

THE VALUE OF UNCERTAINTY UNDER LIMITED COMMITMENT

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In this paper, I analyze an optimal loan contract between a risk-neutral financial intermediary and a risk-averse household, where the household receives a stochastic endowment stream that grows over time and is unable to commit to the contract. I examine the household's welfare in the equilibrium contract, and find that, first, under sufficiently rapid endowment growth, the presence of uncertainty in endowment may improve the household's welfare through relaxation of its commitment problem, and second, regardless of the endowment growth rate, the household's welfare is nonincreasing in the persistence of endowment. Numerical analysis suggests that welfare improvement from uncertainty may occur under reasonable parameters. These results have potentially important practical implications—for example, for developing countries that rely on external borrowing.

Keywords: Long-Term Contracts, Risk Sharing, Consumption Smoothing, Limited Commitment

1. INTRODUCTION

This paper analyzes an optimal loan contract between a risk-neutral financial intermediary and a risk-averse household. The household receives a stochastic endowment stream that grows over time, and discounts the future at a potentially different rate from the financial intermediary. The financial intermediary commits to the contract, but in any period, the household has an outside option to walk away from the contract and to live in autarky thereafter. The problem is thus one of one-sided commitment.

In the canonical model of lending with one-sided commitment, analyzed, for example, in Ljungqvist and Sargent (2004),¹ the two parties of the contract discount the future at the same rate, and endowment is drawn from the same finite discrete distribution in every period. The household's consumption path in an optimal

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contract then obeys the following simple rule: When the household receives the highest-ever endowment, consumption rises to satisfy the household's participation constraint, that is, to prevent the household from renegeing on the contract. Otherwise, consumption stays constant. Thus, the household eventually experiences the highest possible endowment, and consumption is completely stabilized thereafter.

This paper extends the canonical model by adding two sources of nonstationarity—because of endowment growth, the household's value of autarky, and thus its incentive to renege on the contract, in a given state of nature evolves over time, and because of different time-discount factors between the two parties of the contract, constant consumption is not optimal even when the participation constraint does not bind. This paper then examines how the nature of the endowment process affects the household's welfare. The resulting analysis relates to two studies in particular. The first, Krueger and Perri (2006), analyzes risk sharing between two *ex ante* identical risk-averse households and shows that increased idiosyncratic income risk may improve consumption risk sharing by reducing the households' autarky values and thereby relaxing the participation constraints. The second, Krueger and Uhlig (2006), allows an agent to enter a new contract with any competing principal² after renegeing on the original contract, and examines how the relative patience of agents and principals affects the degree of risk sharing achieved.

The key novelty of this paper, with respect to the two studies described in the preceding, lies in addressing two distinct roles of a contract. The first is risk sharing, or consumption smoothing across different states of nature, which is the focus of attention in the limited commitment literature.³ If this is the only role performed by a contract, as with the canonical model, clearly the allocation under no uncertainty is the first-best allocation. Introducing nonstationarity into the model, however, creates a potential second role for a contract, namely intertemporal consumption smoothing, or consumption smoothing over time. That is, an optimal contract attempts to front- or back-load the household's consumption relative to endowment, depending on the relative discount factors of the two sides of the contract, as well as on the endowment growth rate.

The presence of this second role makes the welfare analysis more interesting, because it now involves weighing the two distinct roles of a contract. Such analysis is, however, not explored in existing studies. In Krueger and Perri (2006)'s environment with two symmetric households, there is no gain from intertemporal consumption smoothing, and thus the first-best outcome is perfect risk sharing. Therefore, although greater variance of income may locally improve welfare through better risk sharing, the presence of uncertainty never generates strict welfare improvement with respect to an economy without uncertainty. The situation is different in Krueger and Uhlig (2006), but Krueger and Uhlig (2006) focuses on how the relative patience of agents and principals affects risk sharing, and does not address its impact on welfare through consumption smoothing over time.

In contrast, the present paper considers an environment in which both roles of a contract are relevant, and examines how the variance and persistence of endowment affect the household's welfare. There are two main analytical results. First, under sufficiently rapid endowment growth, the presence of uncertainty in endowment may improve welfare over the deterministic case. This is because the mechanism by which uncertainty relaxes an agent's participation constraints, as discussed in Krueger and Perri (2006), may also improve intertemporal consumption smoothing. This effect is strong under rapid endowment growth, when the household derives high potential benefit from transferring resources from future to present, but has limited capacity to do so because of its temptation to walk away from the contract in the future. Second, regardless of endowment growth, the household's welfare is nonincreasing⁴ in the persistence of endowment. Intuitively, greater persistence of endowment increases the household's value of autarky, and hence tightening the participation constraint, in the state of nature with high endowment, whereas it has the opposite effect in the state of nature with low endowment. The former effect turns out to dominate the latter; hence the optimal contract becomes less efficient as endowment becomes more persistent. The challenge in formalizing this intuition is that assessing the effect of parameters on welfare is not straightforward in nonstationary environments, where the participation constraint in a given state that is, for example, currently slack may bind in the future. One of the contributions of this paper is to overcome this difficulty by examining how, depending on the parameters, the value of walking away from the contract relative to staying in it evolves over time and by providing separate proofs according to these patterns. Finally, this paper conducts numerical analyses and shows, among other results, that welfare improvement from uncertainty can occur under economic fluctuations of plausible magnitude, and that the effect can be sizeable when endowment grows rapidly and is highly variable.

The preceding discussion presumes that the interest rate is exogenous. Thus, it does not extend to a general equilibrium model with *ex ante* identical and infinitely lived households, in which endogenous interest rate adjustments eliminate the benefit from intertemporal consumption smoothing. Addressing the intertemporal consumption smoothing role of a contract is nevertheless important for several reasons. First, analysis of a limited-commitment environment in partial equilibrium, such as has been explored frequently in literature, is interesting in its own right. Second, the environment of this paper can be interpreted as a borrowing problem of a small open economy, as in Worrall (1990). Third, in overlapping-generations models in which intertemporal consumption smoothing through intergenerational transfer is possible, endogenous interest rate adjustments eliminate the benefit from intertemporal consumption smoothing only in the generationally autarkic equilibrium.

In addition to the studies mentioned earlier, this paper is related to the literature on sovereign debt with noncontingent repayment. In particular, Eaton and Gersovitz (1981) considers an example in which the sovereign borrower's output grows over time and examines the implications of output growth rate and income

variability. Eaton and Gersovitz (1981) considers, however, only a one-period debt, which substantially limits the amount of intertemporal consumption smoothing achieved through borrowing. Eaton and Gersovitz (1981) also mainly focuses on the amount of sustainable debt, and does not discuss welfare implications, as in the present paper.

2. MODEL

2.1. Basic Environment

Time is discrete and is indexed by $t = 0, 1, \dots$, and there is a single perishable good. There are two types of agent in the economy. Households are infinitely lived, have a constant relative risk aversion (CRRA) period utility over consumption, and discount the future with discount factor $\beta \in (0, 1)$. Thus, letting C_t be a household’s consumption in period t , the household’s expected lifetime utility is

$$E \sum_{t=0}^{\infty} \beta^t u(C_t), \tag{1}$$

where E is the unconditional expectation operator, and $u(C_t) = C_t^{1-\sigma} / (1 - \sigma)$ ($\sigma > 0, \sigma \neq 1$). Financial intermediaries are risk-neutral, and unlike households, have access to an external financial market, where they can borrow or lend at a constant interest rate $r > 0$.

In period t , a household receives an endowment $Y_t(s_t)$, which is publicly observable, grows over time, and depends on $s_t \in \{H, L\}$, the state in period t . That is,

$$Y_t(s_t) = \begin{cases} (1 + g)^t y_H & \text{if } s_t = H \text{ (high state),} \\ (1 + g)^t y_L & \text{if } s_t = L \text{ (low state),} \end{cases}$$

where $y_H \geq y_L \geq 0$. The state s_t follows a first-order Markov process such that $\Pr(s_{t+1} = H | s_t = H) = 1 - p_H, \Pr(s_{t+1} = L | s_t = L) = 1 - p_L$, where $p_L, p_H \in (0, 1)$. In other words, p_s denotes the probability that the state switches from its current value, s . Given such a transition matrix, the stationary distribution is

$$(\pi_H, \pi_L) = \left(\frac{p_L}{p_H + p_L}, \frac{p_H}{p_H + p_L} \right). \tag{2}$$

The initial state s_0 is drawn from the stationary distribution, or $\Pr(s_0 = s) = \pi_s$ for $s = \{H, L\}$, and thus the unconditional probability of the state being s is π_s for all t . The fluctuation in endowment described earlier is the only source of uncertainty.⁵

A household and a financial intermediary engage in a long-term loan contract before s_0 is realized: the household promises to offer its endowment stream to the financial intermediary, and receives in return a consumption stream. The financial intermediary commits to the contract, whereas in any t , after observing Y_t , the household can walk away from the contract, which results in permanent exclusion

from the loan market. The loan market is competitive; hence, at equilibrium, risk-neutral financial intermediaries make zero expected profit.

Throughout this paper, I make parameter restrictions

$$r > g \geq 0, \tag{3}$$

$$\beta (1 + r)^{1-\sigma} < 1. \tag{4}$$

Because $\beta > 0$ and $\sigma > 0$, (3) and (4) imply that

$$\beta (1 + g)^{1-\sigma} < 1. \tag{5}$$

These conditions guarantee that optimization problems in the model are well defined.

2.2. Optimal Contract

For the remainder of this paper, an *optimal contract* implies a feasible contract, or a contract from which the household never chooses to walk away, that minimizes the financial intermediary’s expected cost Q of providing a given initial promised value (expected discounted lifetime utility) V_0 to the household. Because the financial intermediary’s revenue is determined exogenously, such a contract also maximizes its expected profit. An *equilibrium contract* is thus an optimal contract whose expected cost equals expected revenue.

Hence, letting $s^t = (s_0, s_1, \dots, s_t)$ denote history, an optimal contract is a stream of consumption $\{C(s^t)\}$, $t = 0, 1, \dots$ that solves, for a given V_0 , the following problem:

$$Q(V_0) = \min_{C(s^t) \geq 0, \forall s^t} E \sum_{t=0}^{\infty} \beta^t C(s^t), \tag{6}$$

$$\text{s.t. } E \sum_{t=0}^{\infty} \beta^t u(C(s^t)) \geq V_0, \tag{7}$$

$$E_t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} u(C(s^\tau | s^t)) \mid s^t \right] \geq V_t^{\text{Aut}}(s_t) \quad \forall s^t. \tag{8}$$

Here, $\{C(s^\tau | s^t)\}$, $\tau = t, t + 1, \dots$ is a continuation of $C(s^t)$ following history s^t . Constraint (7) requires that the contract provide the household at least V_0 . The participation constraint (8) requires that following all possible history s^t , the household’s continuation value in the contract be at least as large as the value of autarky $V_t^{\text{Aut}}(s_t)$, given by

$$V_t^{\text{Aut}}(s_t) = E_t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} u(Y_\tau(s_\tau)) \mid s_t \right]. \tag{9}$$

Here, E_t is the expectation operator conditional on information available in period t . Thus, $V_t^{\text{Aut}}(s_t)$ is the household's expected discounted utility from consuming endowment thereafter, in period t , when the state is s_t . Given the assumptions on the processes of $Y_t(s_t)$ and s_t , $V_t^{\text{Aut}}(s_t)$ depends only on s_t , not on the entire history s^t . An optimal contract never terminates, so period t is identified as the t th period of the contract.⁶

Let R denote the expected discounted revenue from a contract. Then

$$R = E \sum_{t=0}^{\infty} (1+r)^{-t} Y_t(s_t). \tag{10}$$

In an equilibrium contract, V_0 is determined such that

$$Q(V_0) = R. \tag{11}$$

2.3. Recursive Optimal Contract

To analyze an optimal contract, it is convenient to consider a recursive formulation of the preceding problem.⁷ Define $\gamma \equiv 1 - \beta(1+g)^{1-\sigma}$. Obviously $\gamma < 1$ and, from (5), $\gamma > 0$. Then, in state $s \in \{H, L\}$, the financial intermediary solves

$$q_s(v) = \min_{c, v'_s, v'_{-s}} c + \frac{1+g}{1+r} [(1-p_s)q_s(v'_s) + p_s q_{-s}(v'_{-s})], \tag{12}$$

$$v = u(c) + (1-\gamma)[(1-p_s)v'_s + p_s v'_{-s}], \tag{13}$$

$$v'_{s'} \geq v_s^{\text{Aut}}, \quad s' = \{s, -s\}, \tag{14}$$

where $-s = H$ if $s = L$, and vice versa. Here, $q_s(v)$ is the financial intermediary's cost of providing the household promised value v when the current state is s , whereas c is the household's current consumption, v'_s is the promised value next period in state s' , and v_s^{Aut} is the value of autarky in state s . Cost and consumption are detrended by $(1+g)^t$, and utility-related terms are detrended by $(1+g)^{(1-\sigma)t}$; for example, $v_s^{\text{Aut}} = (1+g)^{-(1-\sigma)t} V_t^{\text{Aut}}(s) = V_0^{\text{Aut}}(s)$. Henceforth, the analysis proceeds using this recursive formulation, and unless mentioned otherwise, variables refer to the detrended values.⁸

The autarky values in the high and low states, v_H^{Aut} and v_L^{Aut} , are expressed as

$$v_H^{\text{Aut}} = u(y_H) + (1-\gamma)[(1-p_H)v_H^{\text{Aut}} + p_H v_L^{\text{Aut}}], \tag{15}$$

$$v_L^{\text{Aut}} = u(y_L) + (1-\gamma)[p_L v_H^{\text{Aut}} + (1-p_L)v_L^{\text{Aut}}], \tag{16}$$

which yield

$$v_H^{Aut} = \frac{[\gamma + (1 - \gamma) p_L] u(y_H) + (1 - \gamma) p_H u(y_L)}{\gamma [\gamma + (1 - \gamma) (p_H + p_L)]}, \tag{17}$$

$$v_L^{Aut} = \frac{(1 - \gamma) p_L u(y_H) + [\gamma + (1 - \gamma) p_H] u(y_L)}{\gamma [\gamma + (1 - \gamma) (p_H + p_L)]}. \tag{18}$$

Clearly, (17) and (18) imply that $v_H^{Aut} \geq v_L^{Aut}$ because $y_H \geq y_L$. Moreover, $v_H^{Aut} = v_L^{Aut}$ only in special cases, namely, when $y_H = y_L$, or when $\sigma > 1$ and $y_L = 0$ (which implies that $u(y_L) = -\infty$). Thus, unless otherwise noted, the discussion that follows presumes that $v_H^{Aut} > v_L^{Aut}$.

