omega(z_j)-\omega)t)}[M(x,t;z_j)]_2, && z_j\in D_2,\\
\operatorname{res}_{k=\bar z_j}[M(x,t;k)]_2 &= -\mathrm{i}\bar m_j^2 \mathrm{e}^{-2\mathrm{i}(\bar z_jx+(\Omega(\bar z_j)-\omega)t)}[M(x,t;\bar z_j)]_1, && \bar z_j\in D_3,\\
\operatorname{res}_{k=\bar k_j}[M(x,t;k)]_2 &= -\mathrm{i}\bar m_j^1 \mathrm{e}^{2\mathrm{i}(\bar k_jx+(\Omega(\bar k_j)-\omega)t)}[M(x,t;\bar k_j)]_1, && \bar k_j\in D_4,
\end{aligned}\right\} \tag{2.38}$$

where

$$\begin{aligned}
m_j^1 &= (\mathrm{i}b(k_j)\dot a(k_j))^{-1}, &\quad m_j^2 &= -\mathrm{i}\operatorname{res}_{k=z_j} c(k),\\
\bar m_j^1 &= (\mathrm{i}\bar b(\bar k_j)\dot{\bar a}(\bar k_j))^{-1}, &\quad \bar m_j^2 &= \mathrm{i}\operatorname{res}_{k=\bar z_j} \bar c(\bar k).
\end{aligned}$$

Then the solution $q(x,t)$ of the IBV problem (1.1) for the NLS equation is given by

$$q(x,t) = 2\mathrm{i}\lim_{k\to\infty}(kM(x,t;k))_{12}. \tag{2.39}$$

Explanations. Let us explain how direct analysis motivates the introduction of RH_{xt}. Let us define a sectionally meromorphic matrix-valued function

$$M(x,t;k)=\begin{cases}
\left(\dfrac{[\Phi_2(x,t;k)]_1}{a(k)}\mathrm{e}^{\mathrm{i}(kx+(\Omega(k)-\omega)t)} \quad [\Phi_3(x,t;k)]_2\mathrm{e}^{-\mathrm{i}(kx+(\Omega(k)-\omega)t)}\right), & k\in D_1,\\[2ex]
\left(\dfrac{[\Phi_1(x,t;k)]_1}{T_{11}(k)}\mathrm{e}^{\mathrm{i}(kx+(\Omega(k)-\omega)t)} \quad [\Phi_3(x,t;k)]_2\mathrm{e}^{-\mathrm{i}(kx+(\Omega(k)-\omega)t)}\right), & k\in D_2,\\[2ex]
\left([\Phi_3(x,t;k)]_1\mathrm{e}^{\mathrm{i}(kx+(\Omega(k)-\omega)t)} \quad \dfrac{[\Phi_1(x,t;k)]_2}{T_{22}(k)}\mathrm{e}^{-\mathrm{i}(kx+(\Omega(k)-\omega)t)}\right), & k\in D_3,\\[2ex]
\left([\Phi_3(x,t;k)]_1\mathrm{e}^{\mathrm{i}(kx+(\Omega(k)-\omega)t)} \quad \dfrac{[\Phi_2(x,t;k)]_2}{\bar a(\bar k)}\mathrm{e}^{-\mathrm{i}(kx+(\Omega(k)-\omega)t)}\right), & k\in D_4.
\end{cases}$$

In view of (2.4)–(2.7) we have $\det M(x,t;k)\equiv 1$ and $M(x,t;k)$ has the following asymptotic behaviour at infinity:

$$M(x,t;k) = I + O(k^{-1}), \quad \text{as } k\to\infty.$$

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By using the scattering relations (2.13)–(2.17) it is easy to check that the jump condition (2.35) is fulfilled, i.e.

$$M_-(x,t;k) = M_+(x,t;k)J(x,t;k), \quad k \in \Sigma,$$

with the jump matrix $J(x,t;k)$ defined by (2.36) or (2.37).

The residue relations (2.38) can be easily found. Indeed, if $a(k_j) = 0$ for $k_j \in D_1$ then $[\Phi_2(x,t;k_j)]_1 = b^{-1}(k_j)[\Phi_3(x,t;k_j)]_2$. Therefore,

$$\begin{aligned}
\operatorname{res}_{k=k_j}[M(x,t;k)]_1 &= \frac{[\Phi_2(x,t;k_j)]_1}{\dot a(k_j)} \mathrm{e}^{\mathrm{i}\theta(k_j)} \\
&= \frac{[\Phi_3(x,t;k_j)]_2}{b(k_j)\dot a(k_j)} \mathrm{e}^{\mathrm{i}\theta(k_j)} \\
&= \mathrm{i} m_j^1 \mathrm{e}^{2\mathrm{i}\theta(k_j)} [M(x,t;k_j)]_2
\end{aligned}$$

and the first relation in (2.38) is proved. Now, if $T_{11}(z_j) = 0$ for $z_j \in D_2$ then

$$[\Phi_1(x,t;z_j)]_1 = -\bar B(\bar z_j) a^{-1}(z_j)[\Phi_3(x,t;z_j)]_2$$

and

$$\begin{aligned}
\operatorname{res}_{k=z_j}[M(x,t;k)]_1 &= \frac{[\Phi_1(x,t;z_j)]_1}{\dot T_{11}(z_j)} \mathrm{e}^{\mathrm{i}\theta(z_j)} \\
&= \frac{-\bar B(\bar z_j)[\Phi_3(x,t;k_j)]_2}{a(z_j)\dot T_{11}(z_j)} \mathrm{e}^{\mathrm{i}\theta(z_j)} \\
&= \mathrm{i} m_j^2 \mathrm{e}^{2\mathrm{i}\theta(z_j)} [M(x,t;z_j)]_2,
\end{aligned}$$

since $c(k) = -\bar B(\bar k) a^{-1}(k) T_{11}^{-1}(k)$ and $\operatorname{res}_{k=z_j} c(k) = \mathrm{i} m_j^2$. Thus the second relation in (2.38) is also proved. The rest in (2.38) is proved similarly. The proof of the formula (2.39) for the solution $q(x,t)$ is the same as in [**19**].

3. Inverse scattering problem

3.1. Inverse x-scattering problem: reconstruction of initial data

Let us define the spectral functions for the x-scattering problem.

