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Ancient multiple-layer solutions to the Allen–Cahn equation

Manuel del Pino
Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile (delpino@dim.uchile.cl)

Konstantinos T. Gkikas
Centro de Modelamiento Matemático (UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile (kgkikas@dim.uchile.cl)

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We consider the parabolic one-dimensional Allen–Cahn equation

$$u_t = u_{xx} + u(1-u^2), \quad (x,t) \in \mathbb{R} \times (-\infty, 0].$$

The steady state $w(x) = \tanh(x/\sqrt{2})$ connects, as a 'transition layer', the stable phases -1 and $+1$. We construct a solution u with any given number k of transition layers between -1 and $+1$. Mainly they consist of k time-travelling copies of w, with each interface diverging as $t \to -\infty$. More precisely, we find

$$u(x,t) \approx \sum_{j=1}^{k} (-1)^{j-1} w(x - \xi_j(t)) + \tfrac{1}{2}((-1)^{k-1} - 1) \quad \text{as } t \to -\infty,$$

where the functions $\xi_j(t)$ satisfy a first-order Toda-type system. They are given by

$$\xi_j(t) = \frac{1}{\sqrt{2}}\left(j - \frac{k+1}{2}\right)\log(-t) + \gamma_{jk}, \quad j = 1, \ldots, k,$$

for certain explicit constants γ_{jk}.

Keywords: nonlinear parabolic equation; Allen–Cahn equation; ancient solutions

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1. Introduction and statement of the main result

A classical model for phase transitions is the Allen–Cahn equation [1]

$$u_t = \Delta u + f(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $f(u) = -F'(u)$. F is a *balanced bi-stable potential*, i.e. it has exactly two non-degenerate global minimum points, $u = +1$ and $u = -1$. The model is given by

$$F(u) = -\tfrac{1}{4}(1-u^2)^2,$$

so $f(u) = (1-u^2)u$. The constant functions $u = \pm 1$ correspond to the stable equilibria of (1.1). They are idealized as two phases of a material. A solution $u(x)$, whose values lie at all times in $[-1, 1]$, and takes values close to either $+1$ or -1 in most of the space $\mathbb{R}^N$, corresponds to a continuous realization of the phase state of the material, in which the two stable states coexist. There is a large literature on this type of solution (in the static and dynamic cases). The main object is to derive qualitative information on the 'interface region', that is, the walls separating the two phases. A close connection has been established between these walls and minimal surfaces and surfaces evolving by mean curvature in many works. We refer the reader to, for example, [5, 7, 9–13]. On the other hand, the main difference between interfaces and surfaces that evolve mean curvature surfaces is that, in the phase transition model, different components interact, giving rise to interesting patterns of motion.

The aim of this paper is to study multiple-interface interaction in the simplest, one-dimensional, scenario. We shall construct non-stationary solutions defined at all times. In the *ancient regime*, multiple, quite separate, transitions are present, with a dynamical law that is rigorously established. More precisely, we consider the problem of building *ancient solutions* $u(x,t)$ to the one-dimensional Allen–Cahn equation [1]

$$u_t = u_{xx} + u(1-u^2) \quad \text{in } \mathbb{R} \times (-\infty, 0] \tag{1.2}$$

that exhibit a finite number of transitions that connect the values -1 and $+1$.

The building blocks of these solutions are the single-transition-layer equilibrium solutions to (1.2),

$$u'' + u(1-u^2) = 0 \quad \text{in } \mathbb{R}, \qquad \lim_{x\to\infty} u(x) = 1, \qquad \lim_{x\to-\infty} u(x) = -1,$$

which represent a heteroclinic monotone connection between the constant equilibria ± 1 in the phase plane. These solutions are unique up to translations. The unique solution with $u(0) = 0$ will hereafter be denoted by $w(x)$, and is given in closed form by

$$w(x) = \tanh\left(\frac{x}{\sqrt{2}}\right). \tag{1.3}$$

Given an even number k, we want to build a solution $u(x,t)$ to (1.2) that satisfies

$$u(x,t) \approx \pm w(x - \xi_j(t))$$

near each k-ordered, very distant 'transition point' $\xi_j(t)$, $j = 1, \dots, k$. More precisely, we want to find a solution of the form

$$u(t,x) = -1 + \sum_{j=1}^{k} (-1)^{j+1} w(x - \xi_j(t)) + \psi(t,x), \tag{1.4}$$

with

$$\xi_1(t) < \xi_2(t) < \dots < \xi_k(t), \qquad \xi_j(t) = -\xi_{k-j+1}(t), \tag{1.5}$$

