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ERRATUM: CONTINUITY OF HILBERT–KUNZ MULTIPLICITY AND F-SIGNATURE

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Abstract. Unfortunately, there is a mistake in [PS, Lemma 3.10] which invalidates [PS, Theorem 3.12]. We show that the theorem still holds if the ring is assumed to be Gorenstein.

§1. Introduction

Alessandro De Stefani brought to our attention that [PS, Theorem 3.12] is wrong as stated, and an explicit counter-example can be found in [DSS20, Theorem 5.3]. The problem lays in [PS, Lemma 3.10], where the formula for the splitting number was written incorrectly as

$$a_e(R/(\underline{f}+\underline{\epsilon})) = [k:k^{p^e}]\,\lambda\left(\frac{R}{(\underline{f}+\underline{\epsilon},(I_{t_{\underline{\epsilon}}}^{[p^e]}:_R u_{t_{\underline{\epsilon}}}^{p^e}))}\right),$$

while it should be instead

$$a_e(R/(\underline{f}+\underline{\epsilon})) = [k:k^{p^e}]\,\lambda\left(\frac{R}{(\underline{f}+\underline{\epsilon},I_{t_{\underline{\epsilon}}}^{[p^e]}):_R u_{t_{\underline{\epsilon}}}^{p^e}}\right).$$

Below we will present a proof of [PS, Theorem 3.12] under additional Gorenstein hypothesis. In the main case of interest, when $\mathrm{s}(R/(\underline{f}))>0$, this result was generalized in [DSS20, Theorem 5.11] to the $\mathbb{Q}$-Gorenstein case.

It is also likely that the Gorenstein hypothesis is necessary for [PS, Questions 4.5, 4.8] to have positive answers; however, [DSS20, Theorem 5.11] does not provide a counter-example.

§2. Continuity of F-signature

We begin by recalling the following theorem of Huneke and Leuschke.

THEOREM 2.1. *[HL02, Proof of Theorem 11] Let $(R,\mathfrak{m})$ be an F-finite local Gorenstein ring of prime characteristic $p>0$ and dimension d. Suppose $f_1,\dots,f_d$ is a system of parameters of R and u generates the socle mod $(f_1,\dots,f_d)$. Then for each $e\in\mathbb{N}$,*

$$a_e(R) = [k:k^{p^e}]\,\lambda(R/((f_1,\dots,f_d)^{[p^e]}:u^{p^e})).$$

The following corollary is the Gorenstein version of [PS, Lemma 3.10]

COROLLARY 2.2. *Let $(R,\mathfrak{m})$ be a local Gorenstein F-finite ring of prime characteristic p. Let $\underline{f}$ be a regular sequence of length c. Then for any integer e, there exists an integer N such that $a_e(R/(\underline{f})) = a_e(R/(\underline{f}+\underline{\epsilon}))$ for any $\underline{\epsilon}\in(\mathfrak{m}^N)^{\oplus c}$.*

Proof. If $\dim(R)=c$, then $R/(f)$ is artinian and, thus, has a splitting if an only if it is a field. If we take $\underline{\epsilon}\in(\mathfrak{m}^2)^{\oplus c}$, then $(\underline{f})=\mathfrak{m}$ if and only if $(\underline{f}+\underline{\epsilon})=\mathfrak{m}$. So the Frobenius

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splitting numbers of $R/(\underline{f})$ and $R/(\underline{f}+\underline{\epsilon})$ will be the same, either 0 or $[k:k^{p^e}]$, the latter of which occurs if and only if $(\underline{f})=\mathfrak{m}$.

Suppose $\dim(R)=d+c>c$. Let $x_1,\ldots,x_d$ be a system of parameters for $R/(\underline{f})$ and let $u\in R$ generate the socle mod $(\underline{f},x_1,\ldots,x_d)$. By the proof of [PS, Corollary 3.2], there exists an integer N so that for each $\underline{\epsilon}\in(\mathfrak{m}^N)^{\oplus c}$ one has that $(\underline{f}+\underline{\epsilon},x_1,\ldots,x_d)=(\underline{f},x_1,\ldots,x_d)$ and $(\underline{f}+\underline{\epsilon},x_1^{p^e},\ldots,x_d^{p^e})=(\underline{f},x_1^{p^e},\ldots,x_d^{p^e})$. For such choices of $\underline{\epsilon}$ the sequence $x_1,\ldots,x_d$ is a full system of parameters for $R/(\underline{f}+\underline{\epsilon})$, u still generates the socle of $R/(\underline{f}+\underline{\epsilon},x_1,\ldots,x_d)$, and by multiple applications of Theorem 2.1

$$\begin{aligned} a_e(R/(\underline{f})) &= [k:k^{p^e}]\lambda(R/((\underline{f},x_1^{p^e},\ldots,x_d^{p^e}):u^{p^e}))\\ &= [k:k^{p^e}]\lambda(R/((\underline{f}+\underline{\epsilon},x_1^{p^e},\ldots,x_d^{p^e}):u^{p^e}))=a_e(R/(\underline{f}+\underline{\epsilon})). \end{aligned}$$ ∎

