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KOSTANT'S PROBLEM FOR PARABOLIC VERMA MODULES

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Abstract We give a complete combinatorial classification of the parabolic Verma modules in the principal block of the parabolic category $\mathcal{O}$ associated with a minimal or a maximal parabolic subalgebra of the special linear Lie algebra for which the answer to Kostant's problem is positive.

Keywords: parabolic Verma modules; Kostant's problem; Hecke algebras; Harish-Chandra bimodules; projective functors

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1. Introduction and description of the results

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra over the field of complex numbers and M be a $\mathfrak{g}$-module. The (associative) algebra of all linear endomorphisms of M has various interesting (associative) subalgebras. One of these is the image of the universal enveloping algebra $U(\mathfrak{g})$, which is naturally isomorphic to $U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(M)$. The other one is the algebra $\mathcal{L}(M, M)$ of all linear endomorphisms of M, the adjoint action of $\mathfrak{g}$ on which is locally finite. The former algebra is a subalgebra of the latter, and so it is natural to ask, for which M, the inclusion

$$U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(M) \hookrightarrow \mathcal{L}(M, M)$$

is an isomorphism.

For a given M, this is known as *Kostant's problem* (for M), as defined and popularized by Joseph in [13]. It is a well-known and, in general, wide open problem [14]. No complete answer to this problem is known, even for general simple highest weight modules. However, there are various families of modules for which the answer (sometimes positive and sometimes negative) is known, see § 3.5 for a historical overview.

The first non-trivial classical family of modules for which the answer to Kostant's problem was found is the family of Verma modules. Already in [13], it was shown that, for all Verma modules, the answer to the Kostant's problem is positive.

Verma modules are exactly the standard modules with respect to the (essentially unique, see [8]) highest weight structure on the Bernstein–Gelfand–Gelfand (*BGG*) *category* $\mathcal{O}$ for $\mathfrak{g}$, see [3, 10]. Category $\mathcal{O}$ has a number of different generalizations. One of these, proposed in [30], is called *parabolic category* $\mathcal{O}$ and is associated with the choice of a parabolic subalgebra $\mathfrak{p}$ in $\mathfrak{g}$. Just like its ancestor, parabolic category $\mathcal{O}$ is a highest weight category. The standard objects with respect to this structure are *parabolic Verma modules*.

By construction, parabolic Verma modules are quotients of the usual Verma modules. In [12, Section 7.32], one can find an argument that shows that Kostant's problem has a positive answer for any quotient of a Verma module, provided that the latter is projective in $\mathcal{O}$. In particular, Kostant's problem has a positive answer for any parabolic Verma module that is projective in a regular block of parabolic category $\mathcal{O}$. At this stage, it is quite natural to wonder what the answer to Kostant's problem for general parabolic Verma modules will be.

In small ranks and for specific modules, it is sometimes possible to answer Kostant's problem by a direct computation. The smallest non-trivial example of parabolic category $\mathcal{O}$ is for $\mathfrak{g} = \mathfrak{sl}_3$, where we take the parabolic subalgebra corresponding to one of the two simple roots. The regular block of the corresponding parabolic category $\mathcal{O}$ contains three parabolic Verma modules. As we already mentioned above, for the projective parabolic Verma module, the answer to Kostant's problem is positive. For the simple parabolic Verma module, the answer to Kostant's problem is also known to be positive, see [9]. It was fairly surprising for us to find out, by a direct computation, that for the (remaining) third parabolic Verma module, the answer to Kostant's problem is negative. This was clear evidence that determining the answer to Kostant's problem for parabolic Verma modules is a non-trivial problem as one first has to guess for which parabolic Verma modules the answer should be positive and for which it should be negative. So far, there is only one general result in the literature which answers Kostant's problem for a fairly natural general family of modules where both the cases of the positive and the negative answers occur. This is the family of simple highest weight modules over the special linear Lie algebra indexed by fully commutative permutations, see the recent preprint [22]. The main result of the present paper is the second one of this kind.

In the present paper, we study Kostant's problem for parabolic Verma modules over the special linear Lie algebra in two 'opposite' cases. The first case is the case of a minimal parabolic subalgebra, which is the case when the semi-simple part of the parabolic subalgebra is given by one simple root. In this case, parabolic Verma modules are quotients of Verma modules by Verma submodules. These kinds of quotients were studied in [19], where the emphasis was made on a description of the socle for such a quotient. This description is crucial for our proofs. Two other important ingredients in our arguments are as follows:

- An adaptation of Kåhrström's conjectural combinatorial reformulation for Kostant's problem, see [20, Conjecture 1.2].
- An appropriate analogue of the 2-representation theoretic reformulation of Kostant's problem, see [20, Subsection 8.3].

The second case we consider is that of a maximal parabolic subalgebra, which is the case when the semi-simple part of the parabolic subalgebra misses exactly one simple root. In this case, there is a very explicit diagrammatic description of the corresponding parabolic category $\mathcal{O}$ worked out in [5, 6]. To answer Kostant's problem for parabolic Verma modules in this case, we combine this description with the main result of [22]. Surprisingly, in this case, the positive answer is really rare. In fact, there are several infinite families of examples, where the answer is positive only for two parabolic Verma modules, namely, for the projective one and for the simple one, for which the answer is always positive (just like in the $\mathfrak{sl}_3$ example mentioned above).

The paper is organized as follows: In § 2, we collect all relevant preliminaries on category $\mathcal{O}$. In § 3 we give a general perspective on Kostant's problem and its history. In § 4, we study the case of a minimal parabolic subalgebra. Our main result here is Theorem 3 which completely answers Kostant's problem for parabolic Verma modules in this case and also relates it to Kåhrströms combinatorial reformulation. This theorem can be found in § 4.2. In § 5, we study the case of a maximal parabolic subalgebra. Our main result here is Theorem 5 which completely answers Kostant's problem for parabolic Verma modules in this case. This theorem can be found in § 5.5. We finish the paper with some general observations and speculations in § 6.

2. Category $\mathcal{O}$ preliminaries

2.1. Setup

In this paper, we work over the field $\mathbb{C}$ of complex numbers.

As already mentioned above, $\mathfrak{g}$ is a simple finite dimensional complex Lie algebra. We fix a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \tag{1}$$

where $\mathfrak{h}$ is some fixed Cartan subalgebra. Denote by W the Weyl group of $\mathfrak{g}$, and by S the set of simple reflections in W, which corresponds to the triangular decomposition equation (1) above. As usual, we denote by w_0 the longest element of W. Let $\leq$ denote the usual *Bruhat order* on W and ℓ the length function on W.

In this paper, we will mostly consider $\mathfrak{g} = \mathfrak{sl}_n$ with the standard triangular decomposition given by the upper triangular, the diagonal and the lower triangular matrices. In this case, we have $W \cong \mathbf{S}_n$, the symmetric group on $\{1, 2, \dots, n\}$, with S being the set of elementary transpositions.

2.2. Category $\mathcal{O}$

Associated with the triangular decomposition equation (1), we have the BGG category $\mathcal{O}$ defined as the full subcategory of the category of finitely generated $\mathfrak{g}$-modules consisting of all modules on which the action of $\mathfrak{h}$ is diagonalizable and the action of $\mathfrak{n}_+$ is locally finite. We refer to [3, 10] for details.

2.3. Principal block

Consider the principal block $\mathcal{O}_0$ of $\mathcal{O}$. This block is defined as the indecomposable direct summand of $\mathcal{O}$ which contains the trivial $\mathfrak{g}$-module. The isomorphism classes of

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simple objects in $\mathcal{O}_0$ are naturally indexed by the elements of W. For $w \in W$, we have the corresponding simple highest weight module $L_w := L(w \cdot 0)$. Here, $0 \in \mathfrak{h}^*$ denotes the zero weight and $\cdot$ denotes the dot action of W on $\mathfrak{h}^*$. For $\lambda \in \mathfrak{h}^*$, the dot action is defined as follows: $w \cdot \lambda = w(\lambda + \rho) - \rho$. Here, ρ is the half-sum of positive roots.

Furthermore, for $w \in W$, we have the Verma cover Δ_w of L_w and the indecomposable projective cover P_w of L_w. We denote by A the opposite of the endomorphism algebra of a multiplicity-free projective generator of $\mathcal{O}_0$. The algebra A is a finite dimensional and associative algebra. Moreover, the category A-mod of all finite dimensional A-modules is equivalent to the category $\mathcal{O}_0$.

2.4. Parabolic category $\mathcal{O}$

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h} \oplus \mathfrak{n}_+$. The corresponding parabolic category $\mathcal{O}^{\mathfrak{p}}$ is defined as the full subcategory of $\mathcal{O}$ that consists of all objects on which the action of $\mathfrak{p}$ is locally finite, see [30]. We further have the principal block $\mathcal{O}_0^{\mathfrak{p}}$ of $\mathcal{O}^{\mathfrak{p}}$ defined as $\mathcal{O}_0 \cap \mathcal{O}^{\mathfrak{p}}$.