Definition of $a(k)$, $b(k)$. Let $q_0(x) \in \mathcal{S}(\mathbb{R}_+)$. The map

$$\mathbb{S}_x : \{q_0(x)\} \to \{a(k), b(k)\} \tag{3.1}$$

is defined by

$$\begin{pmatrix} b(k) \\ a(k) \end{pmatrix} = \varphi(0;k),$$

with the vector-valued function $\varphi(x;k)$ satisfying the equation

$$\varphi_x(x;k) + \mathrm{i}k\sigma_3\varphi(x;k) = Q_0(x)\varphi(x;k), \quad 0 < x < \infty, \quad \text{for } k \in \mathbb{C}_+,$$

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and the asymptotic condition

$$\lim_{x\to+\infty} \mathrm{e}^{-\mathrm{i}kx}\varphi(x;k) = \begin{pmatrix}0\\1\end{pmatrix},$$

with $Q_0(x)$ given by

$$Q_0(x) = \begin{pmatrix}0 & q_0(x)\\ -\bar{q}_0(x) & 0\end{pmatrix}.$$

Properties of $a(k)$, $b(k)$. *The spectral functions $a(k)$ and $b(k)$ satisfy the following conditions.*

(i) *$a(k)$, $b(k)$ are analytic and bounded for $k \in \mathbb{C}_+$.*

(ii) *$a(k), b(k) \in \mathcal{C}^\infty(\mathbb{R})$.*

(iii) *$|a(k)|^2 + |b(k)|^2 \equiv 1$, $k \in \mathbb{R}$.*

(iv) *$a(k) = 1 + O(k^{-1})$, $b(k) = O(k^{-1})$, $k \to \infty$.*

The Riemann–Hilbert problem RH_x is defined as follows.

Definition of $\mathbf{RH}_{x}$. Let $a(k)$ and $b(k)$ be given as above. Denote by $k_1,\dots,k_n$ the simple zeros of $a(k)$. Find a 2×2 matrix-valued function $M^{(x)}(x;k)$ satisfying (3.2 a)–(3.2g).

(3.2 a) $M^{(x)}(x;k)$ is sectionally meromorphic in $k \in \mathbb{C}\setminus\mathbb{R}$.

(3.2 b) $M^{(x)}(x;k) = I + O(1/k)$, for $k \to \infty$.

(3.2 c) $M^{(x)}(x;k)$ satisfies the jump relation across $\mathbb{R}$:

$$M_-^{(x)}(x;k) = M_+^{(x)}(x;k)J^{(x)}(x;k) \quad \text{for } k \in \mathbb{R},$$

where the jump matrix is given by

$$J^{(x)}(x;k) = \begin{pmatrix} 1 & -\dfrac{b(k)}{\bar{a}(k)}\mathrm{e}^{-2\mathrm{i}kx} \\ \dfrac{-\bar{b}(k)}{a(k)}\mathrm{e}^{2\mathrm{i}kx} & \dfrac{1}{|a(k)|^2}\end{pmatrix} = \begin{pmatrix} 1 & -\bar{r}(k)\mathrm{e}^{-2\mathrm{i}kx} \\ -r(k)\mathrm{e}^{2\mathrm{i}kx} & 1+|r(k)|^2\end{pmatrix}. \tag{3.2 d}$$

(3.2 e) The only singularities of the first column $[M^{(x)}(x;k)]_1$ of $M^{(x)}(x;k)$ are possibly simple poles at $k = k_1,\dots,k_n$, and for the second column $[M^{(x)}(x;k)]_2$ the only singularities are possibly simple poles at $k = \bar{k}_1,\dots,\bar{k}_n$. The associated residues are given by

$$\operatorname{res}_{k=k_j}[M^{(x)}(x;k)]_1 = \frac{\mathrm{e}^{2\mathrm{i}k_jx}}{\dot{a}(k_j)b(k_j)}[M^{(x)}(x;k_j)]_2, \tag{3.2 f}$$

$$\operatorname{res}_{k=\bar{k}_j}[M^{(x)}(x;k)]_2 = \frac{-\mathrm{e}^{-2\mathrm{i}\bar{k}_jx}}{\overline{\dot{a}(\bar{k}_j)b(\bar{k}_j)}}[M^{(x)}(x;\bar{k}_j)]_1. \tag{3.2 g}$$

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Explanations. Let us explain briefly how direct analysis motivates the consideration of RH_x. Let us rewrite (2.13) for $t = 0$ in the form

$$\Phi_2(x,0;k) = \Phi_3(x,0;k)s^{-1}(k),$$

where

$$s^{-1}(k) = \begin{pmatrix} a(k) & -b(k) \\ \bar{b}(\bar{k}) & \bar{a}(\bar{k}) \end{pmatrix}.$$

Let us define the sectionally meromorphic matrix $M^{(x)}(x;k)$ by

$$M^{(x)}(x;k) := \begin{cases} \begin{pmatrix} [\Phi_3(x,0;k)]_1 \mathrm{e}^{\mathrm{i}kx} & \dfrac{[\Phi_2(x,0;k)]_2}{\bar{a}(\bar{k})}\mathrm{e}^{-\mathrm{i}kx} \end{pmatrix} & \text{for } k \in \mathbb{C}_-, \\ \begin{pmatrix} \dfrac{[\Phi_2(x,0;k)]_1}{a(k)}\mathrm{e}^{\mathrm{i}kx} & [\Phi_3(x,0;k)]_2 \mathrm{e}^{-\mathrm{i}kx} \end{pmatrix} & \text{for } k \in \mathbb{C}_+. \end{cases}$$

One can then check (see [**19**, Appendix A.1]) that

- $M^{(x)}(x;k)$ satisfies all properties (3.2) of the Riemann–Hilbert problem RH_x;
- we can recover $q_0(x)$ from $M^{(x)}(x;k)$ by formula $q_0(x) = 2\mathrm{i}\lim_{k\to\infty} kM_{12}^{(x)}(x;k)$.

□

Proposition 3.1. *The map $\mathbb{S}_x$ has an inverse $\mathbb{Q}_x : \{a(k), b(k)\} \mapsto \{q_0(x)\}$ given by*

$$q_0(x) = 2\mathrm{i} \lim_{k\to\infty} kM_{12}^{(x)}(x;k),$$

where $M^{(x)}(x;k)$ is the unique solution of the above Riemann–Hilbert problem RH_x.

Proof. See also [**19**, Appendix A.1]. □

3.2. Inverse t-scattering problem: reconstruction of boundary data

Let us define the spectral functions for the t-scattering problem.

