IMPLEMENTING STOCHASTIC VOLATILITY IN DSGE MODELS: A COMMENT

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We highlight a state variable misspecification with one accepted method to implement stochastic volatility (SV) in DSGE models when transforming the nonlinear state-innovation dynamics to its linear representation. Although the technique is more efficient numerically, we show that it is not exact but only serves as an approximation when the magnitude of SV is small. Not accounting for this approximation error may induce substantial spurious volatility in macroeconomic series, which could lead to incorrect inference about the performance of the model. We also show that, by simply lagging and expanding the state vector, one can obtain the correct state-space specification. Finally, we validate our augmented implementation approach against an established alternative through numerical simulation.

Keywords: Dynamic Equilibrium Economies, Stochastic Volatility, Perturbation

1. INTRODUCTION

Stochastic volatility (SV) has become an important ingredient for many dynamic equilibrium models.[1] These models in general show that SV, manifested in different source of exogenous shocks, can generate realistic comovements in macroeconomic aggregates consistent with economic intuition and data. This is a nontrivial development for general equilibrium modeling in the aftermath of the 2008 crisis, during which heightened economic and policy uncertainty contributed to a prolonged recovery period.

In an early paper that administers SV in the DSGE model, Andreasen (2012) demonstrates how any model with nonlinearities between state variables and innovations, such as those typically encountered in SV dynamics, may be rewritten

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into a standard state-space form where innovations only enter linearly. This allows researchers to employ standard perturbation methods to obtain Taylor series approximations (potentially to arbitrarily high order) of policy functions in DSGE models with SV.[2]

To operationalize the perturbation approach, one still needs to specify the state variables in the model.[3] For each exogenous variable x featuring SV, assuming they are first-order Markovian, it is standard to specify four state variables: the value (x_{t-1}), the level of conditional volatility of x ($\sigma_{x,t-1}$), the innovation to the level ($\epsilon_{x,t}$), and the innovation to the conditional volatility ($\epsilon_{\sigma,t}$). Thus, each structural shock featuring SV leads to two additional states.[4] Interestingly, the replication files associated with the economy described in Andreasen (2012) show an implementation of the SV process that allows one to solve the model with fewer state variables than the four described above. Since state variable reduction is always desirable in numerical methods, the Andreasen (2012) approach is appealing.

This note shows that this reduction in state variables is generally not exact, but it holds only in specific situations. In particular, we show that the set of state variables needs to be conveniently lagged and expanded in order to obtain a valid state-space representation to which one can apply perturbation methods. Although our correction is inconsequential for the results in Andreasen (2012) due to the negligible amount of SV employed in his calibration, not using the proper formulation of the state-space may generate close-to-explosive path in other settings for future work. This note thus raises a warning flag to using the "short-cut" described in Andreasen (2012) when implementing SV in DSGE models.

The rest of the paper proceeds as follows. Section 2 describes four alternative ways to implement SV in DSGE models. Section 3 investigates the quantitative implications of the various implementations in two alternative economies featuring SV. Throughout the paper, we make use of the third- and higher-order perturbation capacities offered by the set of MATLAB programs developed by Schmitt-Grohe and Uribe (2004), Andreasen (2012), and Levintal (2017).

2. DYNAMIC EQUILIBRIUM MODELS WITH SV

2.1. The General Model

Following Schmitt-Grohe and Uribe (2004), we study models with equilibrium conditions of the form

$$E_t[\mathbf{f}(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{x}_t)] = \mathbf{0},$$

where the state vector $\mathbf{x}_t$ has dimension $n_x \times 1$ and the vector $\mathbf{y}_t$ with dimension $n_y \times 1$ contains all the control variables. The vector $\mathbf{x}_t$ is partitioned as $[\mathbf{x}'_{1,t}, \mathbf{x}'_{2,t}]'$, where $\mathbf{x}_{1,t}$ contains endogenous state variables and $\mathbf{x}_{2,t}$ denotes exogenous

state variables. The dimensions of these vectors are $n_{x_1} \times 1$ and $n_{x_2} \times 1$, respectively, where $n_{x_1} + n_{x_2} = n_x$. It is further assumed that

$$\mathbf{x}_{2,t+1} = \mathbf{\Gamma}(\mathbf{x}_{2,t}) + \Lambda \tilde{\eta} \boldsymbol{\varepsilon}_{t+1}, \tag{1}$$

where $\Lambda \geq 0$ is an auxiliary perturbation parameter for the structural innovations $\boldsymbol{\varepsilon}_{t+1}$ with dimension $n_{\varepsilon} \times 1$.

The solution to these class of models—if one exists—is characterized by a policy function describing the evolution of the endogenous state variables and policy functions describing the law of motion of the observables, namely

$$\mathbf{y}_t = \mathbf{g}(\mathbf{x}_t; \Lambda),$$
$$\mathbf{x}_{t+1} = \mathbf{h}(\mathbf{x}_t; \Lambda) + \Lambda \boldsymbol{\eta} \boldsymbol{\varepsilon}_{t+1} \tag{*}$$

with

$$\boldsymbol{\eta} \equiv \begin{bmatrix} \mathbf{0}_{n_{x_1} \times n_{\varepsilon}} \\ \tilde{\eta} \end{bmatrix}.$$

As highlighted in Andreasen (2012), a system with nonlinearities between state variables and innovations can be rewritten into an extended system with only linear innovations, that is, the assumption that innovations only enter linearly in (1) is without loss of generality.

Next we illustrate how to rewrite the nonlinear dynamics implied by SV in a form like (1) containing only linear innovations. The derivation follows closely the steps described in Andreasen (2012)–Appendix. For each step in the derivation, we also provide reference to the specific lines in the associated computer code of the Andreasen (2012) replication files—see the Appendix.