Using this recursive formulation, $Q(V_0)$ can be written as

$$Q(V_0) = \min_{v_H^0, v_L^0} \pi_H q_H(v_H^0) + \pi_L q_L(v_L^0) \tag{19}$$

$$\text{s.t. } V_0 = \pi_H v_H^0 + \pi_L v_L^0, \tag{20}$$

$$v_H^0 \geq v_H^{Aut}, v_L^0 \geq v_L^{Aut}. \tag{21}$$

Krueger (1999) and Krueger and Uhlig (2006)⁹ show that the standard dynamic programming argument can be applied to this type of environment to establish that the cost function q_s is increasing, strictly convex, and continuously differentiable, and therefore so is Q .

2.4. Properties of an Optimal Contract

In this subsection, I present the key properties of an optimal contract.¹⁰

Path of consumption and the promised value. First, consider the continuation problem given by (12)–(14), which concerns choosing current consumption and the next period’s promised values in the two states. Combining the first-order conditions with the envelope condition,

$$\theta_{s'} = \frac{1 + g}{1 + r} \left[\frac{\partial q_{s'}(v_{s'})}{\partial v} - \frac{(1 - \gamma)(1 + r)}{1 + g} \frac{\partial q_s(v)}{\partial v} \right], s' = \{s, -s\}, \tag{22}$$

$$\frac{\partial q_s(v)}{\partial v} = \frac{1}{u'(c)}, \tag{23}$$

where $\theta_{s'}$ is proportional to the Lagrange multiplier for the participation constraint (14) in state s' . Let $\mu \equiv [\beta(1 + r)]^{1/\sigma} - 1$. Then, because $\gamma = 1 - \beta(1 + g)^{1-\sigma}$,

$$\frac{(1 - \gamma)(1 + r)}{1 + g} = \left(\frac{1 + \mu}{1 + g} \right)^\sigma. \tag{24}$$

Also, (4) and $\frac{1+\mu}{1+r} = [\beta(1+r)^{1-\sigma}]^{1/\sigma}$ imply that $r > \mu$. The path of consumption and the promised value can be analyzed using (22)–(24). In the following, for $s \in \{H, L\}$, let c_s^{Aut} be consumption in an optimal contract when the state is s and the promised value is v_s^{Aut} .

First, suppose the participation constraint (14) does not bind in state s' next period. Then $\theta_{s'} = 0$, so (22) yields

$$\frac{\partial q_{s'}(v'_{s'})}{\partial v} = \frac{(1-\gamma)(1+r)}{1+g} \frac{\partial q_s(v_s)}{\partial v}, \tag{25}$$

which determines $v'_{s'}$. In particular, setting $s' = s$ in (25) and noting (24), $g > \mu$ implies $\partial q_s(v'_s)/\partial v < \partial q_s(v_s)/\partial v$, and thus $v'_s < v_s$, because q_s is convex. Conversely, $g = \mu$ implies $v'_s = v_s$, and $g < \mu$ implies $v'_s > v_s$. Thus, when the state remains unchanged and the participation constraint is slack, the promised value falls if $g > \mu$, remains constant if $g = \mu$, and rises if $g < \mu$.

Also, letting $c_{s'}$ be consumption next period in state s' , $\theta_{s'} = 0$ and (22)–(24) yield

$$\frac{c_{s'}}{c} = \left[\frac{(1-\gamma)(1+r)}{1+g} \right]^{1/\sigma} = \frac{1+\mu}{1+g}. \tag{26}$$

Thus, when the participation constraint does not bind, consumption in an optimal contract grows at a rate $\frac{\mu-g}{1+g}$, which corresponds to nondetrended consumption growing at a rate μ .

Next, suppose the participation constraint (14) binds in state s' next period. Then $v'_{s'} = v_{s'}^{\text{Aut}}$, so by definition, $c_{s'} = c_{s'}^{\text{Aut}}$. Because $\theta_{s'} > 0$, (22) and (23) imply that $v'_{s'}$ and $c_{s'}$ exceed the values they would attain if (14) were slack, as determined by (25) and (26).

This argument leads to the following two lemmas.

LEMMA 1. $c_H^{\text{Aut}} \geq c_L^{\text{Aut}}$ with equality if and only if $v_H^{\text{Aut}} = v_L^{\text{Aut}}$.

Proof. See Appendix B. ■

LEMMA 2. Consider an optimal contract whose current consumption is c . In state $s' \in \{H, L\}$ next period, the participation constraint (14) binds if and only if $c(1+\mu)/(1+g) < c_{s'}^{\text{Aut}}$, and consumption $c_{s'}$ equals $\max\{c(1+\mu)/(1+g), c_{s'}^{\text{Aut}}\}$.

Proof. Let s be the current state, and $c'_{s',\text{uc}}$ and $v'_{s',\text{uc}}$ be, respectively, unconstrained (i.e., when (14) is not imposed) optimal consumption and the promised value, in state s' next period. From the preceding argument, $c'_{s',\text{uc}} = c(1+\mu)/(1+g)$, and $v'_{s',\text{uc}}$ is determined by $\frac{\partial q_{s'}(v'_{s',\text{uc}})}{\partial v} = \frac{(1-\gamma)(1+r)}{1+g} \frac{\partial q_s(v_s)}{\partial v}$. Now, (14) binds in state s' next period if and only if $v'_{s',\text{uc}} < v_{s'}^{\text{Aut}}$, and when it does, the promised value $v'_{s'}$ equals $v_{s'}^{\text{Aut}}$. Thus, $v'_{s'} = \max\{v'_{s',\text{uc}}, v_{s'}^{\text{Aut}}\}$. Then from (23), $v'_{s',\text{uc}} < v_{s'}^{\text{Aut}}$ if and only if $c'_{s',\text{uc}} < c_{s'}^{\text{Aut}}$, and $c_{s'} = \max\{c'_{s',\text{uc}}, c_{s'}^{\text{Aut}}\}$. ■

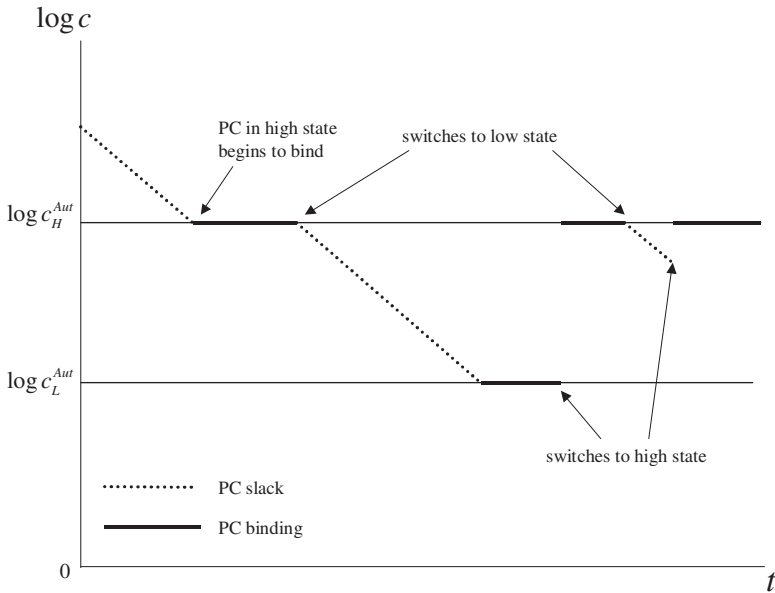


FIGURE 1. Consumption path ($g > \mu$).

Using these results, Figure 1 depicts the path of $\log c$ for $g > \mu$, which turns out to be the more interesting case.¹¹ Here, $\log c$ is denoted by dashed lines when the participation constraint (14) is slack, and by thick solid lines when it is binding. The two solid flat lines correspond to $\log c_H^{Aut}$ and $\log c_L^{Aut}$. Note that from Lemma 2, $\log c \geq \log c_s^{Aut}$ in state s , and that from (26), dashed lines have slope $\log \frac{1+\mu}{1+g} < 0$. Thus, even if the participation constraint is initially slack in both states, it may eventually bind in either state.

Figure 2 shows the path of $\log c$ for $g < \mu$, assuming that initially the state is low and $c \in [c_L^{Aut}, c_H^{Aut})$. Thus, $\log c \geq \log c_L^{Aut}$ in all periods because the dashed lines have slope $\log \frac{1+\mu}{1+g} > 0$, so from Lemma 2, the participation constraint in the low state never binds. Moreover, once $\log c$ reaches $\log c_H^{Aut}$, either from a move along this dashed line or from a jump following the first realization of the high state, the participation constraint in the high state never binds thereafter. The situation for $g = \mu$, including the canonical model in which $g = 0$ and $\beta(1+r) = 1$, is similar. This time, the dashed line becomes flat, so that $\log c$ is constant until the first realization of the high state, when it jumps to $\log c_H^{Aut}$, and remains there.

Initial promised values in the two states. Next, consider the problem given by (19)–(21), which concerns choosing the initial promised values in the two states, v_H^0 and $v_L^0 = \frac{1}{\pi_L}(V_0 - \pi_H v_H^0)$. Let $V^{Aut} \equiv \pi_H v_H^{Aut} + \pi_L v_L^{Aut}$ be the household’s ex ante welfare from autarky, and let V^{FB} be the welfare from the first-best contract, achieved under full commitment. The equilibrium contract does

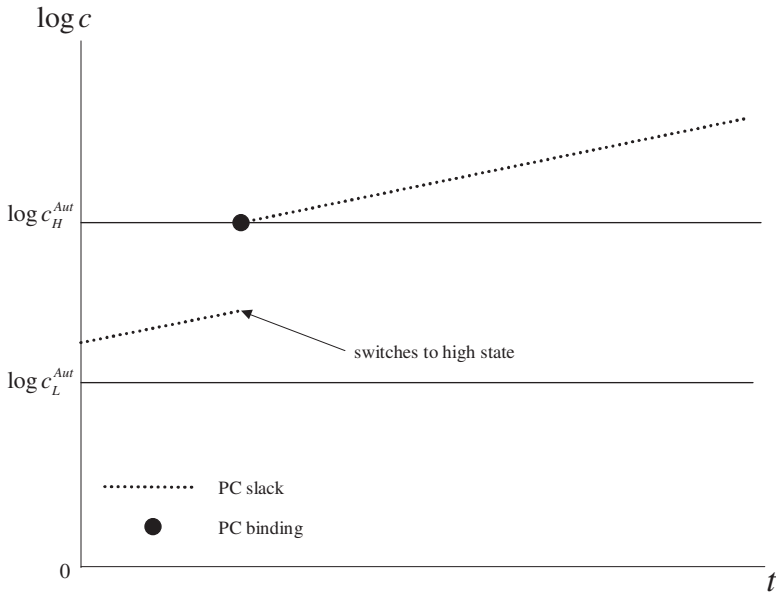


FIGURE 2. Consumption path ($g < \mu$).

no worse than autarky and no better than the first-best contract, so it suffices to consider $V_0 \in [V^{\text{Aut}}, V^{\text{FB}}]$.

From the first-order conditions,

$$\theta_H^0 - \theta_L^0 = \frac{\partial q_H(v_H^0)}{\partial v} - \frac{\partial q_L(v_L^0)}{\partial v}, \tag{27}$$

where $\theta_s^0 \geq 0$ is proportional to the Lagrange multiplier for the initial participation constraint (21) in state s . The condition for v_H^0 and v_L^0 thus depends on whether (21) binds. Clearly, (21) is more likely to bind for small V_0 , and as shown next, it may bind only in the high state.

LEMMA 3. For $V_0 \geq V^{\text{Aut}}$, the initial participation constraint (21) in the low state is slack.

Proof. Because (20) and $V_0 \geq V^{\text{Aut}}$ require $v_s^0 \geq v_s^{\text{Aut}}$ for at least one $s \in \{H, L\}$, (21) may bind in at most one state. Suppose (21) binds in the low state. Then $\theta_H^0 = 0$, $\theta_L^0 > 0$, and $v_L^0 = v_L^{\text{Aut}}$, so from (27), $\frac{\partial q_H(v_H^0)}{\partial v} < \frac{\partial q_L(v_L^{\text{Aut}})}{\partial v}$. Then, because $v_H^{\text{Aut}} \leq v_H^0$ and q_H is convex, $\frac{\partial q_H(v_H^{\text{Aut}})}{\partial v} < \frac{\partial q_L(v_L^{\text{Aut}})}{\partial v}$. Thus, from (23), $c_H^{\text{Aut}} < c_L^{\text{Aut}}$, which contradicts Lemma 1. ■

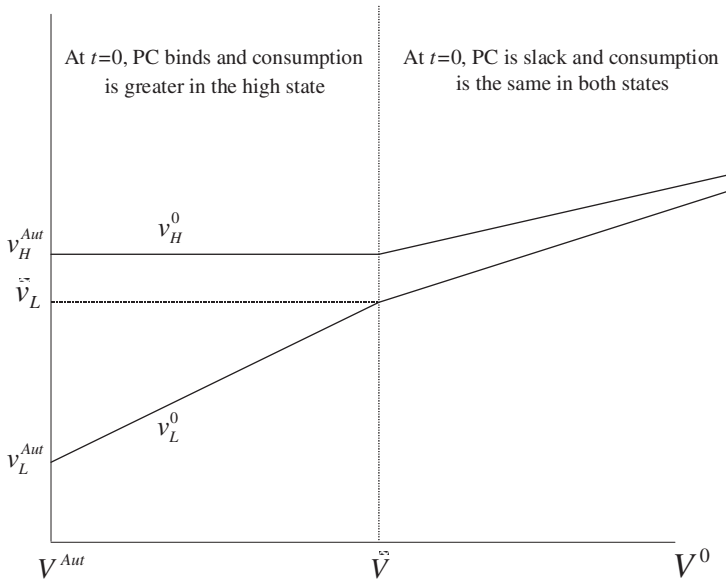


FIGURE 3. v_H^0 and v_L^0 as functions of V^0 .