Definition of $A(k)$, $B(k)$. Let

$$g_0(t) := q(0,t) = a\mathrm{e}^{2\mathrm{i}\omega t},$$
$$g_1(t) := q_x(0,t) = 2\mathrm{i}ab\mathrm{e}^{2\mathrm{i}\omega t} + v_1(t),$$

where $v_1(t) \in \mathcal{S}(\mathbb{R}_+)$. The map

$$\mathbb{S}_t : \{g_0(t), g_1(t)\} \to \{A(k), B(k)\} \tag{3.3}$$

is defined by

$$\begin{pmatrix} -B(k) \\ \bar{A}(\bar{k}) \end{pmatrix} = \lim_{t\to+\infty} \hat{\Phi}(t;k),$$

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where the vector-valued function $\hat{\Phi}(t;k) = [\Phi(t;k)]_2 = [\Psi^{-1}(t;k)\Phi_2(0,t;k)]_2$ satisfies the integral equation

$$\hat{\Phi}(t;k) = [E^{-1}(k)]_2 + \int_0^t \Psi^{-1}(\tau;k)\tilde{Q}_0(\tau;k)\Psi(\tau;k)\hat{\Phi}(\tau;k)\,\mathrm{d}\tau, \quad 0 < t < \infty,$$

with

$$\Psi(t;k) = \mathrm{e}^{\mathrm{i}\omega t\sigma_3}E(k)\mathrm{e}^{-\mathrm{i}\Omega(k)t\sigma_3}, \qquad \tilde{Q}_0(t;k) = \tilde{Q}(0,t;k) - \tilde{Q}_{\mathrm{p}}(t;k).$$

Properties of $A(k)$, $B(k)$. *The spectral functions $A(k)$ and $B(k)$ satisfy the following conditions.*

(i) $A(k)$, $B(k)$ *are analytic and bounded for* $k \in D_+ = D_1 \cup D_3$.

(ii) $A(k), B(k) \in \mathcal{C}^\infty(\Sigma \setminus \{E \cup \bar{E}\})$, $E = -b + \mathrm{i}a$, $\bar{E} = -b - \mathrm{i}a$.

(iii) $A(k)\bar{A}(\bar{k}) + B(k)\bar{B}(\bar{k}) \equiv 1$ *for* $k \in \Sigma$.

(iv)
$$A(k) - \sqrt{\frac{X(k)+k+b}{2X(k)}} \quad \text{and} \quad B(k) - \mathrm{i}\mathrm{e}^{\mathrm{i}\alpha}\sqrt{\frac{k+b-X(k)}{2X(k)}}$$

are bounded for $k \in \bar{D}_+$.

Moreover, for $k \to \infty$,

$$A(k) = \sqrt{\frac{X(k)+k+b}{2X(k)}}[1+O(k^{-1})] \quad \text{and} \quad B(k) = \mathrm{i}\mathrm{e}^{\mathrm{i}\alpha}\sqrt{\frac{k+b-X(k)}{2X(k)}}[1+O(k^{-1})].$$

The Riemann–Hilbert problem RH_t is defined as follows.

Definition of RH_t. Let $A(k)$ and $B(k)$ be as above. We denote by $\kappa_1, \dots, \kappa_m$ the simple zeros of $A(k)$ in $D_+ = D_1 \cup D_3$. Find a 2×2 matrix-valued function $M^{(t)}(t;k)$ satisfying (3.4 *a*)–(3.4 *i*).

(3.4 *a*) $M^{(t)}(t;k)$ is sectionally meromorphic in $k \in \mathbb{C} \setminus \Sigma$ or $k \in \mathfrak{X} \setminus \Sigma$ where the *contour* Σ is defined as follows.

- For the case $\omega \leqslant -3a^2$,
$$\Sigma = \mathbb{R} \cup \Gamma \cup \bar{\Gamma} \cup \gamma \cup \bar{\gamma}$$
in the complex plane (see Figures 7 and 8).

- For the case $\omega > -3a^2$,
$$\Sigma = \mathbb{R}^{\mathrm{upper}} \cup \mathbb{R}^{\mathrm{lower}} \cup \Gamma_{12} \cup \Gamma_{21} \cup \bar{\Gamma}_{12} \cup \bar{\Gamma}_{21}$$
in the Riemann surface $\mathfrak{X}$ (see Figure 9).

(3.4 *b*) $M^{(t)}(t;k) = I + O(1/k)$, for $k \to \infty$.

(3.4 *c*) $M^{(t)}(t;k)$ satisfies the jump relation across Σ:

$$M_-^{(t)}(t;k) = M_+^{(t)}(t;k)J^{(t)}(t;k) \quad \text{for } k \in \Sigma,$$

where the *jump matrix* $J^{(t)}(t;k)$ is as follows.

- For the case $\omega \leqslant -3a^2$,

$$J^{(t)}(t;k) = \begin{cases} \begin{pmatrix} \dfrac{1}{A(k)\bar{A}(\bar{k})} & -\dfrac{B(k)}{A(k)}\mathrm{e}^{-2\mathrm{i}(\Omega(k)-\omega)t} \\ -\dfrac{\bar{B}(\bar{k})}{\bar{A}(\bar{k})}\mathrm{e}^{2\mathrm{i}(\Omega(k)-\omega)t} & 1 \end{pmatrix}, & k \in \mathbb{R} \cup \Gamma \cup \bar{\Gamma}, \\ \begin{pmatrix} \mathrm{e}^{-2\mathrm{i}\Omega^+(k)t} & 0 \\ -\bar{h}(\bar{k})\mathrm{e}^{-2\mathrm{i}\omega t} & \mathrm{e}^{2\mathrm{i}\Omega^+(k)t} \end{pmatrix}, & k \in \gamma, \\ \begin{pmatrix} \mathrm{e}^{-2\mathrm{i}\Omega^+(k)t} & h(k)\mathrm{e}^{2\mathrm{i}\omega t} \\ 0 & \mathrm{e}^{2\mathrm{i}\Omega^+(k)t} \end{pmatrix}, & k \in \bar{\gamma}, \end{cases} \tag{3.4 d}$$

where

$$h(k) := \left(\frac{B(k)}{A(k)}\right)^- - \left(\frac{B(k)}{A(k)}\right)^+ = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\alpha}}{A^+(k)A^-(k)}. \tag{3.4 e}$$

