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Some results on minimizers and stable solutions of a variational problem

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Abstract. We consider the functional

$$\int \frac{|\nabla u|^2}{2} + F(x, u)\, dx$$

in a periodic setting. We discuss whether the minimizers or the stable solutions satisfy some symmetry or monotonicity properties, with special emphasis on the autonomous case when F is x-independent. In particular, we give an answer to a question posed by Victor Bangert when F is autonomous in dimension $n \leq 3$ and in any dimension for non-zero rotation vectors.

1. *Main results*

Given $F \in C^{2,\alpha}_{\text{loc}}(\mathbb{R}^{n+1})$ for some $\alpha \in (0, 1)$, and a bounded open set $\Omega \subset \mathbb{R}^n$, we consider the energy functional $\mathcal{E}_\Omega$ on the space

$$\mathcal{D}(\Omega) = \{u \in L^\infty(\Omega) \text{ with } \nabla u \in L^2(\Omega, \mathbb{R}^n)\}$$

defined by

$$\mathcal{E}_\Omega(u) = \int_\Omega \frac{|\nabla u(x)|^2}{2} + F(x, u(x))\, dx.$$

This functional is very important for applications since many classical physical models are particular cases. We just mention here that when $n = 1$, the functional includes the case of the Lagrangian action of a pendulum and that, in any dimension, it can be seen as the continuous limit of well-known discrete models for crystal dislocations, such as the ones dealt with in [**Aub83, Mat82**]. Also, the scalar Ginzburg–Landau–Allen–Cahn functional may be reduced to this functional in many cases of interest (see [**JGV09**], for example).

We say that $u : \mathbb{R}^n \to \mathbb{R}$ is a *minimizer*† if, for any bounded open set $\Omega \subset \mathbb{R}^n$, we have that $u \in \mathcal{D}(\Omega)$ and $\mathcal{E}_\Omega(u) \leq \mathcal{E}_\Omega(u + \varphi)$ for any $\varphi \in C_0^\infty(\Omega)$, that is u is a minimizer if its energy increases under compact perturbations (the size of the domain and the size of the perturbation may be taken arbitrarily large).

It is easily seen that if u is a minimizer, then it satisfies the Euler–Lagrange equation associated to the energy functional, that is

$$\Delta u(x) + f(x, u(x)) = 0, \tag{1}$$

where $f(x, r) = -\partial_r F(x, r)$. Also, $u \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ (though u may not be in $L^\infty(\mathbb{R}^n)$).

We also say that u has a *rotation vector* $\rho \in \mathbb{R}^n$ if the map $x \mapsto u(x) - \rho \cdot x$ belongs to $L^\infty(\mathbb{R})$.

We say that u is *Birkhoff*‡ if for any $k = (k', k_{n+1}) \in \mathbb{Z}^n \times \mathbb{Z} = \mathbb{Z}^{n+1}$ we have that either $u(x + k') + k_{n+1} \geq u(x)$ for any $x \in \mathbb{R}^n$, or that $u(x + k') + k_{n+1} \leq u(x)$ for any $x \in \mathbb{R}^n$.

We say that F is *integer periodic* if for any $k = (k', k_{n+1}) \in \mathbb{Z}^n \times \mathbb{Z} = \mathbb{Z}^{n+1}$ we have that $F(x + k', r + k_{n+1}) = F(x, r)$ for any $(x, r) \in \mathbb{R}^2$.

Finally, we say that F is *autonomous* if $F(x, r) = F(r)$, that is, F does not depend on the space variables x.

The construction of Birkhoff solutions of a given rotation vector in an integer-periodic setting is a classical result in dynamical systems (see [**Aub83, Mat82**]). The extension to the PDE case has been the topic of many recent research papers in the field. First of all, a very important result, proven in [**Mos86**] is that, if F is integer periodic, for any $\rho \in \mathbb{R}^n$, there exists a minimizer u_ρ which has a rotation vector ρ and is Birkhoff. Such a result inspired a broad investigation on Birkhoff minimizers for this problem and for related ones (for instance, see [**Aue01, Ban89, Ban90, CdlL01, Val04**] and references therein). The case of Birkhoff solutions that are not minimizing have also been dealt with (see [**Bes05, dlLV07**]). In spite of the organization given by these Birkhoff solutions, the system may also exhibit chaotic behaviors, as shown in [**AJM02, AM05, Rab04, RS03, RS04**] for instance. Thus, equation (1) somehow bridges some features typically arising in dynamical systems with the theory of elliptic PDEs.