Let $W_{\mathfrak{p}}$ be the parabolic subgroup of W corresponding to $\mathfrak{p}$. Denote by $W^{\mathfrak{p}}_{\mathrm{short}}$ the set of the shortest representatives in the cosets $W_{\mathfrak{p}} \setminus W$. The category $\mathcal{O}_0^{\mathfrak{p}}$ is the Serre subcategory of $\mathcal{O}_0$ generated by the simple objects L_w, where $w \in W^{\mathfrak{p}}_{\mathrm{short}}$. The category $\mathcal{O}_0^{\mathfrak{p}}$ is equivalent to $A_{\mathfrak{p}}$-mod for a certain quotient $A_{\mathfrak{p}}$ of A.

For $w \in W^{\mathfrak{p}}_{\mathrm{short}}$, we denote by $\Delta^{\mathfrak{p}}_w$ the corresponding parabolic Verma module and by $P^{\mathfrak{p}}_w$ the corresponding indecomposable projective module in $\mathcal{O}_0^{\mathfrak{p}}$.

2.5. Projective functors

Following [4], for every $w \in W$, there is a unique, up to isomorphism, indecomposable projective endofunctor θ_w of the category $\mathcal{O}_0$ such that $\theta_w\, P_e \cong P_w$. Any indecomposable projective endofunctor of $\mathcal{O}_0$ is isomorphic to θ_w, for some $w \in W$.

Consider the monoidal category $\mathscr{P}$ of all projective endofunctors of $\mathcal{O}_0$. By Soergel's combinatorial description, see [32], the monoidal category $\mathscr{P}$ is monoidally equivalent to the monoidal category of Soergel bimodules over the coinvariant algebra of the Weyl group W.

2.6. Twisting functors

For $w \in W$, we denote by $\top_w$ the corresponding twisting functor, see [1, 18]. For $x, y \in W$ such that $\ell(xy) = \ell(x) + \ell(y)$, we have $\top_x \Delta_y \cong \Delta_{xy}$. The main, for us, properties of twisting functors are that they functorially commute with projective functors, that they are acyclic on Verma modules and that the corresponding left derived functors are equivalences, see [1]. In what follows, we denote by $\mathbb{L}\top_w$ the left derived functor of the functor $\top_w$.

2.7. Graded lift

By [31], the algebra A is Koszul, and hence, it is equipped with the corresponding Koszul $\mathbb{Z}$-grading. We therefore have the category ${}^{\mathbb{Z}}\mathcal{O}_0$ of finite dimensional $\mathbb{Z}$-graded

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A-modules. We denote by $\langle\_\rangle$ the functor which shifts the grading with the convention that $\langle 1\rangle$ shifts degree 0 to degree -1.

All structural modules in the category $\mathcal{O}_0$ admit graded lifts, see [34] for details. These graded lifts are unique, up to isomorphism and grading shift, for all indecomposable modules.

We fix standard graded lifts of all indecomposable structural modules and will use for these standard graded lifts the same notation as for the corresponding ungraded objects in $\mathcal{O}_0$, abusing notation. Similarly, we have the standard graded lifts of the indecomposable projective functors, see [34]. We denote by $\mathscr{P}^{\mathbb{Z}}$ the corresponding monoidal category of graded projective endofunctors of ${}^{\mathbb{Z}}\mathcal{O}_0$.

The algebra $A_{\mathfrak{p}}$ is a graded quotient of A, and we have the corresponding graded analogue ${}^{\mathbb{Z}}\mathcal{O}_0^{\mathfrak{p}}$ of the category $\mathcal{O}_0^{\mathfrak{p}}$.

2.8. Hecke algebra combinatorics

Consider the Hecke algebra $\mathbf{H} = \mathbf{H}(W, S)$ of W in the normalization of [33]. It is an algebra over the commutative ring $\mathbb{Z}[v, v^{-1}]$. This algebra has the standard basis $\{H_w : w \in W\}$ as well as the Kazhdan–Lusztig basis $\{\underline{H}_w : w \in W\}$, defined in [17].

The Grothendieck group $\mathbf{Gr}[{}^{\mathbb{Z}}\mathcal{O}_0]$ of the category ${}^{\mathbb{Z}}\mathcal{O}_0$ is isomorphic, as a $\mathbb{Z}[v, v^{-1}]$-module, to $\mathbf{H}$ by sending $[\Delta_w]$ to H_w. This isomorphism intertwines $\langle 1\rangle$ and multiplication with v^{-1} and sends $[P_w]$ to $\underline{H}_w$, see [2, 7].

The split Grothendieck ring $\mathbf{Gr}_{\oplus}[\mathscr{P}^{\mathbb{Z}}]$ of the monoidal category $\mathscr{P}^{\mathbb{Z}}$ is isomorphic to the algebra $\mathbf{H}$ by sending $[\theta_w]$ to $\underline{H}_w$. The defining action of $\mathscr{P}^{\mathbb{Z}}$ on the category $\mathcal{O}_0^{\mathbb{Z}}$ decategorifies to the right regular representation of $\mathbf{H}$, see [32].

Denote by $\leq_L$, $\leq_R$ and $\leq_J$ the left, right and two-sided Kazhdan-Lusztig preorders on W, respectively. The equivalence classes associated with these preorders are called left, right and two-sided cells. Each left and each right cell contains a distinguished involution, which is called the Duflo involution. For symmetric groups, all involutions are, in fact, Duflo involutions.

The singletons $\{e\}$ and $\{w_0\}$ are two-sided cells. Furthermore, we have the *small* two-sided cell $\mathcal{J}_s$ containing all simple reflections. In the case of the symmetric group $\mathbf{S}_n$, the small cell contains exactly $n-1$ left and $n-1$ right cells and consists of all elements having exactly one unique expression. Multiplying the elements of this small two-sided cell with w_0, we obtain the *penultimate* two-sided cell $\mathcal{J}_p$ (also referred to as the *subregular* cell in the literature).

The set $W^{\mathfrak{p}}_{\text{short}}$ is a union of right cells. If $w_0^{\mathfrak{p}}$ is the longest elements of $W_{\mathfrak{p}}$, then $W^{\mathfrak{p}}_{\text{short}}$ is the principal ideal with respect to $\leq_R$, generated by the element $w_0^{\mathfrak{p}}w_0$. Furthermore, for $w \in W^{\mathfrak{p}}_{\text{short}}$, the projective module $P^{\mathfrak{p}}(w)$ is injective if and only if w and $w_0^{\mathfrak{p}}w_0$ belong to the same right cell. If $\mathfrak{p}$ is a minimal parabolic, that is, $W_{\mathfrak{p}}$ contains just one simple reflection s, the element $w_0^{\mathfrak{p}}w_0 = sw_0$ belongs to $\mathcal{J}_p$.

2.9. Bigrassmannian permutations and socles of cokernels of inclusions of Verma modules

Recall that an element $w \in W$ is called *bigrassmannian*, provided that there is a unique simple reflection s such that $sw < w$ and there is a unique simple reflection t (note that

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t might be equal to s) such that $wt < w$. By [19, Theorem 1], in the case of the algebra $\mathfrak{sl}_n$, for $w \in \mathbf{S}_n$, the module $\Delta_e/(\Delta_w\langle -\ell(w)\rangle)$ has a simple socle if and only if w is bigrassmannian.

Furthermore, the map $w \mapsto \mathrm{Soc}(\Delta_e/(\Delta_w\langle -\ell(w)\rangle))$ is a bijection between the set of all bigrassmannian elements in $\mathbf{S}_n$ and graded subquotients of Δ_e of the form $L_w\langle i\rangle$, where $w \in \mathcal{J}_p$. If $w \in \mathbf{S}_n$ is not bigrassmannian, then the socle of $\Delta_e/(\Delta_w\langle -\ell(w)\rangle$ is given by the elements that correspond, under the above bijection, to those bigrassmannian elements in the Bruhat interval $[e, w]$ that are maximal among all bigrassmannian elements in this interval with respect to the Bruhat order.

3. Kostant's problem

3.1. Harish–Chandra bimodules

Recall from [12, Kapitel 6], that a $\mathfrak{g}$–$\mathfrak{g}$-bimodule M is called a *Harish–Chandra bimodule* provided that it is finitely generated as a bimodule and, additionally, that the adjoint action of $\mathfrak{g}$ on M is locally finite with finite multiplicities for composition factors. For example, for any $M \in \mathcal{O}_0$, the bimodule $U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(M)$ is a Harish–Chandra bimodule.

Another example can be given as follows: for two objects M and N in $\mathcal{O}_0$, the space $\mathcal{L}(M, N)$ of all linear maps from M to N on which the adjoint action of $\mathfrak{g}$ is locally finite forms a Harish–Chandra bimodule. In the case $M = N$, we have a natural inclusion $U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(M) \subset \mathcal{L}(M, M)$.

3.2. Classical Kostant's problem

As we have already mentioned in § 1, for a $\mathfrak{g}$-module M, the corresponding Kostant's problem, as formulated in [13], is the following:

Kostant's Problem. Is the embedding

$$U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(M) \hookrightarrow \mathcal{L}(M, M)$$

an isomorphism?

We will denote by $\mathbf{K}(M) \in \{\texttt{true}, \texttt{false}\}$ the logical value of the claim 'the embedding $U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(M) \hookrightarrow \mathcal{L}(M, M)$ is an isomorphism'.