- For the case $\omega > -3a^2$,

$$J^{(t)}(t;k) = \begin{cases} \begin{pmatrix} 1 & \dfrac{B(k)}{A(k)}\mathrm{e}^{-2\mathrm{i}(\Omega(k)-\omega)t} \\ \dfrac{\bar{B}(\bar{k})}{\bar{A}(\bar{k})}\mathrm{e}^{2\mathrm{i}(\Omega(k)-\omega)t} & \dfrac{1}{A(k)\bar{A}(\bar{k})} \end{pmatrix}, \\ \qquad k \in \Gamma_{12} \cup \bar{\Gamma}_{21} \cup \mathbb{R}^{\text{lower}}, \\ \begin{pmatrix} \dfrac{1}{A(k)\bar{A}(\bar{k})} & -\dfrac{B(k)}{A(k)}\mathrm{e}^{-2\mathrm{i}(\Omega(k)-\omega)t} \\ -\dfrac{\bar{B}(\bar{k})}{\bar{A}(\bar{k})}\mathrm{e}^{2\mathrm{i}(\Omega(k)-\omega)t} & 1 \end{pmatrix}, \\ \qquad k \in \Gamma_{21} \cup \bar{\Gamma}_{12} \cup \mathbb{R}^{\text{upper}}. \end{cases} \tag{3.4 f}$$

(3.4 *g*) The first column of $M^{(t)}(t;k)$ can have simple poles at $k = \bar{\kappa}_1, \dots, \bar{\kappa}_m$, and the second column of $M^{(t)}(t;k)$ can have simple poles at $k = \kappa_1, \dots, \kappa_m$. The associated residues are given by

$$\operatorname{res}_{k=\bar{\kappa}_j}[M^{(t)}(t;k)]_1 = -\mathrm{e}^{2\mathrm{i}(\Omega(\bar{\kappa}_j)-\omega)t}\frac{\bar{B}(\bar{\kappa}_j)}{\dot{\bar{A}}(\bar{\kappa}_j)}[M^{(t)}(t;\bar{\kappa}_j)]_2, \tag{3.4 h}$$

$$\operatorname{res}_{k=\kappa_j}[M^{(t)}(t;k)]_2 = \mathrm{e}^{-2\mathrm{i}(\Omega(\kappa_j)-\omega)t}\frac{B(\kappa_j)}{\dot{A}(\kappa_j)}[M^{(t)}(t;\kappa_j)]_1. \tag{3.4 i}$$

Explanations. Let us explain how direct analysis motivates the consideration of RH_t. The scattering relations of the t-problem are

$$\frac{1}{\bar{A}(\bar{k})}[\Phi_1(0,t;k)]_1 = [\Phi_2(0,t;k)]_1 - \frac{\bar{B}(\bar{k})}{\bar{A}(\bar{k})}[\Phi_2(0,t;k)]_2, \tag{3.5 a}$$

$$\frac{1}{A(k)}[\Phi_1(0,t;k)]_2 = \frac{B(k)}{A(k)}[\Phi_2(0,t;k)]_1 + [\Phi_2(0,t;k)]_2, \quad k \in \Sigma, \tag{3.5 b}$$

where the functions $[\Phi_2(0,t;k)]_{1,2}$ are entire, and the functions $[\Phi_1(0,t;k)]_{1,2}$ are analytic in the domains $D_- = D_2 \cup D_4$ and $D_+ = D_1 \cup D_3$, respectively. Starting from these scattering relations (3.5) we can define the sectionally meromorphic matrix-valued function

$$M^{(t)}(t;k) = \begin{cases} \begin{pmatrix} [\Phi_2(0,t;k)]_1 \mathrm{e}^{\mathrm{i}(\Omega(k)-\omega)t} & \dfrac{[\Phi_1(0,t;k)]_2}{A(k)} \mathrm{e}^{-\mathrm{i}(\Omega(k)-\omega)t} \end{pmatrix}, \\ \qquad\qquad k \in D_+ = D_1 \cup D_3, \\ \begin{pmatrix} \dfrac{[\Phi_1(0,t;k)]_1}{\bar{A}(\bar{k})} \mathrm{e}^{\mathrm{i}(\Omega(k)-\omega)t} & [\Phi_2(0,t;k)]_2 \mathrm{e}^{-\mathrm{i}(\Omega(k)-\omega)t} \end{pmatrix}, \\ \qquad\qquad k \in D_- = D_2 \cup D_4. \end{cases}$$

This matrix has unit determinant, and satisfies the asymptotics (3.4 b):

$$M^{(t)}(t;k) = I + O(k^{-1}) \quad \text{for } k \to \infty.$$

Proof of the residue relations. The first column of $M^{(t)}(t;k)$ can have poles at $k = \bar{\kappa}_j$, $j = 1, \dots, m$, where $\{\bar{\kappa}_j\}_{j=1}^m$ are the zeros of $\bar{A}(\bar{k})$, $k \in D_-$. For simplicity we *suppose* that these zeros are simple. The associated residues are

$$\begin{aligned} \operatorname{res}_{k=\bar{\kappa}_j}[M^{(t)}(t;k)]_1 &= \frac{[\Phi_1(0,t;\bar{\kappa}_j)]_1}{\dot{\bar{A}}(\bar{\kappa}_j)} \mathrm{e}^{2\mathrm{i}(\Omega(\bar{\kappa}_j)-\omega)t} \\ &= -\frac{\bar{B}(\bar{\kappa}_j)}{\dot{\bar{A}}(\bar{\kappa}_j)} \mathrm{e}^{2\mathrm{i}(\Omega(\bar{\kappa}_j)-\omega)t} [\Phi_2(0,t;\bar{\kappa}_j)]_2 \\ &= -\frac{\bar{B}(\bar{\kappa}_j)}{\dot{\bar{A}}(\bar{\kappa}_j)} \mathrm{e}^{2\mathrm{i}(\Omega(\bar{\kappa}_j)-\omega)t} [M^{(t)}(t;\bar{\kappa}_j)]_2. \end{aligned}$$

The second column of $M^{(t)}(t;k)$ can have simple poles at $k = \kappa_j$, and

$$\operatorname{res}_{k=\kappa_j}[M^{(t)}(t;k)]_2 = \frac{B(\kappa_j)}{\dot{A}(\kappa_j)} \mathrm{e}^{-2\mathrm{i}(\Omega(\kappa_j)-\omega)t} [M^{(t)}(t;\kappa_j)]_1,$$

where the set $\{\kappa_1, \kappa_2, \dots, \kappa_m\}$ is the set of simple zeros of $A(k)$ in D_+. Thus we have proved the residue relations (3.4 h), (3.4 i).