where the perturbation function $\psi(t,x)$ goes to zero uniformly as $t \to -\infty$ and satisfies the orthogonality conditions

$$\int_{\mathbb{R}} \psi(t,x) w'(x - \xi_i(t))\,\mathrm{d}x = 0 \quad \text{for all } i = 1, \dots, k, \quad t < -T, \tag{1.6}$$

for a suitable large $T > 0$. We shall establish the existence of a solution with this characteristic. In fact, as we shall see, the interface dynamic is driven mainly by the following system of differential equations (a first-order Toda system)

$$\frac{1}{\beta}\xi_j' - \exp(-\sqrt{2}(\xi_{j+1}-\xi_j)) + \exp(-\sqrt{2}(\xi_j-\xi_{j-1})) = 0, \quad j = 1,\dots,k, \quad t \in (-\infty, 0]. \tag{1.7}$$

The dynamic law of interface interaction was formally derived in a related Neumann problem by Fusco and Hale [8] (see also [2, 3]). In [4] Chen *et al*. built a solution with two transition layers travelling in opposite directions (our $k = 2$ case). Their argument was based on barriers, and it is not clear to us how to extend it to multiple transitions. In [6] the first-order Toda system appears in the construction of ancient solutions for the Yamabe flow.

More precisely, we shall find

$$\xi_j(t) = \xi_j^0(t) + h_j(t), \quad j = 1,\dots,k,$$

for some suitable parameter functions $h_j(t)$, such that these functions will decay in $|t|$ as $t \to -\infty$ for all $j = 1,\dots,k$, and the functions ξ_j^0 solve the first-order Toda system

$$\frac{1}{\beta}\xi_j' - \exp(-\sqrt{2}(\xi_{j+1}-\xi_j)) + \exp(-\sqrt{2}(\xi_j-\xi_{j-1})) = 0, \quad j = 1,\dots,k, \quad t \in (-\infty, 0], \tag{1.8}$$

with the conventions

$$\xi_{k+1} = \infty \quad \text{and} \quad \xi_0 = -\infty,$$

where $T_0 > 0$ and

$$\beta = \frac{6\int_{\mathbb{R}} \mathrm{e}^{2x/\sqrt{2}}(1-w^2(x))w'(x)\,\mathrm{d}x}{\int_{\mathbb{R}}(w'(x))^2\,\mathrm{d}x}. \tag{1.9}$$

We shall see that a solution of the above system is given by

$$\xi_j^0(t) = \frac{1}{\sqrt{2}}\left(j - \frac{k+1}{2}\right)\log(-2\sqrt{2}\beta t) + \gamma_{jk}, \quad j = 1,\dots,k, \tag{1.10}$$

for certain explicit constants γ_{jk}.

Our main result is as follows.

Theorem 1.1. *Let $k \geqslant 2$ be an even integer and let ξ_j^0 be the solution (1.10) of the Toda system (1.8). Then there exist a number $T > 0$ and a solution $u(t,x)$ to (1.2) defined on $(-\infty, -T] \times \mathbb{R}$, of the form (1.4)–(1.6), with*

$$\xi_j(t) = \xi_j^0(t) + h_j(t), \quad j = 1,\dots,k,$$

where the functions $\psi(t,x)$ and $h_j(t)$ tend to zero in suitable uniform norms as $t \to -\infty$.

If k is odd, a similar construction can be made with slightly different asymptotic configurations. For notational simplicity we shall only consider the case of an even k in this paper.

2. The first approximation

We want to solve the following problem:

$$u_t = u_{xx} + f(u) \quad \text{in } (-\infty, -T) \times \mathbb{R},$$

where $f(u) = u(1 - u^2)$, and T is a large positive number whose value can be adjusted at different steps.

Let $k \geqslant 2$ be an even integer. We set

$$w_i(t, x) = w(x - \xi_i(t)),$$

where the functions $\xi_i(t)$ are ordered and symmetric:

$$\xi_1(t) < \xi_2(t) < \cdots < \xi_{k/2}(t) < 0 < \xi_{k/2+1} < \cdots < \xi_k(t), \qquad \xi_j(t) = -\xi_{k-j+1}(t).$$

We set $\xi(t) = (\xi_1(t), \ldots, \xi_k(t))^{\mathrm{T}}$ and write

$$\xi(t) := \xi^0(t) + h(t), \tag{2.1}$$

where

$$\xi_j^0 = \frac{1}{\sqrt{2}}\left(j - \frac{k+1}{2}\right)\log(-2\sqrt{2}\beta t) + \gamma_j,$$