Now, let $\tilde{V} \equiv \pi_H v_H^{Aut} + \pi_L \tilde{v}_L$, where \tilde{v}_L is defined by

$$\frac{\partial q_H(v_H^{Aut})}{\partial v} = \frac{\partial q_L(\tilde{v}_L)}{\partial v}. \tag{28}$$

Given (23), \tilde{v}_L is such that if the state is low and the promised value is \tilde{v}_L , then current consumption is c_H^{Aut} . Because $c_H^{Aut} \geq c_L^{Aut}$, (23) implies that $\tilde{v}_L \geq v_L^{Aut}$; if $v_H^{Aut} > v_L^{Aut}$, then $c_H^{Aut} > c_L^{Aut}$, so $\tilde{v}_L > v_L^{Aut}$. The condition for v_H^0 and v_L^0 is then described as follows.

If $V_0 \in [V^{Aut}, \tilde{V})$, (21) binds in the high state, so $\theta_H^0 > 0$ and $\theta_L^0 = 0$. Thus, from (27),

$$\frac{\partial q_H(v_H^0)}{\partial v} > \frac{\partial q_L(v_L^0)}{\partial v}, \quad v_H^0 = v_H^{Aut}, \tag{29}$$

so from (23), initial consumption is higher in the high state. Conversely, if $V_0 \in [\tilde{V}, V^{FB}]$, (21) does not bind, so $\theta_H^0 = \theta_L^0 = 0$. Thus, (27) yields

$$\frac{\partial q_H(v_H^0)}{\partial v} = \frac{\partial q_L(v_L^0)}{\partial v}, \tag{30}$$

so from (23), initial consumption is the same in both states.

Figure 3 shows how v_H^0 and v_L^0 vary with V_0 . For $V_0 \in [V^{Aut}, \tilde{V}]$, v_H^0 is constant at v_H^{Aut} ; hence v_L^0 increases linearly from v_L^{Aut} to \tilde{v}_L . For $V_0 \geq \tilde{V}$, both v_H^0 and v_L^0 increase with V_0 .

As shown above, for $V_0 \in [\tilde{V}, V^{FB}]$, consumption in the initial period, $t = 0$, is the same in both states. Then, from Lemma 2, whether the participation constraint (14) binds in the next period, $t = 1$, is independent of s_0 . Moreover, the promised value in state s' next period is v_s^{Aut} if (14) binds, and is determined by (25) otherwise. Thus, given (30), the promised value next period depends only on s_1 , and not on s_0 . Lemma 4 generalizes this result.

LEMMA 4. *For any $t \in \{0, 1, \dots\}$, suppose the participation constraint in neither state binds up to period t . Then, consumption in period t is the same in both states, and the promised values in period $t + 1$ depend only on s_{t+1} and not on s^t , the history up to period t .*

Proof. See Appendix B. ■

3. UNCERTAINTY AND WELFARE

I now discuss how endowment uncertainty, in terms of variance and persistence, affects the household’s welfare. More precisely, I first consider an increase in the mean-preserving spread by setting y_H and y_L as

$$y_H = 1 + \frac{\pi_L}{\pi_H} \alpha, \tag{31}$$

$$y_L = 1 - \alpha, \tag{32}$$

and varying α in $\alpha \in [0, 1]$. Then, because the unconditional probability of state s is π_s for all t , the unconditional expectation of $Y_t(s_t)$ equals $(1 + g)^t$ for all t , which in turn implies that $R = \frac{1+r}{r-g}$. Second, to analyze the effect of persistence, I assume that

$$p_H = p \cdot \tilde{p}_H, \tag{33}$$

$$p_L = p \cdot \tilde{p}_L, \tag{34}$$

where \tilde{p}_H, \tilde{p}_L are positive constants and $0 \leq p \leq \min\{\frac{1}{\tilde{p}_H}, \frac{1}{\tilde{p}_L}\}$. Then, as p varies, p_H and p_L vary proportionately, keeping π_H and π_L unchanged.

Given these assumptions, the variance of endowment y_s is $\alpha^2 \frac{\pi_L}{\pi_H} = \alpha^2 \frac{\tilde{p}_H}{\tilde{p}_L}$, whereas the first lagged autocorrelation of y_s is $1 - p_H - p_L = 1 - p(\tilde{p}_H + \tilde{p}_L)$. Thus, a rise in α and p , respectively, raises the variance of y_s and lowers the persistence of y_s , independent of the other parameter. Therefore, the effect of variance and persistence on the household’s welfare can be analyzed by varying, respectively, α and p .

Note that the channel through which α and p affect the household’s welfare is affecting the household’s values of autarky, and accordingly the participation constraints, in the two states. As an implication, V^{FB} is independent of α and p .

3.1. Welfare Effect of Variance of Endowment

Role of growth rate. I first discuss the effect of α . Let $V^U(\alpha)$ denote the household's ex ante welfare, or expected lifetime utility before realization of s_0 , from an equilibrium contract, and let $V^C \equiv V^U(0)$ be the corresponding welfare without uncertainty. Also, I write V^{Aut} as $V^{\text{Aut}}(\alpha)$ when emphasizing its dependence on α .

In what follows, I examine whether the presence of uncertainty can be welfare-improving, formally defined in the following sense.

DEFINITION 1. *Uncertainty is welfare-improving if $V^U(\alpha) > V^C$ for some $\alpha \in (0, 1]$.*

The following proposition describes key relations between endowment growth and welfare-improving uncertainty.

PROPOSITION 1. (1) *When $g \leq \mu$, $V^C = V^{\text{FB}}$; thus uncertainty can be welfare-improving only when $g > \mu$. (2) *When $g > \mu$, $V^C = V^{\text{Aut}}(0)$; when $\alpha = 0$, consumption and endowment streams coincide in an equilibrium contract.**

Proof. See Appendix B. ■

Proposition 1(1) implies that $g > \mu$ is a necessary condition for uncertainty to be welfare-improving, because if $g \leq \mu$, the first-best consumption path, which satisfies $C_{t+1}/C_t = 1 + \mu$ and $\sum_{t=0}^{\infty} (1+r)^{-t} C_t = R$, can be sustained under certainty. In stark contrast, Proposition 1(2) states that when $g > \mu$, an equilibrium contract does no better than autarky when there is no endowment uncertainty.

Intuitively, when $g \leq \mu$, an optimal contract seeks to backload consumption relative to endowment. Therefore, under certainty, the household is a lender and has no incentive to renege on the contract; hence the first-best consumption path can be achieved. Conversely, when $g > \mu$, an optimal contract seeks to frontload consumption, so consumption is initially relatively high compared to endowment, and later becomes relatively low. But because the household cannot commit to the contract, the contract must always provide the household at least its value of autarky. Thus, in the absence of uncertainty, the financial intermediary is unable to generate profits during what should be the low-consumption phase; hence the high-consumption phase cannot exist under the zero-profit condition—in other words, the household is a borrower with no incentive for repayment. Thus it cannot borrow at all.

When there is uncertainty, however, the risk-sharing role makes the contract valuable to the household even in the low-consumption phase, which allows the financial intermediary to make profits in the late stages of the contract. As a result, some frontloading of consumption can be achieved, which may lead to welfare improvement over the certainty case. In the following, I derive a simple, sufficient condition for uncertainty to be welfare-improving.

Local welfare improvement from α . The analysis proceeds by examining how α affects the values of autarky, written as $v_H^{Aut}(\alpha)$ and $v_L^{Aut}(\alpha)$ when emphasizing their dependence on α . Further, let $v^{Aut} \equiv v_H^{Aut}(0) = v_L^{Aut}(0)$. I begin with a condition under which uncertainty locally improves welfare.¹²

LEMMA 5. *If $g > \mu$, there exists $\alpha_0 \in [0, 1)$ such that $V^U(\alpha)$ is increasing in α for $\alpha \in [\alpha_0, 1)$.*

Proof. Let $g > \mu$. Then Lemma 2 implies that consumption grows at a rate $\frac{\mu-g}{1+g} < 0$ when the participation constraint does not bind, and thus the participation constraint in either state may eventually bind. Therefore, if the participation constraint in one state is relaxed without that in the other state being tightened, an optimal contract achieves a more efficient allocation with the same cost. Thus, a sufficient condition for $V^U(\alpha)$ to be locally increasing in α is that $\partial v_s^{Aut}(\alpha)/\partial\alpha \leq 0$ for $s = \{H, L\}$, with strict inequality for at least one s . I show later that there exists $\alpha_1 \in (0, 1)$ such that this sufficient condition holds for any $\alpha \in [\alpha_1, 1)$. Thus, $V^U(\alpha)$ is increasing in α at least for $\alpha \in [\alpha_1, 1)$, and thus there must exist $\alpha_0 \in [0, \alpha_1]$ such that $V^U(\alpha)$ is increasing in α for $\alpha \in [\alpha_0, 1)$.

Differentiating (17) and (18) with respect to α and noting that $\pi_L/\pi_H = p_H/p_L$,

$$\frac{\partial v_H^{Aut}}{\partial\alpha} = \frac{p_H [\gamma + (1 - \gamma) p_L] u'(y_H) - (1 - \gamma) p_L u'(y_L)}{p_L \gamma [\gamma + (1 - \gamma) (p_H + p_L)]}, \tag{35}$$

$$\frac{\partial v_L^{Aut}}{\partial\alpha} = \frac{(1 - \gamma) p_H u'(y_H) - [\gamma + (1 - \gamma) p_H] u'(y_L)}{\gamma [\gamma + (1 - \gamma) (p_H + p_L)]}. \tag{36}$$

Note from (31) and (32) that $\alpha = 0$ implies that $y_H = y_L = 1$, whereas $\alpha = 1$ implies that $y_H = 1 + \frac{\pi_L}{\pi_H}$ and $y_L = 0$. Then, because $\gamma \in (0, 1)$, (35) implies $\partial v_H^{Aut}/\partial\alpha > 0$ at $\alpha = 0$, but $\partial v_H^{Aut}/\partial\alpha < 0$ at $\alpha = 1$ as $u'(y_L) = u'(0) = \infty$. Further, (35) yields $\partial^2 v_H^{Aut}/\partial\alpha^2 < 0$, so there is a unique $\alpha \in (0, 1)$ such that $\partial v_H^{Aut}/\partial\alpha = 0$, denoted as α_1 . Thus, $\partial v_H^{Aut}/\partial\alpha \leq 0$ if and only if $\alpha \geq \alpha_1$. On the other hand, (36) implies $\partial v_L^{Aut}/\partial\alpha < 0$ for any $\alpha \in [0, 1]$, because $u'(y_L) \geq u'(y_H) > 0$. Thus, for $\alpha \geq \alpha_1$, $\partial v_H^{Aut}/\partial\alpha \leq 0$ and $\partial v_L^{Aut}/\partial\alpha < 0$, so the sufficient condition holds, completing the proof. ■

Figure 4 illustrates how v_L^{Aut} and v_H^{Aut} vary with α . As shown in the preceding, v_L^{Aut} always decreases in α , because a rise in α lowers current autarky consumption, $1 - \alpha$. In contrast, v_H^{Aut} initially increases in α , but decreases in α beyond a certain value, denoted as α_1 . This is because a rise in α has two opposing effects on v_H^{Aut} : to increase current consumption, and to worsen the future outcome in the low state. For sufficiently large α , the second effect dominates, so that v_H^{Aut} decreases in α , and a rise in α lowers both v_L^{Aut} and v_H^{Aut} , increasing the household's welfare $V^U(\alpha)$. Krueger and Perri (2006) resorts to this mechanism to show that a rise in household income dispersion may lead to a fall in consumption dispersion.

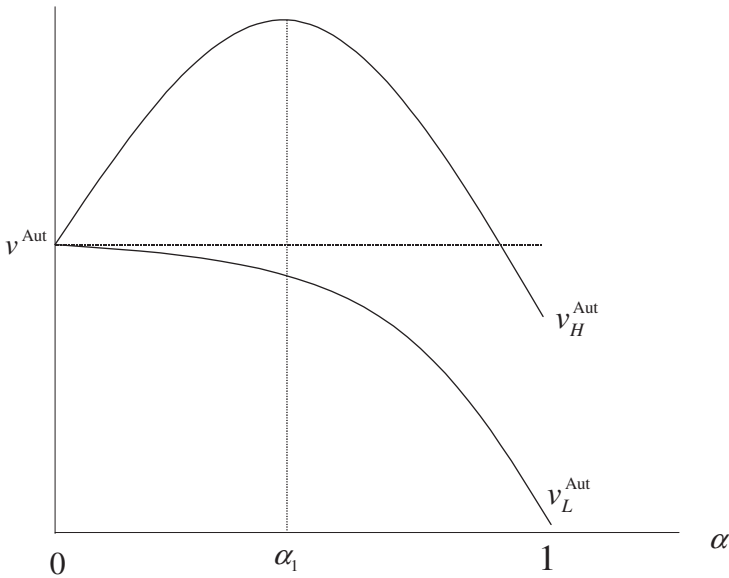


FIGURE 4. Autarky values and α .

Sufficient condition for welfare improvement. The discussion thus far suggests that whether uncertainty is welfare-improving can be checked by comparing $V^C = V^U(0)$ and $V^U(1)$. This comparison delivers a sufficient condition, as summarized in the following proposition.

PROPOSITION 2. *Uncertainty is welfare-improving if $g > \mu$ and $v_H^{\text{Aut}}(1) \leq v^{\text{Aut}}$.*

Proof. As stated in Proposition 1(1), $g > \mu$ is a necessary condition for uncertainty to be welfare-improving. Uncertainty is welfare-improving if, in addition, the participation constraint is weakly less stringent at $\alpha = 1$ than at $\alpha = 0$ in both states, and strictly so in at least one state. This is true if $v_H^{\text{Aut}}(1) \leq v^{\text{Aut}}$, because $v_L^{\text{Aut}}(1) < v^{\text{Aut}}$ always holds, because $\partial v_L^{\text{Aut}}/\partial \alpha < 0$ for $\alpha \in [0, 1]$. ■

Note that for any $r > 0$ satisfying (4), there exists $g > \mu$ satisfying (3), because $r > \mu$, as argued before. When the sufficient condition stated in Proposition 2 is satisfied, $V^U(1) > V^U(0) = V^C$. Then the continuity of $V^U(\alpha)$, which follows from that of Q , guarantees the existence of $\alpha_2 \in (\alpha_0, 1)$ such that $V^U(\alpha) > V^C$ for all $\alpha > \alpha_2$. This sufficient condition can be written as a condition for g that allows easier economic interpretation.