3.3. Kåhrström's conjecture

The following conjecture, formulated in [19, Conjecture 1.2], is due to Johan Kåhrström:

Conjecture 1. *Let d be an involution in the symmetric group $\boldsymbol{S}_n$. Then, the following assertions are equivalent:*

(1) $\mathbf{K}(L_d) = \texttt{true}$.
(2) *For $x, y \in W$ such that $x \neq y$, $\theta_x L_d \neq 0$ and $\theta_y L_d \neq 0$, we have $\theta_x L_d \not\cong \theta_y L_d$.*

(3) *For* $x, y \in W$ *such that* $x \neq y$, $\theta_x L_d \neq 0$ *and* $\theta_y L_d \neq 0$, *we have* $[\theta_x L_d] \neq [\theta_y L_d]$ *in* $\mathbf{Gr}[\mathcal{O}_0^{\mathbb{Z}}]$.

(4) *For* $x, y \in W$ *such that* $x \neq y$, $\theta_x L_d \neq 0$ *and* $\theta_y L_d \neq 0$, *we have* $[\theta_x L_d] \neq [\theta_y L_d]$ *in* $\mathbf{Gr}[\mathcal{O}_0]$.

3.4. Kostant's problem and 2-representation theory

Recall that the monoidal category $\mathscr{P}$ acts on $\mathcal{O}_0$. Given $M \in \mathcal{O}_0$, we can consider its annihilator $\mathrm{Ann}_{\mathscr{P}}(M)$. By definition, this is the left monoidal ideal of $\mathscr{P}$ consisting of all morphisms in $\mathscr{P}$ whose evaluation at M is zero. The quotient $\mathscr{P}/\mathrm{Ann}_{\mathscr{P}}(M)$ is, naturally, a birepresentation of $\mathscr{P}$.

Alternatively, the additive closure $\mathrm{add}(\mathscr{P}\, M)$ is also a birepresentation of $\mathscr{P}$. Mapping

$$\mathscr{P} \ni \theta \mapsto \theta(M) \in \mathrm{add}(\mathscr{P}\, M)$$

defines an injective morphism of birepresentation from $\mathscr{P}/\mathrm{Ann}_{\mathscr{P}}(M)$ to $\mathrm{add}(\mathscr{P}\, M)$. The arguments from [20, Subsection 8.3] show that this morphism is an equivalence if and only if $\mathbf{K}(M) = \mathtt{true}$.

In general, finitary birepresentations of finitary bicategories can be described in terms of (co)algebra objects in a certain abelianization of the bicategory, see [23]. In the case of the bicategory $\mathscr{P}$, the abelianization in question is the category of Harish–Chandra bimodules. Furthermore, the fact that the birepresentation $\mathrm{add}(\mathscr{P}\, M)$ of $\mathscr{P}$ is given by the algebra $\mathcal{L}(M, M)$ follows from [12, Subsection 6.8]. The principal birepresentation of $\mathscr{P}$, which is the natural left action of $\mathscr{P}$ on $\mathscr{P}$, corresponds to the identity 1-morphism of $\mathscr{P}$ by [23, Formula (4.2)]. In the world of Harish–Chandra bimodules, the algebra $U(\mathfrak{g})$ surjects onto the identity 1-morphism of $\mathscr{P}$. Consequently, the induced action of $\mathscr{P}$ on $\mathscr{P}/\mathrm{Ann}_{\mathscr{P}}(M)$ corresponds to the algebra given by the quotient of the identity 1-morphism of $\mathscr{P}$ (and hence also of $U(\mathfrak{g})$) by the annihilator of M. The algebra embedding $U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(M) \to \mathcal{L}(M, M)$ is a manifestation of the fact that $\mathscr{P}/\mathrm{Ann}_{\mathscr{P}}(M)$ is a sub-birepresentation of $\mathrm{add}(\mathscr{P}\, M)$, and therefore, the positive answer to Kostant's problem for M reformulates into the property of this natural embedding being an equivalence.

This reformulation allows us to connect the answers to Kostant's problem for modules related by twisting functors.

Lemma 2. *Let* $M \in \mathcal{O}$ *and* $w \in W$ *be such that* $\mathbb{L}\top_w(M) \cong \top_w(M) \in \mathcal{O}$. *Then,* $\mathbf{K}(M) = \mathbf{K}(\top_w(M))$.

Proof. Since $\top_w$ functorially commutes with projective functors, see § 2.6, under the assumptions of our lemma, we have $\mathrm{Ann}_{\mathscr{P}}(M) = \mathrm{Ann}_{\mathscr{P}}(\top_w(M))$ as well as an equivalence $\mathrm{add}(\mathscr{P}M) \cong \mathrm{add}(\mathscr{P}\top_w M)$ of birepresentations of $\mathscr{P}$. Therefore, the claim of the lemma follows from the birepresentation-theoretical reformulation of Kostant's problem described above. □

We refer the reader to [24, 27, 28] for more details on representations of finitary bicategories.

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3.5. Known results on Kostant's problem

Below, we present a list of the results on Kostant's problem which we could find in the existing literature.

- In the case of symmetric groups, the value $\mathbf{K}(L_w)$ is constant on Kazhdan–Lusztig left cells, as shown in [29, Theorem 61].
- Let $w_0^{\mathfrak{p}}$ denote the longest element in $W_{\mathfrak{p}}$. Then, $\mathbf{K}(L_{w_0^{\mathfrak{p}} w_0}) = \texttt{true}$, see [9, Theorem 4.4] and [12, Section 7.32].
- If $s \in W^{\mathfrak{p}}$ is a simple reflection, then $\mathbf{K}(L_{s w_0^{\mathfrak{p}} w_0}) = \texttt{true}$, see [25, Theorem 1].
- [15, Theorem 1.1] relates the answer to Kostant's problem for certain highest weight $\mathfrak{g}$-modules to the answer to Kostant's problem for certain highest weight modules over the Levi quotients of $\mathfrak{p}$.
- [16, Theorem 1] proposes a module-theoretic characterization of the equality $\mathbf{K}(L_w) = \texttt{true}$.
- In [16, Section 4], one can find a complete answer to Kostant's problem for simple highest weight modules over $\mathfrak{sl}_n$, where $n = 2, 3, 4, 5$, and a partial answer for the same problem for $\mathfrak{sl}_6$. Some further $\mathfrak{sl}_6$-answers were computed in [15, Section 6]. The highest weight $\mathfrak{sl}_6$-story was completed in [20, Subsection 10.1].
- The paper [22] provides a complete answer to Kostant's problem for simple highest weight $\mathfrak{sl}_n$-modules indexed by fully commutative permutations.

4. Main results: minimal parabolic subalgebras

4.1. Preliminaries

For $n \geq 1$, we consider the symmetric group $\mathbf{S}_n$. We view elements of $\mathbf{S}_n$ as functions on $\underline{n} := \{1, 2, \ldots, n\}$ which we compose from right to left.

Fix $k \in \{1, 2, \ldots, n-1\}$. For $1 \leq i < j \leq n$, denote by $\tau_{i,j}^{n,k}$ the unique element of $\mathbf{S}_n$ such that

- $\tau_{i,j}^{n,k}(i) = k$;
- $\tau_{i,j}^{n,k}(j) = k+1$;
- for all $s < t$ in $\underline{n} \setminus \{i, j\}$, we have $\tau_{i,j}^{n,k}(s) < \tau_{i,j}^{n,k}(t)$.

We denote by $\mathcal{X}_{n,k}$ the set of all these $\tau_{i,j}^{n,k}$. Clearly, $|\mathcal{X}_{n,k}| = \binom{n}{2}$. We further split $\mathcal{X}_{n,k}$ into two disjoint subsets: $\mathcal{X}_{n,k} = \mathcal{X}_{n,k}^{+} \coprod \mathcal{X}_{n,k}^{-}$, where

$$\mathcal{X}_{n,k}^{+} = \{\tau_{i,i+1}^{n,k} : i = 1, 2, \ldots, n-1\} \quad \text{and} \quad \mathcal{X}_{n,k}^{-} = \mathcal{X}_{n,k} \setminus \mathcal{X}_{n,k}^{+}.$$

Here is an example for $n=4$ and $k=2$, where in the first row, we list all elements in $\mathcal{X}_{4,2}^{+}$, and in the second row, we list all elements in $\mathcal{X}_{4,2}^{-}$ (for convenience, we colour in magenta the important edges in these graphs which go to $k=2$ and $k+1=3$):

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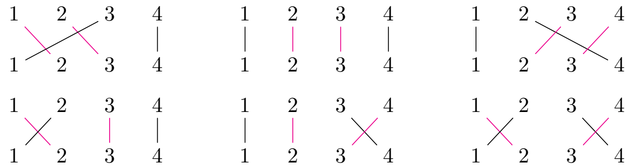

We denote by G_k the centralizer of $(k, k+1)$ in $\mathbf{S}_n$. The group G_k consists of all elements of $\mathbf{S}_n$, which leave $\{k, k+1\}$ invariant; in particular, G_k is the direct product of the symmetric groups on $\{k, k+1\}$ and $\underline{n} \setminus \{k, k+1\}$. Hence $|G_k| = 2 \cdot (n-2)!$. Let $\hat{G}_k$ denote the subgroup of G_k consisting of all elements which fix both k and $k+1$. Then, $\hat{G}_k$ is naturally isomorphic to the symmetric group on $\underline{n} \setminus \{k, k+1\}$ and $|\hat{G}_k| = (n-2)!$.