Proof of the jump relations. First of all we find the jump of $[\Phi_1(0,t;k)]_2/A(k)$ over the contour $\bar\gamma$. Thus we have

$$\begin{aligned}\frac{[\Phi_1^-]_2}{A^-(k)}-\frac{[\Phi_1^+]_2}{A^+(k)}&=\frac{1}{A^+A^-}[\det([\Phi_2]_1,[\Phi_1^+]_2)[\Phi_1^-]_2-\det([\Phi_2]_1,[\Phi_1^-]_2)[\Phi_1^+]_2]\\&=\frac{1}{A^+A^-}\det([\Phi_1^-]_2,[\Phi_1^+]_2)[\Phi_2]_1\\&=\frac{B^-A^+-B^+A^-}{A^-A^+}[\Phi_2]_1\\&=h(k)[\Phi_2(0,t;k)]_1,\end{aligned}$$

where

$$h(k)=\left(\frac{B(k)}{A(k)}\right)^- - \left(\frac{B(k)}{A(k)}\right)^+ = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\alpha}}{A^+(k)A^-(k)}.$$

Then it is easy to verify that, for $k\in\bar\gamma$,

$$\begin{aligned}&\begin{pmatrix}[\Phi_2]_1\mathrm{e}^{\mathrm{i}\theta(k)t} & \dfrac{[\Phi_1]_2}{A(k)}\mathrm{e}^{-\mathrm{i}\theta(k)t}\end{pmatrix}_-\\&\qquad=\begin{pmatrix}[\Phi_2]_1\mathrm{e}^{\mathrm{i}\theta(k)t} & \dfrac{[\Phi_1]_2}{A(k)}\mathrm{e}^{-\mathrm{i}\theta(k)t}\end{pmatrix}_+\begin{pmatrix}\mathrm{e}^{-2\mathrm{i}\Omega^+(k)t} & h(k)\mathrm{e}^{2\mathrm{i}\omega t}\\ 0 & \mathrm{e}^{2\mathrm{i}\Omega^+(k)t}\end{pmatrix},\end{aligned}$$

with $\theta(k)=\Omega(k)-\omega$, and similarly, for $k\in\gamma$,

$$\begin{aligned}&\begin{pmatrix}\dfrac{[\Phi_1]_1}{\bar A(\bar k)}\mathrm{e}^{\mathrm{i}\theta(k)t} & [\Phi_2]_2\mathrm{e}^{-\mathrm{i}\theta(k)t}\end{pmatrix}_-\\&\qquad=\begin{pmatrix}\dfrac{[\Phi_1]_1}{\bar A(\bar k)}\mathrm{e}^{\mathrm{i}\theta(k)t} & [\Phi_2]_2\mathrm{e}^{-\mathrm{i}\theta(k)t}\end{pmatrix}_+\begin{pmatrix}\mathrm{e}^{2\mathrm{i}\Omega^+(k)t} & 0\\ -\bar h(\bar k)\mathrm{e}^{-2\mathrm{i}\omega t} & \mathrm{e}^{-2\mathrm{i}\Omega^+(k)t}\end{pmatrix}.\end{aligned}$$

For $k\in(\kappa_-,+\infty)\cup\Gamma\cup\bar\Gamma$ or for $k\in\mathbb{R}^{\text{upper}}\cup\Gamma_{21}\cup\bar\Gamma_{12}$ one can find that the relation

$$\begin{aligned}&\begin{pmatrix}\dfrac{[\Phi_1]_1}{\bar A(\bar k)}\mathrm{e}^{\mathrm{i}\theta(k)t} & [\Phi_2]_2\mathrm{e}^{-\mathrm{i}\theta(k)t}\end{pmatrix}_-\\&\qquad=\begin{pmatrix}[\Phi_2]_1\mathrm{e}^{\mathrm{i}\theta(k)t} & \dfrac{[\Phi_1]_2}{A(k)}\mathrm{e}^{-\mathrm{i}\theta(k)t}\end{pmatrix}_+\begin{pmatrix}1+\dfrac{B(k)\bar B(\bar k)}{A(k)\bar A(\bar k)} & -\dfrac{B(k)}{A(k)}\mathrm{e}^{-2\mathrm{i}\theta(k)t}\\ -\dfrac{\bar B(\bar k)}{\bar A(\bar k)}\mathrm{e}^{2\mathrm{i}\theta(k)t} & 1\end{pmatrix}\end{aligned}$$

is valid. The rest is proved similarly. These relations give rise to the jump matrix $J^{(t)}(t;k)$ defined by equations (3.4 d)–(3.4 f).

Thus $M^{(t)}(t;k)$ satisfies all properties (3.4) of the Riemann–Hilbert problem RH_t.

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Moreover, one can check that $g_0(t)$ and $g_1(t)$ can be recovered from $M^{(t)}(t;k)$ by the following formulae:

$$g_0(t) = 2\mathrm{i} \lim_{k\to\infty} kM_{12}^{(t)}(t;k),$$
$$g_1(t) = \lim_{k\to\infty} [4k^2 M_{12}^{(t)}(t;k) + 2\mathrm{i}g_0(t)kM_{22}^{(t)}(t;k)].$$

□

Proposition 3.2. *The map* $\mathbb{S}_t$ *has an inverse* $\mathbb{Q}_t : \{A(k), B(k)\} \mapsto \{g_0(t), g_1(t)\}$ *given by*

$$g_0(t) = 2\mathrm{i} \lim_{k\to\infty} kM_{12}^{(t)}(t;k),$$
$$g_1(t) = \lim_{k\to\infty} [4k^2 M_{12}^{(t)}(t;k) + 2\mathrm{i}g_0(t)kM_{22}^{(t)}(t;k)],$$

where $M^{(t)}(t;k)$ *is the unique solution of the Riemann–Hilbert problem* RH_t.

Proof. The proof is almost the same as in [**19**, Appendix A.2]. □

3.3. Inverse xt-scattering problem: reconstruction of the solution $q(x,t)$

Under the assumption that for the given function $q_0(x)$ there exists a function $v_1(t)$ such that the spectral functions $A(k)$ and $B(k)$ of the t-problem together with the spectral functions $a(k)$ and $b(k)$ of the x-problem satisfy the global relation (2.29), we can prove that $q(x,t)$ arising from the basic RH_{xt} problem satisfies the NLS equation in the quarter plane, and that its initial and boundary values coincide with $q_0(x)$, $g_0(t) = a\mathrm{e}^{2\mathrm{i}\omega t}$ and $g_1(t) = 2\mathrm{i}ab\mathrm{e}^{2\mathrm{i}\omega t} + v_1(t)$.

The Riemann–Hilbert problem RH_{xt} is defined as follows.