Other than minimality, a variational condition that is often interesting to look at is stability. If u is a solution of (1), we say that it is *stable*§ if

$$\int_{\mathbb{R}^n} |\nabla \psi(x)|^2 + \partial_r^2 F(x, u(x))(\psi(x))^2 \, dx \ \geq 0$$

for any compactly supported $\psi \in C^\infty(\mathbb{R}^n)$.

From the variational point of view, a solution is stable if the second variation of the energy is non-negative. In particular, minimizers are stable solutions. For other properties of stable solutions, see [**AAC01, FSV08**], for example.

This paper has been motivated by the above-mentioned results and by the following problem posed by [**Ban89**] (see the very last line there).

† What we call here simply a minimizer is also known in the literature as a *local minimizer* or *class A minimizer*.
‡ In the literature, the Birkhoff property also occurs under the names *non-intersection*, *non-self-intersection* or *self-conforming* property.
§ The definition of stability we use here is classical, but different from other famous stability conditions such as Ljapunov stability, structural stability, etc.

Question 1. Let F be integer periodic. Let u be a minimizer with rotation vector ρ. Then, is u Birkhoff?

In this generality, Question 1 is still open. As far as we know, the state of the art on it is the following.

- Question 1 has a positive answer in any dimension n if $(-\rho, 1)$ is rationally independent (i.e. $\rho \cdot m' = m_{n+1}$ with $m = (m', m_{n+1}) \in \mathbb{Z}^n \times \mathbb{Z}$ implies $m = 0$). This is proved in [**Ban89**, Theorem 8.4].
- Question 1 has a positive answer in dimension $n \le 7$ if $\rho = 0$ and, for instance, $F(x, r) = 1 - \cos(2\pi r)$. This follows by the results of [**Sav09**] (and the arguments in the proof of Corollary 4 here).
- Question 1 has a negative answer in dimension $n = 9$. Indeed, as observed in [**JGV09**], the example built in [**dPKW08, dPKW09**] provides a negative answer to Question 1, with F independent of x and $\rho = 0$.

So, in this paper, we would like to give some rigidity and symmetry results that also provide some partial answers to Question 1. The techniques we use come from a different, but related, subject, namely the 1D-symmetry of minimal solutions of the Ginzburg–Landau–Allen–Cahn phase transitions, which is the content of a famous problem set in [**DG79**] (see [**AAC01, AC00, BCN97, Far07, FSV08, FV11, GG98, Sav09**] and the review [**FV09**] for more details on this). In this respect, it is convenient to introduce the following notation: we say that $u : \mathbb{R}^n \to \mathbb{R}$ is 1D if there exist $\varpi \in S^{n-1}$ and $u_\star : \mathbb{R} \to \mathbb{R}$ such that $u(x) = u_\star(\varpi \cdot x)$, that is u is 1D if it depends only on one variable, up to rotations. With a slight abuse of the terminology, in the above notation, if $u_\star$ is (strictly) monotone, we say that u is (strictly) *monotone*.

We prove the following 1D-results for the case in which F is autonomous.

THEOREM 2. (Stable solutions in $\mathbb{R}^2$) *Let $n = 2$. Let F be autonomous and u be a stable solution of (1) with rotation vector ρ. Then, u is* 1D. *Also, u is either constant or strictly monotone, and $u(x) = u_\star(\varpi \cdot x)$, with $\rho = \pm|\rho|\, \varpi$.*

THEOREM 3. (Minimal solutions in $\mathbb{R}^3$ when $\rho = 0$) *Let $n = 3$. Let F be autonomous bounded from below and attaining its minimum. Let u be a minimizer with rotation vector $\rho = 0$. Then, u is* 1D. *Also, u is either constant or strictly monotone.*

In particular, from Theorems 2 and 3, one can obtain the following corollary.