β is as defined in (1.9) and γ_j are constants which we shall determine later. In addition, $h(t)$ satisfies

$$\|h(t)\|_{L^\infty} + \|(t-1)h'(t)\|_{L^\infty} \leqslant 1 \quad \text{and} \quad \lim_{t \to -\infty} |h(t)| + |h'(t)| = 0.$$

We seek a solution of the form

$$u(t, x) = \sum_{j=1}^{k} (-1)^{j+1} w_j(t, x) - 1 + \psi(t, x). \tag{2.2}$$

Set

$$z(t, x) = \sum_{j=1}^{k} (-1)^{j+1} w_j(t, x) - 1. \tag{2.3}$$

We would like ψ to satisfy

$$\psi_t = \psi_{xx} + f'(z(t, x))\psi + E + N(\psi) - \sum_{i=1}^{k} c_i(t) w'(x - \xi_i(t)),$$
$$(t, x) \in (-\infty, -T) \times \mathbb{R}, \tag{2.4}$$

and

$$\int_{\mathbb{R}} \psi(t, x) w'(x - \xi_i(t))\, \mathrm{d}x = 0 \quad \text{for all } i = 1, \ldots, k, \quad t < -T, \tag{2.5}$$

where

$$E = \sum_{j=1}^{k} (-1)^{j+1} w'(x - \xi_j(t))\xi_j'(t) + f(z(t, x)) - \sum_{j=1}^{k} (-1)^{j+1} f(w_j(t, x)), \tag{2.6}$$

$$N(\psi) = f(\psi(t, x) + z(t, x)) - f(z(t, x)) - f'(z(t, x))\psi$$

and $c_i(t)$ have been chosen such that ψ satisfies the orthogonality condition (2.5), i.e. in such a way that the following (nearly diagonal) system holds:

$$\begin{aligned}\sum_{i=1}^{k} c_i(t)\int_{\mathbb{R}} w'(x-\xi_i(t))w'(x-\xi_j(t))\,\mathrm{d}x \\ = \int_{\mathbb{R}} (\psi_{xx}(t,x) + f'(z(t,x))\psi(t,x))w'(x-\xi_j(t))\,\mathrm{d}x \\ - \xi_j'(t)\int_{\mathbb{R}} \psi(t,x)w''(x-\xi_j(t))\,\mathrm{d}x + \int_{\mathbb{R}} (E+N(\psi))w'(x-\xi_j(t))\,\mathrm{d}x, \\ \text{for all } i=1,\dots,k, \quad t<-T. \end{aligned} \tag{2.7}$$

Later we shall choose $h(t)$ such that $c_i(t) = 0$ for all $i = 1,\dots,k$.

In the remainder of the paper we use the following notation.

Notation 2.1.

(i) We set

$$\xi = \xi^0 + h, \tag{2.8}$$

where $h\colon \mathbb{R} \mapsto \mathbb{R}^k$ is a function that satisfies

$$\sup_{t\leqslant -1} |h(t)| + \sup_{t\leqslant -1} |t||h'(t)| < 1. \tag{2.9}$$

(ii) We define

$$z(t,x) = \sum_{j=1}^{k} (-1)^{j+1} w(x-\xi_j(t)) - 1.$$

In the following lemma we find a bound for the error term $E = E(t,x)$ in (2.6).

Lemma 2.2. *Let $T_0 > 1$ and $0 < \sigma < \sqrt{2}$. We define*

$$\begin{aligned}\Phi(t,x) = \exp(\sigma(-x+\xi^0_{j-1}(t))) + \exp(\sigma(x-\xi^0_{j+1}(t))), \\ \text{if } \tfrac{1}{2}(\xi^0_j(t)+\xi^0_{j-1}(t)) \leqslant x \leqslant \tfrac{1}{2}(\xi^0_j(t)+\xi^0_{j+1}(t)), \quad j=1,\dots,k, \end{aligned} \tag{2.10}$$

with $\xi^0_0 = -\infty$ and $\xi^0_{k+1} = \infty$. Then there exists a uniform constant $C > 0$ that depends only on k, such that

$$|E(t,x)| \leqslant C\Phi(t,x) \quad \text{for all } (t,x) \in (-\infty,-T_0]\times\mathbb{R},$$

where E is the error term in (2.6) and ξ satisfies the assumptions (2.8) and (2.9).