Definition of RH_{xt}. Find a 2×2 matrix-valued function $M(x,t;k)$ satisfying properties (3.6 a)–(3.6 d):

(3.6 a) $M(x,t;k)$ is sectionally meromorphic in $k \in \mathbb{C} \setminus \Sigma$ or $k \in \mathcal{X} \setminus \Sigma$.

(3.6 b) Its first column $[M(x,t;k)]_1$ has simple poles at $k_j \in D_1$ and $z_j \in D_2$; the second column $[M(x,t;k)]_2$ has simple poles at $\bar{k}_j \in D_4$ and $\bar{z}_j \in D_3$. The associated residues satisfy the relations (2.38).

(3.6 c) $M(x,t;k)$ satisfies the jump condition

$$M_-(x,t;k) = M_+(x,t;k)J(x,t;k), \quad \text{for } k \in \Sigma,$$

where the jump matrix $J(x,t;k)$ is defined in terms of the spectral functions by (2.36) for the case $\omega \leqslant -3a^2$ or (2.37) for the case $\omega > -3a^2$.

(3.6 d) $M(x,t;k)$ has the asymptotics

$$M(x,t;k) = I + O\left(\frac{1}{k}\right), \quad \text{for } k \to \infty.$$

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Theorem 3.3. *Let $q_0(x) \in \mathcal{S}(\mathbb{R}_+)$. Suppose that the functions $g_0(t) = a\mathrm{e}^{2\mathrm{i}\omega t}$ and $g_1(t) = 2\mathrm{i}ab\mathrm{e}^{2\mathrm{i}\omega t} + v_1(t)$ are such that the spectral functions $\{a(k), b(k), A(k), B(k)\}$ satisfy the global relation (2.29):*

$$b(k)A(k) - a(k)B(k) = 0, \quad k \in D_1.$$

Then

(i) *the above Riemann–Hilbert problem RH_{xt} has a unique solution $M(x, t; k)$;*

(ii) *if we define $q(x, t)$ in terms of this solution by*

$$q(x, t) = 2\mathrm{i} \lim_{k\to\infty} (kM(x, t; k))_{12}, \tag{3.7}$$

then

(a) *$q(x, t)$ solves the NLS equation (1.1 a),*

(b) *with*

$$q(x, 0) = q_0(x), \quad q(0, t) = g_0(t) \quad \text{and} \quad q_x(0, t) = g_1(t).$$

Proof. The 'singular' RH problem above can be mapped to a 'regular' RH problem (i.e. to an RH problem for holomorphic functions), coupled with a system of algebraic equations. The unique solvability of the relevant algebraic equations, the proof of the associated vanishing lemma, and therefore, the solvability of the corresponding RH problem are based on the symmetry properties of $J(x, t; k)$ (see [**19**]).

Proof that $q(x, t)$ solves the NLS equation. It is straightforward to prove that if $M(x, t; k)$ solves the above RH problem and if $q(x, t)$ is defined by (3.7) then $q(x, t)$ solves the NLS equation. This proof is based on ideas of the dressing method [**13**].

Proof that $\mathrm{RH}_{xt}|_{t=0} \sim \mathrm{RH}_x$. It means that the Riemann–Hilbert problem $\mathrm{RH}_{xt}|_{t=0}$ is equivalent to the problem RH_x in the following sense: there exists a sectionally meromorphic matrix $G(x; k)$ such that

$$M^{(x)}(x; k) = M(x, 0; k)G(x; k)$$

and

$$G(x; k) = I + \frac{D^1}{k} + \frac{D^2}{k^2} + \cdots + O\left(\frac{\mathrm{e}^{-C(k)x}}{|k|}\right), \quad k \to \infty,$$

where D^1, D^2, etc., are diagonal and independent on x matrices, and $C(k)$ is positive and grows for $k \to \infty$.

To prove this equivalence let us put

$$\hat{M}(x; k) = M(x, 0; k)G(x; k),$$

where $M(x,0;k)$ is defined by (3.6 c), (3.6 d) and

$$G(x;k)=\begin{cases}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & \text{for } k\in D_1\cup D_4,\\ \begin{pmatrix}1 & 0\\ -c(k)\mathrm{e}^{2\mathrm{i}kx} & 1\end{pmatrix} & \text{for } k\in D_2,\\ \begin{pmatrix}1 & \bar{c}(\bar{k})\mathrm{e}^{-2\mathrm{i}kx}\\ 0 & 1\end{pmatrix} & \text{for } k\in D_3.\end{cases}$$

We now show that $\hat{M}(x;k)$ is meromorphic in $k\in\mathbb{C}_+$. Indeed, for $k\in\Gamma$,

$$\begin{aligned}\hat{M}_-(x;k)&=M_-(x,0;k)G_-(x;k)\\ &=M_+(x,0;k)\begin{pmatrix}1 & 0\\ c(k)\mathrm{e}^{2\mathrm{i}kx} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -c(k)\mathrm{e}^{2\mathrm{i}kx} & 1\end{pmatrix}\\ &=M_+(x,0;k)\\ &=\hat{M}_+(x;k),\end{aligned}$$

and, for $k\in\gamma$,

$$\begin{aligned}\hat{M}_-(x;k)&=M_-(x,0;k)G_-(x;k)\\ &=M_-(x,0;k)\begin{pmatrix}1 & 0\\ -c^-(k)\mathrm{e}^{2\mathrm{i}kx} & 1\end{pmatrix}\\ &=M_+(x,0;k)\begin{pmatrix}1 & 0\\ f(k)\mathrm{e}^{2\mathrm{i}kx} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -c^-(k)\mathrm{e}^{2\mathrm{i}kx} & 1\end{pmatrix}\\ &=\hat{M}_+(x;k)\begin{pmatrix}1 & 0\\ c^+(k)\mathrm{e}^{2\mathrm{i}kx} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ f(k)\mathrm{e}^{2\mathrm{i}kx} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -c^-(k)\mathrm{e}^{2\mathrm{i}kx} & 1\end{pmatrix}\\ &=\hat{M}_+(x;k)\begin{pmatrix}1 & 0\\ (f(k)+c^+(k)-c^-(k))\mathrm{e}^{2\mathrm{i}kx} & 1\end{pmatrix}\\ &=\hat{M}_+(x;k),\end{aligned}$$

where we used the definition $f(k):=c^-(k)-c^+(k)$. Thus $\hat{M}(x;k)$ is continuous in $k\in\mathbb{C}_+$ with exception of poles, hence it is meromorphic in $\mathbb{C}_+$. We now investigate the residue conditions:

$$\operatorname{res}_{k=k_j}\hat{M}(x;k)=\mathrm{i}m_j^1\mathrm{e}^{2\mathrm{i}k_jx}\hat{M}(x;k_j),\quad k_j\in D_1,$$

$$\begin{aligned}\operatorname{res}_{k=z_j}\hat{M}(x;k)&=\operatorname{res}_{k=z_j}[M(x,0;k)-c(k)\mathrm{e}^{2\mathrm{i}kx}M(x,0;k)]\\ &=\operatorname{res}_{k=z_j}c(k)\mathrm{e}^{2\mathrm{i}z_jx}M(x,0;z_j)-\operatorname{res}_{k=z_j}c(k)\mathrm{e}^{2\mathrm{i}z_jx}M(x,0;z_j)\\ &=0,\quad z_j\in D_2,\end{aligned}$$

$$\begin{aligned}\operatorname{res}_{k=k_j}\hat{M}(x;k) &= \operatorname{res}_{k=k_j}[M(x,0;k) - c(k)\mathrm{e}^{2\mathrm{i}kx}M(x,0;k)] \\ &= -\operatorname{res}_{k=k_j} c(k)\mathrm{e}^{2\mathrm{i}k_jx}M(x,0;k_j) \\ &= \mathrm{i}m_j^1\mathrm{e}^{2\mathrm{i}k_jx}\hat{M}(x;k_j), \quad k_j \in D_2,\end{aligned}$$

where we used

$$\begin{aligned}\operatorname{res}_{k=k_j} c(k) &= -\frac{\bar{B}(\bar{k}_j)}{T_{11}(k_j)\dot{a}(k_j)} \\ &= -\frac{1}{b(k_j)\dot{a}(k_j)} \\ &= -\mathrm{i}m_j^1,\end{aligned}$$

which follows from $a(k_j) = 0$ and

$$\begin{aligned}T_{11}(k_j) &= [\bar{A}(\bar{k})a(k) + \bar{B}(\bar{k})b(k)]|_{k=k_j} \\ &= \bar{B}(\bar{k}_j)b(k_j).\end{aligned}$$

Hence $\hat{M}(x;k)$ is meromorphic in $k \in \mathbb{C}_+$ with poles at $k = k_j \in \mathbb{C}_+$ $(a(k_j) = 0)$. By symmetry $M(x;k)$ is meromorphic in $k \in \mathbb{C}_-$ with poles at $k = \bar{k}_j$. For $k \in [\kappa_+, \infty)$ the following jump condition is valid:

$$\begin{aligned}\hat{M}_-(x;k) &= M_-(x,0;k) \\ &= M_+(x,0;k)J(x,0;k) \\ &= \hat{M}_+(x;k)J^{(x)}(x;k).\end{aligned}$$

Similarly, taking into account the relation $c(k) - \rho(k) = -r(k)$, for $k \in (-\infty, \kappa_+)$,

$$\begin{aligned}\hat{M}_-(x;k) &= M_-(x,0;k)\begin{pmatrix}1 & \bar{c}(\bar{k})\mathrm{e}^{-2\mathrm{i}kx} \\ 0 & 1\end{pmatrix} \\ &= M_+(x,0;k)\begin{pmatrix}1 & -\bar{\rho}(\bar{k})\mathrm{e}^{-2\mathrm{i}kx} \\ -\rho(k)\mathrm{e}^{2\mathrm{i}kx} & 1+|\rho(k)|^2\end{pmatrix}\begin{pmatrix}1 & \bar{c}(\bar{k})\mathrm{e}^{-2\mathrm{i}kx} \\ 0 & 1\end{pmatrix} \\ &= \hat{M}_+(x;k)\begin{pmatrix}1 & 0 \\ c(k)\mathrm{e}^{2\mathrm{i}kx} & 1\end{pmatrix}\begin{pmatrix}1 & -\bar{\rho}(\bar{k})\mathrm{e}^{-2\mathrm{i}kx} \\ -\rho(k)\mathrm{e}^{2\mathrm{i}kx} & 1+|\rho(k)|^2\end{pmatrix}\begin{pmatrix}1 & \bar{c}(\bar{k})\mathrm{e}^{-2\mathrm{i}kx} \\ 0 & 1\end{pmatrix} \\ &= \hat{M}_+(x;k)\begin{pmatrix}1 & -\bar{r}(\bar{k})\mathrm{e}^{-2\mathrm{i}kx} \\ -r(k)\mathrm{e}^{2\mathrm{i}kx} & 1+|r(k)|^2\end{pmatrix}.\end{aligned}$$

Hence $\hat{M}(x;k) \equiv M^{(x)}(x;k)$ where the sectionally meromorphic matrix $M^{(x)}(x;k)$ is characterized by all conditions (3.2). This proves that $\mathrm{RH}_{xt}|_{t=0}$ and RH_x are equivalent Riemann–Hilbert problems.

Proof that $q(x,0) = q_0(x)$. By using this equivalence $\mathrm{RH}_{xt}|_{t=0} \sim \mathrm{RH}_x$ we obtain the initial condition (1.1 *b*). Indeed, the equality $q(x,0) = q_0(x)$ follows by comparing the large k asymptotic formulae for $M(x,0;k)$ and $M^{(x)}(x;k)$, respectively.

Proof that $\mathrm{RH}_{xt}|_{x=0} \sim \mathrm{RH}_t$. Now we prove that in an analogous sense as above $\mathrm{RH}_{xt}|_{x=0}$ is equivalent to RH_t under the *assumption* that *the global relation is valid*. Let

$$N(t;k) = M(0,t;k)H(t;k),$$

where $M(0,t;k)$ is defined by (3.6 c), (3.6 d) and

$$H(t;k) = \begin{cases} \begin{pmatrix} a(k) & 0 \\ 0 & \dfrac{1}{a(k)} \end{pmatrix} & \text{for } k \in D_1, \\ \begin{pmatrix} \dfrac{T_{11}(k)}{\bar{A}(\bar{k})} & -b(k)\mathrm{e}^{-2\mathrm{i}\theta(k)t} \\ 0 & \dfrac{\bar{A}(\bar{k})}{T_{11}(k)} \end{pmatrix} & \text{for } k \in D_2, \\ \begin{pmatrix} \dfrac{A(k)}{T_{22}(k)} & 0 \\ \bar{b}(\bar{k})\mathrm{e}^{2\mathrm{i}\theta(k)t} & \dfrac{T_{22}(k)}{A(k)} \end{pmatrix} & \text{for } k \in D_3, \\ \begin{pmatrix} \dfrac{1}{\bar{a}(\bar{k})} & 0 \\ 0 & \bar{a}(\bar{k}) \end{pmatrix} & \text{for } k \in D_4, \end{cases}$$