2.2. The SV Process

As an illustration, consider a neoclassical growth model with SV. Using standard notation, the equilibrium conditions are given by

$$C_t^{-\gamma} = \beta E_t \big[C_{t+1}^{-\gamma} \big(A_{t+1} \alpha K_{t+1}^{\alpha-1} + 1 - \delta \big) \big],$$
$$C_t + K_{t+1} = A_t K_t^{\alpha} + (1 - \delta) K_t,$$

so that in the notation of the previous section we have $y_t = C_t$ and $x_{1,t} = K_t$. Notice K_t is determined in period $t-1$, following the convention of Schmitt-Grohe and Uribe (2004).

Let's assume that the exogenous level of productivity $x_{2,t} = A_t$ evolves according to[5]

$$\log(A_{t+1}) = \rho_a \log(A_t) + \sigma_{a,t+1} \varepsilon_{a,t+1}, \tag{2}$$
$$\log(\sigma_{a,t+1}) = \rho_\sigma \log(\sigma_{a,t}) + \varepsilon_{\sigma,t+1}, \tag{3}$$

where $\varepsilon_{a,t} \sim N(0, 1)$ and $\varepsilon_{\sigma,t} \sim N(0, \mathrm{Var}(\varepsilon_{\sigma,t}))$.

Andreasen (2012)—Appendix shows that these equations describing the SV process can be equivalently rewritten as[6]

$$\log(A_t) = \sigma_{a,t} \log V_t, \tag{4}$$

$$\log V_{t+1} = \rho_a \frac{\sigma_{a,t}}{\sigma_{a,t+1}} \times \log V_t + \varepsilon_{a,t+1}, \tag{5}$$

$$\log(\sigma_{a,t+1}) = \rho_\sigma \log(\sigma_{a,t}) + \varepsilon_{\sigma,t+1}. \tag{6}$$

As is standard in the literature, we use log-transformed variables, namely we replace

$$A_t, V_t, \sigma_{a,t} \longrightarrow e^{a_t}, e^{v_t}, e^{\tilde{\sigma}_{a,t}},$$

where lowercase letters denote log variables, that is, $a_t = \log A_t$, $v_t = \log V_t$, and $\tilde{\sigma}_{a,t} = \log \sigma_{a,t}$.[7] Equations (4)–(6) now read

$$a_t = e^{\tilde{\sigma}_{a,t}} v_t, \tag{7}$$

$$v_{t+1} = \rho_a \frac{e^{\tilde{\sigma}_{a,t}}}{e^{\tilde{\sigma}_{a,t+1}}} \times v_t + \varepsilon_{a,t+1}, \tag{8}$$

$$\tilde{\sigma}_{a,t+1} = \rho_\sigma \tilde{\sigma}_{a,t} + \varepsilon_{\sigma,t+1}. \tag{9}$$

2.3. Definition of State Variables: The "Direct" Specification

So far we acknowledge (one of) the contribution in Andreasen (2012) that shows how the system in equations (2) and (3), which is characterized by nonlinearities between state variables and innovations, can be rewritten as in equations (7)–(9), where innovations only enter linearly.

However, to eventually obtain a solution via perturbation method we need to list the state variables. Inspection of the MATLAB code provided by Andreasen (2012) suggests to use[8]

$$\mathbf{x}_{2,t} = [v_t, \ \tilde{\sigma}_{a,t}]'. \tag{10}$$

We dub this choice of state variables the "Direct" specification.

Next, we show that this particular choice yields a system of equations which is *not* in the form of (*). To see this, define also $y_t = a_t$. It is straightforward to see that, given the choice of state variables in equation (10), $y_t = g(x_t; \Lambda)$ is satisfied with $g(a, b; \Lambda) = e^a \times b$, and $a \equiv \sigma_{a,t}$, $b \equiv v_t$. The state equation in (*) becomes

$$\mathbf{x}_{2,t+1} = \begin{bmatrix} v_{t+1} \\ \tilde{\sigma}_{a,t+1} \end{bmatrix} = \begin{bmatrix} \underbrace{\rho_a \dfrac{e^{\tilde{\sigma}_{a,t}}}{e^{\tilde{\sigma}_{a,t+1}}} \times v_t}_{h_1(a,\,b)\,=\,\rho_a e^{(a-a')} \times b} \\ \underbrace{\rho_\sigma \tilde{\sigma}_{a,t}}_{h_2(a,b)\,=\,\rho_\sigma \times a + 0 \times b} \end{bmatrix} + \Lambda \begin{bmatrix} \varepsilon_{a,t+1} \\ \varepsilon_{\sigma,t+1} \end{bmatrix}, \tag{11}$$

where $a' \equiv \tilde{\sigma}_{a,t+1}$.

Clearly not all the variables on the right-hand side of (11) are adapted at time t. So we exploit the dynamics of $\mathbf{x}_{2,t+1}$ to replace these $(t+1)$-variables with their time t value. We have

$$\rho_a \frac{e^{\tilde{\sigma}_{a,t}}}{e^{\tilde{\sigma}_{a,t+1}}} \times v_t = \rho_a e^{\tilde{\sigma}_{a,t} - \tilde{\sigma}_{a,t+1}} \times v_t$$
$$= \rho_a e^{\tilde{\sigma}_{a,t}(1-\rho_\sigma) - \varepsilon_{\sigma,t+1}} \times v_t,$$

where $\tilde{\sigma}_{a,t} - \tilde{\sigma}_{a,t+1} = \tilde{\sigma}_{a,t}(1-\rho_\sigma) - \varepsilon_{\sigma,t+1}$. Now observe that $\varepsilon_{\sigma,t+1}$ is not a state variable in the "Direct" specification—see equation (10). It cannot even be singled out because of the interaction term, that is, $e^{-\varepsilon_{\sigma,t+1}} \times v_t$. We conclude that the "Direct" specification leads to a system of equations which is *not* in the form described in equation (*).