COROLLARY 4. *Question 1 has a positive answer in dimension $2 \le n \le 3$ if F is autonomous and $\rho = 0$.*

In any dimension, and when $\rho \ne 0$, we have the following result.

THEOREM 5. (Minimal solutions in $\mathbb{R}^n$ when $\rho \ne 0$) *Let $0 \le m \le n$ and suppose that F does not depend on $(x_1, \ldots, x_m)$, that is there exists $G : \mathbb{R}^{n-m} \times \mathbb{R}$ for which*

$$F(x_1, \ldots, x_n, r) = G(x_{m+1}, \ldots, x_n, r) \tag{2}$$

for any $(x, r) \in \mathbb{R}^{n+1}$.

Then, Question 1 has a positive answer when $(\rho_1, \ldots, \rho_m) \ne 0$ and $(\rho_{m+1}, \ldots, \rho_n, -1)$ is rationally independent.

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The case in which $m = 0$ in Theorem 5 reduces to [**Ban89**, Theorem 8.4]. The case in which $m = n$ in Theorem 5 is interesting in itself and gives the following corollary.

COROLLARY 6. *Let F be autonomous. Then, Question 1 has a positive answer in any dimension n when $\rho \neq 0$.*

By Corollary 4, we see that the case $\rho = 0$ is to some extent the 'most delicate' in the framework of Question 1. Indeed, as remarked above, Question 1 has a negative answer with $n = 9$, F autonomous and $\rho = 0$ (see [**dPKW08, dPKW09, JGV09**]), hence Theorem 5 is, in a sense, optimal.

An immediate consequence of Theorem 5 and Corollary 4 is the following corollary.

COROLLARY 7. *Question 1 has a positive answer in dimension $2 \leq n \leq 3$ if F is autonomous, for any rotation vector ρ.*

When F is not autonomous, it is not conceivable to expect u to be 1D. Nevertheless, some of the above results can be adapted to deal with the case in which the dependence of F on the space variable is not complete.

THEOREM 8. (Monotonicity in $\mathbb{R}^2$) *Let $n = 2$. Let $F(x_1, x_2, r) = F(x_1, r)$, that is suppose that F does not depend on the second space variable. Let u be a stable solution of (1) with rotation vector ρ. Then, either $\partial_{x_2} u(x) > 0$ for any $x \in \mathbb{R}^2$, or $\partial_{x_2} u(x) < 0$ for any $x \in \mathbb{R}^2$, or u is* 1D.

Solutions of (1) which are monotone in one direction, and for which the nonlinearity is independent of this variable, are stable (see [**AAC01, FSV08**], for instance), and some interesting examples of such solutions in $\mathbb{R}^2$ have recently been constructed in [**AM05**] when the nonlinearity is Allen–Cahn-type and periodic with respect to, say, x_1. In this sense, our Theorem 8 may be seen as a counterpart of [**AM05**, Theorem 1.2].

The extension of Theorem 8 in dimension three (in analogy with Theorem 3) is given by the following result.

THEOREM 9. (Monotonicity in $\mathbb{R}^3$ when $\rho = 0$) *Let $n = 3$. Let $F(x_1, x_2, x_3, r) = \mu(x_1, x_2) g(r)$ with $\mu \in L^\infty(\mathbb{R}^2)$ and suppose that $F(x, r) \geq 0$ for any $(x, r) \in \mathbb{R}^4$ and that $g(r_\star) = 0$ for some $r_\star \in \mathbb{R}$. Let u be a minimizer with rotation vector $\rho = 0$. Then, either $\partial_{x_3} u(x) > 0$ for any $x \in \mathbb{R}^3$, or $\partial_{x_3} u(x) < 0$ for any $x \in \mathbb{R}^3$, or $\partial_{x_3} u(x) = 0$ for any $x \in \mathbb{R}^3$.*

It is worth recalling that in the ODE case of the standard pendulum, the bounded stable solutions are monotone in time (it can be explicitly checked that they are either equilibria or heteroclinics); thus, Theorem 9 may be seen as an extension of this elementary fact to the PDE case.