Proof. First, note that there exists a positive constant $c := c(\gamma_1,\dots,\gamma_k,\beta) > 0$ such that the following inequality holds:

$$\sup_{x\in\mathbb{R}} \left\{ \frac{w'(x-\xi_j(t))}{\Phi(t,x)} \right\} \leqslant c|t|^{\sigma/\sqrt{2}} \quad \text{for all } j=1,\dots,k.$$

Using the fact that

$$|\xi_j'| \leqslant C_1(\beta,k)|t|^{-1},$$

we obtain that there exists a positive constant $C_2 = C_2(k)$ such that

$$\sup_{x\in\mathbb{R}}\left\{\frac{w'(x-\xi_j(t))}{\Phi(t,x)}\right\}|\xi_j'| \leqslant C_2|t|^{\sigma/\sqrt{2}-1} \quad \text{for all } j=1,\dots,k.$$

Now, let

$$\tfrac{1}{2}(\xi_j^0(t)+\xi_{j-1}^0(t)) \leqslant x \leqslant \tfrac{1}{2}(\xi_j^0(t)+\xi_{j+1}^0(t)), \quad j=1,\dots,k,$$

with $\xi_0^0=-\infty$ and $\xi_{k+1}^0=\infty$.

If $i\leqslant j-1$, by our assumptions on ξ_i there exists a uniform constant $C>0$ such that

$$|w(x-\xi_i(t))-1| \leqslant C\exp(\sqrt{2}(-x+\xi_{j-1}^0(t))).$$

Similarly, if $i\geqslant j+1$,

$$|w(x-\xi_i(t))+1| \leqslant C\exp(\sqrt{2}(x-\xi_{j+1}(t))).$$

We set

$$g=\sum_{i=1}^{j-1}(-1)^{i+1}(w(x-\xi_i)-1)+\sum_{i=j+1}^{k}(-1)^{i+1}(w(x-\xi_i)+1).$$

Then

$$\begin{aligned}
&\left|f(g+(-1)^{j+1}w(x-\xi_j(t)))-\sum_{i=1}^{k}(-1)^{i+1}f(w_i(t,x))\right| \\
&\quad=\left|f(g)+(-1)^{j+1}f(w_j(t,x))-(-1)^{j+1}3g^2w_j-3gw_j^2-\sum_{i=1}^{k}(-1)^{i+1}f(w_i(t,x))\right| \\
&\quad\leqslant C|g|+\left|f(g)-\sum_{i=1,\,i\neq j}^{k}(-1)^{i+1}f(w_i(t,x))\right| \\
&\quad\leqslant C\left(\sum_{i=1}^{j-1}|w(x-\xi_i)-1|+\sum_{i=j+1}^{k}(-1)^{i+1}|w(x-\xi_i)+1|\right).
\end{aligned}$$

Combining the above and using the properties of ξ, we obtain the desired result. □

3. The linear problem

This section is devoted to finding a solution to the linear parabolic problem

$$\psi_t=\psi_{xx}+f'(z(t,x))\psi+h(t,x)-\sum_{j=1}^{k}c_i(t)w'(x-\xi_j(t)) \quad \text{in } (-\infty,-T_0]\times\mathbb{R}, \tag{3.1}$$

$$\int_{\mathbb{R}}\psi(t,x)w'(x-\xi_i(t))\,\mathrm{d}x=0 \quad \text{for all } i=1,\dots,k, \quad t\in(-\infty,-T_0], \tag{3.2}$$

for a bounded function h, and $T_0>0$ fixed sufficiently large.

The $c_i(t)$ are exactly those that make the above relations consistent. Namely, by definition, for each $t < -T_0$, they solve the following linear system of equations:

$$\begin{aligned}\sum_{i=1}^{k} c_i(t) \int_{\mathbb{R}} w'(x-\xi_i(t))w'(x-\xi_j(t))\,\mathrm{d}x \\ = \int_{\mathbb{R}} (\psi_{xx} + f'(z)\psi)w'(x-\xi_j(t))\,\mathrm{d}x \\ - \xi_j'(t)\int_{\mathbb{R}} \psi(t,x)w''(x-\xi_j(t))\,\mathrm{d}x \\ + \int_{\mathbb{R}} hw'(x-\xi_j(t))\,\mathrm{d}x, \quad j=1,\dots,k. \end{aligned} \tag{3.3}$$

This system can indeed be solved uniquely, since if T_0 is taken to be sufficiently large, the matrix with coefficients

$$\int_{\mathbb{R}} w'(x-\xi_i(t))w'(x-\xi_j(t))\,\mathrm{d}x$$

is nearly diagonal.