THEOREM 2.3. *Let $(R,\mathfrak{m},k)$ be a Gorenstein F-finite ring of prime characteristic p and dimension $d+c$. If $\underline{f}$ is a parameter sequence of length c such that $\hat{R}/(\underline{f})\hat{R}$ is reduced, then for any $\delta>0$, there exists an integer $N>0$ such that for any $\underline{\epsilon}\in(\mathfrak{m}^N)^{\oplus c}$*

$$|\mathrm{s}(R/(\underline{f}))-\mathrm{s}(R/(\underline{f}+\underline{\epsilon}))|<\delta.$$

Proof. We may assume R is complete. By the Cohen–Gabber theorem [GO08], we may choose parameters $x_1,\ldots,x_d\in R$ such that $x_1,\ldots,x_d$ is a system of parameters for $R/(\underline{f})$ and $R/(\underline{f})$ is module finite and generically separable over the regular local ring $A:=k[[x_1,\ldots,x_d]]$. Thus we have short exact sequence

$$0\to R/(\underline{f})[A^{1/p}]\to(R/(\underline{f}))^{1/p}\to M\to 0$$

and $0\neq c=\mathrm{D}_A(R/(\underline{f}))$ annihilates M.

Let M and $M_{\underline{\epsilon}}$ be as in the proof of [PS, Theorem 3.5], so that there are isomorphisms $R/(\underline{f}+\underline{\epsilon})[A^{1/p}]\cong\oplus^{p^d[k:k^p]}(R/(\underline{f}+\underline{\epsilon}))$ and short exact sequences

$$0\to R/(\underline{f}+\underline{\epsilon})[A^{1/p}]\to(R/(\underline{f}+\underline{\epsilon}))^{1/p}\to M_{\underline{\epsilon}}\to 0.$$

Apply the exact functor $(-)^{1/p^e}$ to the above to get the exact sequence

$$0\to\bigoplus^{p^d[k:k^p]}\left((R/(\underline{f}+\underline{\epsilon})^{1/p^e})\right)\to(R/(\underline{f}+\underline{\epsilon}))^{1/p^{e+1}}\to M_{\underline{\epsilon}}^{1/p^e}\to 0.$$

By [PT18, Lemma 2.1]

$$\operatorname{freerank}\left((R/(\underline{f}+\underline{\epsilon}))^{1/p^{e+1}}\right)\le p^d[k:k^p]\operatorname{freerank}\left((R/(\underline{f}+\underline{\epsilon}))^{1/p^e}\right)+\mu((M_{\underline{\epsilon}})^{1/p^e}),$$

or, equivalently, $a_{e+1}(R/(\underline{f}+\underline{\epsilon}))\le p^d[k:k^p]a_e(R/(\underline{f}+\underline{\epsilon}))+\mu((M_{\underline{\epsilon}})^{1/p^e})$. The proof of [PS, Theorem 3.5] gives the existence of a constant C such that for all $\underline{\epsilon}\in(\mathfrak{m}^N)^{\oplus c}$

$$\mu((N_{\underline{\epsilon}})^{1/p^e})=\lambda(N_{\underline{\epsilon}}/\mathfrak{m}^{[p^e]}N_{\underline{\epsilon}})\le Cp^{e(d-1)}.$$

Let L and $L_{\underline{\epsilon}}$ be as in the proof [PS, Theorem 3.5]. Similarly, we can also bound

$$p^d[k:k^p]\operatorname{freerank}\left((R/(\underline{f}+\underline{\epsilon}))^{1/p^e}\right)-\operatorname{freerank}\left((R/(\underline{f}+\underline{\epsilon}))^{1/p^{e+1}}\right)\le\lambda(L_{\underline{\epsilon}}/\mathfrak{m}^{[p^e]}L_{\underline{\epsilon}})$$

and once again obtain constant C, independent of $\underline{\epsilon}$, such that

$$|a_{e+1}(R/(\underline{f}+\underline{\epsilon}))-p^d[k:k^p]a_e(R/(\underline{f}+\underline{\epsilon}))|<Cp^{e(d-1)}.$$

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Since $\operatorname{rank} R^{1/p^e} = p^{ed}[k : k^{p^e}]$ [Kun76, Proposition 2.3], as explained in [PS, Corollary 3.6], this gives us a uniform convergence statement: there exist $D, N > 0$ such that for all $\underline{\epsilon} \in (\mathfrak{m}^N)^{\oplus c}$

$$\left| \mathrm{s}(R/(\underline{f}+\underline{\epsilon})) - \frac{1}{\operatorname{rank} R^{1/p^e}} a_e(R/(\underline{f}+\underline{\epsilon})) \right| < \frac{D}{p^e}.$$

The statement now follows by employing Corollary 2.2 and following the proof of [PS, Corollary 3.7]. ∎

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