4.2. Formulation

For $k \in \{1, 2, \dots, n-1\}$, consider the parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_k$ of $\mathfrak{sl}_n$ corresponding to the simple reflection $(k, k+1)$. The composition $\circ$ in $\mathbf{S}_n$ gives rise to a bijection

$$\hat{G}_k \times \mathcal{X}_{n,k} \to (\mathbf{S}_n)^{\mathfrak{p}}_{\text{short}}.$$

Theorem 3. *For $w \in W^{\mathfrak{p}}_{\text{short}}$, the following assertions are equivalent:*

(a) $\mathbf{K}(\Delta^{\mathfrak{p}}_w) = \mathtt{true}$.
(b) *For $x, y \in W$ such that $x \neq y$, $\theta_x \Delta^{\mathfrak{p}}_w \neq 0$ and $\theta_y \Delta^{\mathfrak{p}}_w \neq 0$, we have $\theta_x \Delta^{\mathfrak{p}}_w \not\cong \theta_y \Delta^{\mathfrak{p}}_w$ (as ungraded modules).*
(c) *For all $x, y \in W$ such that $x \neq y$, $\theta_x \Delta^{\mathfrak{p}}_w \neq 0$ and $\theta_y \Delta^{\mathfrak{p}}_w \neq 0$, we have $[\theta_x \Delta^{\mathfrak{p}}_w] \neq [\theta_y \Delta^{\mathfrak{p}}_w]\langle i \rangle$, for $i \in \mathbb{Z}$, in $\mathbf{Gr}[\mathcal{O}^{\mathbb{Z}}_0]$.*
(d) *For all $x, y \in W$ such that $x \neq y$, $\theta_x \Delta^{\mathfrak{p}}_w \neq 0$ and $\theta_y \Delta^{\mathfrak{p}}_w \neq 0$, we have $[\theta_x \Delta^{\mathfrak{p}}_w] \neq [\theta_y \Delta^{\mathfrak{p}}_w]$ in $\mathbf{Gr}[\mathcal{O}_0]$.*
(e) $w \in \hat{G}_k \circ \mathcal{X}^+_{n,k}$.
(f) *The annihilator of $\Delta^{\mathfrak{p}}_w$ in $U(\mathfrak{g})$ is a primitive ideal.*

We would like to point out the subtle difference between Theorem 3(c) and Conjecture 1(3), namely, the appearance of all possible graded shifts in Theorem 3(c). Theorem 3 provides a complete answer to Kostant's problem for parabolic Verma modules in the above setup.

4.3. Positive cases

Let $w \in \hat{G}_k \circ \mathcal{X}^+_{n,k}$. For this w, we will prove that all assertions in Theorem 3 hold. Note that Theorem 3(e) is obvious. Also note the obvious implications Theorem 3(d) $\Rightarrow$ Theorem 3(c) $\Rightarrow$Theorem 3(b).

Assume that $w = \sigma \circ \tau^{n,k}_{i,i+1}$, for some $\sigma \in \hat{G}_k$ and some $i \in \{1, 2, \dots, n-1\}$. We have the short exact sequence

$$0 \to \Delta_{(i,i+1)} \to \Delta_e \to \Delta^{\mathfrak{p}_i}_e \to 0.$$

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Since $\Delta_e^{\mathfrak{p}_i}$ is a quotient of Δ_e, we have $\mathbf{K}(\Delta_e^{\mathfrak{p}_i}) = \mathtt{true}$ by [12, Section 7.32].

Note that $\tau_{i,i+1}^{n,k} \circ (i,i+1) = (k,k+1) \circ \tau_{i,i+1}^{n,k}$ and that $\sigma \circ (i,i+1) = (i,i+1) \circ \sigma$ since $\sigma \in \hat{G}_k$. From the definition of $\hat{G}_k$, it also follows that $\ell((k,k+1)w) = \ell(w)+1$. Therefore, applying $\top_w$ to the above short exact sequence, we obtain the isomorphisms $\top_w \Delta_{(i,i+1)} \cong \Delta_{(k,k+1)w}$ and $\top_w \Delta_e \cong \Delta_w$ and, additionally, that the non-zero map $\Delta_{(i,i+1)} \hookrightarrow \Delta_e$ is mapped to a non-zero map $\Delta_{(k,k+1)w} \to \Delta_w$, which is automatically injective as any non-zero homomorphism between the Verma modules is injective. Therefore, from the right exactness of $\top_w$, we obtain a short exact sequence

$$0 \to \Delta_{(k,k+1)w} \to \Delta_w \to \Delta_w^{\mathfrak{p}} \to 0.$$

Since $\top_w$ is acyclic on Verma modules, it follows that $\Delta_w^{\mathfrak{p}} \cong \top_w \Delta_e^{\mathfrak{p}_i} \cong \mathbb{L}\top_w \Delta_e^{\mathfrak{p}_i}$. Therefore, we can now apply Lemma 2 and conclude that $\mathbf{K}(\Delta_w^{\mathfrak{p}}) = \mathtt{true}$, giving Theorem 3(a).

The non-zero objects of the form $\theta_x\, \Delta_e^{\mathfrak{p}_i}$ are exactly the indecomposable projective objects in $\mathcal{O}_0^{\mathfrak{p}_i}$. Since the latter category is a highest weight category, it has a finite global dimension, and thus, the images of the indecomposable projective objects form a basis in the Grothendieck group of this category. In particular, these images are linearly independent and therefore different.

As $\mathbb{L}\top_w$ is a derived equivalence, the classes of the non-zero modules of the form $\top_w \theta_x\, \Delta_e^{\mathfrak{p}_i}$ are also linearly independent in the corresponding Grothendieck group. In particular, they are different, which implies Theorem 3(d) (and thus also both Theorem 3(c) and Theorem 3(b), see above).

It remains to prove Theorem 3(f). The construction of twisting functors via localization (see [18]) immediately implies that $\mathrm{Ann}_{U(\mathfrak{g})}(\top_w M) \supset \mathrm{Ann}_{U(\mathfrak{g})}(M)$. The right adjoints of twisting functors, which can be realized via Enright completion functors when acting on Verma modules, have the same property. This means that $\mathrm{Ann}_{U(\mathfrak{g})}(\Delta_w^{\mathfrak{p}}) = \mathrm{Ann}_{U(\mathfrak{g})}(\Delta_e^{\mathfrak{p}_i})$. The fact that the latter annihilator coincides with the annihilator of the (simple) socle of $\Delta_e^{\mathfrak{p}_i}$ is a standard fact, for example, it follows from [15, Proposition 5.1]. Therefore, $\mathrm{Ann}_{U(\mathfrak{g})}(\Delta_e^{\mathfrak{p}_i})$ (and thus also $\mathrm{Ann}_{U(\mathfrak{g})}(\Delta_w^{\mathfrak{p}})$) is a primitive ideal.

This completes the proof of all positive cases.

4.4. Negative cases

Let $w \in \hat{G}_k \circ \mathcal{X}_{n,k}^-$. For this w, we will prove that all assertions in Theorem 3 fail. Note that ¬Theorem 3(e) is obvious. Also note the obvious implication ¬Theorem 3(b) ⇒ ¬Theorem 3(d).

The argument with twisting functors used in the previous subsection reduces the present consideration to the case $w \in \mathcal{X}_{n,k}^-$, that is, $w = \tau_{i,j}^{n,k}$, for some $j \geq i+2$. We note that this w is not bigrassmannian. Let us use the results of [19] to analyse the subquotients of $\Delta^{\mathfrak{p}}_{\tau_{i,j}^{n,k}} = \Delta_{\tau_{i,j}^{n,k}} / \Delta_{(k,k+1)\tau_{i,j}^{n,k}}$ of the form L_x, where $x \in \mathcal{J}_p$ (recall that $\mathcal{J}_p$ denotes the penultimate cell, see § 2.8). We will loosely call such subquotients *penultimate*.

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To understand the penultimate subquotients in $\Delta^{\mathfrak{p}}_{\tau^{n,k}_{i,j}}$, we just need to compare such subquotients for the modules $\Delta_e/\Delta_{\tau^{n,k}_{i,j}}$ and for $\Delta_e/\Delta_{(k,k+1)\tau^{n,k}_{i,j}}$. By [19, Theorem 1], the difference between these two sets is in bijection with the bigrassmannian elements in the Bruhat complement $[e,(k,k+1)\tau^{n,k}_{i,j}]\setminus[e,\tau^{n,k}_{i,j}]$. Taking into account the very explicit forms of $\tau^{n,k}_{i,j}$ and $(k,k+1)\tau^{n,k}_{i,j}$, we can determine all bigrassmannian elements in this Bruhat complement. This, however, will require consideration of a number of different cases.

To simplify our notation, we denote $\tau^{n,k}_{i,j}$ simply by τ. We also denote the elementary transposition $(l,l+1)$ by s_l.