2.4. Definition of State Variables: Fixing the "Direct" Specification

We show that to put equations (8) and (9) in the form of equation (*), we must expand the state vector. First, lag equations (8) and (9),

$$v_t = \rho_a \frac{e^{\tilde{\sigma}_{a,t-1}}}{e^{\tilde{\sigma}_{a,t}}} \times v_{t-1} + \varepsilon_{a,t}, \tag{12}$$

$$\tilde{\sigma}_{a,t} = \rho_\sigma \tilde{\sigma}_{a,t-1} + \varepsilon_{\sigma,t}. \tag{13}$$

Next, specify the following state vector:

$$\mathbf{x}_{2,t} = [v_{t-1}, \ \tilde{\sigma}_{a,t-1}, \ \varepsilon_{a,t}, \ \varepsilon_{\sigma,t}]'. \tag{14}$$

We dub this choice of state variables the "Lagged Direct" specification. Importantly, equations (12) and (13) are in the form described in equation (*):

$$\mathbf{x}_{2,t+1} = \begin{bmatrix} v_t \\ \tilde{\sigma}_{a,t} \\ \varepsilon_{a,t+1} \\ \varepsilon_{\sigma,t+1} \end{bmatrix} = \begin{bmatrix} \rho_a \frac{e^{\tilde{\sigma}_{a,t-1}}}{e^{\tilde{\sigma}_{a,t}}} \times v_{t-1} + \varepsilon_{a,t} \\ \rho_\sigma \tilde{\sigma}_{a,t-1} + \varepsilon_{\sigma,t} \\ 0 \\ 0 \end{bmatrix} + \Lambda \begin{bmatrix} 0 \\ 0 \\ u_{a,t+1} \\ u_{\sigma,t+1} \end{bmatrix}.$$

2.5. The Fernández-Villaverde et al. (2015a) Specification

Fernández-Villaverde et al. (2015a) propose an implementation of SV in DSGE models which is alternative to Andreasen (2012) (see Sections 2.2 and 2.3). Recall equations (2) and (3) reported here for reader convenience:

$$\log(A_{t+1}) = \rho_a \log(A_t) + \sigma_{a,t+1} \varepsilon_{a,t+1},$$
$$\log(\sigma_{a,t+1}) = \rho_\sigma \log(\sigma_{a,t}) + \varepsilon_{\sigma,t+1}.$$

Fernández-Villaverde et al. (2015a) specification consists of the following steps. Lag the above system of equations

$$\log(A_t) = \rho_a \log(A_{t-1}) + \sigma_{a,t}\varepsilon_{a,t}, \qquad (15)$$

$$\log(\sigma_{a,t}) = \rho_\sigma \log(\sigma_{a,t-1}) + \varepsilon_{\sigma,t}, \qquad (16)$$

and log-transform the variables,

$$A_t, \ \sigma_{a,t} \longrightarrow e^{a_t}, \ e^{\tilde{\sigma}_{a,t}},$$

so that equations (15) and (16) now read

$$a_t = \rho_a a_{t-1} + e^{\tilde{\sigma}_{a,t}}\varepsilon_{a,t}, \qquad (17)$$

$$\tilde{\sigma}_{a,t} = \rho_\sigma \tilde{\sigma}_{a,t-1} + \varepsilon_{\sigma,t}. \qquad (18)$$

Fernández-Villaverde et al. (2015a) choose the following state vector:

$$\mathbf{x}_{2,t} = [a_{t-1}, \ \tilde{\sigma}_{a,t-1}, \ \varepsilon_{a,t}, \ \varepsilon_{\sigma,t}]'. \qquad (19)$$

We can finally verify that equations (17) and (18) are in the form described in equation (*):

$$
\mathbf{x}_{2,t+1} = \begin{bmatrix} a_t \\ \tilde{\sigma}_{a,t} \\ \varepsilon_{a,t+1} \\ \varepsilon_{\sigma,t+1} \end{bmatrix} = \begin{bmatrix} \underbrace{\rho_a a_{t-1} + e^{\tilde{\sigma}_{a,t}}\varepsilon_{a,t}}_{h_1(a,\,b,\,c,\,d)\,=\,\rho_a a + e^{b'}c} \\ \underbrace{\rho_\sigma \tilde{\sigma}_{a,t-1} + \varepsilon_{\sigma,t}}_{h_2(a,\,b,\,c,\,d)\,=\,0\times a + \rho_\sigma \times b + 0 \times c + d} \\ 0 \\ 0 \end{bmatrix} + \Lambda \begin{bmatrix} 0 \\ 0 \\ u_{a,t+1} \\ u_{\sigma,t+1} \end{bmatrix}.
$$

We dub this choice of state variables the "FGR" specification.