PROPOSITION 3. *If $\sigma > 1$ or $\frac{1-\beta(1+r)^{1-\sigma}(1-p_L)}{1-\beta(1+r)^{1-\sigma}(1-p_H-p_L)} \leq \frac{1}{(1+p_H/p_L)^{1-\sigma}}$, there exists $g^* < r$ such that uncertainty is welfare-improving for $g > g^*$.*

Proof. If $\sigma > 1$, then $v_H^{\text{Aut}}(1) = -\infty < v^{\text{Aut}}$, so the sufficient condition given in Proposition 2 holds for $g > g^* = \mu$. If $\sigma < 1$, then the condition $v_H^{\text{Aut}}(1) \leq v^{\text{Aut}}$ becomes

$$\frac{1 - \beta(1 + g)^{1-\sigma}(1 - p_L)}{1 - \beta(1 + g)^{1-\sigma}(1 - p_H - p_L)} \leq \frac{1}{\left(1 + \frac{p_H}{p_L}\right)^{1-\sigma}}. \tag{37}$$

For $\sigma < 1$, the left-hand side (LHS) of (37) is decreasing in g , and there exists \tilde{g} such that (37) holds for $g \geq \tilde{g}$. The condition $\frac{1 - \beta(1+r)^{1-\sigma}(1-p_L)}{1 - \beta(1+r)^{1-\sigma}(1-p_H-p_L)} < \frac{1}{(1+p_H/p_L)^{1-\sigma}}$, which holds at least for $\beta(1+r)^{1-\sigma}$ close to 1, guarantees $\tilde{g} < r$. Accordingly, the sufficient condition holds for $g > g^* = \max\{\mu, \tilde{g}\}$, where such a g satisfying (3) exists because $g^* < r$. ■

Proposition 3 implies that uncertainty is welfare-improving for sufficiently large g , under some technical condition that makes such values of g compatible with parameter restriction (3). For $\sigma > 1$, $v_H^{\text{Aut}}(1) < v^{\text{Aut}}$ always holds, so the condition $g > \mu$ (which clearly holds for sufficiently large g), necessary for uncertainty to be welfare-improving, is also sufficient. For $\sigma < 1$, a rise in g reduces, in a relative sense, the household’s autarky value in the high state by increasing the importance of future utility; thus $v_H^{\text{Aut}}(1) \leq v^{\text{Aut}}$ holds for large enough g .

It should be noted, however, that $v_H^{\text{Aut}}(1) \leq v^{\text{Aut}}$ is not a necessary condition for uncertainty to be welfare-improving. This is because, even if $v_H^{\text{Aut}}(1) > v^{\text{Aut}}$, relaxation of the participation constraint in the low state may more than offset its tightening in the high state. Thus, sharper predictions on welfare gains from uncertainty require numerical analysis, which is addressed in Section 4.

3.2. Welfare Effect of Persistence of Endowment

I now turn to the effect of p . The following corollary to Proposition 3 illustrates how p affects the sufficient condition for uncertainty to be welfare-improving.

COROLLARY 1. *g^* is nonincreasing in p .*

The proof is immediate from observing that the LHS of (37) is decreasing in p , whereas p_H/p_L on the right-hand side (RHS) is independent of p . When the endowment process is highly persistent (i.e., low p), the value of autarky in the high state is high, requiring a large value of g for this value to be smaller than the value of autarky under no uncertainty.

The following proposition describes the direct effect of p on welfare and is one of the main findings of this paper.

PROPOSITION 4. *$V^U(\alpha)$ is nondecreasing in p . If $g > \mu$, it is increasing in p , unless $\alpha \leq \left(1 + \frac{1+\mu}{1+g} \frac{\pi_L}{\pi_H}\right)^{-1} \left(\frac{g-\mu}{1+g}\right)$, or $\sigma > 1$ and $\alpha = 1$.*

Proof. See Appendix B. ■

Proposition 4 implies that, regardless of the relative magnitude of g and μ , $V^U(\alpha)$ is always nondecreasing in p , or equivalently, nonincreasing in the persistence of endowment. It follows from (17) and (18) that $\pi_H \frac{\partial v_H^{\text{Aut}}}{\partial p} = -\pi_L \frac{\partial v_L^{\text{Aut}}}{\partial p} \leq 0$, with strict inequality if $v_H^{\text{Aut}} > v_L^{\text{Aut}}$. Thus, a larger p relaxes the participation constraint in the high state and tightens it in the low state. The positive welfare effect of p indicates that the former effect dominates the latter. If $g \leq \mu$, this result is immediate because, as suggested by Figure 2, the participation constraint in the low state never binds in the equilibrium contract; hence there is no cost from tightening it.¹³ Thus, $V^U(\alpha)$ is nondecreasing in p , and it is in fact increasing in p , so long as the participation constraint in the high state binds at $t = 0$. The situation differs for $g > \mu$, where the participation constraint in the low state binds in the future with positive probability (except for the case of $\sigma > 1$ and $\alpha = 1$, which implies that $v_L^{\text{Aut}} = -\infty$). Thus, proving the result for this case requires a much more detailed analysis, which involves formally evaluating the cost (benefit) of tightening (relaxing) the participation constraints that are currently slack but that may eventually bind. Intuitively, the proof proceeds by establishing different properties of $\partial Q(V_0)/\partial p$ for relatively small and large V_0 , and by consolidating these implications to show that $\partial Q(V_0)/\partial p \leq 0$.

That the household's welfare is never increasing in the persistence of endowment is an interesting result, and that this can be proved even for $g > \mu$ is remarkable, given the complexity caused by repeatedly binding participation constraints. Note that the situation in which the state deterministically alternates between high and low¹⁴ can be considered a limiting case as $p \rightarrow 1/\bar{p}_H = 1/\bar{p}_L$. Because V^C is independent of p , such an environment has the highest chance that $V^U(\alpha) > V^C$.

4. NUMERICAL ANALYSIS

This section sets forth the numerical analysis.¹⁵ The analysis allows examination of the exact relation between α and $V^U(\alpha)$, which cannot be done analytically. It also provides an idea of the magnitude of the welfare effect of α and p , or the variance and persistence of endowment, under realistic parameters.

In all examples that follow, the model period corresponds to one quarter, $r = 0.01$, and $\beta = 1/(1+r)$. This implies that $\mu = 0$, so $g > \mu$ for any $g > 0$.

4.1. Welfare Effect of Variance of Endowment

I first show how the household's welfare varies with α . The parameter values are $g = 0.005$ (2% annual growth) and $p_H = p_L = 0.06$, and for risk aversion σ , two different values, 0.5 and 2, are used for comparison. To facilitate quantitative evaluation of welfare, Figure 5 plots consumption-equivalent variations of welfare gain of $V^U(\alpha)$, relative to V^C . More specifically, the vertical axis corresponds to

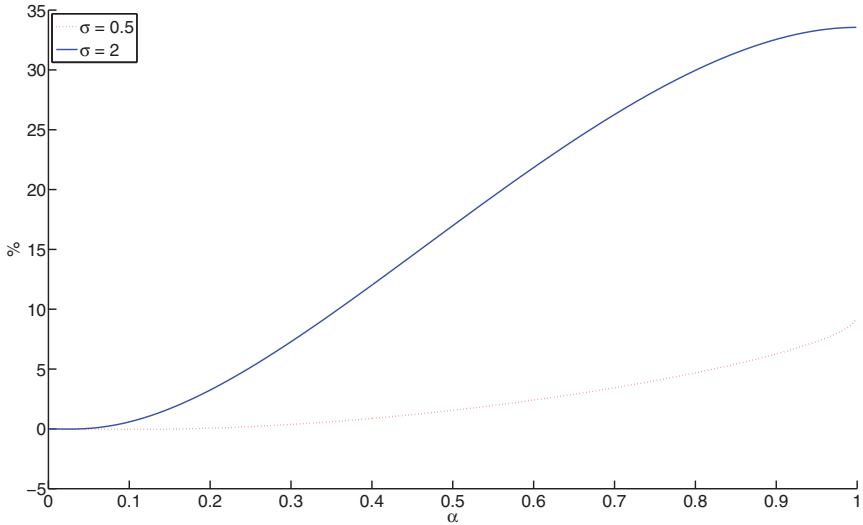


FIGURE 5. Welfare gain relative to certainty case (consumption-equivalent variations in %).

δ expressed as a percentage, where δ solves

$$V^U(\alpha) = \sum_{t=0}^{\infty} (1 - \gamma)^t u(1 + \delta). \tag{38}$$

Figure 5 shows that for both $\sigma = 0.5$ and 2 , $V^U(\alpha) < V^C$ for α close to 0 , and $V^U(\alpha) > V^C$ for sufficiently large α . Also, the range of α for which $V^U(\alpha) > V^C$ is wider for $\sigma = 2$. Figure 5 also shows that when $\sigma = 0.5$, the impact of α on welfare, either positive or negative, is not so large. However, the welfare gain from uncertainty can be substantial when $\sigma = 2$ —in this example, close to 34% of flow consumption in the certainty case, as α approaches 1. These effects of σ can be understood as follows. A rise in σ reduces the household’s autarky values under uncertainty through an increase in risk aversion, and increases the benefit of consumption frontloading through a reduction in intertemporal elasticity of substitution (IES). Both these channels increase the value of uncertainty. To examine the strength of each channel, I compute corresponding consumption-equivalent variations for the first-best welfare, V^{FB} , and find the value to be 13% and 34% for $\sigma = 0.5$ and 2 , respectively. These values represent the maximum benefit of intertemporal consumption reallocation, determined by the IES. Figure 5 indicates that as $\alpha \rightarrow 1$, this maximum benefit is achieved for $\sigma = 2$, whereas in the case of $\sigma = 0.5$, the welfare gain in terms of consumption is approximately 9%, considerably lower than in the first best contract. This difference

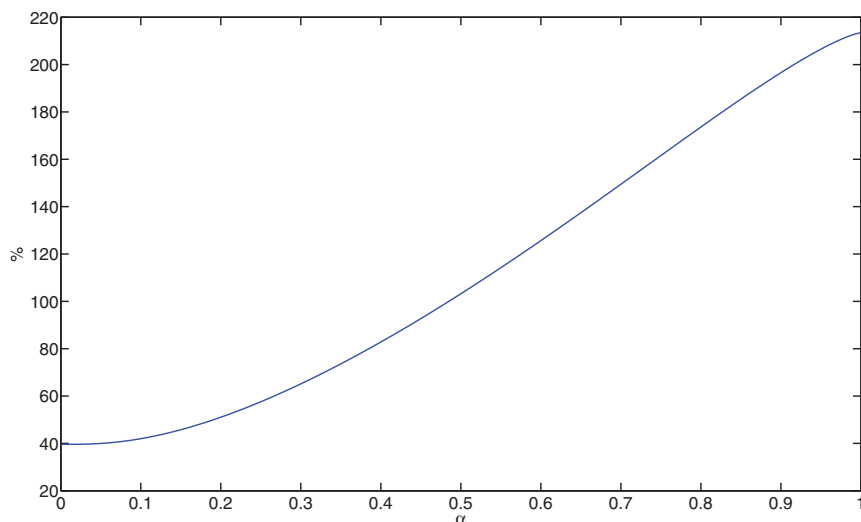


FIGURE 6. Welfare cost of 2% lower growth.

arises because as $\alpha \rightarrow 1$, participation constraints cease to bind if and only if $\sigma > 1$.

At the country level, however, α cannot be very large. For example, for U.S. quarterly real GDP per capita between 1969Q1 and 2012Q2, the standard deviation is 1.60% and the first autocorrelation coefficient is 0.88. Assuming high and low states to be symmetric, this corresponds to $\alpha = 0.016$, and as assumed for Figure 5, $p_H = p_L = 0.06$.¹⁶ For values of α of this magnitude, the presence of uncertainty indeed leads to welfare losses for both values of σ , although they are negligible, as observed from Figure 5.

The welfare gain from uncertainty, however, can be considerably greater than in the example preceding when the difference between g and $\mu = [\beta(1+r)]^{1/\sigma} - 1$ is large, because this creates a greater benefit from intertemporal consumption smoothing. Given the restriction $r > g$, this occurs when g is close to r . For $\alpha = 0.016$, $p_H = p_L = 0.06$, and $\sigma = 2$, welfare turns out to be higher than under no uncertainty for g greater than approximately 0.0085, or 3.4% annual growth rate. Therefore, business cycle fluctuations may still improve welfare, especially in developing countries with fast-growing and volatile output. These are typically also the countries that are most appropriate to analyze using a small open economy setup, which assumes an exogenous interest rate.