4.5. Case 1: $k=i$

In this case, $\tau = s_{k+1}s_{k+2}\dots s_{j-1}$ and the Bruhat complement $[e,s_k\tau]\setminus[e,\tau]$ contains the following bigrassmannian elements: s_k, s_ks_{k+1},..., $s_k\tau$. Note that there are at least two such elements and that they all have different right descents.

Consequently, the module $\Delta^{\mathfrak{p}}_{\tau^{n,k}_{i,j}}$ contains at least two (pairwise) non-isomorphic penultimate subquotients. From [19, Proposition 22], it follows that they appear in different degrees. Let L_x and L_y be two of them (living in some degrees). Note that both x and y belong to the same Kazhdan–Lusztig right cell, namely, the unique right cell inside $\mathscr{J}_p$, which indexes some simples in $\mathcal{O}^{\mathfrak{p}}_0$. Since $x\neq y$, these two elements belong to different Kazhdan–Lusztig left cells. In particular, from [27, Lemma 12], it follows that $\theta_{x^{-1}}$ kills all simples in $\mathcal{O}^{\mathfrak{p}}_0$ except for L_x, while $\theta_{y^{-1}}$ kills all simples in $\mathcal{O}^{\mathfrak{p}}_0$ except for L_y.

Therefore, up to graded shift, $\theta_{x^{-1}}\Delta^{\mathfrak{p}}_w$ is isomorphic to $\theta_{x^{-1}}L_x$. Similarly, $\theta_{y^{-1}}\Delta^{\mathfrak{p}}_w$ is isomorphic to $\theta_{y^{-1}}L_y$. However, we have $\theta_{x^{-1}}L_x\cong P^{\mathfrak{p}}_d\cong\theta_{y^{-1}}L_y$, where d is the Duflo involution in the right cell of x, see [26, Section 3]. Note that the latter cell coincides with the right cell of y. This shows that Theorem 3(b), Theorem 3(c) and Theorem 3(d) fail.

Since Theorem 3(b) fails, the birepresentations $\mathscr{P}/\mathrm{Ann}_{\mathscr{P}}(\Delta^{\mathfrak{p}}_w)$ and $\mathrm{add}(\mathscr{P}\,\Delta^{\mathfrak{p}}_w)$ cannot be equivalent, and hence, Theorem 3(a) fails as well.

Finally, since L_x and L_y belong to different left cells inside the same two-sided cell, their annihilators are incomparable primitive ideals of minimal Gelfand–Kirillov dimension (among all other primitive ideals for $\mathcal{O}^{\mathfrak{p}}_0$). Therefore, the annihilator of $\Delta^{\mathfrak{p}}_w$ must be contained in the intersection of these two ideals and hence cannot be a primitive ideal. This shows that Theorem 3(f) fails and completes Case 1.

4.6. Case 2: $k+1=j$

This case follows from Case 1 using the symmetry of the root system.

4.7. Case 3: $i>k$

In this case, $\tau = s_{k+1}s_{k+2}\dots s_{j-1}s_ks_{k+1}\dots s_{i-1}$ and the Bruhat complement $[e,s_k\tau]\setminus[e,\tau]$ contains the following bigrassmannian elements: $s_ks_{k+1}\dots s_i$, $s_ks_{k+1}\dots s_{i+1}$...,

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$s_k s_{k+1} \dots s_{j-1}$. Note that there are at least two such elements (since $j \neq i+1$). Applying to them the arguments from Case 1, we complete the proof.

4.8. Case 4: $j < k + 1$

This case follows from Case 3 using the symmetry of the root system.

4.9. Case 5: $i < k$ and $j > k + 1$

In this case, $\tau = s_{k-1}s_{k-2}\dots s_i s_{k+1}s_{k+2}\dots s_{j-1}$ and the Bruhat complement $[e, s_k\tau] \setminus [e, \tau]$ contains the following bigrassmannian elements: s_k, $s_k s_{k+1}, \dots, s_k s_{k+1}\dots s_{j-1}$, $s_k s_{k-1}, \dots\ s_k s_{k-1}\dots s_i$. Clearly, there are at least two such elements. Applying to them the arguments from Case 1, we complete the proof.

4.10. Asymptotic

From Theorem 3, we see that the ratio of positive vs negative cases equals

$$\frac{|\mathcal{X}^+_{n,k}|}{|\mathcal{X}^-_{n,k}|} = \frac{n-1}{\binom{n}{2} - (n-1)}.$$

This goes to 0 when n goes to infinity, which aligns with the results of [22, Section 6]. It is also an interesting observation that this ratio does not depend on k.

5. Main results: maximal parabolic subalgebras

5.1. Setup

For $1 \leq k < n$, set $m = n - k$. Let $\mathfrak{q} = \mathfrak{q}_k$ be the unique parabolic subalgebra of $\mathfrak{sl}_n$ whose Levi factor is $\mathfrak{sl}_k \oplus \mathfrak{sl}_m$, with $\mathfrak{sl}_k$ adjusted at the top left corner. This means that the only simple reflection that is missed by $\mathfrak{q}_k$ is $(k, k+1)$.

5.2. Weights and oriented cup diagrams

We will now present a concise version of the combinatorial diagrammatic description of the category $\mathcal{O}_0^{\mathfrak{q}}$ from [5, 6], so we refer the reader to these paper for all technical details.

We denote by $\mathcal{W}_{n,k}$ the set of all words of length n in the alphabet with two letters, $\vee$ and $\wedge$, in which the letter $\wedge$ appears exactly k times. The letter $\vee$ should be thought of as the head of an arrow pointing down, while the letter $\wedge$ should be thought of as the head of an arrow pointing up. For example, we have

$$\mathcal{W}_{4,2} = \{\wedge\wedge\vee\vee,\ \wedge\vee\wedge\vee,\ \wedge\vee\vee\wedge,\ \vee\wedge\wedge\vee,\ \vee\wedge\vee\wedge,\ \vee\vee\wedge\wedge\}.$$

We call the unique word in $\mathcal{W}_{n,k}$ which starts with k wedges (i.e., $\wedge$) *dominant*. Hence, in the above example, the dominant word is $\wedge\wedge\vee\vee$.

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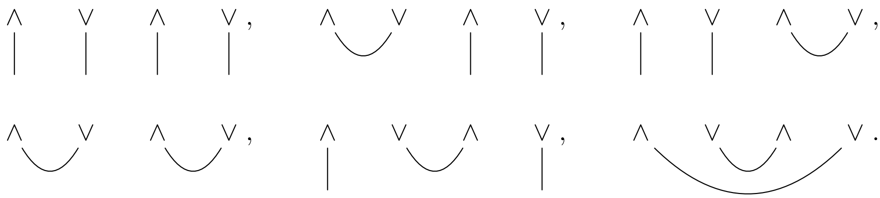
The group $\mathbf{S}_n$ acts on $\mathcal{W}_{n,k}$ by permuting the positions of the letters in a word. This action induces a natural bijection Φ between $W^{\mathfrak{q}}_{\text{short}}$ and $\mathcal{W}_{n,k}$, which sends $w \in W^{\mathfrak{q}}_{\text{short}}$ to the image of the dominant word under w^{-1}.

Given an element $\lambda \in \mathcal{W}_{n,k}$, we can form an *oriented cup diagram* by attaching to letters of λ ends of vertically falling strings and cups that altogether form a planar and oriented diagram. In particular, the requirement to be oriented means that the two ends of a cup should be attached to different letters (i.e., one to a $\wedge$ and the other one to a $\vee$). For example, here are the six oriented cup diagrams that can be drawn for the elements $\wedge \vee \wedge \vee$:



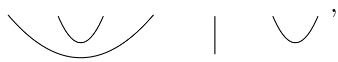
The *degree* of an oriented cup diagram is the number of cups oriented clockwise.

5.3. Standard modules

An oriented cup diagram is called *admissible*, provided that this diagram does not contain any vertical strand under $\vee$ to the left of any vertical strand under $\wedge$. For example, for the element $\wedge \vee \wedge \vee$, out of the six oriented cup diagrams above, all but the first one (i.e., but the one with four vertical strands) are admissible.

For each $\lambda \in \mathcal{W}_{n,k}$, there is a unique admissible oriented cup diagram $\mathbf{d}(\lambda)$ for λ of degree zero. It can be constructed recursively using the following algorithm. If λ does not contain any $\vee$ to the left of some $\wedge$, then $\mathbf{d}(\lambda)$ consists of vertical strings. Otherwise, λ must contain a subword $\vee\wedge$ which we connect by a cup which becomes oriented counterclockwise. We can now remove this cup and proceed recursively.

Conversely, given an unoriented cup diagram $\mathbf{d}$ with at most $\min(k, m)$ cups, there is a unique $\lambda \in \mathcal{W}_{n,k}$ such that, removing the orientation from $\mathbf{d}(\lambda)$, we obtain $\mathbf{d}$. In order to construct λ, we orient each cup in $\mathbf{d}$ counterclockwise. After orienting cups, the admissibility condition will uniquely determine positions of the remaining $\wedge$s and $\vee$s. For example, if $n=7$ and $k=3$, then, for the diagram,

$\mathbf{d} =$  ,

we get the corresponding element $\vee \vee \wedge \wedge \vee \vee \wedge \in \mathcal{W}_{7,3}$.

The following proposition is a reformulation (with adaptation to our terminology) of [5, Theorem 5.1].