By exchanging the roles of x_2 and x_3 in Theorem 9, we also obtain the following result.

COROLLARY 10. *Let the assumptions of Theorem 9 and suppose that μ depends only on x_1 (i.e. it is independent of both x_2 and x_3). Then, for any $i \in \{2, 3\}$, we have that either $\partial_{x_i} u(x) > 0$ for any $x \in \mathbb{R}^3$, or $\partial_{x_i} u(x) < 0$ for any $x \in \mathbb{R}^3$, or $\partial_{x_i} u(x) = 0$ for any $x \in \mathbb{R}^3$.*

Remark 11. We observe that no periodicity for F is needed for Theorems 2, 3, 8 and 9. Also, more general energy functionals may be dealt with using the techniques discussed here (for instance, one can replace the term $|\nabla u|^2$ in the functional with $\Lambda_2(|\nabla u|)$, as defined in [**FSV08**] with $p = 2$).

The rest of the paper is devoted to proofs of the results presented above.

2. *Proof of Theorem 2*

We define $v(x) = u(x) - \rho \cdot x$. Then, by (1), we have that $-\Delta v(x) = f(u(x))$ and the latter quantity is bounded for any $x \in \mathbb{R}^n$. Accordingly, by elliptic regularity theory, we have that $|\nabla v| \in L^\infty(\mathbb{R}^n)$ and so

$$|\nabla u| \le |\rho| + |\nabla v| \in L^\infty(\mathbb{R}^n). \tag{3}$$

Then, the fact that u is 1D follows from, for instance, [**FSV08**, Theorem 1.1].

So, we will write $u(x) = u_\star(\varpi \cdot x)$ for some $\varpi \in S^{n-1}$. We now show that $u_\star$ is either constant or strictly monotone. For this, we observe that, since u is stable, so is $u_\star$. Then, there exists a positive function φ solution of $\ddot{\varphi}(t) + q(t)\varphi(t) = 0$ for any $t \in \mathbb{R}$ (see [**FCS80, MP78**] or [**AAC01**, Proposition 4.2], for example), where $q(t) = f'(u_\star(t))$. Let $w = \dot{u}_\star$. Notice that $w \in L^\infty(\mathbb{R})$ as a result of (3). Since also $\ddot{w}(t) + q(t)w(t) = 0$, we deduce from [**BCN97**, Theorem 1.8] (applied here with $m = 1$) that w is proportional to φ, hence either $\{\dot{u}_\star = 0\} = \mathbb{R}$ or $\{\dot{u}_\star = 0\} = \varnothing$. Accordingly, $u_\star$ is either constant or strictly monotone, as desired.

Now, take any $\eta \in S^{n-1}$ with $\varpi \cdot \eta = 0$. By definition,

$$\begin{aligned} +\infty &> \sup_{x \in \mathbb{R}^n} |u_\star(\varpi \cdot x) - \rho \cdot x| \\ &\ge \sup_{t>0} |u_\star(\varpi \cdot \eta t) - \rho \cdot \eta t| \\ &\ge \sup_{t>0} |\rho \cdot \eta| t - |u_\star(0)|. \end{aligned}$$

Therefore, $\rho \cdot \eta = 0$.

Accordingly, $\varpi^\perp \subseteq \rho^\perp$, that is either $\rho = 0$ or $\varpi = \pm\rho/|\rho|$.

3. *Proof of Theorem 3*

Suppose that

$$\min_{r \in \mathbb{R}} F(r) = F(r_o).$$

Let

$$G(r) = F(r) - F(r_o).$$

Notice that $G \ge 0$ and $G' = -f$. Then, by proceeding as in [**CC95**, Lemma 1], we see that there exists $C > 0$ such that

$$\int_{B_R} |\nabla u(x)|^2 + G(u(x))\, dx \le CR^2, \tag{4}$$

for any $R > 0$. From (4), one obtains that u is 1D, either by the arguments of [**AAC01**] or by [**FSV08**, Lemma 5.2] (indeed, (4) here implies [**FSV08**, (5.1)] and a and λ_1 of [**FSV08**] are both equal to 1 in this setting).