Our aim is to build a linear operator $\psi = A(h)$ that defines a solution of (3.1), (3.2) which is bounded for a norm suitably adapted to our setting.

Let $\mathcal{C}_\Phi((s,t)\times\mathbb{R})$ is the space of continuous functions with norm

$$\|u\|_{\mathcal{C}_\Phi((s,t)\times\mathbb{R})} = \left\|\frac{u}{\Phi}\right\|_{L^\infty((s,t)\times\mathbb{R})},$$

where Φ is as defined in (2.10).

Proposition 3.1. *There exist positive numbers T_0 and C such that for each $h \in \mathcal{C}_\Phi((-\infty,0)\times\mathbb{R})$ there exists a solution of problem (3.1), (3.2) $\psi = A(h)$ that defines a linear operator of h and satisfies the estimate*

$$\|\psi\|_{\mathcal{C}_\Phi((-\infty,t)\times\mathbb{R})} \leqslant C\|h\|_{\mathcal{C}_\Phi((-\infty,t)\times\mathbb{R})} \quad \textit{for all } t \leqslant -T_0. \tag{3.4}$$

We obtain the proof via intermediate steps that we state and prove next. Let $g(t,x) \in \mathcal{C}_\Phi((-\infty,-T)\times\mathbb{R})$. For $T>0$ and $s<-T$ we consider the Cauchy problem

$$\left.\begin{aligned}\psi_t = \psi_{xx} + f'(z(t,x))\psi + g(t,x) \quad \text{in } (s,-T]\times\mathbb{R},\\ \psi(s,x) = 0 \quad \text{in } \mathbb{R},\end{aligned}\right\} \tag{3.5}$$

which is uniquely solvable. We call its solution $T^s(t,x)$. By standard regularity theory we have $T^s \in C^{0,\alpha}((s,-T)\times\mathbb{R})$.

3.1. *A priori* estimates for the solution of problem (3.5)

In this subsection we shall establish *a priori* estimates for the solutions T^s of (3.5) that are independent of s.

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LEMMA 3.2. *Let $T^s \in \mathcal{C}_\Phi((s,-T)\times\mathbb{R})$ be a solution of problem (3.5). Then $g(t,x) \in \mathcal{C}_\Phi((s,-T)\times\mathbb{R})$ satisfies*

$$\int_{\mathbb{R}} g(t,x)w'(x-\xi_j(t))\,\mathrm{d}x = -\int_{\mathbb{R}} (T^s_{xx}(t,x) + f'(z(t,x))T^s(t,x))w'(x-\xi_j(t))\,\mathrm{d}x + \xi_j'(t)\int_{\mathbb{R}} T^s(t,x)w''(x-\xi_j(t))\,\mathrm{d}x \quad \text{for all } i=1,\dots,k, \quad s<t<-T. \tag{3.6}$$

Then there exists a uniform constant $T_0>0$ such that for any $t\in(s,-T_0]$ the following estimate is valid:

$$\|T^s\|_{\mathcal{C}_\Phi((s,t)\times\mathbb{R})} \leqslant C\|g\|_{\mathcal{C}_\Phi((s,t)\times\mathbb{R})}, \tag{3.7}$$

where $C>0$ is a uniform constant.

Proof. We note here that the assumption (3.6) implies

$$\int_{\mathbb{R}} T^s(t,x)w'(x-\xi_i(t))\,\mathrm{d}x = 0 \quad \text{for all } i=1,\dots,k, \quad s<t<-T. \tag{3.8}$$

Indeed, since T^s is the solution of (3.5), using $w'(x-\xi_j(t))$ as a test function, for any $t\in(s,-T]$ we have

$$\int_{\mathbb{R}} T^s_t w'(x-\xi_j(t))\,\mathrm{d}x = -\int_{\mathbb{R}} T^s_x w''(x-\xi_j(t))\,\mathrm{d}x + \int_{\mathbb{R}} f'(z(t,x))T^s w'(x-\xi_j(t))\,\mathrm{d}x + \int_{\mathbb{R}} g(t,x)\,\mathrm{d}x.$$

But, by (3.6), for all $t\in(s,-T]$ we have

$$\int_{\mathbb{R}} g(t,x)w'(x-\xi_j(t))\,\mathrm{d}x = \int_{\mathbb{R}} T^s_x(t,x)w''(x-\xi_j(t))\,\mathrm{d}x - \int_{\mathbb{R}} f'(z(t,x))T^s(t,x)w'(x-\xi_j(t))\,\mathrm{d}x + \xi_j'(t)\int_{\mathbb{R}} T^s(t,x)w''(x-\xi_j(t))\,\mathrm{d}x.$$