Proposition 4. *Let $x, y \in W^{\mathfrak{q}}_{\text{short}}$ and $i \in \mathbb{Z}_{\geq 0}$. Then, the simple module L_y appears as a simple subquotient of the parabolic Verma module $\Delta^{\mathfrak{q}}_x$ in degree i (and then, necessarily, with multiplicity one) if and only if there is an admissible oriented cup diagram of degree*

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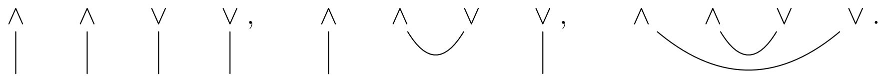
i for $\Phi(x)$ such that the associated unoriented cup diagram coincides with the unoriented cup diagram associated with $\mathbf{d}(\Phi(y))$.

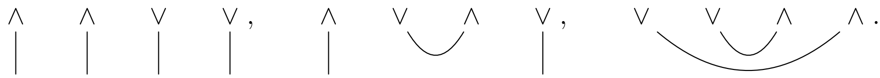
Let us consider the example of $n=4$ and $k=2$. In this example, we have

$$W^{\mathfrak{q}}_{\text{short}} = \{e, s_2, s_2s_1, s_2s_3, s_2s_1s_3, s_2s_1s_3s_2\}.$$

For the element $x=e$, here are the corresponding admissible oriented cup diagrams:



The corresponding degrees, from left to right, are 0, 1 and 2. For each underlying unoriented cup diagram, we can now write the corresponding words for which the diagram becomes of degree zero:



We see that the first word corresponds to e, the second to s_2 and the third to $s_2s_1s_3s_2$. We conclude that $\Delta^{\mathfrak{q}}_e$ has L_e in degree 0, and then it has L_{s_2} in degree 1 and, finally, $L_{s_2s_1s_3s_2}$ in degree 2.

For the element $x = s_2$, we saw five admissible oriented cup diagrams in § 5.2. From this, we conclude that $\Delta^{\mathfrak{q}}_{s_2}$ has L_{s_2} in degree 0, then $L_{s_2s_1}$, $L_{s_2s_3}$ and $L_{s_2s_1s_3s_2}$ in degree 1 and, finally, $L_{s_2s_1s_3}$ in degree 2.

5.4. Thin standard modules

Set $\mathtt{a} = \min(k, m)$. For $w \in W^{\mathfrak{q}}_{\text{short}}$, we will say that the module $\Delta^{\mathfrak{q}}_w$ is *thin*, provided that there is a unique $y \in W^{\mathfrak{q}}_{\text{short}}$ such that

- L_y is a subquotient of $\Delta^{\mathfrak{q}}_w$;
- the diagram $\mathbf{d}(\Phi(y))$ contains exactly $\mathtt{a}$ cups.

We note that, for $w \in W^{\mathfrak{q}}_{\text{short}}$, the number of cups in $\mathbf{d}(\Phi(w))$ coincides with the value of Lusztig's **a**-function on w, see [21]. Hence, the value $\mathtt{a}$ is the maximum value which the **a**-function attains at the elements of $W^{\mathfrak{q}}_{\text{short}}$. By a result of Irving, see [11, Proposition 4.3], any socular constituent L_y of any $\Delta^{\mathfrak{q}}_w$, where $w \in W^{\mathfrak{q}}_{\text{short}}$, has the property that the diagram $\mathbf{d}(\Phi(y))$ contains exactly $\mathtt{a}$ cups. Consequently, a thin parabolic Verma module must have simple socle.

It is easy to check, by a direct computation, that in our running example $n=4$ and $k=2$, the list of all thin parabolic Verma modules looks as follows:

$$\Delta^{\mathfrak{q}}_e, \quad \Delta^{\mathfrak{q}}_{s_2s_1}, \quad \Delta^{\mathfrak{q}}_{s_2s_3}, \quad \Delta^{\mathfrak{q}}_{s_2s_1s_3s_2}.$$

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5.5. Formulation

Let n and k be as above. Denote by $\mathcal{Y}_{n,k}$ the set of all elements $w \in W^{\mathfrak{q}}_{\text{short}}$ such that the word $\Phi(w)$ has the following property:

- if $k < n-k$, then all $\vee$s in w appear next to each other;
- if $k > n-k$, then all $\wedge$s in w appear next to each other;
- if $k = n-k$, then either all $\vee$s or all $\wedge$s in w appear next to each other.

For example, if $n=5$ and $k=2$, then $k = 2 < 3 = n-k$ and

$$\mathcal{Y}_{5,2} = \{\wedge\wedge\vee\vee\vee, \quad \wedge\vee\vee\vee\wedge, \quad \vee\vee\vee\wedge\wedge\}.$$

At the same time, if $n=4$ and $k=2$, then $k = n-k$ and we have

$$\mathcal{Y}_{4,2} = \{\wedge\wedge\vee\vee, \quad \wedge\vee\vee\wedge, \quad \vee\wedge\wedge\vee, \quad \vee\vee\wedge\wedge\}.$$

We can now formulate our main result for maximal parabolic subalgebras.

Theorem 5. *Let n and k be as above.*

(a) *If $k \in \{1, n-1, \frac{n}{2}\}$, then the only $w \in W^{\mathfrak{q}}_{\text{short}}$ for which $\mathbf{K}(\Delta^{\mathfrak{q}}_w) = \texttt{true}$ are $w = e$ and $w = w_0^{\mathfrak{q}} w_0$.*
(b) *If $k \neq \frac{n}{2}$, then, for $w \in W^{\mathfrak{q}}_{\text{short}}$, we have $\mathbf{K}(\Delta^{\mathfrak{q}}_w) = \texttt{true}$ if and only if $w \in \mathcal{Y}_{n,k}$.*

We note that the case $k \in \{1, n-1\}$ is covered by both statements of Theorem 5. The next two subsections are dedicated to prove Theorem 5.

5.6. Non-thin parabolic Verma modules are Kostant negative

If $\Delta^{\mathfrak{q}}_w$ is not thin, for some $w \in W^{\mathfrak{q}}_{\text{short}}$, then there exist different $x, y \in W^{\mathfrak{q}}_{\text{short}}$ of maximal **a**-value such that both L_x and L_y are subquotients of $\Delta^{\mathfrak{q}}_w$ (necessarily with multiplicity 1). Let d be the unique involution in the Kazhdan–Lusztig right cell of x (which coincides with the corresponding cell for y). Then,

$$\theta_{x^{-1}} \Delta^{\mathfrak{q}}_w \cong \theta_{x^{-1}} L_x \cong \theta_d L_d \cong \theta_{y^{-1}} L_y \cong \theta_{y^{-1}} \Delta^{\mathfrak{q}}_w.$$

Since $\theta_{x^{-1}} \not\cong \theta_{y^{-1}}$ while $\theta_{x^{-1}} \Delta^{\mathfrak{q}}_w \cong \theta_{y^{-1}} \Delta^{\mathfrak{q}}_w$, from § 3.4, it follows that $\mathbf{K}(\Delta^{\mathfrak{q}}_w) = \texttt{false}$.

5.7. Kostant's problem for thin parabolic Verma modules

After the observation in the previous subsection, we are left to consider thin parabolic Verma modules. We start with a classification of such modules.

Proposition 6. *For $w \in W^{\mathfrak{q}}_{\text{short}}$, the module $\Delta^{\mathfrak{q}}_w$ is thin if and only if $w \in \mathcal{Y}_{n,k}$.*

Proof. For $\lambda \in \mathcal{W}_{n,k}$, define the *signature* of λ as the vector $(a_1, a_2, \ldots, a_l)$ of positive integers where

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- λ starts on the left with a_1 equal letters $\vee$ (or $\wedge$);
- λ continues with a_2 equal letters $\wedge$ (respectively, $\vee$); and
- so on.

For example, the signature of $\vee\wedge\wedge\vee\wedge\wedge\wedge$ equals $(1,2,1,3)$. The number l will be called the *flip number*, which is 4 in our example.

It is straightforward to check that all λ have the flip number at most two index thin parabolic Verma modules. So, let us take some λ with flip number $l>2$. Assume that there exists $1<i<l$ such that $a_i<a_{i+1}+a_{i-1}$. We claim that in this case, λ indexes a parabolic Verma module that is not thin.

To prove this, we need to construct at least two different oriented cup diagrams with $\min(k,n-k)$ cups for this λ. Take any connected subword $\wedge\vee$ or $\vee\wedge$ in which one of the letters is in the a_i-part of λ, connect the two letters in this subword by a cup and remove it. In this way, we reduce a_i by 1 and we also reduce either a_{i+1} or a_{i-1} by 1. We can do this recursively such that we reduce a_i to 1 while keeping the new $a_{i\pm1}$s positive. This gives a connected subword of the form either $\vee\wedge\vee$ or $\wedge\vee\wedge$. Here, we clearly see that we can connect the middle letter by a cup either to the letter on the right or to the letter on the left. This gives rise to two different oriented cup diagrams as desired.