2.6. The Fernández-Villaverde and Levintal (2017) Specification

Fernández-Villaverde and Levintal (2017) propose yet another implementation of SV which is similar to the implementation of Fernández-Villaverde et al. (2015a) discussed above. In contrast to Fernández-Villaverde et al. (2015a), however, the state vector $\mathbf{x}_{2,t}$ only consists of just three variables. In particular, the state vector is defined as follows:

$$\mathbf{x}_{2,t} = [a_{t-1}, \ \tilde{\sigma}_{a,t}, \ \varepsilon_{a,t}]'. \qquad (20)$$

With this choice, the state equation can still be written in the form described in equation (*):

$$
\mathbf{x}_{2,t+1} = \begin{bmatrix} a_t \\ \tilde{\sigma}_{a,t+1} \\ \varepsilon_{a,t+1} \end{bmatrix} = \begin{bmatrix} \underbrace{\rho_a a_{t-1} + e^{\tilde{\sigma}_{a,t}}\varepsilon_{a,t}}_{h_1(a,\,b,\,c)\,=\,\rho_a a + e^{b'}c} \\ \underbrace{\rho_\sigma \tilde{\sigma}_{a,t}}_{h_2(a,\,b,\,c)\,=\,0\times a + \rho_\sigma \times b + 0 \times c} \\ 0 \end{bmatrix} + \Lambda \begin{bmatrix} 0 \\ u_{\sigma,t+1} \\ u_{a,t+1} \end{bmatrix}.
$$

For obvious reasons, we dub this choice of state variables the "FL" specification. Due to the reduction in the size of the state vector, the "FL" implementation is computationally more efficient than the "FGR" approach. Importantly, however, the "FL" approach can only be implemented in perturbation packages that allow the exogenous state variable ($\tilde{\sigma}_{a,t}$) to be an autoregressive process, as in Schmitt-Grohe and Uribe (2004) and the extended packages by Andreasen (2012) and Levintal (2017).[9]

2.7. A Brief Comparison of SV Implementations

We have described four ways to implement SV in DSGE models: the "Direct," the "Lagged Direct,', the "FGR," and the "FL" approaches (see Sections 2.3, 2.4, 2.5, and 2.6, respectively).

The "Direct" and "Lagged Direct" specifications reshuffle the SV equations (2) and (3) in the form suggested by Andreasen (2012), that is, (7)–(9). The "FGR" and "FL" approaches work directly with equations (2) and (3) instead.

The "Lagged Direct" and "FGR" specifications have the same number of state variables—see (14) and (19). This is higher than the number of state variables in the "Direct" specification—see equation (10). Specifically, if one has n processes with SV, then the number of state variables is higher by $2n$. For example, in the model economy of Fernández-Villaverde et al. (2011) we have $n = 2$ variables with SV, and 4 more states in the "Lagged Direct" and "FGR" specifications than in the "Direct" one. This consideration makes the "Direct" specification appealing. However, we show in Section 2.3 that the Direct approach is not consistent with the form described in equation (∗). Finally, in terms of computational efficiency, the "FL" approach is superior to the "Lagged Direct" and the "FGR" specifications. However, as discussed above, the implementation of "FL" requires a perturbation package that allows the exogenous state variables to be defined as autoregressive processes.

The next section quantifies the implications for the various SV specifications in the context of two alternative economies featuring SV.

3. A QUANTITATIVE EVALUATION OF THE DIFFERENT SV SPECIFICATIONS

3.1. Volatility in Productivity

We first consider the model economy described in Andreasen (2012). Our purpose is to verify to what extent the "Direct" specification used in that paper alters its conclusion. To this end, we compare the "Direct" specification against the "Lagged Direct" one. In the interest of space, we do not report results for the "FGR" nor for the more efficient "FL" specification since they both numerically coincide with those obtained from the "Lagged Direct" approach.
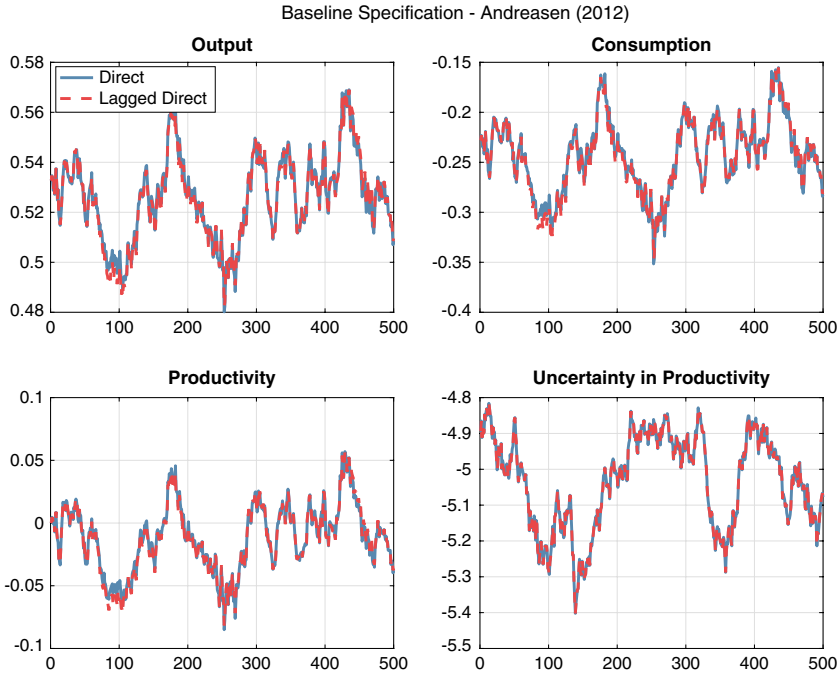
Baseline Specification - Andreasen (2012)

FIGURE 1. Andreasen (2012) economy: simulation. We simulate the Andreasen (2012) economy for 500 periods (after a period of burn-in to eliminate the effect of initial conditions) at the benchmark calibration parameter values and follow the results for the deviations of output and consumption (top panels) with respect to the steady state when we have a third-order approximation. The bottom panels display the path for the productivity level (left) and its time-varying volatility (right).

Figure 1 compares the paths of output, consumption, productivity level, and productivity uncertainty as implied by the "Direct" specification (solid line) with that implied by the "Lagged Direct" specification (dashed line). We observe no significant differences. We also verify that the variance decomposition in the two SV specifications does not change.

We argue that the two specifications yield similar results due to the chosen value for the volatility-of-volatility (vol-of-vol) parameter, that is, the volatility of $\varepsilon_{\sigma,t+1}$. Indeed, Figure 2 shows that the difference between the two alternative specifications gets larger for output, consumption, and productivity level as we increase the vol-of-vol parameter from 0.0265 to 0.2265.