Thus, combining all the above, we obtain that

$$\int_{\mathbb{R}} T^s_t w'(x-\xi_j(t))\,\mathrm{d}x = \xi_j'(t)\int_{\mathbb{R}} T^s(t,x)w''(x-\xi_j(t))\,\mathrm{d}x \iff \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} T^s w'(x-\xi_j(t))\,\mathrm{d}x = 0 \iff \int_{\mathbb{R}} T^s w'(x-\xi_j(t))\,\mathrm{d}x = c.$$

Using the fact that $T^s(s,x)=0$, by above the equality we deduce that T^s satisfies the orthogonality condition (3.8).

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Set

$$A_j^{(s,t)} = \{(\tau, x) \in (s,t) \times \mathbb{R}\colon \tfrac{1}{2}(\xi_j^0(\tau) + \xi_{j-1}^0(\tau)) < x < \tfrac{1}{2}(\xi_j^0(\tau) + \xi_{j+1}^0(\tau))\}$$

with $\xi_0^0 = -\infty$, $\xi_{k+1}^0 = \infty$ and

$$A_{j,R}^{(s,t)} = \{(\tau, x) \in (s,t) \times \mathbb{R}\colon |x - \xi_j^0(\tau)| < R + 1\}.$$

We shall prove (3.7) by contradiction. Let $\{s_i\}$ and $\{\bar{t}_i\}$ be sequences such that $s_i < \bar{t}_i \leqslant -T_0$, and $s_i \downarrow -\infty$, $\bar{t}_i \downarrow -\infty$. We assume that there exists g_i satisfying (3.6) and such that ψ_i solves (3.5) with $s = s_i$, $-T = \bar{t}_i$ and $g = g_i$. Finally, we assume that

$$\left.\begin{aligned} \left\| \frac{\psi_i}{\Phi} \right\|_{L^\infty((s_i,\bar{t}_i)\times\mathbb{R})} &= 1, \\ \left\| \frac{g_i}{\Phi} \right\|_{L^\infty((s_i,\bar{t}_i)\times\mathbb{R})} &\to 0, \end{aligned}\right\} \tag{3.9}$$

First, note that we can assume

$$s_i + 1 < \bar{t}_i.$$

Indeed, let $\lambda > 0$. Then the function $\psi_i = \mathrm{e}^{\lambda(t-s_i)} v_i(t,x)$ satisfies

$$\left.\begin{aligned} v_t = v_{xx} + (-\lambda + f'(z(t,x)))v + \mathrm{e}^{-\lambda(t-s_i)} g(t,x) \quad &\text{in } (s, -T] \times \mathbb{R}, \\ v(s,x) = 0 \quad &\text{in } \mathbb{R}. \end{aligned}\right\} \tag{3.10}$$

Let $M > 0$ be sufficiently large. Set

$$\phi_j(t,x) = M(\exp(\sigma(-x + \xi_{j+1}^0(t))) + \exp(\sigma(x - \xi_{j-1}^0(t)))).$$

Next observe that there exists $C > 0$ independent of t such that

$$\Phi(t,x) \leqslant C\phi_j(t,x) \quad \text{for all } (t,x) \in (-\infty, -1) \times \mathbb{R}. \tag{3.11}$$

Now, since $|f'(z(t,x))| \leqslant C_0$, where C_0 does not depend on t, we can choose $\lambda > 2C_0$ independent of t such that for any $(t,x) \in (s,-1] \times \mathbb{R}$ the function ϕ_j satisfies

$$(\phi_j)_t - (\phi_j)_{xx} + (\lambda - f'(z(t,x)))\phi_j \geqslant c_1\phi_j(t,x) \geqslant c_2 M\Phi(t,x) \geqslant \mathrm{e}^{-\lambda(t-s_i)} g_i(t,x),$$

where $c_1, c_2 > 0$ are independent of t and

$$M = \frac{1}{c_2} \left\| \frac{g_i}{\Phi} \right\|_{L^\infty((s_i,\bar{t}_i)\times\mathbb{R})}.$$

Thus, we can use ϕ_j as a barrier to obtain

$$|v_i(t,x)| \leqslant \phi_j(t,x) \implies |\psi_i(t,x)| \leqslant C\mathrm{e}^{\lambda(t-s_i)}\Phi(t,x) \left\| \frac{g_i}{\Phi} \right\|_{L^\infty((s_i,\bar{t}_i)\times\mathbb{R})}. \tag{3.12}$$

Thus, by the above inequality, we can choose $s_i + 1 < \bar{t}_i$.