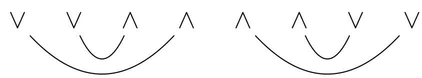
So, if λ indexes a thin parabolic Verma module, then $a_i\geq a_{i+1}+a_{i-1}$, for all $1<i<l$. In particular, $a_i\geq a_{i+1}$ and $a_i\geq a_{i-1}$. Hence, $l\leq 3$. As the case $l\leq 2$ is already settled, it remains to consider the case $l=3$.

In the case $a_2<a_1+a_3$, the above argument implies that λ indexes a parabolic Verma module that is not thin. If $a_2\geq a_1+a_3$, we get $\lambda\in\mathcal{Y}_{n,k}$ by definition. It is easy to see that such λ do indeed index thin parabolic Verma modules. □

Now, we need to consider two cases. Recall that, to avoid trivial cases, we assume $1\leq k\leq n-1$. Let w be such that $\lambda=\Phi(w)$. Note that if the flip number of λ is equal to 2, then either $w=e$ or $w=w_0^{\mathfrak{q}}w_0$. In the first case, the corresponding parabolic Verma module is a quotient of the projective Verma module, and hence, it is Kostant positive. In the second case, the corresponding parabolic Verma module is simple and Kostant positive by § 3.5, second bullet. Thus, we assume that the flip number of λ is equal to 3.

Case 1. Assume $k=\frac{n}{2}$. In this case, we have $a_2=a_1+a_3$. This implies that any admissible oriented cup diagram for λ with k cups must have two non-nested cups next to each other. By the main result of [22], the corresponding simple module L_u, which is the socle of our parabolic Verma module, is Kostant negative.

Let us now analyse the module $\Delta_w^{\mathfrak{q}}$ in more detail. There is a unique admissible oriented cup diagram for λ with k cups. Any other admissible oriented cup diagrams for λ are obtained from this one by replacing some outer clockwise oriented cups by vertical strings. Here is an example, for $n=8$ and $k=4$, of an original cup diagram



and the cup diagrams that can be obtained from it by replacing some outer clock-wise oriented cups:

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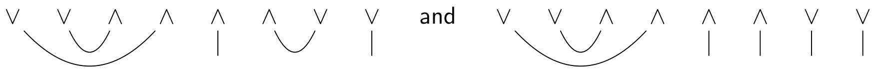
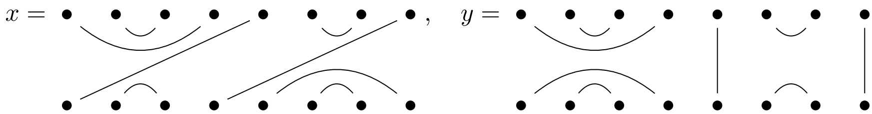


In [22, Section 5.5], one can find an explicit construction of two different elements x and y in W such that $\theta_x L_u \cong \theta_y L_u$. In the case of the above (generic) example, these elements x and y are given by the following diagrams:



Note the outer right lower cup in x. For all admissible oriented cup diagrams for λ, except the original one, this cup hits vertical strands. This means that θ_x kills the corresponding simple module; in other words, $\theta_x L_u \cong \theta_x \Delta^{\mathfrak{q}}_w$. Similar arguments apply to general w leading to the same conclusion $\theta_x L_u \cong \theta_x \Delta^{\mathfrak{q}}_w$.

Since $\theta_x L_u \cong \theta_y L_u$ and $\theta_y L_u$ is a submodule of $\theta_y \Delta^{\mathfrak{q}}_w$, we thus have a non-zero degree zero morphism from $\theta_x \Delta^{\mathfrak{q}}_w$ to $\theta_y \Delta^{\mathfrak{q}}_w$. As θ_x and θ_y are indecomposable and non-isomorphic and the endomorphism algebra of projective functors is positively graded, it follows that this degree zero zero map from $\theta_x \Delta^{\mathfrak{q}}_w$ to $\theta_y \Delta^{\mathfrak{q}}_w$ cannot be the evaluation of some map from θ_x to θ_y at $\Delta^{\mathfrak{q}}_w$. From § 3.4, we therefore conclude that the answer to Kostant's problem for $\Delta^{\mathfrak{q}}_w$ is negative.

Case 2. Assume $k \neq \frac{n}{2}$. In this case, we have $a_2 > a_1 + a_3$. This implies that any oriented cup diagram for λ with $\min(k, n-k)$ cups must have a vertical strand that separates the two sets of nested cups. By the main result of [22], the corresponding simple module L_u, which is the socle of our parabolic Verma module, is Kostant positive. For different x and y in W, we have an injective restriction map from $\mathrm{Hom}_{\mathfrak{g}}(\theta_x \Delta^{\mathfrak{q}}_w, \theta_y \Delta^{\mathfrak{q}}_w)$ to $\mathrm{Hom}_{\mathfrak{g}}(\theta_x L_u, \theta_y L_u)$.

Since L_u is Kostant positive, $\mathrm{Hom}_{\mathscr{P}}(\theta_x, \theta_y)$ surjects on the latter and hence also on the former. This implies that $\Delta^{\mathfrak{q}}_w$ is Kostant positive.

This completes the proof.

5.8. Example: the extreme maximal parabolic

Consider the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{sl}_n$ corresponding to the choice of all simple roots but the first one. Then, the Levi factor of $\mathfrak{p}$ is isomorphic to $\mathfrak{sl}_{n-1}$. Setting $s_i = (i, i+1)$, we have

$$W^{\mathfrak{p}}_{\text{short}} = \{e, s_1, s_1 s_2, \ldots, s_1 s_2 s_3 \cdots s_{n-1}\}.$$

It is well-known, see, for example, [35], that the corresponding category $\mathcal{O}^{\mathfrak{p}}_0$ is equivalent to the category of modules over the following quiver

$$n-1 \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} n-2 \underset{\beta_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} \ldots \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 0$$

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with the following relations:

$$\alpha_1\beta_1 = 0, \quad \alpha_i\alpha_{i+1} = 0, \quad \beta_i\beta_{i-1} = 0, \quad \alpha_i\beta_i = \beta_{i-1}\alpha_{i-1}.$$

Here, the vertex 0 corresponds to e, and, for $i>0$, the vertex i corresponds to $s_1s_2\cdots s_i$.

In this relation, the parabolic Verma modules are the standard modules with the (unique) quasi-hereditary structure.

$$\Delta_0 = \begin{array}{c} 0 \\ \downarrow \\ 1 \end{array}, \qquad \Delta_1 = \begin{array}{c} 1 \\ \downarrow \\ 2 \end{array}, \qquad \Delta_{n-2} = \begin{array}{c} n-2 \\ \downarrow \\ n-1 \end{array}, \qquad \Delta_{n-1} = n-1 .$$

The **a**-value of e is zero, while the **a**-value of all other elements in $(\mathbf{S}_n)^{\mathfrak{p}}_{\text{short}}$ is 1. It follows that the only thin parabolic Verma modules are Δ_0 and Δ_{n-1}. From Theorem 5, we obtain that these two parabolic Verma modules are the only Kostant positive parabolic Verma modules in this case.

In fact, in this case, one can extend Theorem 5 in the following way (which is an analogue of Theorem 3).

Proposition 7. *For $w \in W^{\mathfrak{p}}_{\text{short}}$, the following assertions are equivalent:*

(a) $\mathbf{K}(\Delta^{\mathfrak{p}}_w) = \texttt{true}$.
(b) *For $x, y \in W$ such that $x \neq y$, $\theta_x\Delta^{\mathfrak{p}}_w \neq 0$ and $\theta_y\Delta^{\mathfrak{p}}_w \neq 0$, we have $\theta_x\Delta^{\mathfrak{p}}_w \not\cong \theta_y\Delta^{\mathfrak{p}}_w$ (as ungraded modules).*
(c) *For all $x, y \in W$ such that $x \neq y$, $\theta_x\Delta^{\mathfrak{p}}_w \neq 0$ and $\theta_y\Delta^{\mathfrak{p}}_w \neq 0$, we have $[\theta_x\Delta^{\mathfrak{p}}_w] \neq [\theta_y\Delta^{\mathfrak{p}}_w]\langle i\rangle$, for $i \in \mathbb{Z}$, in $\mathbf{Gr}[\mathcal{O}^{\mathbb{Z}}_0]$.*
(d) *For all $x, y \in W$ such that $x \neq y$, $\theta_x\Delta^{\mathfrak{p}}_w \neq 0$ and $\theta_y\Delta^{\mathfrak{p}}_w \neq 0$, we have $[\theta_x\Delta^{\mathfrak{p}}_w] \neq [\theta_y\Delta^{\mathfrak{p}}_w]$ in $\mathbf{Gr}[\mathcal{O}_0]$.*
(e) $w \in \{e, s_1s_2\cdots s_{n-1}\}$.
(f) *The annihilator of $\Delta^{\mathfrak{p}}_w$ in $U(\mathfrak{g})$ is a primitive ideal.*

Proof. We start with $w = e$. Then, $\mathbf{K}(\Delta^{\mathfrak{p}}_e) = \texttt{true}$ by Theorem 5, which gives Claim (a). Applying (pairwise non-isomorphic) indecomposable projective functors to $\Delta^{\mathfrak{p}}_e$, one gets either zero or (pairwise non-isomorphic) indecomposable projectives in $\mathcal{O}^{\mathfrak{p}}_0$. As the latter is a highest weight category, the images of these indecomposable projectives in the (graded) Grothendieck group are linearly independent. This implies Claims (b), (c) and (d). Since L_e is the trivial module, the annihilator of $\Delta^{\mathfrak{p}}_e$ coincides with the annihilator of L_{s_1} and hence is a primitive ideal, giving Claim (f).