where $\theta(k) := \Omega(k) - \omega$. The following relations between the jump matrices $J^{(t)}(t;k)$ and $J(0,t;k)$ are easy to check:

$$\begin{aligned} J(0,t;k)H_2(t;k) &= H_1(t;k)J^{(t)}(t;k), && k \in \Gamma, \\ H_2^{-1}(t;k)J^{-1}(0,t;k) &= (J^{(t)})^{-1}(t;k)H_3^{-1}(t;k), && k \in (-\infty,\kappa_+), \\ J(0,t;k)H_{2-}(t;k) &= H_{2+}(t;k)J^{(t)}(t;k), && k \in \gamma, \\ J(0,t;k)H_{3-}(t;k) &= H_{3+}(t;k)J^{(t)}(t;k), && k \in \bar{\gamma}, \\ J(0,t;k)H_4(t;k) &= H_3(t;k)J^{(t)}(t;k), && k \in \bar{\Gamma}, \\ J(0,t;k)H_4(t;k) &= H_1(t;k)J^{(t)}(t;k), && k \in (\kappa_+,\infty). \end{aligned}$$

It is clear, for example, that the first relation for $k \in \Gamma$,

$$\begin{aligned} &\begin{pmatrix} 1 & 0 \\ c(k)\mathrm{e}^{2\mathrm{i}\theta} & 1 \end{pmatrix} \begin{pmatrix} \dfrac{T_{11}(k)}{\bar{A}(\bar{k})} & -b(k)\mathrm{e}^{-2\mathrm{i}\theta(k)t} \\ 0 & \dfrac{\bar{A}(\bar{k})}{T_{11}(k)} \end{pmatrix} \\ &\qquad = \begin{pmatrix} a(k) & 0 \\ 0 & \dfrac{1}{a(k)} \end{pmatrix} \begin{pmatrix} \dfrac{1}{A(k)\bar{A}(\bar{k})} & -\dfrac{B(k)}{A(k)}\mathrm{e}^{-2\mathrm{i}\theta} \\ -\dfrac{\bar{B}(\bar{k})}{\bar{A}(\bar{k})}\mathrm{e}^{2\mathrm{i}\theta} & 1 \end{pmatrix}, \end{aligned}$$

is satisfied due to the global relation $a(k)B(k) - b(k)A(k) = 0$ (2.29), the definitions of the functions $c(k) = -\bar{B}(\bar{k})/a(k)T_{11}(k)$ (2.21), $T_{11}(k) = a(k)\bar{A}(\bar{k}) + b(k)\bar{B}(\bar{k}) = a(k)/A(k)$ (2.19), (2.31). One can also verify all other relations by using the definitions of $f(k)$ (2.22), $h(k)$ (3.4 f), $T_{22}(k)$ (2.20), determinant relations $|a(k)|^2 + |b(k)|^2 = 1$, $A(k)\bar{A}(\bar{k}) + B(k)\bar{B}(\bar{k}) = 1$ and the equations (2.30)–(2.34). Using the arguments of [**19**] it can be verified that $N(t;k)$ has no poles at the points k_j. For the points κ_j it is easy to obtain the correct relations for the corresponding residues of the matrix $N(t;k)$. We omit similar considerations in the case of the RH-problem on the Riemann surface $\mathcal{X}$. This concludes the proof that $\mathrm{RH}_{xt}|_{x=0}$ is equivalent to RH_t.

Proof that $q(0,t) = g_0(t)$ and $q'_x(0,t) = g_1(t)$. Using

$$M^{(t)}(t;k) = N(t;k) = I + \frac{N^1(t)}{k} + \frac{N^2(t)}{k^2} + \cdots, \quad k \to \infty,$$

as well as

$$M(x,t;k) = I + \frac{M^1(x,t)}{k} + \frac{M^2(x,t)}{k^2} + \cdots, \quad k \to \infty,$$

it follows that

$$\begin{aligned} 2g_0(t) &= 2\mathrm{i}N^1_{12}(t), \\ g_1(t) &= 4N^2_{12}(t) + 2\mathrm{i}v(t)N^1_{22}(t), \\ q(x,t) &= 2\mathrm{i}M^1_{12}(x,t), \\ q_x(x,t) &= 4M^2_{12}(x,t) + 2\mathrm{i}q(x,t)M^1_{22}(x,t). \end{aligned}$$

Using the relations $M^{(t)}(t;k) = M(0,t;k)H(t;k)$, as well as the asymptotic expansion

$$H(t;k) = I + \frac{D^1}{k} + \frac{D^2}{k^2} + \cdots + O\left(\frac{\mathrm{e}^{-C(k)t}}{|k|}\right),$$

where D^1, D^2, etc., are diagonal and independent on t matrices, and $C(k)$ is positive and grows for $k \to \infty$, we find

$$N^1(t) = M^1(0,t) + D^1, \qquad N^2(t) = M^2(0,t) + M^1(0,t)D^1 + D^2.$$

These relations yield the Dirichlet boundary values

$$g_0(t) = 2\mathrm{i}N^1_{12}(t) = 2\mathrm{i}(M^1_{12}(0,t) + D^1_{12}) = 2\mathrm{i}M^1_{12}(0,t) = q(0,t),$$

as well as the Neumann boundary condition:

$$\begin{aligned} g_1(t) &= 4N^2_{12}(t) + 2\mathrm{i}g_0(t)N^1_{22}(t) \\ &= 4M^2_{12}(0,t) + 4(M^1(0,t)D^1)_{12} + 4D^2_{12} + 2\mathrm{i}g_0(t)(M^1_{22}(0,t) + D^1_{22}) \\ &= 4M^2_{12}(0,t) + 2\mathrm{i}q(0,t)M^1_{22}(0,t) + 4M^1_{12}(0,t)D^1_{22} + 2\mathrm{i}q(0,t)D^1_{22} \\ &= q_x(0,t) + 2\mathrm{i}(q(0,t) - 2\mathrm{i}M^1_{12}(0,t))D^1_{22} \\ &= q_x(0,t). \end{aligned}$$

□

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As was mentioned in the introduction, the justification of the structure of the Dirichlet to Neumann map will be studied in a forthcoming paper.

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