Also, u is either constant or strictly monotone (see §2).

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4. *Proof of Corollary 4*

From Theorems 2 and 3, we know that u is 1D and it is either constant or strictly monotone. If it is constant, we are done. Therefore, without loss of generality, we can assume that $u(x) = u_\star(\varpi \cdot x)$, with $u_\star$ strictly increasing and $\ddot{u}_\star(t) = F'(u_\star(t))$ for any $t \in \mathbb{R}$.

Hence, by energy conservation, there exists $c \in \mathbb{R}$ such that

$$\frac{|\dot{u}_\star(t)|^2}{2} - F(u_\star(t)) = c \quad \text{for any } t \in \mathbb{R}. \tag{5}$$

We define

$$c_o = c + \min_{r\in\mathbb{R}} F(r).$$

Notice that

$$\frac{|\dot{u}_\star(t)|^2}{2} = c + F(u_\star(t)) \geq c + \min_{r\in\mathbb{R}} F(r) = c_o. \tag{6}$$

We claim that

$$|u_\star(t) - u_\star(s)| < 1 \quad \text{for any } t, s \in \mathbb{R}. \tag{7}$$

To prove (7), we argue by contradiction: fix $s \in \mathbb{R}$ and suppose, say, that there exists $t_+ \in \mathbb{R}$ such that

$$u_\star(t_+) - u_\star(s) = 1. \tag{8}$$

By the integer periodicity of F, there exists

$$m \in [u_\star(s), u_\star(s) + 1) \tag{9}$$

such that

$$\min_{r\in\mathbb{R}} F(r) = F(m).$$

By (8) and (9), there exists $t_\star \in \mathbb{R}$ for which

$$u_\star(t_\star) = m.$$

We claim that

$$\dot{u}_\star(t_\star) \neq 0. \tag{10}$$

Indeed, notice that $F'(u_\star(t_\star)) = F'(m) = 0$ by the minimality of m, so, if $\dot{u}_\star(t_\star) = 0$, then $u_\star$ would be constant by the uniqueness theorem for ODEs.

This proves (10). From (5) and (10),

$$c_o = c + F(m) = c + F(u_\star(t_\star)) = \frac{|\dot{u}_\star(t_\star)|^2}{2} > 0.$$

Thus, by (6), $|\dot{u}_\star(t)| \geq \sqrt{2c_o} > 0$ for any $t \in \mathbb{R}$, and so $u_\star$ cannot be bounded. In particular, ρ cannot be 0. This contradiction proves (7).

From (7), we obtain that, if $(k', k_{n+1}) \in \mathbb{Z}^{n+1}$ and $k_{n+1} > 0$ (i.e. $k_{n+1} \geq 1$), we have that

$$\begin{aligned} u(x + k') + k_{n+1} &= u_\star(\varpi \cdot (x + k')) + k_{n+1} > u_\star(\varpi \cdot x) - 1 + k_{n+1} \\ &\geq u_\star(\varpi \cdot x) = u(x). \end{aligned}$$

Analogously, if $(k', k_{n+1}) \in \mathbb{Z}^{n+1}$ and $k_{n+1} < 0$, we have

$$u(x + k') + k_{n+1} \leq u(x).$$

Finally, if $k_{n+1} = 0$, since $u_\star$ is increasing, we have that, if $\varpi \cdot k \geq 0$, then

$$u(x + k') + k_{n+1} = u_\star(\varpi \cdot (x + k')) \geq u_\star(\varpi \cdot x) = u(x)$$

and, analogously, if $\varpi \cdot k \leq 0$, then

$$u(x + k') + k_{n+1} \leq u(x).$$

The above observations give that u is Birkhoff, proving Corollary 4.

5. *Proof of Theorem 5*

If $m = 0$, then Theorem 5 reduces to [**Ban89**, Theorem 8.4], so we may suppose that $m \geq 1$.