To prove by contradiction we need the following assertion.

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Assertion 3.3. *Let $R>0$. Then we have*

$$\lim_{i\to\infty}\left\|\frac{\psi_i}{\Phi}\right\|_{L^\infty(A_{j,R}^{(s_i,\bar t_i)})}=0 \quad \textit{for all } j=1,\dots,k. \tag{3.13}$$

Let us first assume that (3.13) is valid. Set

$$\mu_{i,j}:=\left\|\frac{\psi_i}{\Phi}\right\|_{L^\infty(A_{j,R}^{(s_i,\bar t_i)})}\xrightarrow{i\to\infty}0 \quad \text{for all } j=1,\dots,k,$$

and let

$$\tfrac12(\xi_j(t)+\xi_{j-1}(t))\leqslant x\leqslant \tfrac12(\xi_j(t)+\xi_{j+1}(t)), \quad j=1,\dots,k,$$

with $\xi_0=-\infty$ and $\xi_{k+1}=\infty$.

If $n\leqslant j-1$, then by our assumptions on ξ_n we have

$$|w(x-\xi_n(t))-1|\leqslant C\exp(\sqrt2(-x+\xi_{n-1}(t)))\leqslant C\exp(-\tfrac{\sqrt2}{2}(\xi_j-\xi_{j-1}(t)))\leqslant\frac{C}{\sqrt t}.$$

Similarly, if $n\geqslant j+1$,

$$|w(x-\xi_n(t))+1|\leqslant 2\exp(\sqrt2(x-\xi_{n+1}(t)))\leqslant\frac{C}{\sqrt t}.$$

Moreover, if we assume that $|x-\xi_j(t)|>R+1$, then we have that

$$|w(x-\xi_j(t))|\geqslant w(R).$$

Combining all the above for any $0<\varepsilon<\sqrt2$, there exist $i_0\in\mathbb N$ and $R>0$ such that

$$-f'(z(t,x))\geqslant 2-\varepsilon^2 \quad \text{for all } t\leqslant\bar t_i,\ x\in\mathbb R\setminus\bigcup_{j=1}^k A_{j,R}^{(s_i,t_i)} \text{ and } i\geqslant i_0. \tag{3.14}$$

Consider the function

$$\begin{aligned}\bar\phi_{i,j}(t,x)&=M(\exp(\sigma(-x+\xi_{j+1}^0(t)))\\&\quad+\exp(\sigma(x-\xi_{j-1}^0(t))))\left(\left\|\frac{g_i}{\Phi}\right\|_{L^\infty((s_i,\bar t_i)\times\mathbb R)}+\sup_{1\leqslant j\leqslant k}\mu_{i,j}\right),\end{aligned} \tag{3.15}$$

where $M>1$ is sufficiently large and does not depend on s_i or $\bar t_i$.

Let $\varepsilon>0$ be such that $2-\varepsilon^2>\sigma^2$. Then we can choose i_0 such that, for any $i>i_0$ and for all $(t,x)\in((s_i,t_i)\times\mathbb R)\setminus\bigcup_{j=1}^k A_{j,R}^{(s_i,\bar t_i)}$, the function $\bar\phi_{i,j}(t,x)$ satisfies

$$(\bar\phi_{i,j})_t-(\bar\phi_{i,j})_{xx}-f'(z(t,x))\bar\phi_{i,j}\geqslant c_1\bar\phi_{i,j}\geqslant c_2M\Phi(t,x)\geqslant g_i(t,x), \tag{3.16}$$

where the constants $0<c_2<c_1<1$ are independent of t, $M\geqslant 1/c_2$ and we have used (3.11).

Let $0\leqslant\eta\leqslant1$ be a smooth function in $C_0^\infty\mathbb R$ such that $\eta=1$ if $|x|<1$ and $\eta=2$ if $|x|>2$. Set $\zeta=\eta^2(t/R)\max(\psi_i(t,x)-\phi_{i,j}(t,x),0)$.