4.2. Cost of Slower Growth

The discussion thus far suggests that the difference in endowment growth rates may have a disproportionately large effect on welfare. Figure 6 shows this more explicitly. Here, $p_H = p_L = 0.06$, as in Figure 5, and $\sigma = 2$. The vertical

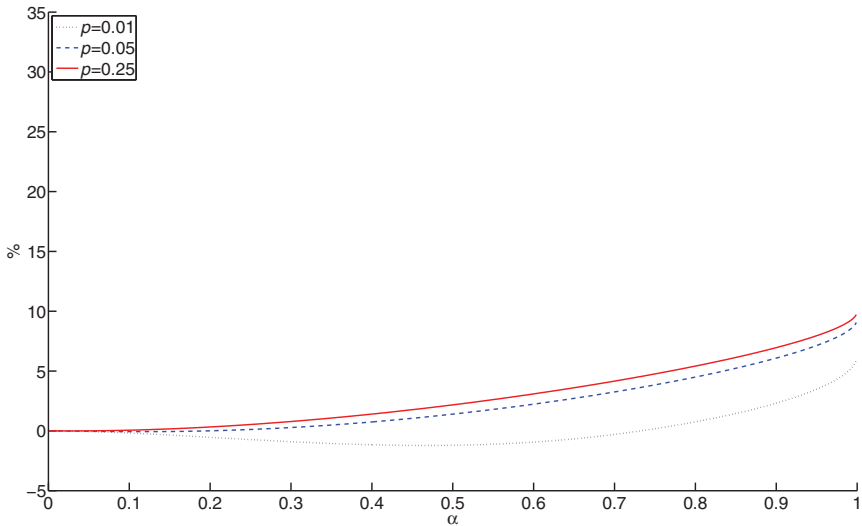


FIGURE 7. Welfare effect of p ($\sigma = 0.5$).

axis corresponds to the difference in welfare for $g = 0.0075$ and $g = 0.0025$, for different values of α , where welfare is measured as the permanent percentage increase in endowment for $g = 0.0025$ under certainty. Accordingly, Figure 6 shows how the cost of 2% lower annual endowment growth, converted into the fraction of consumption flows in a deterministic economy with 1% annual growth, varies with α . Note that the cost becomes substantial when the variance of endowment is high. In this example, the cost of a 2% lower growth rate, for α close to 1, is more than five times higher than that for $\alpha = 0$, or under no uncertainty. This is because slow growth has the indirect cost of damping, or even eliminating, the positive effect of uncertainty on welfare. Put differently, uncertainty may be a beneficial commitment device under fast growth, but it becomes a pure nuisance as the growth rate declines.

4.3. Welfare Effect of Persistence of Endowment

I now turn to the impact of persistence of endowment on welfare. Figures 7 and 8 plot the welfare gain of uncertainty, again measured in terms of consumption as in Figure 5, for $p = \{0.01, 0.05, 0.25\}$ and $\tilde{p}_H = \tilde{p}_L = 1$ (i.e., $p_H = p_L = \{0.01, 0.05, 0.25\}$). The remaining parameters are the same as in Figure 5: $g = 0.005$, and $\sigma = \{0.5, 2\}$. As stated in Proposition 4, $V^U(\alpha)$ is independent of p for α very close to 0 for both values of σ , as well as for $\alpha = 1$ when $\sigma = 2 > 1$; for all other values of α , $V^U(\alpha)$ is increasing in p .

Interestingly, for both values of σ , raising the switching probability by five times, from 0.01 to 0.05, has a much larger impact on welfare than increasing it from 0.05 to 0.25. Intuitively, when endowment is highly persistent, the incentive

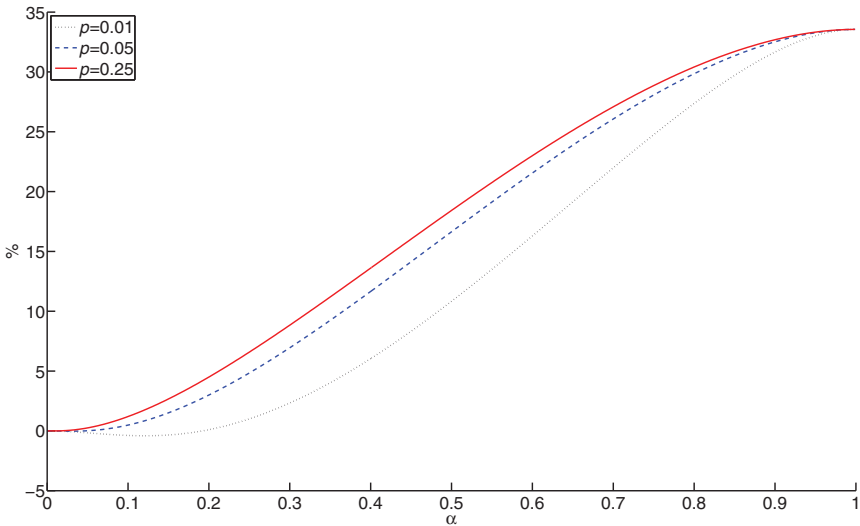


FIGURE 8. Welfare effect of p ($\sigma = 2$).

to walk away from the contract in the high state becomes very strong, because switching to the low state is rather unlikely. As a result, the path of consumption cannot deviate much from that of endowment. In such an environment, lowering the persistence of endowment has a large positive impact on welfare through enabling more intertemporal consumption smoothing.

5. CONCLUSION

In this paper, I have examined the welfare implications of endowment uncertainty in a contracting problem between a risk-neutral financial intermediary and a risk-averse household, where the latter cannot commit to the contract. Allowing for endowment growth and different discount factors between the two sides of the contract leads to a number of novel implications. Through both analytical and numerical analysis, I have shown that if the growth rate of the household's endowment is sufficiently high, the presence of uncertainty in the household's endowment may improve its welfare compared to the case of deterministic growth. This is because, in such an environment, uncertainty makes the contract valuable by imparting to it the role of risk sharing, which in turn helps to achieve better intertemporal consumption smoothing by relaxing the household's commitment problem. I have also shown that slower growth may have a disproportionately large effect on welfare, through undermining the positive effect of uncertainty on welfare. Finally, I have shown that the household's welfare is always nonincreasing in the persistence of endowment, and strictly so for most parameter values.

There are several potential extensions and applications of this paper. In terms of theory, the present model assumes CRRA utility for the household, which forces

the IES to be the reciprocal of the degree of risk aversion. Conducting welfare analysis under recursive preferences [Epstein and Zin (1989); Weil (1990)], which allow disentanglement of these factors, would be an interesting extension. On the application side, the model framework may be used to explore the implications of growth with respect to topics to which a limited commitment approach has been applied, such as international lending under risk of default and implicit insurance arrangements in village economies.¹⁷

NOTES

1. Ljungqvist and Sargent (2004, Ch. 19) refers to the canonical model as the “villager-moneylender” model and analyzes it in detail. Ljungqvist and Sargent (2004) attributes the model to Thomas and Worrall (1988) and Kocherlakota (1996), although it is closer to the sovereign debt model of Worrall (1990). Worrall (1990) also briefly discusses the case where the two sides of the contract have different discount factors.

2. The agent and the principal in Krueger and Uhlig (2006) correspond, respectively, to the household and the financial intermediary in this paper.

3. There is a vast literature on the limited commitment environment. Thomas and Worrall (1988) explores optimal wage contracts to which neither the firm nor the worker can commit. Kocherlakota (1996) examines efficient risk sharing between two symmetric agents who lack commitment ability. Kehoe and Levine (1993) and Alvarez and Jermann (2000) discuss how optimal allocations under limited commitment can be decentralized. Of the two roles mentioned here, only risk sharing plays a part in these papers.

4. Throughout this paper, increasing (decreasing) implies strictly increasing (decreasing), and nondecreasing (nonincreasing) implies weakly increasing (decreasing).

5. Because the contract of a given household has no effect on the contracts of other households, in this paper it suffices to analyze a contract between a single household and a single financial intermediary. The cross-sectional distribution of household endowment is therefore irrelevant.

6. With this terminology, $t = 0$ corresponds to the initial, or the 0th, period of the contract.

7. The recursive formulation here largely follows the “villager-moneylender” model in Ljungqvist and Sargent (2004, Ch. 19), which corresponds to the case of $g = 0$, $\beta(1+r) = 1$, and an i.i.d. endowment process. However, I define conditional value functions and promised values as in Krueger and Uhlig (2006), which turns out to be more convenient for some of the analyses that follow. The idea of using promised utilities as state variables dates back to Abreu et al. (1986) and Spear and Srivastava (1987).

8. For example, I simply state consumption and promised value to refer to the detrended values of these variables. When referring to nondetrended variables, I explicitly state, e.g., “nondetrended consumption.”

9. As discussed in these papers, using the standard dynamic programming argument requires adequately bounding the range of promised values to make the one-period return function bounded. However, as analyzed in Section 2.4, the promised values may grow without bound, depending on the parameters, which may appear to be a problem. In such case, however, the participation constraints never bind in the future for sufficiently large promised values. The cost functions can thus be computed directly using the unconstrained optimal growth rate of consumption, which allows sidestepping this issue.

10. More details on the derivations of key equations in this subsection are provided in Appendix A.

11. In Figure 1 and subsequent figures, “PC” stands for “participation constraint.”

12. More details on the derivations of some of the equations and on some of the claims to be presented in Lemma 5 and Proposition 4 are available upon request.

13. Kehoe and Levine (2001) shows, in a stationary environment (corresponding to $g = \mu = 0$ here), a similar result in the limited commitment environment with two symmetric agents.

14. Eaton and Gersovitz (1981) provides an analysis of noncontingent sovereign debt for such an environment.

15. Instead of computing q_s from (12)–(14) via the value function iteration, I use the properties of an optimal contract, discussed in Section 2.4, to obtain the set of equations that define $Q(V_0)$ and solve these equations for the exact solution. This procedure has several computational advantages, because it does not require a minimization operation and is free of the need to choose the appropriate range and number of grids. The precise procedure for computing $Q(V_0)$ and V^U , which is tedious but straightforward, is available upon request. Although the analysis that follows assumes $g > \mu$, this computational procedure is especially useful for the $g < \mu$ case, where the ever-increasing promised value causes difficulty in setting the grid range.

16. The quarterly real GDP per capita series is logged and filtered using the Hodrick–Prescott filter, with a smoothing parameter of 1,600, before these statistics are computed. Model parameters α , p_H , and p_L are such that assuming $p_H = p_L$, the percentage deviation of y_s from its steady state value of 1 will have the same standard deviation and first lagged autocorrelation as in the data.

17. See, e.g., Kehoe and Perri (2002) for the former application, and Ligon et al. (2002) for the latter.

18. More details on the derivations of some of the equations in Appendix B, which are straightforward but lengthy, are available upon request.

19. For \hat{V}_0 defined here, let \hat{N} be the counterpart of N , that is, the first t such that $\hat{v}_L^t = v_L^{\text{Aut}}$. Then $\hat{N} = N$ or $\hat{N} = N - 1$; the latter occurs when the sequence $\{\hat{v}_L^t\}_{t=0}^{N-1}$ satisfies (B.14) with $\hat{v}_L^{N-1} = v_L^{\text{Aut}}$.

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APPENDIX A: FIRST-ORDER AND ENVELOPE CONDITIONS

I provide further details on the first-order and envelope conditions of the two problems in Section 2.4, which yields key equations, (22), (23), and (27). Further, I combine these conditions to obtain several equations that are used in the proofs in Appendix B.

A.1. CONTINUATION PROBLEM

The Lagrangian for the problem given by (12)–(14) can be written as

$$\begin{aligned} \mathcal{L} = & c + \frac{1+g}{1+r} [(1-p_s)q_s(v'_s) + p_s q_{-s}(v'_{-s})] \\ & + \lambda \{v - u(c) - (1-\gamma)[(1-p_s)v'_s + p_s v'_{-s}]\} \\ & + (1-p_s)\theta_s(v_s^{\text{Aut}} - v'_s) + p_s \theta_{-s}(v_{-s}^{\text{Aut}} - v'_{-s}), \end{aligned} \tag{A.1}$$

where λ is the Lagrange multiplier for the promise-keeping constraint (13), and $\theta_{s'}$, $s' \in \{s, -s\}$, is proportional to the Lagrange multiplier for the participation constraint (14) in state s' .

Therefore, the first-order conditions are

$$1 = \lambda u'(c), \tag{A.2}$$

$$\frac{1+g}{1+r} \frac{\partial q_{s'}(v'_{s'})}{\partial v} = \lambda(1-\gamma) + \theta_{s'}, s' \in \{s, -s\}, \tag{A.3}$$

and the envelope condition with respect to v is

$$\frac{\partial q_s(v)}{\partial v} = \lambda. \tag{A.4}$$

Combining (A.3) and (A.4) yields (22), whereas combining (A.2) and (A.4) yields (23).

Moreover, by complementary slackness,

$$\theta_{s'}(v_{s'}^{\text{Aut}} - v'_{s'}) = 0, s' \in \{s, -s\}. \tag{A.5}$$

Finally, the envelope condition with respect to p is

$$\begin{aligned} \frac{\partial q_s(v)}{\partial p} &= \frac{1+g}{1+r} \left[(1-p_s) \frac{\partial q_s(v'_s)}{\partial p} + p_s \frac{\partial q_{-s}(v'_{-s})}{\partial p} \right] \\ &\quad - \tilde{p}_s \frac{1+g}{1+r} (q_s(v'_s) - q_{-s}(v'_{-s})) + \tilde{p}_s (1-\gamma) \lambda (v'_s - v'_{-s}) \\ &\quad - \tilde{p}_s \theta_s (v_s^{\text{Aut}} - v'_s) + \tilde{p}_s \theta_{-s} (v_{-s}^{\text{Aut}} - v'_{-s}) + (1-p_s) \theta_s \frac{\partial v_s^{\text{Aut}}}{\partial p} + p_s \theta_{-s} \frac{\partial v_{-s}^{\text{Aut}}}{\partial p}. \end{aligned} \tag{A.6}$$

Substituting (22), (A.4), and (A.5) into (A.6) yields

$$\begin{aligned} \frac{\partial q_s(v)}{\partial p} &= \frac{1+g}{1+r} \left[(1-p_s) \frac{\partial q_s(v'_s)}{\partial p} + p_s \frac{\partial q_{-s}(v'_{-s})}{\partial p} \right] \\ &\quad - \tilde{p}_s \frac{1+g}{1+r} (q_s(v'_s) - q_{-s}(v'_{-s})) + \tilde{p}_s (1-\gamma) \frac{\partial q_s(v)}{\partial v} (v'_s - v'_{-s}) \\ &\quad + (1-p_s) \left[\frac{1+g}{1+r} \frac{\partial q_s(v'_s)}{\partial v} - (1-\gamma) \frac{\partial q_s(v)}{\partial v} \right] \frac{\partial v_s^{\text{Aut}}}{\partial p} \\ &\quad + p_s \left[\frac{1+g}{1+r} \frac{\partial q_{-s}(v'_{-s})}{\partial v} - (1-\gamma) \frac{\partial q_s(v)}{\partial v} \right] \frac{\partial v_{-s}^{\text{Aut}}}{\partial p}. \end{aligned} \tag{A.7}$$

