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Minimizing topological entropy for maps of the circle

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Abstract. For each $n \geq 2$, we find the minimum value of the topological entropies of all continuous self-maps of the circle having a fixed point and a point of least period n, and we exhibit a map with this minimal entropy.

1. *Introduction*

This paper is concerned with the following problem.

For each $n \geq 2$, find a continuous map f_n of the circle to itself, having a fixed point and a point of least period n, minimal in the sense that $\text{ent}(f_n) \leq \text{ent}(f)$ for every continuous map f of the circle having a fixed point and a point of least period n. Determine $\text{ent}(f_n)$.

Here $\text{ent}(\cdot)$ denotes topological entropy.

The solution to the analogous entropy-minimizing problem for maps of the interval was discovered in the course of investigations having to do with Šarkovskii's theorem. Recall the Šarkovskii ordering $\lhd$ of the positive integers

$$3 \lhd 5 \lhd \cdots \lhd 2\cdot 3 \lhd 2\cdot 5 \lhd \cdots \lhd 2^2\cdot 3 \lhd 2^2\cdot 5 \lhd \cdots \lhd 2^2 \lhd 2^1 \lhd 2^0.$$

Šarkovskii's theorem [**4**], [**5**], [**3**] states that, if a continuous map of a compact interval to itself (or to the reals) has a point of least period n, then it has a point of least period m for every $m \rhd n$.

P. Štefan [**5**] has described a set of constructions which, for each $n \geq 2$, yields a map g_n of the interval having a point of least period n but no point of least period m for any $m \lhd n$. It turns out [**5**], [**3**] that the maps g_n are the solution to the entropy-minimizing problem for maps of the interval: $\text{ent}(g_n) \leq \text{ent}(g)$ for every continuous map g of the interval having a point of least period n. (If g maps a compact interval I into the reals, then $\text{ent}(g)$ is defined to be $\text{ent}(g|I')$, where $I' = \bigcap_{j\geq 0} g^{-j}(I)$. Note that I' need not be an interval.)

The topological entropy of g_n is given by the formula

$$\text{ent}(g_n) = \log \sigma_n,$$

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where σ_n is defined as follows. For $k \geq 3$, let $L_k(x) = x^k - 2x^{k-2} - 1$ and let λ_k denote the largest root of L_k. For $n = 2^s k$ where k is odd, let $\sigma_n = 1$ if $k = 1$ and $\sigma_n = (\lambda_k)^{2^{-s}}$ if $k \geq 3$.

The result corresponding to Šarkovskii's theorem for maps of the circle is the following [**2**]: if a continuous map of the circle to itself has a fixed point and a point of least period n, then either (a) it has a point of least period m for every $m \rhd n$ or (b) it has a point of least period m for every $m > n$.

The examples g_n of Štefan can be extended to maps of the circle without changing the set of least periods or the topological entropy. L. Block [**1**] has constructed maps f_n of the circle having a fixed point and a point of least period n, but no point of least period m for any m, $1 < m < n$. The topological entropy of f_n is given by the formula

$$\text{ent}(f_n) = \log \mu_n,$$

where μ_n is the largest root of $M_n(x) = x^{n+1} - x^n - x - 1$.

THEOREM A. *If a continuous map of the circle has a fixed point and a point of least period* $n \geq 2$, *then* $\text{ent}(f) \geq \min\{\log \mu_n, \log \sigma_n\}$.

This result was proved in [**3**] except for certain maps of degree -1. See § 2 for details.

In light of the constructions described above, in order to solve the problem stated at the beginning of this paper we need only determine which is smaller, μ_n or σ_n.

THEOREM B. *Let* $2 \leq n = 2^s k$ *where* k *is odd.*

(1) *If* $s = 0$, *then* $\mu_n < \sigma_n$ *except that* $\mu_3 = \sigma_3$.

(2) *If* $1 \leq s \leq 6$, *then* $\sigma_n < \mu_n$ *when* $k \leq 2s + 3$ *and* $\mu_n < \sigma_n$ *when* $k \geq 2s + 5$.

(3) *If* $s \geq 7$, *then* $\sigma_n < \mu_n$ *when* $k \leq 2s + 5$ *and* $\mu_n < \sigma_n$ *when* $k \geq 2s + 7$.

2. *Proof of theorem A*

We prove theorem A by restricting our attention to maps of the circle of a fixed degree, denoted by $\deg(\cdot)$, and using the following result.

THEOREM [**3**]. *Let* f *be a continuous map of the circle having a fixed point and a point of least period* $n \geq 2$.