However, for the range of values commonly used in the literature,[10] the "Lagged Direct" and "Direct" specifications yield extremely similar results for the model economy described in Andreasen (2012). Hence we conclude that using the correct "Lagged Direct" specification does not alter the conclusion in Andreasen (2012).

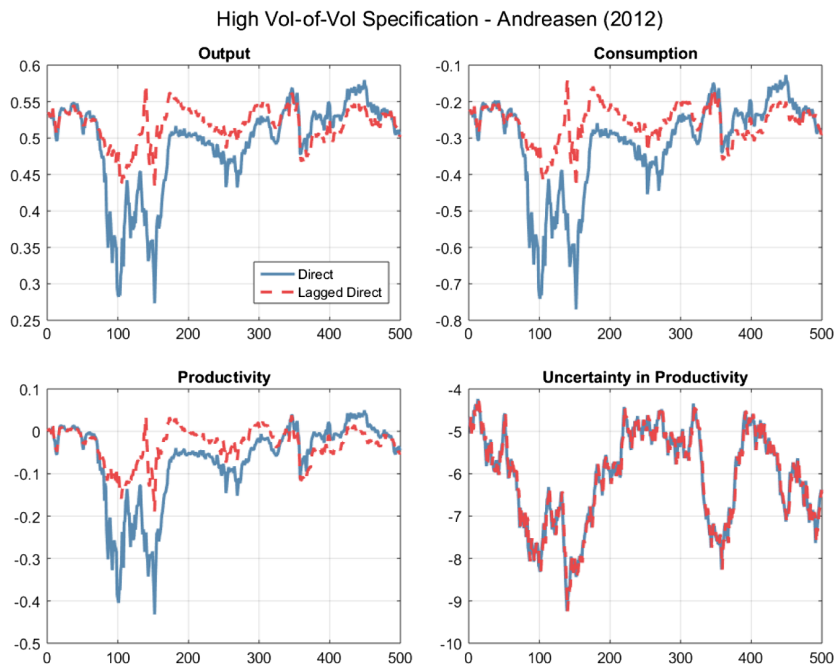High Vol-of-Vol Specification - Andreasen (2012)

FIGURE 2. Andreasen (2012) economy: simulation. We simulate the Andreasen (2012) economy for 500 periods (after a period of burn-in to eliminate the effect of initial conditions) at the benchmark calibration parameter values except for the vol-of-vol which is set to 0.2265 compared to the original value 0.0265 and follow the results for the deviations of output and consumption (top panels) with respect to the steady state when we have a third-order approximation. The bottom panels display the path for the productivity level (left) and its time-varying volatility (right).

3.2. Volatility in Interest Rates

We next implement the "Direct" and "Lagged Direct" SV specifications in the model economy of Fernández-Villaverde et al. (2011). In this model, a country faces a real interest rate, r_t, on loans denominated in US dollars. This real rate is decomposed into the international risk-free real rate plus a country-specific spread:

$$r_t = r + \varepsilon_{tb,t} + \varepsilon_{r,t},$$

where r is the mean of the international risk-free real rate plus the mean of the country spread. The term $\varepsilon_{tb,t}$ equals the international risk-free real rate subtracted from its mean, and $\varepsilon_{r,t}$ equals the country spread subtracted from its mean. Both $\varepsilon_{tb,t}$ and $\varepsilon_{r,t}$ follow AR(1) processes described by

$$\varepsilon_{tb,t} = \rho_{tb}\varepsilon_{tb,t-1} + e^{\tilde{\sigma}_{tb,t}}u_{tb,t},$$

$$\varepsilon_{r,t} = \rho_r\varepsilon_{r,t-1} + e^{\tilde{\sigma}_{r,t}}u_{r,t},$$
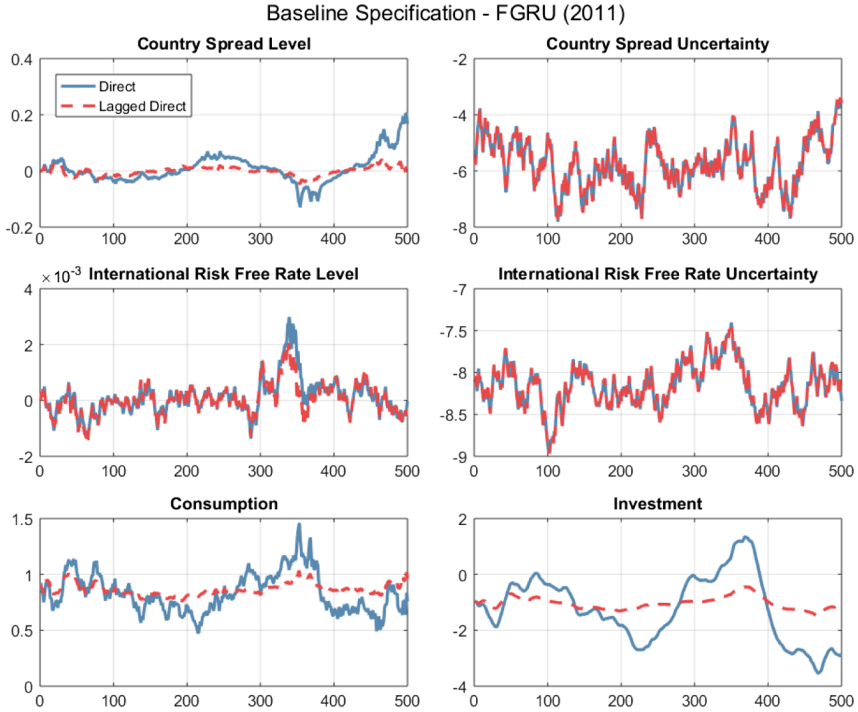
FIGURE 3. Fernández-Villaverde et al. (2011) economy: simulation. We simulate the Argentinian economy for 500 periods (after a period of burn-in to eliminate the effect of initial conditions) at the benchmark calibration parameter values and follow the results for the deviations of consumption and investment (bottom panels) with respect to the steady state when we have a third-order approximation. The top four panels display the path for the country spread (top row) and risk-free interest rate (middle row) level (left panels) and volatility (right panels) variables.