A.2. INITIAL PROBLEM

The Lagrangian for the problem given by (19)–(21) can be written as

$$\begin{aligned} \mathcal{L} &= \pi_H q_H(v_H^0) + \pi_L q_L(v_L^0) + \lambda^0 (V_0 - \pi_H v_H^0 - \pi_L v_L^0) \\ &\quad + \pi_H \theta_H^0 (v_H^{\text{Aut}} - v_H^0) + \pi_L \theta_L^0 (v_L^{\text{Aut}} - v_L^0), \end{aligned} \tag{A.8}$$

where λ^0 is the Lagrange multiplier for the promise-keeping constraint (20), and θ_H^0 and θ_L^0 are proportional to the Lagrange multipliers for the participation constraint (21). The first-order conditions are

$$\frac{\partial q_H(v_H^0)}{\partial v} = \lambda^0 + \theta_H^0, \tag{A.9}$$

$$\frac{\partial q_L(v_L^0)}{\partial v} = \lambda^0 + \theta_L^0, \tag{A.10}$$

which yield (27). Further, the envelope condition with respect to p is

$$\frac{\partial Q(V_0)}{\partial p} = \pi_H \frac{\partial q_H(v_H^0)}{\partial p} + \pi_L \frac{\partial q_L(v_L^0)}{\partial p} + \pi_H \theta_H^0 \frac{\partial v_H^{\text{Aut}}}{\partial p} + \pi_L \theta_L^0 \frac{\partial v_L^{\text{Aut}}}{\partial p}. \tag{A.11}$$

APPENDIX B: PROOFS

B.1. PROOF OF LEMMA 1

If $v_H^{Aut} = v_L^{Aut}$, then there is no reason to make consumption contingent on realization of the state; hence $c_H^{Aut} = c_L^{Aut}$. Thus, let $v_H^{Aut} > v_L^{Aut}$ in the following, and write¹⁸

$$v_H^{Aut} = u(c_H^{Aut}) + (1 - \gamma) [(1 - p_H) v'_H + p_H v'_L], \tag{B.1}$$

$$v_L^{Aut} = u(c_L^{Aut}) + (1 - \gamma) [p_L v''_H + (1 - p_L) v''_L]. \tag{B.2}$$

Here, $v'_s, (v''_s), s' \in \{H, L\}$, is the optimal promised value next period in state s' , when the current state is high (low) and the promised value is $v_H^{Aut} (v_L^{Aut})$.

Now, suppose the claim is false, so that $c_L^{Aut} \geq c_H^{Aut}$. Then (23) implies that

$$\frac{\partial q_L(v_L^{Aut})}{\partial v} = \frac{1}{u'(c_L^{Aut})} \geq \frac{1}{u'(c_H^{Aut})} = \frac{\partial q_H(v_H^{Aut})}{\partial v}. \tag{B.3}$$

Given (25), (B.3) implies $v''_H \geq v'_H$ and $v''_L \geq v'_L$. In the following, I use these results to derive contradictions separately for $g > \mu$ and $g \leq \mu$.

Suppose $g > \mu$. Then $c_H^{Aut} (1 + \mu) / (1 + g) < c_H^{Aut} \leq c_L^{Aut}$. Thus, Lemma 2 implies that if the current state is high and the promised value is v_H^{Aut} , the participation constraint binds in both states next period; hence $v'_H = v_H^{Aut}$ and $v'_L = v_L^{Aut}$. But then,

$$\begin{aligned} v_H^{Aut} &= u(c_H^{Aut}) + (1 - \gamma) [(1 - p_H) v_H^{Aut} + p_H v_L^{Aut}] \\ &< u(c_H^{Aut}) + (1 - \gamma) v_H^{Aut}, \end{aligned}$$

hence $u(c_H^{Aut}) > \gamma v_H^{Aut}$. On the other hand, $v''_H \geq v_H^{Aut}$ and $v''_L \geq v_L^{Aut}$, which implies that

$$\begin{aligned} v_L^{Aut} &\geq u(c_L^{Aut}) + (1 - \gamma) [p_L v_H^{Aut} + (1 - p_L) v_L^{Aut}] \\ &> u(c_L^{Aut}) + (1 - \gamma) v_L^{Aut}, \end{aligned}$$

hence $\gamma v_L^{Aut} > u(c_L^{Aut})$. Thus, $\gamma v_L^{Aut} > u(c_L^{Aut}) \geq u(c_H^{Aut}) > \gamma v_H^{Aut}$, which is a contradiction.

Next, suppose that $g \leq \mu$. Then, if the current state is low and the promised value is v_L^{Aut} , consumption grows forever from c_L^{Aut} at a rate $\frac{\mu - g}{1 + g} \geq 0$, regardless of the subsequent realization of the state, which in turn implies that $v''_H = v'_L$. This is seen by noting that given $c_L^{Aut} \geq c_H^{Aut}$ and $\frac{1 + \mu}{1 + g} \geq 1$, applying Lemma 2 repeatedly implies that the participation constraint never binds in either state. But then $c_L^{Aut} \geq c_H^{Aut}$, $v''_L = v''_H \geq v'_H$, and $v''_H = v''_L \geq v'_L$, and thus (B.1) and (B.2) imply $v_L^{Aut} \geq v_H^{Aut}$, which is a contradiction. ■

B.2. PROOF OF LEMMA 4

As argued in the main text, the claim holds for $t = 0$. Suppose, for some $\tau \in \{0, 1, \dots\}$, that the participation constraint in neither state binds up to period $\tau + 1$, and that the claim holds for $t = \tau$. Then consumption in period τ can be denoted as c^τ , and the promised values in period $\tau + 1$ can be denoted as $v_s^{\tau+1}, s \in \{H, L\}$.

In period $\tau + 1$, the participation constraint in neither state binds by hypothesis, so from Lemma 2, consumption is $c^\tau (1 + \mu) / (1 + g)$ in both states. From (23), this implies that $\partial q_H (v_H^{\tau+1}) / \partial v = \partial q_L (v_L^{\tau+1}) / \partial v$; hence from the same argument as for the claim for $t = 0$, the promised value in period $\tau + 2$ depends only on $s_{\tau+2}$. The claim thus holds for $t = \tau + 1$, so by induction, it holds for any $t \in \{0, 1, \dots\}$. ■

B.3. PROOF OF PROPOSITION 1

Let $\alpha = 0$. Because there is no uncertainty, I drop the subscript indicating states, to write, for example, y, v^0 , and v^{Aut} . Then the continuation problem (12)–(14) becomes

$$q(v) = \min_{c, v'} c + \frac{1 + g}{1 + r} q(v'), \tag{B.4}$$

$$\text{s.t. } v = u(c) + (1 - \gamma)v', \tag{B.5}$$

$$v' \geq v^{Aut}. \tag{B.6}$$

The initial problem (19)–(21) becomes trivial, with $v^0 = V_0$ and $Q(V_0) = q(v^0)$. Because $V_0 \geq V^{Aut}(0)$ in an equilibrium contract and $V^{Aut}(0) = v^{Aut}$, it follows that $v^0 \geq v^{Aut}$. Combining the first-order and envelope conditions of the problem (B.4)–(B.6) and noting (24),

$$\theta = \frac{1 + g}{1 + r} \left[\frac{\partial q(v')}{\partial v} - \left(\frac{1 + \mu}{1 + g} \right)^\sigma \frac{\partial q(v)}{\partial v} \right], \tag{B.7}$$

where θ is the Lagrange multiplier on the participation constraint (B.6).

First, suppose that $g \leq \mu$. If $\theta = 0$, then (B.7) implies that $\partial q(v') / \partial v \geq \partial q(v) / \partial v$ and thus $v' \geq v$, because q is convex. Thus, the participation constraint (B.6) never binds if $v^0 \geq v^{Aut}$, which holds as argued earlier. Therefore, the first-best contract can be sustained under certainty and thus uncertainty is never welfare-improving, which proves Proposition 1(1).

Next, suppose that $g > \mu$. If $\theta = 0$, then (B.7) implies that $\partial q(v') / \partial v < \partial q(v) / \partial v$ and thus $v' < v$. Thus, if $v = v^{Aut}$, the participation constraint (B.6) binds in the next period and $v' = v^{Aut}$. But then $v^{Aut} = u(c^{Aut}) + (1 - \gamma)v^{Aut}$ and thus $c^{Aut} = y = 1$, which implies that for $V_0 = V^{Aut}(0)$, $Q(V_0) = q(v^{Aut}) = R$. Therefore, an optimal contract with $V_0 = V^{Aut}(0)$, in which consumption always equals endowment, costs R , and is thus an equilibrium contract. Thus, $V^C = V^{Aut}(0)$, which proves Proposition 1(2). ■

B.4. PROOF OF PROPOSITION 4

As argued in Section 2.3, if $\alpha = 0$ or $\sigma > 1$ and $\alpha = 1$, then $v_H^{Aut} = v_L^{Aut}$. In this special case, clearly p has no effect on $V^U(\alpha)$. The proof that follows assumes that $v_H^{Aut} > v_L^{Aut}$, which implies that $\tilde{v}_L > v_L^{Aut}$ and thus $\tilde{V} > V^{Aut}$, and resorts to different arguments for $g \leq \mu$ and $g > \mu$.

Proof for $g \leq \mu$. Suppose $g \leq \mu$. Then the participation constraint in the low state never binds for $V_0 \geq V^{Aut}$, as will be explained. From Lemma 3, it does not bind for $t = 0$. From Lemma 2, the same applies for $t > 0$, because initial consumption is no less than c_L^{Aut} and $(1 + \mu) / (1 + g) \geq 1$. That $V^U(\alpha)$ is nondecreasing in p is then immediate, because a rise in p relaxes the participation constraint in the high state because $\partial v_H^{Aut} / \partial p < 0$,

without any cost from tightening it in the low state. Indeed, $V^U(\alpha)$ is increasing in p if $V^U(\alpha) \in [V^{Aut}, \tilde{V}]$, in which case the initial participation constraint in the high state binds.

Proof for $g > \mu$. Now, suppose $g > \mu$. Then, as argued in Section 2.4, the participation constraint in either state may eventually bind, which complicates the proof. To prove the claim, it suffices to show that $\partial Q(V_0)/\partial p \leq 0$, with strict inequality for $\alpha \in \left((1 + \frac{1+\mu}{1+g} \frac{\pi_L}{\pi_H})^{-1} (\frac{g-\mu}{1+g}), 1 \right)$. To see this, write $Q(V_0)$ and $V^U(\alpha)$ as $Q(V_0, p)$ and $V^U(\alpha, p)$, to emphasize their dependence on p . Then, for any $\tilde{p} > p > 0$, $\partial Q(V_0)/\partial p < 0$ and (11) imply that $R = Q(V^U(\alpha, p), p) = Q(V^U(\alpha, \tilde{p}), \tilde{p}) < Q(V^U(\alpha, \tilde{p}), p)$; hence $V^U(\alpha, p) < V^U(\alpha, \tilde{p})$.

The first step is to obtain an operational expression for $\partial Q(V_0)/\partial p$. From (2), (33), and (34),

$$\frac{\pi_H}{\pi_L} = \frac{p_L}{p_H} = \frac{\tilde{p}_L}{\tilde{p}_H}. \tag{B.8}$$

Also, (17) and (18) yield

$$-\frac{1}{\tilde{p}_H} \frac{\partial v_H^{Aut}}{\partial p} = \frac{1}{\tilde{p}_L} \frac{\partial v_L^{Aut}}{\partial p} = \frac{1 - \gamma}{1 - (1 - \gamma)(1 - p_H - p_L)} (v_H^{Aut} - v_L^{Aut}), \tag{B.9}$$

which allows converting the terms involving $\partial v_L^{Aut}/\partial p$ and $v_H^{Aut} - v_L^{Aut}$ into those involving $\partial v_H^{Aut}/\partial p$.

Now, (B.8) and the first equality in (B.9) imply that $\pi_H \frac{\partial v_H^{Aut}}{\partial p} = -\pi_L \frac{\partial v_L^{Aut}}{\partial p}$, and thus

$$\pi_H \theta_H^0 \frac{\partial v_H^{Aut}}{\partial p} + \pi_L \theta_L^0 \frac{\partial v_L^{Aut}}{\partial p} = \pi_H (\theta_H^0 - \theta_L^0) \frac{\partial v_H^{Aut}}{\partial p}, \tag{B.10}$$

so noting (27) and substituting into (A.11),

$$\frac{\partial Q(V_0)}{\partial p} = \pi_H \frac{\partial q_H(v_H^0)}{\partial p} + \pi_L \frac{\partial q_L(v_L^0)}{\partial p} + \pi_H \left(\frac{\partial q_H(v_H^0)}{\partial v} - \frac{\partial q_L(v_L^0)}{\partial v} \right) \frac{\partial v_H^{Aut}}{\partial p}. \tag{B.11}$$

The remaining proof proceeds as follows. Lemmas A1 and A2 resort to (B.11) to show different properties of $\partial Q(V_0)/\partial p$ for relatively low and high values of V_0 , taking advantage of the fact that $v_H^0 = v_H^{Aut}$ for $V_0 \in [V^{Aut}, \tilde{V}]$, and that the initial participation constraint is slack for $V_0 \in [\tilde{V}, V^{FB}]$. By noting that both these properties hold at $V_0 = \tilde{V}$, Lemmas A3–A6 show that $\partial Q(V_0)/\partial p \leq 0$ for $V_0 \in [V^{Aut}, \tilde{V}]$. The result easily extends to $V_0 \in [\tilde{V}, V^{FB}]$.

I begin by defining the following variables.

DEFINITION B.1. For any $V_0 \in [V^{Aut}, \tilde{V}]$, let $\{v_L^t\}_{t=0}^\infty$ be the path of promised values in the optimal contract when $s_t = L$ for all $t = 0, 1, \dots$, and let $N \geq 0$ be the first t such that $v_L^t = v_L^{Aut}$. In particular, let $\tilde{N} > 0$ be the value of N when $V_0 = \tilde{V}$.