In fact, by possibly adding a spurious variable x_0, we may suppose that

$$m \geq 2. \tag{11}$$

Indeed, if $m = 1$, we consider u as a function of $(n + 1)$ variables $(x_0, x_1, \ldots, x_n)$ and F as a function of $(n + 2)$ variables $(x_0, x_1, \ldots, x_n, r)$, though independent of the spurious variable x_0, i.e. we set

$$u_0(x_0, x_1, \ldots, x_n) := u(x_1, \ldots, x_n) \quad \text{and} \quad F_0(x_0, x_1, \ldots, x_n, r) := F(x_1, \ldots, x_n, r).$$

In this way, u_0 is a minimizer of the functional

$$v(x_0, x_1, \ldots, x_n) \longmapsto \int \frac{|\nabla v(x_0, x_1, \ldots, x_n)|^2}{2} + F_0((x_0, x_1, \ldots, x_n), v(x_0, x_1, \ldots, x_n))\, dx_0 \cdots dx_n$$

and u_0 has rotation vector $(0, \rho) \in \mathbb{R}^{n+1}$.

So, if $\rho_1 \neq 0$, we have that $(0, \rho_1) \neq 0$, hence u_0 follows under the assumptions of Theorem 5 with $m = 2$ (and n replaced by $n + 1$). Accordingly, if Theorem 5 holds for $m = 2$, we deduce that u_0 is Birkhoff, hence so is u.

These considerations prove that we may suppose that (11) holds.

Now, given $a = (a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1}$, we call $\underline{a} = (a_1, \ldots, a_m)$ and $\overline{a} = (a_{m+1}, \ldots, a_{n+1})$. Notice that $a = (\underline{a}, \overline{a})$.

Analogously, if $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, we set $\widetilde{b} = (b_{m+1}, \ldots, b_n)$. Hence, with a slight abuse of THE notation, we write $b = (\underline{b}, \widetilde{b})$.

We prove that

$$\begin{aligned} &\text{for any } \omega \in \mathbb{R}^{n+1} \text{ with } \underline{\omega} \neq 0 \text{ and } \overline{\omega} \text{ rationally independent} \\ &\quad \text{and any } \epsilon > 0, \text{ there exists a rotation } R_{\omega,\epsilon} \text{ of } \mathbb{R}^m \text{ such that} \\ &\quad \|R_{\omega,\epsilon} - \mathrm{Id}\| \leq \epsilon \text{ and } (R_{\omega,\epsilon}\underline{\omega}, \overline{\omega}) \text{ is rationally independent.} \end{aligned} \tag{12}$$

To check this, given $k \in \mathbb{Z}^{n+1}$, with

$$\underline{k} \neq 0, \tag{13}$$

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we set

$$\mathcal{S} = \{v \in \mathbb{R}^m \text{ s.t. } |v| = |\underline{\omega}|\}$$

and

$$\mathcal{B}_k = \{v \in \mathcal{S} \text{ s.t. } (v, \overline{\omega}) \cdot k = 0\}.$$

Notice that $\mathcal{B}_k$ is the intersection between the sphere $\mathcal{S}$ and an affine plane of dimension $m-1$, from (11) and (13), and therefore its $(m-1)$-dimensional Hausdorff measure on $\mathcal{S}$ vanishes, i.e.

$$\mathcal{H}^{m-1}(\mathcal{B}_k) = 0. \tag{14}$$

Now, we define

$$\mathcal{B} = \{v \in \mathcal{S} \text{ s.t. } \exists k \in \mathbb{Z}^{n+1} \text{ with } \underline{k} \neq 0 \text{ s.t. } (v, \overline{\omega}) \cdot k = 0\}.$$

Then,

$$\mathcal{H}^{m-1}(\mathcal{B}) = \mathcal{H}^{m-1}\Bigg(\bigcup_{\substack{k \in \mathbb{Z}^{n+1} \\ \underline{k} \neq 0}} \mathcal{B}_k\Bigg) = 0,$$

as a result of (14).