(a) *If* $|\deg(f)| \geq 2$, *then* $\text{ent}(f) \geq \log |\deg(f)|$.

(b) *If* $\deg(f) = 0$, *then* $\text{ent}(f) \geq \log \sigma_n$.

(c) *If* $\deg(f) = 1$, *then* $\text{ent}(f) \geq \min\{\log \mu_n, \log \sigma_n\}$.

(d) *If* $\deg(f) = -1$ *and* n *is odd, then* $\text{ent}(f) \geq \log \sigma_n$.

It is an elementary fact (see § 3) that $\mu_n, \sigma_n < 2$. Therefore to prove theorem A it suffices to show that (d) holds when n is even.

LEMMA 1. *Let* f *be a continuous map of the circle having a fixed point and a point of least period* $n \geq 3$. *If* f *has no point of least period* $n + 1$, *then* $\text{ent}(f) \geq \log \sigma_n$.

Proof. By theorem A_1 of [**2**], the hypotheses of theorem A_2 of [**2**] are satisfied. Then, as in the proof of theorem A_2, there is a proper closed subinterval K of the circle, containing the orbit of a point of least period n, a homeomorphism h from

K onto a compact subinterval I of the reals, and a continuous map g from I into the reals such that for all $x \in K$, $f(x) \in K$ if and only if $g(h(x)) \in I$, and in this case $h(f(x)) = g(h(x))$. In particular, g has a point of least period n.

Let $K' = \bigcap_{j\geq 0} f^{-j}(K)$ and $I' = \bigcap_{j\geq 0} g^{-j}(I)$. Then $f|K'$ and $g|I'$ are topologically conjugate (via the appropriate restriction of h) and by definition, ent $(g) =$ ent $(g|I')$. Then

$$\text{ent}\,(f) \geq \text{ent}\,(f|K') = \text{ent}\,(g|I') = \text{ent}\,(g) \geq \log \sigma_n,$$

the last inequality by Štefan's results. □

We now complete the proof of theorem A. Suppose f is a map of the circle of degree -1 having a fixed point and a point of least period n, where n is even. We may assume that $n > 2$ for, if $n = 2$, then $\sigma_n = 1$ and there is nothing to prove. If f has no point of least period $n+1$, then by lemma 1, ent $(f) \geq \log \sigma_n$. If f has a point of least period $n+1$, then since $n+1$ is odd, ent $(f) \geq \log \sigma_{n+1}$. But it is another elementary fact (see § 3) that $\sigma_m > \sqrt{2}$ if m is odd, and $\sigma_m < \sqrt{2}$ if m is even. Thus $\log \sigma_{n+1} > \log \sigma_n$. □

3. *Proof of theorem B*

We begin by listing some elementary facts about the polynomials

$$M_n(x) = x^{n+1} - x^n - x - 1 \quad (n \geq 2)$$

and

$$L_k(x) = x^k - 2x^{k-2} - 1 \quad (k \geq 3).$$

M_n is increasing on $(1, \infty)$. Since $M_n(1) < 0$ and $M_n(2) > 0$, M_n has a unique root μ_n in $(1, \infty)$ and

$$1 < \mu_n < 2. \tag{1}$$

$$(\mu_n)^n = \frac{\mu_n + 1}{\mu_n - 1}. \tag{2}$$

L_k' has exactly one root in $(0, \infty)$ and L_k' changes from negative to positive at this root. Since $L_k(0) < 0$, $L_k(\sqrt{2}) < 0$ and $L_k(2) > 0$, L_k has a unique root λ_k in $(0, \infty)$ and

$$\sqrt{2} < \lambda_k < 2. \tag{3}$$

Recall that for $n = 2^s k$ where k is odd, $\sigma_n = 1$ if $k = 1$ and $\sigma_n = (\lambda_k)^{2^{-s}}$ if $k \geq 3$.

$$\sigma_n > \sqrt{2} \quad \text{if } n \text{ is odd} \quad \text{and} \quad \sigma_n < \sqrt{2} \quad \text{if } n \text{ is even.} \tag{4}$$

LEMMA 2. *Let $n = 2^s k$ where k is odd. If $k \geq 2s + 7$, then $\mu_n < \sigma_n$.*

Proof. Let $q = 2^{-(s+1)}$. By (3), $\sigma_n > 2^q$. On the other hand,

$$M_n(2^q) > 0 \quad \text{if} \quad 2^{k/2} > \frac{2^q + 1}{2^q - 1}.$$

But

$$\frac{2^x + 1}{2^x - 1} < \frac{4}{x} \quad \text{whenever} \quad 0 < x \leq 1.$$