where $u_{r,t}$ and $u_{tb,t}$ are normally distributed random variables with mean zero and unit variance. The process for interest rates displays SV. In particular, the standard deviations $\sigma_{tb,t}$ and $\sigma_{r,t}$ follow an AR(1) process:

$$\tilde{\sigma}_{tb,t} = (1 - \rho_{\sigma_{tb}})\sigma_{tb} + \rho_{\sigma_{tb}}\tilde{\sigma}_{tb,t-1} + \eta_{tb}u_{\sigma_{tb},t}, \tag{21}$$

$$\tilde{\sigma}_{r,t} = (1 - \rho_{\sigma_r})\sigma_r + \rho_{\sigma_r}\tilde{\sigma}_{r,t-1} + \eta_r u_{\sigma_r,t}, \tag{22}$$

where both $u_{\sigma_{tb},t}$ and $u_{\sigma_r,t}$ are normally distributed random variables with mean zero and unit variance.

Figure 3 shows, for various variables of interest, two simulated paths, one obtained under the "Direct" specification and the other under the "Lagged Direct" specification. The variables of interest are the level of the risk-free and country spread interest rates, $\varepsilon_{tb,t}$ and $\varepsilon_{r,t}$ (left panels, top, and mid), their volatility $\tilde{\sigma}_{tb,t}$ and $\tilde{\sigma}_{r,t}$ (right panels, top, and mid), and the consumption and investment series
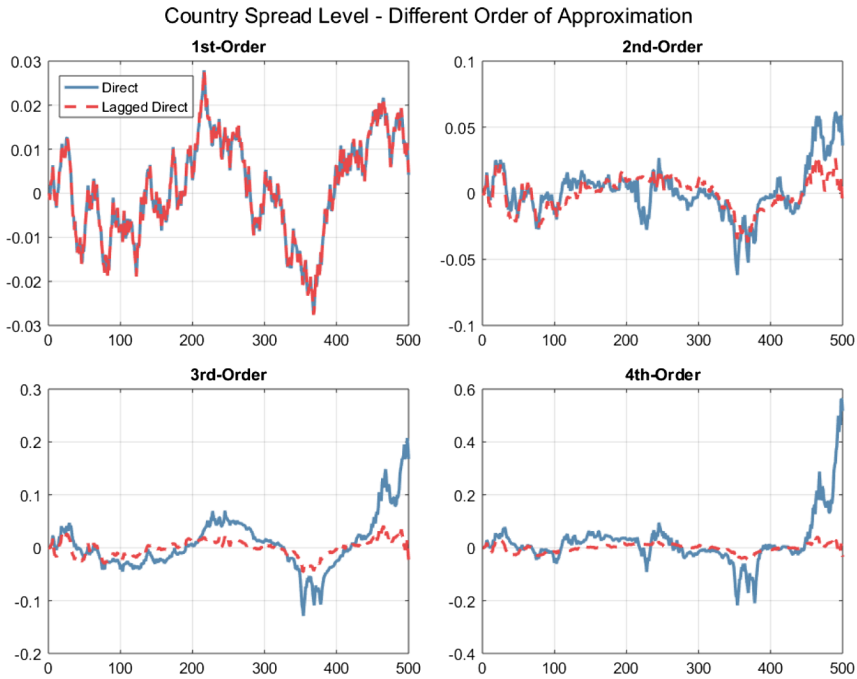
FIGURE 4. Fernández-Villaverde et al. (2011) economy: simulation for different order of approximations. We simulate the Fernández-Villaverde et al. (2011b) Argentinian economy for 500 periods (after a period of burn-in to eliminate the effect of initial conditions) at the benchmark calibration parameter values and follow the results for the deviations of consumption and investment (bottom panels) with respect to the steady state when we have a first-, second-, third-, and fourth-order approximations. The top fourth panels display the path for the country spread (top row) and risk-free interest rate (middle row) level (left panels) and volatility (right panels) variables.

(bottom panels, left, and right). As in the previous subsection, we do not report results for the "FGR" nor the "FL" specification since they numerically coincide with those obtained from the "Lagged Direct" approach.

Two things are noteworthy. First, the two implementations imply no difference in the path of the time-varying *volatilities*. This is to be expected since each of the (log) volatilities follows a standard AR(1) process which does not involve nonlinearities, and thus its implementation is standard. Second, the difference between the path under the "Direct" and "Lagged Direct" specifications is large for the *level* variables, and remarkably so for the consumption and investment series (bottom panels).