As a consequence, given $\epsilon > 0$, there exists

$$\underline{\omega}_\epsilon \in \mathcal{S} \backslash \mathcal{B} \tag{15}$$

such that

$$|\underline{\omega}_\epsilon - \underline{\omega}| \leq \epsilon^2. \tag{16}$$

Since $\underline{\omega} \neq 0$, we may suppose that $\underline{\omega}_\epsilon \neq 0$ as well, and so we can take $R_{\omega,\epsilon}$ to be a rotation in $\mathbb{R}^m$ sending $\underline{\omega}$ to $\underline{\omega}_\epsilon$ (we recall that $|\underline{\omega}| = |\underline{\omega}_\epsilon|$, since both the vectors belong to the sphere $\mathcal{S}$). By (16), $\|R_{\omega,\epsilon} - \mathrm{Id}\| \leq \epsilon$ if ϵ is small.

Furthermore

$$(R_{\omega,\epsilon}\underline{\omega}, \overline{\omega}) \cdot k = (\underline{\omega}_\epsilon, \overline{\omega}) \cdot k.$$

Hence, if the above quantity vanishes for some $k \in \mathbb{Z}^{n+1}$, we deduce from (15) that $\underline{k} = 0$. Therefore, $\overline{\omega} \cdot \overline{k} = 0$ and so, since $\overline{\omega}$ is rationally independent, we have that $\overline{k} = 0$, that is $k = 0$, and this gives that $(R_{\omega,\epsilon}\underline{\omega}, \overline{\omega})$ is rationally independent, thus proving (12).

Now, we apply (12) to $\omega = (-\rho, 1)$. Notice that, in this case, $\underline{\omega} = -(\rho_1, \ldots, \rho_m) \neq 0$ and $\overline{\omega} = -(\rho_{m+1}, \ldots, \rho_n, -1)$ is rationally independent by our assumptions. Hence, we obtain that, given $\epsilon > 0$, there exists a rotation R_ϵ on $\mathbb{R}^m$ such that $(-R_\epsilon \underline{\rho}, -\widetilde{\rho}, 1)$ is rationally independent and

$$\lim_{\epsilon \to 0^+} R_\epsilon = \mathrm{Id}. \tag{17}$$

We define

$$v_\epsilon(x) = u(R_\epsilon^T \underline{x}, \widetilde{x}).$$

Then, in the light of (2), we have that v_ϵ is a minimizer too, and has rotation vector $(R_\epsilon \underline{\rho}, \widetilde{\rho})$. Consequently, by [**Ban89**, Theorem 8.4], v_ϵ is Birkhoff. Now, take any $k = (k', k_{n+1}) \in \mathbb{Z}^n \times \mathbb{Z} = \mathbb{Z}^{n+1}$. We have that, for an infinitesimal sequence of ϵ,

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either $v_\epsilon(x+k')+k_{n+1}\geq v_\epsilon(x)$ for any $x\in\mathbb{R}^n$, or that $v_\epsilon(x+k')+k_{n+1}\leq v_\epsilon(x)$ for any $x\in\mathbb{R}^n$, that is either

$$u(\overline{y}+R_\epsilon^T\overline{k},\,\widetilde{y}+\widetilde{k})+k_{n+1}\geq u(y)$$

for any $y\in\mathbb{R}^n$, or

$$u(\overline{y}+R_\epsilon^T\overline{k},\,\widetilde{y}+\widetilde{k})+k_{n+1}\leq u(y)$$

for any $y\in\mathbb{R}^n$. So, sending $\epsilon\to 0^+$ and using (17), we obtain that u is Birkhoff, as desired.

6. *Proof of Theorem 8*

As in §2, we see that $|\nabla u|\in L^\infty(\mathbb{R}^2)$. Consequently, for any $R>0$,

$$\int_{B_R}|\nabla u(x)|^2\,dx\leq\|\nabla u\|_{L^\infty(\mathbb{R}^2)}R^2. \tag{18}$$

Also, since u is stable, there exists a positive function φ solution of

$$\Delta\varphi(x)+\partial_r f(x_1,u(x))\varphi(x)=0$$

for any $x\in\mathbb{R}^n$ (see† [**FCS80, MP78**] or [**AAC01**, Proposition 4.2], for example).