Clearly, N is nondecreasing in V_0 and $N \in \{0, 1, \dots, \tilde{N}\}$, where $N = 0$ if and only if $V_0 = V^{Aut}$. Thus, $\tilde{N} > 0$ follows from $\tilde{V} > V^{Aut}$. Because $g > \mu$, the argument in Section 2.4 implies that $\{v_L^t\}_{t=0}^\infty$ is a nonincreasing sequence, where $v_L^t = v_L^{Aut}$ for $t = N, N + 1, \dots$, and for $N > 0$, $v_L^t > v_L^{t+1}$ for $t = 0, 1, \dots, N - 1$.

These variables play key roles in the proof of Lemma A1, which concerns $V_0 \in [V^{Aut}, \tilde{V}]$.

LEMMA A1. For any $V_0 \in [V^{\text{Aut}}, \tilde{V}]$, $\partial Q(V_0)/\partial p$ is nondecreasing in V_0 , and is increasing in V_0 in the range of V_0 such that $N > 1$.

Proof. Note that if the current state is low and the promised value is $v \leq \tilde{v}_L$, then current consumption c satisfies $c \leq c_H^{\text{Aut}}$; hence given $g > \mu$ and Lemma 2, the participation constraint in the high state binds next period.

Take any $V_0 \in (V^{\text{Aut}}, \tilde{V}]$. Then $v_H^0 = v_H^{\text{Aut}}$ and $v_L^0 \leq \tilde{v}_L$, as shown in Section 2.4. In the following, for any $\hat{V}_0 \in [V^{\text{Aut}}, V_0)$, write \hat{v}_H^0 and \hat{v}_L^0 to denote v_H^0 and v_L^0 . Because $v_H^0 = \hat{v}_H^0 = v_H^{\text{Aut}}$, (B.11) yields

$$\begin{aligned} \frac{\partial Q(V_0)}{\partial p} - \frac{\partial Q(\hat{V}_0)}{\partial p} &= \pi_L \left(\frac{\partial q_L(v_L^0)}{\partial p} - \frac{\partial q_L(\hat{v}_L^0)}{\partial p} \right) \\ &\quad - \pi_H \left(\frac{\partial q_L(v_L^0)}{\partial v} - \frac{\partial q_L(\hat{v}_L^0)}{\partial v} \right) \frac{\partial v_H^{\text{Aut}}}{\partial p}. \end{aligned} \tag{B.12}$$

The proof proceeds by evaluating (B.12). To make this evaluation feasible, I first consider \hat{V}_0 sufficiently close to V_0 such that the path of \hat{v}_L^t exhibits a pattern similar to that of v_L^t , and then extend the result to smaller \hat{V}_0 by repeating the argument.

First, suppose $N > 1$. When $s_t = L$ for all $t = 0, 1, \dots$, the participation constraint in the low state does not bind, at least up to $N - 1$, because $v_L^{N-1} > v_L^{\text{Aut}}$. Thus, (25) implies that

$$\frac{1 + g}{1 + r} \frac{\partial q_L(v_L^{t+1})}{\partial v} = (1 - \gamma) \frac{\partial q_L(v_L^t)}{\partial v}, \quad t = 0, 1, \dots, N - 2. \tag{B.13}$$

Now, take any $\hat{V}_0 < V_0$ such that the sequence $\{\hat{v}_L^t\}_{t=0}^{N-1}$ satisfies¹⁹

$$\frac{1 + g}{1 + r} \frac{\partial q_L(\hat{v}_L^{t+1})}{\partial v} = (1 - \gamma) \frac{\partial q_L(\hat{v}_L^t)}{\partial v}, \quad t = 0, 1, \dots, N - 2. \tag{B.14}$$

Then $\{v_L^t\}_{t=0}^{N-1}$ and $\{\hat{v}_L^t\}_{t=0}^{N-1}$ are decreasing sequences. Moreover, $\tilde{v}_L \geq v_L^t > \hat{v}_L^t \geq v_L^{\text{Aut}}$ for $t = 0, 1, \dots, N - 1$, and $v_L^t = \hat{v}_L^t = v_L^{\text{Aut}}$ for $t = N, N + 1, \dots$

I evaluate the RHS of (B.12) recursively. Let $s = L$ in (A.7). If $v = v_L^{N-1}$, then the participation constraint in the high state binds next period and thus $v_H^t = v_H^{\text{Aut}}$, whereas by definition, $v_L^t = v_L^N = v_L^{\text{Aut}}$. Similarly, if $v = \hat{v}_L^{N-1}$, then $v_H^t = v_H^{\text{Aut}}$ and $v_L^t = \hat{v}_L^N = v_L^{\text{Aut}}$, so

$$\begin{aligned} \frac{\partial q_L(v_L^{N-1})}{\partial p} - \frac{\partial q_L(\hat{v}_L^{N-1})}{\partial p} &= -\tilde{p}_L (1 - \gamma) (v_H^{\text{Aut}} - v_L^{\text{Aut}}) \left(\frac{\partial q_L(v_L^{N-1})}{\partial v} - \frac{\partial q_L(\hat{v}_L^{N-1})}{\partial v} \right) \\ &\quad - (1 - p_L) (1 - \gamma) \left(\frac{\partial q_L(v_L^{N-1})}{\partial v} - \frac{\partial q_L(\hat{v}_L^{N-1})}{\partial v} \right) \frac{\partial v_L^{\text{Aut}}}{\partial p} \\ &\quad - p_L (1 - \gamma) \left(\frac{\partial q_L(v_L^{N-1})}{\partial v} - \frac{\partial q_L(\hat{v}_L^{N-1})}{\partial v} \right) \frac{\partial v_H^{\text{Aut}}}{\partial p} \\ &= \frac{\pi_H}{\pi_L} \left(\frac{\partial q_L(v_L^{N-1})}{\partial v} - \frac{\partial q_L(\hat{v}_L^{N-1})}{\partial v} \right) \frac{\partial v_H^{\text{Aut}}}{\partial p}, \end{aligned} \tag{B.15}$$

where the second equality follows by substituting for $v_H^{Aut} - v_L^{Aut}$ and $\partial v_L^{Aut} / \partial p$ from (B.9) and noting (B.8).

Again, let $s = L$ in (A.7). If $v = v_L^{N-2}$, then $v'_H = v_H^{Aut}$ and $v'_L = v_L^{N-1}$, whereas if $v = \hat{v}_L^{N-2}$, then $v'_H = v_H^{Aut}$ and $v'_L = \hat{v}_L^{N-1}$. Thus, in the expressions for $\partial q_L(v_L^{N-2}) / \partial p$ and $\partial q_L(\hat{v}_L^{N-2}) / \partial p$, the terms involving $\partial v_L^{Aut} / \partial p$ cancel out given (B.13) and (B.14), so that

$$\begin{aligned} & \frac{\partial q_L(v_L^{N-2})}{\partial p} - \frac{\partial q_L(\hat{v}_L^{N-2})}{\partial p} \\ &= (1 - p_L) \frac{1 + g}{1 + r} \left(\frac{\partial q_L(v_L^{N-1})}{\partial p} - \frac{\partial q_L(\hat{v}_L^{N-1})}{\partial p} \right) - \tilde{p}_L \frac{1 + g}{1 + r} (q_L(v_L^{N-1}) - q_L(\hat{v}_L^{N-1})) \\ & \quad + \tilde{p}_L (1 - \gamma) \left[\frac{\partial q_L(v_L^{N-2})}{\partial v} (v_L^{N-1} - v_H^{Aut}) - \frac{\partial q_L(\hat{v}_L^{N-2})}{\partial v} (\hat{v}_L^{N-1} - v_H^{Aut}) \right] \\ & \quad - p_L (1 - \gamma) \left(\frac{\partial q_L(v_L^{N-2})}{\partial v} - \frac{\partial q_L(\hat{v}_L^{N-2})}{\partial v} \right) \frac{\partial v_H^{Aut}}{\partial p}. \end{aligned} \tag{B.16}$$

Using (B.8), (B.9), and (B.13)–(B.15), one may rewrite (B.16) as

$$\begin{aligned} & \frac{\partial q_L(v_L^{N-2})}{\partial p} - \frac{\partial q_L(\hat{v}_L^{N-2})}{\partial p} = \frac{\pi_H}{\pi_L} \left(\frac{\partial q_L(v_L^{N-2})}{\partial v} - \frac{\partial q_L(\hat{v}_L^{N-2})}{\partial v} \right) \frac{\partial v_H^{Aut}}{\partial p} \\ & \quad + \tilde{p}_L \frac{1 + g}{1 + r} \left[\frac{\partial q_L(v_L^{N-1})}{\partial v} (v_L^{N-1} - v_L^{Aut}) - (q_L(v_L^{N-1}) - q_L(v_L^{Aut})) \right] \\ & \quad - \tilde{p}_L \frac{1 + g}{1 + r} \left[\frac{\partial q_L(\hat{v}_L^{N-1})}{\partial v} (\hat{v}_L^{N-1} - v_L^{Aut}) - (q_L(\hat{v}_L^{N-1}) - q_L(v_L^{Aut})) \right]. \end{aligned} \tag{B.17}$$

On the RHS of (B.17), the strict convexity of q_L and $v_L^{N-1} > \hat{v}_L^{N-1} \geq v_L^{Aut}$ imply that the last two lines combined are strictly positive. This can be observed in Figure B.1, in which the length of $CE (= AE - AC)$ is $\frac{\partial q_L(v_L^{N-1})}{\partial v} (v_L^{N-1} - v_L^{Aut}) - (q_L(v_L^{N-1}) - q_L(v_L^{Aut}))$, and the length of $CD (= BD - BC)$ is $\frac{\partial q_L(\hat{v}_L^{N-1})}{\partial v} (\hat{v}_L^{N-1} - v_L^{Aut}) - (q_L(\hat{v}_L^{N-1}) - q_L(v_L^{Aut}))$. Therefore,

$$\frac{\partial q_L(v_L^{N-2})}{\partial p} - \frac{\partial q_L(\hat{v}_L^{N-2})}{\partial p} > \frac{\pi_H}{\pi_L} \left(\frac{\partial q_L(v_L^{N-2})}{\partial v} - \frac{\partial q_L(\hat{v}_L^{N-2})}{\partial v} \right) \frac{\partial v_H^{Aut}}{\partial p}. \tag{B.18}$$

Following the same argument and noting (B.18), one obtains

$$\begin{aligned} & \frac{\partial q_L(v'_L)}{\partial p} - \frac{\partial q_L(\hat{v}'_L)}{\partial p} > \frac{\pi_H}{\pi_L} \left(\frac{\partial q_L(v'_L)}{\partial v} - \frac{\partial q_L(\hat{v}'_L)}{\partial v} \right) \frac{\partial v_H^{Aut}}{\partial p} \\ & \quad + \tilde{p}_L \frac{1 + g}{1 + r} \left[\frac{\partial q_L(v_L^{t+1})}{\partial v} (v_L^{t+1} - v_L^{Aut}) - (q_L(v_L^{t+1}) - q_L(v_L^{Aut})) \right] \end{aligned}$$

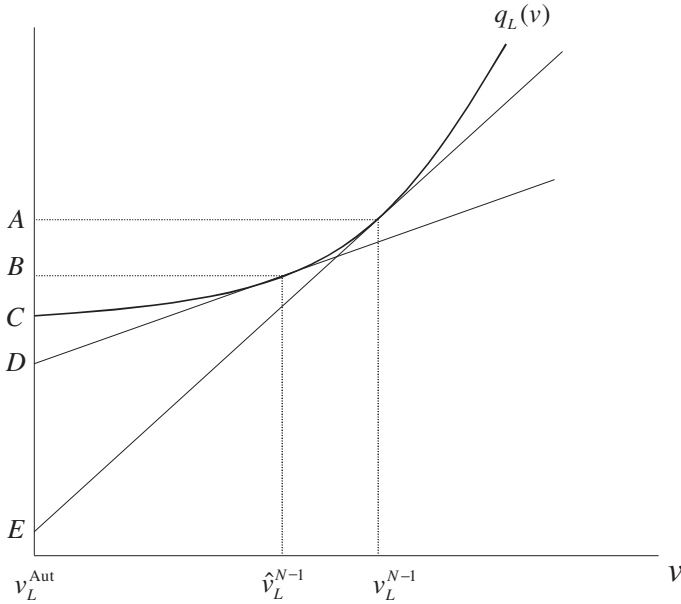


FIGURE B.1. Proof of Lemma A1.

$$\begin{aligned}
 & -\tilde{p}_L \frac{1+g}{1+r} \left[\frac{\partial q_L(\hat{v}_L^{t+1})}{\partial v} (\hat{v}_L^{t+1} - v_L^{\text{Aut}}) - (q_L(\hat{v}_L^{t+1}) - q_L(v_L^{\text{Aut}})) \right] \\
 & > \frac{\pi_H}{\pi_L} \left(\frac{\partial q_L(v_L^t)}{\partial v} - \frac{\partial q_L(\hat{v}_L^t)}{\partial v} \right) \frac{\partial v_H^{\text{Aut}}}{\partial p}
 \end{aligned}$$

for $t = N - 3$, and similarly for $t = N - 4, N - 5, \dots, 0$. Thus, the inequality for $t = 0$ and (B.12) yield

$$\frac{\partial Q(V_0)}{\partial p} - \frac{\partial Q(\hat{V}_0)}{\partial p} > 0. \tag{B.19}$$

This argument shows (B.19) for \hat{V}_0 , whose associated sequence $\{\hat{v}_L^t\}_{t=0}^{N-1}$ satisfies (B.14); note that $\hat{v}_L^{N-1} = v_L^{\text{Aut}}$ for the smallest such \hat{V}_0 . To extend this result, set V_0 to this smallest \hat{V}_0 , redefine variables such as N and \hat{V}_0 accordingly, and repeat the preceding argument for $N > 1$. It then follows that (B.19) holds for any $V_0 \in (V^{\text{Aut}}, \tilde{V}]$ and $\hat{V}_0 \in [V^{\text{Aut}}, V_0)$ such that $\frac{1+g}{1+r} \frac{\partial q_L(\hat{v}_L^{t+1})}{\partial v} = (1-\gamma) \frac{\partial q_L(\hat{v}_L^t)}{\partial v}$ holds at least for $t = 0$.