Figure 4 shows that the difference induced by the "Direct" and "Lagged Direct" specifications increases with the order of approximation.[11] This is consistent with the literature on perturbation methods showing the necessity of higher-order terms for the purpose of capturing effects of SV.
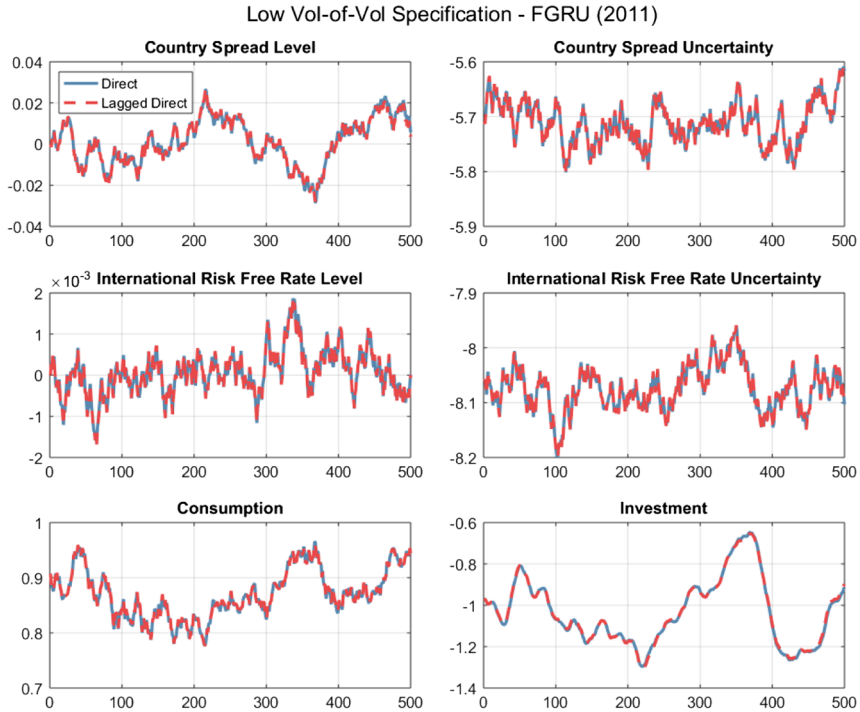
FIGURE 5. Fernández-Villaverde et al. (2011) economy: simulation with low vol-of-vol. We simulate the Fernández-Villaverde et al. (2011b) Argentinian economy for 500 periods (after a period of burn-in to eliminate the effect of initial conditions) at the benchmark calibration parameter values except for the vol-of-vol parameters of the country spread and the international risk-free rate which are both set to 0.02 compared to the original values 0.46 and 0.13, respectively. We then follow the results for the deviations of consumption and investment (bottom panels) with respect to the steady state when we have a third-order approximation. The top fourth panels display the path for the country spread (top row) and risk-free interest rate (middle row) level (left panels) and volatility (right panels) variables.

Finally, Figure 5 shows that these differences disappear as the vol-of-vol parameters, η_r and η_{tb}, are reduced in magnitude. In particular, for this experiment we set both the vol-of-vol parameters of the country spread and the international risk-free rate to 0.02, a value much lower than those used in the original paper (namely $\eta_r = 0.46$ and $\eta_{tb} = 0.13$). We conclude that for the small open emerging economies described in Fernández-Villaverde et al. (2011), the "Lagged Direct" specification must be used to achieve the correct conclusions.

4. CONCLUSION

This note discusses subtle issues arising from the implementation of SV in the DSGE model described by Andreasen (2012). We show that, to correct the

procedure, it is enough to augment the vector of exogenous state variables. Although our corrected version of the SV implementation does not alter the conclusion in Andreasen (2012), we show that for alternative parametrization and economies, the incorrect specification may generate substantial spurious variability in macroeconomic series. The error induced by the incorrect specification is increasing with the order of approximation and with the value of the vol-of-vol parameter.

NOTES

1. For an incomplete list of recent literature, see Bloom (2009), Fernández-Villaverde et al. (2011), Bloom et al. (2012), Fernández-Villaverde et al. (2015b), and Basu and Bundick (2017). In a related paper, Bretscher et al. (2017) argue that increased level of risk aversion can further amplify the effectiveness of SV shocks.

2. See Judd (1998). Many publicly available packages implement perturbation methods—see, for example, Dynare [Adjemian et al. (2011)], the library of numerical routines developed by Schmitt-Grohe and Uribe (2004) and recently extended by Andreasen et al. (2017), and the AIM software routines developed by Swanson et al. (2006).

3. Note that some packages, like Dynare, automatically specify the state variables.

4. This is the way Dynare implements SV. See also Fernández-Villaverde et al. (2015a).

5. We neglect the steady state to ease notation.

6. See lines 61–65 and 73–74 in `NK_Rotemberg_SV_model.m`.

7. The log-transformation is also implemented in the MATLAB code. See line 92 in `NK_Rotemberg_SV_model.m`.

8. See lines 88–89 in `NK_Rotemberg_SV_model.m`.

9. Since Dynare assumes that the exogenous state variables are i.i.d. shocks, the "FL" approach cannot be implemented.

10. The literature has found values of the vol-of-vol in productivity ranging from 0.01 to 0.03—see, for example, Justiniano and Primiceri (2008) and Kung (2015).

11. den Haan and de Wind (2010) investigate the behavior of higher-order perturbation solutions with and without pruning. To simulate the economy we adopt the pruning scheme proposed by Andreasen et al. (2017). Our conclusion does not change when we instead use an unpruned state-space system.