We define $\psi=\partial_{x_2}u$, and we also observe that ψ is a solution of

$$\Delta\psi(x)+\partial_r f(x_1,u(x))\psi(x)=0$$

for any $x\in\mathbb{R}^n$. As a consequence, if we set $\sigma=\psi/\varphi$, we have that σ is a solution of

$$\operatorname{div}(\varphi^2\nabla\sigma)=0 \tag{19}$$

in $\mathbb{R}^n$.

Furthermore, from (18),

$$\int_{B_R}(\varphi\sigma)^2\,dx=\int_{B_R}|\partial_{x_2}u|^2\,dx\leq\|\nabla u\|_{L^\infty(\mathbb{R}^2)}R^2. \tag{20}$$

Then, from the Liouville-type result of [**BCN97**] (for example, see the version‡ in [**AAC01**, Theorem 3.1]), we conclude that σ is constant. This and the maximum principle yield the desired result.

† Let us describe the construction of such a positive solution φ. The idea is that the stability condition implies that the first eigenvalue of the Schrödinger operator $\Delta-\partial^2F$ in B_R is positive. As a consequence, one considers φ_R to be a positive eigenfunction with constant boundary datum c_R. The value c_R is adjusted so as to make $\varphi_R(0)=1$. Then, one sends $R\to+\infty$ and obtains the desired φ, using elliptic regularity theory to pass to the limit and the Harnack inequality to be sure that the limit remains positive.

‡ The Liouville-type result of [**BCN97**] is a very powerful tool for proving symmetry. The idea of the classical Liouville theorem is that bounded harmonic functions need to be constant. The brillant version of it given by [**BCN97**] is that solutions σ of (19) need to be constant if the energy grows not more than quadratic, according to (20). The proof of the Liouville-type result of [**BCN97**] is based on the choice of the 'right' test function (somewhat inspired by the classical Caccioppoli inequality) and on some simple, but smart, integral bounds.

7. *Proof of Theorem 9*

The proof is a simple modification of an argument in [**CC95**] and of the proof of Theorem 8 above.

Let $R > 1$ and $\psi_R \in C_0^\infty(B_R, [0, 1])$, with $\psi_R(x) = 1$, for any $x \in B_{R-1}$ and $\|\nabla\psi_R\|_{L^\infty(\mathbb{R}^3)} \le 2$.

Let $u_R(x) = u(x) + (r_\star - u(x))\psi_R(x)$. Notice that, if $x \in B_{R-1}$, then

$$F(x, u_R(x)) = \mu(x_1, x_2)g(r_\star) = 0$$

and

$$|\nabla u_R(x)| \le |\nabla u(x)| + |r_\star - u(x)||\nabla\psi_R(x)| + |\nabla u(x)| \le K,$$

for a suitable K, depending on $r_\star$, $\|u\|_{L^\infty(\mathbb{R}^3)}$ and $\|\nabla u\|_{L^\infty(\mathbb{R}^3)}$, but independent of R.

As a consequence, by the minimality of u, we have that

$$\begin{aligned}\int_{B_R} |\nabla u(x)|^2\,dx &\le \int_{B_R} |\nabla u(x)|^2 + F(x, u(x))\,dx\\ &\le \int_{B_R} |\nabla u_R(x)|^2 + F(x, u_R(x))\,dx\\ &= \int_{B_R\setminus B_{R-1}} |\nabla u_R(x)|^2 + F(x, u_R(x))\,dx\\ &\le \Big(K^2 + \sup_{(x_1,x_2)\in\mathbb{R}^2} |\mu(x_1, x_2)| \sup_{|r|\le|r_\star|+\|u\|_{L^\infty(\mathbb{R}^3)}} |g(r)|\Big)|B_R\setminus B_{R-1}|\\ &\le CR^2 \qquad (21)\end{aligned}$$

for a suitable $C > 0$, possibly depending on $r_\star$, $\|u\|_{L^\infty(\mathbb{R}^3)}$ and $\|\nabla u\|_{L^\infty(\mathbb{R}^3)}$, but independent of R.

Then, the proof of Theorem 9 follows by repeating verbatim the argument in §6, but replacing (18) with (21).

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