Next consider $w = s_1s_2\cdots s_{n-1}$. Again, $\mathbf{K}(\Delta^{\mathfrak{p}}_{w_0}) = \texttt{true}$ by Theorem 5, which gives Claim (a). Applying (pairwise non-isomorphic) indecomposable projective functors to $L_{w^{\mathfrak{p}}_0 w_0}$ one gets either zero or (pairwise non-isomorphic) indecomposable tilting modules in $\mathcal{O}^{\mathfrak{p}}_0$. As the latter is a highest weight category, the images of these indecomposable projectives in the (graded) Grothendieck group are linearly independent. This implies Claims (b), (c) and (d). Since $\Delta^{\mathfrak{p}}_w$ is simple, its annihilator is a primitive ideal, giving Claim (f).

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For $w \notin \{e, s_1 s_2 \cdots s_{n-1}\}$, the corresponding $\Delta^{\mathfrak{p}}_w$ is not thin. It has top L_w and socle L_{ws_i}, for some i. We have $\theta_{s_i} L_w = 0$ and $\theta_{s_{i-1}} L_{ws_i} = 0$, while $\theta_{s_i} L_{ws_i} \cong \theta_{s_{i-1}s_i} L_w$. This implies that Claims (a)–(d) fail. Also, as the annihilators of L_w and L_{ws_i} are incomparable primitive ideals, it follows that the annihilator of $\Delta^{\mathfrak{p}}_w$ is contained in the intersection of these two primitive ideals and hence is not a primitive ideal. Therefore, Claim (f) fails as well.

This completes the proof. □

5.9. Asymptotic

From Theorem 5, we see that the ratio of positive vs negative cases equals $\frac{2}{\binom{n}{k}-2}$, for $k = 1, n-1, \frac{n}{2}$. For $k \neq \frac{n}{2}$, this ratio equals

$$\frac{\min(k, n-k) + 1}{\binom{n}{k} - (\min(k, n-k) + 1)}.$$

This goes to 0 when n goes to infinity, which again aligns with the results of [22, Section 6].

6. Upshot: general observations and speculations

6.1. Preliminary definitions

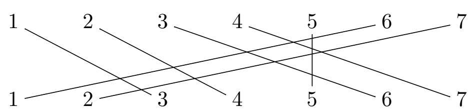
For a composition $\mu \models n$, we denote by $\mathbf{c}(\mu) \vdash n$ the corresponding partition of n. The partition $\mathbf{c}(\mu)$ is obtained from μ by ordering the parts of the latter in a weakly decreasing order. For two compositions $\mu, \nu \models n$, we write $\mu \sim \nu$, provided that $\mathbf{c}(\mu) = \mathbf{c}(\nu)$.

There is a natural bijection between compositions of n and parabolic subalgebras of $\mathfrak{sl}_n$ given as follows, for $\mu = (\mu_1, \mu_2, \dots)$, the corresponding subalgebra $\mathfrak{p}_\mu$ has diagonal blocks of size $\mu_1, \mu_2, \dots$ reading along the main diagonal from north-west to south-east.

For $\mu = (\mu_1, \mu_2, \dots, \mu_k) \models n$, define the sets

$$X_1 = \{1, 2, \dots, \mu_1\}, X_2 = \{\mu_1 + 1, \mu_1 + 2, \dots, \mu_1 + \mu_2\} \quad \text{and so on.}$$

We will call these sets the *blocks of* μ. Denote by G_μ the subgroup of $\mathbf{S}_n$ consisting of all permutations π satisfying the following property: for each $i \in \{1, 2, \dots, k\}$, there is some $j \in \{1, 2, \dots, k\}$ such that $|X_i| = |X_j|$ and π sends the elements of X_i to the elements of X_j preserving the natural order between these elements. Here is an example of an element in $G_{(2,2,1,2)}$, where $X_1 = \{1, 2\}$, $X_2 = \{3, 4\}$, $X_3 = \{5\}$ and $X_4 = \{6, 7\}$:



Let $\mu, \nu \models n$ be two compositions such that $\mu \sim \nu$. Let $X_1, X_2, \dots, X_K$ be the blocks of μ and $Y_1, Y_2, \dots, Y_K$ be the blocks of ν. There is a unique $\tau \in S_k$ such that

- $|X_i| = |Y_{\tau(i)}|$, for all i;

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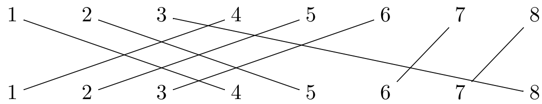
- if $i < j$ and $|X_i| = |X_j|$, then $\tau(i) < \tau(j)$.

We denote by $\omega_{\mu,\nu}$ the unique element of $\mathbf{S}_n$ which sends, for each i, the elements of X_i to the elements of $Y_{\tau(i)}$ preserving the natural order among these elements. For example,

$\omega_{(2,1,3,2),(3,2,2,1)} =$



note that $\omega_{\mu,\nu} G_\mu = G_\nu \omega_{\mu,\nu}$.

The parabolic subgroup W_μ of $\mathbf{S}_n$ corresponding to the composition μ is the product $S_{X_1} \times S_{X_2} \times \cdots \times S_{X_k}$ of symmetric groups.

6.2. Some positive cases

For $\mu \models n$, consider $\mathfrak{p} = \mathfrak{p}_\mu$.

Proposition 8. *For $w \in (\mathbf{S}_n)^{\mathfrak{p}}_{\text{short}}$, we have $\mathbf{K}(\Delta^{\mathfrak{p}}_w) =$ `true` if there is $\nu \models n$ such that $\mu \sim \nu$ and $w \in G_\mu \omega_{\nu,\mu}$.*

Proof. The proof is similar to the one in § 4.3. □

Note that, if ν and ν' are different compositions of n such that $\mu \sim \mu$ and $\nu' \sim \mu$, then the sets $G_\mu \omega_{\nu,\mu}$ and $G_\mu \omega_{\nu',\mu}$ are disjoint.

6.3. Some negative cases

For $\mu \models n$, consider $\mathfrak{p} = \mathfrak{p}_\mu$. Let w_0^μ be the longest element in the parabolic subgroup of W corresponding to $\mathfrak{p}$. Let $\mathcal{R}$ denote the right Kazhdan–Lusztig cell of W containing the element $w_0^\mu w_0$.

For $w \in (\mathbf{S}_n)^{\mathfrak{p}}_{\text{short}}$, we say that the parabolic Verma module $\Delta^{\mathfrak{p}}_w$ is *thin*, provided that there is a unique $u \in \mathcal{R}$ such that $[\Delta^{\mathfrak{p}}_w : L_u] \neq 0$; moreover, $[\Delta^{\mathfrak{p}}_w : L_u] = 1$.

The following proposition is a general off-shot from the arguments applied in § 5.

Proposition 9. *For $w \in (\mathbf{S}_n)^{\mathfrak{p}}_{\text{short}}$, we have $\mathbf{K}(\Delta^{\mathfrak{p}}_w) =$ `false` provided that $\Delta^{\mathfrak{p}}_w$ is not thin.*

Proof. For $u \in \mathcal{R}$, we have $\theta_{u^{-1}} \Delta^{\mathfrak{p}}_w$ is isomorphic to a direct sum of $[\Delta^{\mathfrak{p}}_w : L_u]$ copies of $\theta_d L_d$, where d is the Duflo involution in $\mathcal{R}$. From § 3.4, it follows that Kostant positivity of $\Delta^{\mathfrak{p}}_w$ implies that $\theta_{u^{-1}} \Delta^{\mathfrak{p}}_w$ is either indecomposable or zero. In particular, we must have $[\Delta^{\mathfrak{p}}_w : L_u] \leq 1$, for each such u.

If we have two different $u, \hat{u} \in \mathcal{R}$ such that $[\Delta^{\mathfrak{p}}_w : L_u] = [\Delta^{\mathfrak{p}}_w : L_{\hat{u}}] = 1$, then we have $\theta_{u^{-1}} \Delta^{\mathfrak{p}}_w \cong \theta_{\hat{u}^{-1}} \Delta^{\mathfrak{p}}_w$, which implies that $\Delta^{\mathfrak{p}}_w$ is Kostant negative, again by § 3.4. This completes the proof. □

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6.4. Some speculations

The above results reduce the study of Kostant problem for the parabolic Verma module to the case of thin parabolic Verma modules that are not covered by § 6.2. Each thin parabolic Verma module has simple socle. The results of § 5 suggest that, for a thin parabolic Verma module $\Delta_w^{\mathfrak{p}}$ with socle L_u, we have $\mathbf{K}(\Delta_w^{\mathfrak{p}}) = \mathbf{K}(L_u)$. We do not know whether this is true in general and if yes, how to prove this. In any case, this is only a reduction to the still open problem of determining $\mathbf{K}(L_u)$.

The above results also suggest that classification of thin parabolic Verma modules is an interesting and important problem.

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