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APPENDIX: THE PUBLISHED REPLICATION FILES

This Appendix shows the MATLAB code of the Andreasen (2012) replication file
NK_Rotemberg_SV_model.m posted at https://sites.google.com/site/mandreasendk/home-1. Section 2.3 highlights the potential issues associated with this implementation.

```
1   function [f,x,xp,y,yp] = NK_Rotemberg_SV_model
2   % This function reports the equations for the New Keynesian model with
3   % Rothenberg prices and SV in f, and it reports the state vector
4   %(x and xp) and the control vector (y and yp)
5
6     %Define parameters
7     syms  GAMA NU BETTA ALFA THETA ETA XI DELTA PHI_pai PHI_y RHOA RHOG
8        RHOR RHOSIGA RHOSIGG ...
9           AA Ass Gss Kss Yss PAIss Rss SIGAss MUZss
10
11  %Define variables
12  syms p_cu   p2_cu   r_cu   y_cu   c_cu   n_cu   evf_cu   pai_cu   a_cu   varsdf_cu
13     ...
14        p_cup p2_cup r_cup y_cup c_cup n_cup evf_cup pai_cup a_cup
15           varsdf_cup ...
16        r_ba1   va_cu   siga_cu   g_cu   epsR_cu   ...
17        r_ba1p  va_cup  siga_cup  g_cup  epsR_cup
18
19  % Defining muz − ONLY a deterministic trend. For a stochastic trend the
20  % coding must be changed
21  muz_cu  = MUZss;
```

```
22   muz_cup = MUZss;
23   % EQ2: FOC for household with respect to labour
24   w_cu  = (1−NU)∗c_cu/(NU∗(1−n_cu));
25
26   % EQ4: FOC for frims with respect to labour
27   mc_cu  = w_cu/((1−THETA)∗a_cu∗Kss^THETA∗n_cu^(−THETA));
28
29   % EQ1 The expression for minus the value function
30   % Here we use the fact that the trend is deterministic
31   mvf_cup = −(c_cup^(NU∗(1−GAMA)))∗(1−n_cu)^((1−NU)∗(1−GAMA))/(1−GAMA)−BETTA
32      ∗muz_cup^(NU∗(1−GAMA))∗AA∗evf_cup^(1/(1−ALFA)));
33
34   % The ratio of lambda_cup/lambda_cu
35   mu_la_cup = (AA∗evf_cu^(1/(1−ALFA))/mvf_cup)^ALFA∗muz_cup^(NU∗(1−GAMA)−1)
36      ...
37                  ∗(c_cup/c_cu)^(NU∗(1−GAMA)−1)∗((1−n_cup)/(1−n_cu))^((1−NU)
                        ∗(1−GAMA));
38
39   % The equations in f
40   % The expected value of the value function
41   f1 = −evf_cu + (mvf_cup/AA)^(1−ALFA);
42
43   % EQ3: The one period bond price
44   f2 = − p_cu + BETTA∗mu_la_cup∗1/pai_cup;
45   41
46   % The one period interest rate
47   f3 = −r_cu + 1/p_cu;
48
49   % EQ5: The FOC firms with respect to prices
50   f4 = − mc_cu + (ETA−1)/ETA ...
51                  − BETTA∗(AA∗evf_cu^(1/(1−ALFA))/mvf_cup)^ALFA∗muz_cup^(NU
                        ∗(1−GAMA)) ...
52                  ∗(c_cup/c_cu)^(NU∗(1−GAMA)−1)∗((1−n_cup)/(1−n_cu))^((1−NU)
                        ∗(1−GAMA)) ...
53                  ∗XI/ETA∗(pai_cup/PAIss−1)∗pai_cup∗y_cup/(PAIss∗y_cu)  ...
54                  + XI/ETA∗(pai_cu/PAIss−1)∗pai_cu/PAIss;
55
56   % EQ6: The Taylor rule
57   f5 = −log(r_cu/Rss) +RHOR∗log(r_ba1/Rss) + PHI_pai∗log(pai_cu/PAIss) +
        PHI_y∗log(y_cu/Yss) + epsR_cu;
58
59   % EQ7: The output level
60   f6 = −y_cu + a_cu∗Kss^THETA∗n_cu^(1−THETA);
61
62   % EQ8: The budget resource constraint
63   f7 = −y_cu + c_cu + g_cu + DELTA∗Kss;
64
65   % Defining technolog
66   f8 = −log(a_cu/Ass) + siga_cu ∗log(va_cu);
67
68   % Technology shocks: the process for va_cu
69   f9 = −log(va_cup) + RHOA∗siga_cu/siga_cup∗log(va_cu);
70
71   % Defining government spending
72   f10 = −log(g_cup/Gss) + RHOG∗log(g_cu/Gss);
73
74   % Monetary policy shocks
75   f11 = −epsR_cup;
```

```
76    % The process for volatility in technology
77    f12 = −log(siga_cup/SIGAss) + RHOSIGA∗log(siga_cu/SIGAss);
78
79    % The link for r_ba1
80    f13 = −r_ba1p + r_cu;
81
82    % The variance of the nominal stochastic discount factor
83    f14 = −varsdf_cu + (BETTA∗mu_la_cup∗1/pai_cup)^2−(1/r_cu)^2;
84
85    %Creating function f
86    f = [f1;f2;f3;f4;f5;f6;f7;f8;f9;f10;f11;f12;f13;f14];
87
88    % Define the variables which needs to be log−transformed
89    y  = [p_cu, r_cu ,y_cu, c_cu, n_cu, evf_cu, pai_cu ,a_cu ];
90    yp = [p_cup,r_cup,y_cup,c_cup,n_cup,evf_cup,pai_cup,a_cup];
91    x  = [r_ba1 ,va_cu, siga_cu ,g_cu ];
92    xp = [r_ba1p,va_cup,siga_cup,g_cup];
93
94    %Make f a function of the logarithm of the state and control vector
95    f = subs(f, [x,y,xp,yp], exp([x,y,xp,yp]));
96
97    % Define the vector of controls , y, and states , x
98    y  = [p_cu, r_cu ,y_cu, c_cu, n_cu, evf_cu, pai_cu ,a_cu ,varsdf_cu];
99    yp = [p_cup,r_cup,y_cup,c_cup,n_cup,evf_cup,pai_cup,a_cup,varsdf_cup];
100   x  = [r_ba1 ,va_cu, siga_cu ,g_cu ,epsR_cu];
101   xp = [r_ba1p,va_cup,siga_cup,g_cup,epsR_cup];
```