Finally, suppose $N = 1$. Take any $\hat{V}_0 \in [V^{\text{Aut}}, V_0)$. The argument leading to (B.15) does not hinge on $N > 1$, so setting $N = 1$ in (B.15) and substituting into (B.12),

$$\frac{\partial Q(V_0)}{\partial p} - \frac{\partial Q(\hat{V}_0)}{\partial p} = 0. \tag{B.20}$$

Summarizing these results, $\partial Q(V_0)/\partial p$ is nondecreasing in V_0 for all $V_0 \in [V^{\text{Aut}}, \tilde{V}]$, and is increasing in V_0 in the range of V_0 such that $N > 1$. ■

The next lemma concerns $V_0 \in [\tilde{V}, V^{FB}]$ and is stated using the variables defined next.

DEFINITION B.2. For any $V_0 \in [\tilde{V}, V^{FB}]$, let $\{v_H^t, v_L^t\}_{t=0}^M$ be the path of promised values in the high and low states up to the M th period of the optimal contract and $V_0^t \equiv \pi_H v_H^t + \pi_L v_L^t$, where $M > 0$ is the first t such that the participation constraint in the high state binds.

For $V_0 \in [\tilde{V}, V^{FB}]$, the initial participation constraint is slack, so $M > 0$. Then, from Lemma 4, the promised values for $t = 0, 1, \dots, M$ depend only on s_t . Thus $\{v_H^t, v_L^t\}_{t=0}^M$ are well defined, and for $V_0 = \tilde{V}$, $\{v_L^t\}_{t=0}^M$ in Definition B.2 are consistent with those in Definition B.1.

LEMMA A2. For any $V_0 \in [\tilde{V}, V^{FB}]$, $\frac{\partial Q(V_0)}{\partial p} = (\frac{1+g}{1+r})^M \frac{\partial Q(V_0^M)}{\partial p}$ and $V_0^M \in [V^{Aut}, \tilde{V}]$.

Proof. Take any $V_0 \in [\tilde{V}, V^{FB}]$. Then v_H^0 and $v_L^0 = \frac{1}{\pi_L}(V_0 - \pi_H v_H^0)$ satisfy (30). Substituting (30) into (B.11) and using (A.7) to evaluate $\partial q_H(v_H^0)/\partial p$ and $\partial q_L(v_L^0)/\partial p$,

$$\frac{\partial Q(V_0)}{\partial p} = \frac{1+g}{1+r} \left[\pi_H \frac{\partial q_H(v_H^1)}{\partial p} + \pi_L \frac{\partial q_L(v_L^1)}{\partial p} + \pi_H \left(\frac{\partial q_H(v_H^1)}{\partial v} - \frac{\partial q_L(v_L^1)}{\partial v} \right) \frac{\partial v_H^{Aut}}{\partial p} \right]. \tag{B.21}$$

Now, because $V_0 \in [\tilde{V}, V^{FB}]$, initial consumption is the same in both states. Then at $t = 1$, if the participation constraint in the high state binds, Lemma 2 implies that consumption must be greater in the high state; hence, from (23),

$$\frac{\partial q_H(v_H^1)}{\partial v} > \frac{\partial q_L(v_L^1)}{\partial v}, \quad v_H^1 = v_H^{Aut}. \tag{B.22}$$

Otherwise, (25) and (30) imply that

$$\frac{\partial q_H(v_H^1)}{\partial v} = \frac{\partial q_L(v_L^1)}{\partial v}. \tag{B.23}$$

Comparing (B.22) and (B.23) with (29) and (30), one observes that for an optimal contract that provides the household V_0^1 instead of V_0 , the initial promised values in the high and low state are, respectively, v_H^1 and v_L^1 . Thus, from (B.11),

$$\frac{\partial Q(V_0^1)}{\partial p} = \pi_H \frac{\partial q_H(v_H^1)}{\partial p} + \pi_L \frac{\partial q_L(v_L^1)}{\partial p} + \pi_H \left(\frac{\partial q_H(v_H^1)}{\partial v} - \frac{\partial q_L(v_L^1)}{\partial v} \right) \frac{\partial v_H^{Aut}}{\partial p}. \tag{B.24}$$

From (B.21) and (B.24), $\frac{\partial Q(V_0)}{\partial p} = \frac{1+g}{1+r} \frac{\partial Q(V_0^1)}{\partial p}$. This proves the first part of the claim if $M = 1$.

If $M > 1$, take any $t \in \{2, 3, \dots, M\}$. From the definition of M and Lemma 4, consumption in the $(t - 1)$ th period is the same in both states; hence from (23), $\partial q_L(v_L^{t-1})/\partial v = \partial q_H(v_H^{t-1})/\partial v$. Combining with $v_H^{t-1} \geq v_H^{Aut}$ and (28), one obtains $v_L^{t-1} \geq \tilde{v}_L$ because

$$\frac{\partial q_L(v_L^{t-1})}{\partial v} = \frac{\partial q_H(v_H^{t-1})}{\partial v} \geq \frac{\partial q_H(v_H^{Aut})}{\partial v} = \frac{\partial q_L(\tilde{v}_L)}{\partial v}, \tag{B.25}$$

hence $V_0^{t-1} \in [\tilde{V}, V^{FB}]$. Thus, letting V_0^{t-1} and V_0^t play the role of V_0 and V_0^1 in the preceding argument, $\frac{\partial Q(V_0^{t-1})}{\partial p} = \frac{1+g}{1+r} \frac{\partial Q(V_0^t)}{\partial p}$ for any $t \in \{2, 3, \dots, M\}$; hence $\frac{\partial Q(V_0)}{\partial p} = (\frac{1+g}{1+r})^M \frac{\partial Q(V_0^M)}{\partial p}$.

For the second part of the claim, $V_0^M \geq V^{Aut}$ is obvious. For $t = M$, the participation constraint in the high state binds, and thus $v_H^M = v_H^{Aut}$, and given Lemma 2, consumption is greater in the high state. Thus from (23), $\partial q_L(v_L^M)/\partial v < \partial q_H(v_H^M)/\partial v$, so combining with (28),

$$\frac{\partial q_L(v_L^M)}{\partial v} < \frac{\partial q_H(v_H^M)}{\partial v} = \frac{\partial q_H(v_H^{Aut})}{\partial v} = \frac{\partial q_L(\tilde{v}_L)}{\partial v}. \tag{B.26}$$

Therefore, $v_L^M < \tilde{v}_L$ and thus $V_0^M < \tilde{V}$, completing the proof. ■

Next, Lemma A3 describes an important implication of $\tilde{N} = 1$. Using this result, Lemma A4 obtains the parameter conditions that determine whether $\tilde{N} > 1$ or $\tilde{N} = 1$, which affects $\partial Q(V_0)/\partial p$, as shown in Lemmas A5 and A6.

LEMMA A3. *If $\tilde{N} = 1$, then $v_H' = v_H^{Aut}$ and $v_L' = v_L^{Aut}$ in (B.1), and $c_H^{Aut} = y_H = 1 + \frac{\pi_L}{\pi_H} \alpha$.*

Proof. Suppose $\tilde{N} = 1$. Let $V_0 = \tilde{V}$, and let $\{v_H^t, v_L^t\}_{t=0}^M$ be as in Definition B.2. From the definition of \tilde{V} , initial consumption is c_H^{Aut} in both states. Thus, given $g > \mu$ and Lemma 2, the participation constraint in the high state binds at $t = 1$; hence $M = 1$ and $v_H^1 = v_H^{Aut}$. Further, $v_L^1 = v_L^{Aut}$ from the definition of \tilde{N} .

Thus, $v_H^1 = v_H^{Aut}$ and $v_L^1 = v_L^{Aut}$ regardless of s_0 , and in particular, for $s_0 = H$. Because $v_H^0 = v_H^{Aut}$, this implies that if the current state is high and the promised value is v_H^{Aut} , then the promised value in state $s' \in \{H, L\}$ next period is $v_{s'}^{Aut}$. Thus, $v_H' = v_H^{Aut}$ and $v_L' = v_L^{Aut}$ in (B.1), and comparing the resulting expression with (15) yields $c_H^{Aut} = y_H = 1 + \frac{\pi_L}{\pi_H} \alpha$. ■

LEMMA A4. *Let $\tilde{\alpha} \equiv (1 + \frac{1+\mu}{1+g} \frac{\pi_L}{\pi_H})^{-1} (\frac{g-\mu}{1+g})$. Then $\tilde{N} > 1$ if $\alpha \in (\tilde{\alpha}, 1)$ or $\sigma \in (0, 1)$ and $\alpha = 1$, and $\tilde{N} = 1$ if $\alpha \in [0, \tilde{\alpha}]$.*

Proof. The proof is made by deriving the condition under which $\tilde{N} = 1$. For $V_0 = \tilde{V}$, initial consumption is c_H^{Aut} in both states. Then, because $v_L^{\tilde{N}-1} > v_L^{Aut}$ and $v_L^{\tilde{N}} = v_L^{Aut}$, Lemma 2 implies that \tilde{N} is the smallest t such that $c_H^{Aut} (\frac{1+\mu}{1+g})^t \leq c_L^{Aut}$. Thus, $\tilde{N} = 1$ is equivalent to $c_H^{Aut} (\frac{1+\mu}{1+g}) \leq c_L^{Aut}$. It remains to express this condition using the parameters of the model.

Given $g > \mu$, Lemma 2 implies that if the current state is low and the promised value is v_L^{Aut} , the participation constraint binds in both states next period. Thus, in (B.2), $v_H'' = v_H^{Aut}$ and $v_L'' = v_L^{Aut}$, so comparing the resulting expression with (16) yields $c_L^{Aut} = y_L = 1 - \alpha$.

Now, if $\tilde{N} = 1$, then $c_H^{Aut} = y_H = 1 + \frac{\pi_L}{\pi_H} \alpha$ from Lemma A3. Substituting for c_H^{Aut} and c_L^{Aut} in $c_H^{Aut} (\frac{1+\mu}{1+g}) \leq c_L^{Aut}$ yields $\alpha \leq (1 + \frac{1+\mu}{1+g} \frac{\pi_L}{\pi_H})^{-1} (\frac{g-\mu}{1+g}) = \tilde{\alpha}$. Thus, $\tilde{N} = 1$ if $\alpha \leq \tilde{\alpha}$, and $\tilde{N} > 1$ otherwise; the case of $\sigma > 1$ and $\alpha = 1$ is excluded by assuming $v_H^{Aut} > v_L^{Aut}$. ■

LEMMA A5. *If $\tilde{N} > 1$, then $\partial Q(V_0)/\partial p < 0$ for all $V_0 \in [V^{Aut}, \tilde{V}]$.*

Proof. Suppose $\tilde{N} > 1$. Let $V_0 = \tilde{V}$, and let $V_0^1 < V_0$ be as in Definition B.2. Then $M = 1$, and

$$\frac{1+g}{1+r} \frac{\partial Q(\tilde{V})}{\partial p} > \frac{1+g}{1+r} \frac{\partial Q(V_0^1)}{\partial p} = \frac{\partial Q(\tilde{V})}{\partial p}, \tag{B.27}$$

where the inequality is from Lemma A1, and the equality is from Lemma A2. Thus, given $r > g$ [equation (3)], $\partial Q(\tilde{V})/\partial p < 0$. Then from Lemma A1, $\partial Q(V_0)/\partial p < 0$ for any $V_0 \in [V^{Aut}, \tilde{V}]$. ■

LEMMA A6. *If $\tilde{N} = 1$, then $\partial Q(V_0)/\partial p = 0$ for all $V_0 \in [V^{Aut}, \tilde{V}]$.*

Proof. Suppose $\tilde{N} = 1$. Then $N = 1$ for any $V_0 \in (V^{Aut}, \tilde{V}]$; hence (B.20) holds for any $V_0 \in (V^{Aut}, \tilde{V}]$ and $\hat{V}_0 \in [V^{Aut}, V_0)$. Thus, $\partial Q(V_0)/\partial p$ is independent of V_0 for $V_0 \in (V^{Aut}, \tilde{V}]$, so it suffices to show that $\partial Q(V_0)/\partial p = 0$ for any one $V_0 \in [V^{Aut}, \tilde{V}]$.

Let $V_0 = V^{Aut}$. Clearly, $v_H^0 = v_H^{Aut}$ and $v_L^0 = v_L^{Aut}$. First, let $s = L$ and $v = v_L^{Aut}$ in (A.7). From $g > \mu$ and Lemma 2, the participation constraint binds in both states next period; hence $v'_H = v_H^{Aut}$ and $v'_L = v_L^{Aut}$. Next, let $s = H$ and $v = v_H^{Aut}$ in (A.7). Then v'_H and v'_L coincide with those in (B.1), so from $\tilde{N} = 1$ and Lemma A3, $v'_H = v_H^{Aut}$ and $v'_L = v_L^{Aut}$. Substituting the resulting expressions into (B.11) and rearranging using (B.8) and (B.9) yields

$$\frac{\partial Q(V^{Aut})}{\partial p} = \frac{1 + g}{1 + r} \frac{\partial Q(V^{Aut})}{\partial p}. \tag{B.28}$$

Thus, given $r > g$ [equation (3)], $\partial Q(V_0)/\partial p = 0$ for $V_0 = V^{Aut}$, completing the proof. ■

To complete the proof of Proposition 4, take any $V_0 \in [\tilde{V}, V^{FB}]$. From Lemma A2, $\frac{\partial Q(V_0)}{\partial p} = (\frac{1+g}{1+r})^M \frac{\partial Q(V_0^M)}{\partial p}$ and $V_0^M \in [V^{Aut}, \tilde{V})$. Thus, if $\alpha \in (\tilde{\alpha}, 1)$ or $\sigma \in (0, 1)$ and $\alpha = 1$, then $\tilde{N} > 1$ from Lemma A4, so $\frac{\partial Q(V_0)}{\partial p} = (\frac{1+g}{1+r})^M \frac{\partial Q(V_0^M)}{\partial p} < 0$ from Lemma A5. If $\alpha \in [0, \tilde{\alpha}]$, then $\tilde{N} = 1$ from Lemma A4, so $\frac{\partial Q(V_0)}{\partial p} = (\frac{1+g}{1+r})^M \frac{\partial Q(V_0^M)}{\partial p} = 0$ from Lemma A6. ■