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(To see this, look at $F(x)=(4-x)2^x-x-4$; $F(0)=0$ and $F'(x)>0$ if $0\le x\le 1$.) Thus

$$\frac{2^q+1}{2^q-1}<2^{s+3},$$

and hence $M_n(2^q)>0$ if $k>2(s+3)$. Therefore

$$\mu_{2^sk}<2^q<\sigma_{2^sk}$$

for all odd $k\ge 2s+7$. □

We shall find it desirable to use the polynomials

$$T_s(x)=x^{2^{s+1}}-x-1 \quad (s\ge 0).$$

T_s is increasing on $(1,\infty)$. Since $T_s(1)<0$ and $T_s(2)>0$, T_s has a unique root τ_s in $(1,\infty)$ and

$$(\tau_s)^{2^{s+1}}=\tau_s+1. \tag{5}$$

LEMMA 3. *Let $n=2^sk$ where $k\ge 3$ is odd. Then $\sigma_n-\mu_n$ has the same sign (positive, negative, zero) as $(\tau_s+1)^{k-2}(\tau_s-1)^2-1$.*

Proof. Suppose $\sigma_n-\mu_n>0$. Let $r=2^s$. Then $\lambda_k>(\mu_n)^r$ and hence $L_k((\mu_n)^r)<0$. Writing μ in place of μ_n and using (2), we have

$$L_k(\mu^r)=\frac{2}{\mu^{2r}(\mu-1)}T_s(\mu).$$

Hence $T_s(\mu)<0$ and so $\mu<\tau_s$. Writing τ in place of τ_s, using (2) and (5) and the fact that $G(x)=((x+1)/(x-1))^2$ is decreasing on $(1,\infty)$, we have

$$\left(\frac{\tau+1}{\tau-1}\right)^2<\left(\frac{\mu+1}{\mu-1}\right)^2=\mu^{2n}<\tau^{2n}=(\tau+1)^k.$$

Thus $(\tau_s+1)^{k-2}(\tau_s-1)^2>1$.

The same argument goes through with all the inequalities reversed or all replaced by equalities. □

An immediate consequence of lemma 3 is

LEMMA 4. *Let $k\ge 3$ be odd. If $\mu_{2^sk}<\sigma_{2^sk}$, then $\mu_{2^sl}<\sigma_{2^sl}$ for all odd $l>k$.*

LEMMA 5. *If $\sigma_{2^s(2s+5)}<\mu_{2^s(2s+5)}$, then $\sigma_{2^t(2t+5)}<\mu_{2^t(2t+5)}$ for all $t>s$.*

Proof.† It suffices to show that the result holds for $t=s+1$. Let $\alpha=\tau_s$ and $\beta=\tau_{s+1}$. Using (5), we have that $T_{s+1}(\alpha)=\alpha(\alpha+1)>0$, and so $\beta<\alpha$. Using (5) again,

$$(\beta^{2^{s+1}})^2=\beta^{2^{s+2}}=\beta+1<\alpha+1=\alpha^{2^{s+1}}.$$

Thus $\beta^2<\alpha$ and hence $(\beta^2-1)^2<(\alpha-1)^2$. Then

$$(\beta+1)^{2(s+1)+3}(\beta-1)^2=(\beta+1)^{2s+3}(\beta^2-1)^2<(\alpha+1)^{2s+3}(\alpha-1)^2<1,$$

the last inequality by lemma 3. By lemma 3 again, $\sigma_{2^{s+1}(2(s+1)+5)}<\mu_{2^{s+1}(2(s+1)+5)}$. □

We now complete the proof of theorem B.

Since $M_3(x)=(x^2-x-1)(x^2+1)$ and $L_3(x)=(x^2-x-1)(x+1)$, $\mu_3=\lambda_3=\sigma_3$. Since $L_5(1.5)<0<M_5(1.5)$, $\mu_5<1.5<\lambda_5=\sigma_5$. Then (1) follows from lemma 2.

† This proof is due to M. L. Ginsberg (personal communication).

It can be checked (using, for example, a hand-held calculator) that if $1 \leq s \leq 6$, then

$$\sigma_{2^s(2s+3)} < \mu_{2^s(2s+3)}$$

and

$$\mu_{2^s(2s+5)} < \sigma_{2^s(2s+5)},$$

and if $s = 7$, then

$$\sigma_{2^s(2s+5)} < \mu_{2^s(2s+5)}.$$

Then (2) follows from lemma 4 and (3) follows from lemmas 2, 4 and 5. □

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