Note that by (3.12) we can choose $M > 0$ such that

$$\max(\psi_i(t,x) - \phi_{i,j}(t,x), 0) = 0 \quad \text{for all } (t,x) \in \overline{\bigcup_{j=1}^{k} A_{j,R}^{(s_i,\bar{t}_i)}}.$$

Thus, by (3.14) and (3.16) we can easily obtain

$$\int_{s_i}^{\bar{t}_i}\int_{\mathbb{R}} (\psi_i - \phi_{i,j})_t \zeta \,\mathrm{d}x\,\mathrm{d}t + \int_{s_i}^{\bar{t}_i}\int_{\mathbb{R}} (\psi_i - \phi_{i,j})_x \zeta_x \,\mathrm{d}x\,\mathrm{d}t \\ + (2-\varepsilon^2)\int_{s_i}^{\bar{t}_i}\int_{\mathbb{R}} (\psi_i - \phi_{i,j})\zeta \,\mathrm{d}x\,\mathrm{d}t \leqslant 0.$$

By the above inequality and using standard arguments, we obtain

$$|\psi_i(t,x)| \leqslant |\bar{\phi}_{i,j}(t,x)| \quad \text{for all } (t,x) \in ((s_i,\bar{t}_i)\times\mathbb{R}) \setminus \bigcup_{j=1}^{k} A_{j,R}^{(s_i,\bar{t}_i)}, \\ j = 1,\dots,k, \quad i \geqslant i_0.$$

Thus, we have

$$|\psi_i(t,x)| \leqslant |\bar{\phi}_{i,j}(t,x)| \quad \text{for all } (t,x) \in (s_i,\bar{t}_i)\times\mathbb{R}, \quad j = 1,\dots,k.$$

Hence, by (3.10) we can easily obtain that

$$1 = \left\|\frac{\psi_i}{\varPhi}\right\|_{L^\infty((s_i,\bar{t}_i)\times\mathbb{R})} \leqslant M\left(\left\|\frac{g_i}{\varPhi}\right\|_{L^\infty((s_i,\bar{t}_i)\times\mathbb{R})} + \sup_{1\leqslant j\leqslant k}\mu_{i,j}\right),$$

which is clearly a contradiction if we choose i large enough.

Proof of assertion 3.3. We shall prove assertion 3.3 by contradiction in four steps.

Let us first set out the contradiction and give some notation. We assume that (3.13) is not valid. Then there exists $j \in \{1,\dots,k\}$ and $\delta > 0$ such that

$$\left\|\frac{\psi_i}{\varPhi}\right\|_{L^\infty(A_{j,R}^{(s_i,\bar{t}_i)})} > \delta > 0 \quad \text{for all } i \in \mathbb{N}.$$

Let $(t_i, y_i) \in A_{j,R}^{(s_i,\bar{t}_i)}$ such that

$$\left|\frac{\psi_i(t_i,y_i)}{\varPhi(t_i,y_i)}\right| > \delta. \tag{3.17}$$

We observe here that, by the definition of $\varPhi$,

$$\varPhi(t_i,y_i) = \exp(\sigma(-y_i + \xi_{j-1}(t_i))) + \exp(\sigma(y_i - \xi_{j+1}(t_i))). \tag{3.18}$$

We set $y = x + \xi_j(t+t_i)$, $y_i = x_i + \xi_j(t_i)$ and

$$\phi_i(t,x) = \frac{\psi_i(t+t_i, x + x_i + \xi_j(t+t_i))}{\varPhi(t_i, x_i + \xi_j(t_i))}.$$

Then ϕ_i satisfies

$$\left.\begin{aligned}(\phi_i)_t = (\phi_i)_{xx} - \xi_j'(t+t_i)(\phi_i)_x \\ + f'(z(t+t_i, x+x_i+\xi_j(t+t_i)))\phi_i \\ + \frac{g_i(t+t_i, x+x_i+\xi_j(t+t_i))}{\Phi(t_i, x_i+\xi_j(t_i))} \quad \text{in } (s_i - t_i, 0]\times\mathbb{R}, \\ \phi_i(s_i-t_i, x) = 0 \quad \text{in } \mathbb{R}.\end{aligned}\right\} \tag{3.19}$$

Also set

$$\begin{aligned}B_{t_i,n,j} = \{(t,x)\in(s_i-t_i,0]\times\mathbb{R}\colon \\ \tfrac{1}{2}(\xi_n^0(t+t_i)+\xi_{n-1}^0(t+t_i)) - \xi_j(t+t_i) - x_i \leqslant x \\ \leqslant \tfrac{1}{2}(\xi_n^0(t+t_i)+\xi_{n+1}^0(t+t_i)) - \xi_j(t+t_i) - x_i\}\end{aligned}$$

and

$$B_{t_i,n,j}^M = B_{t_i,n,j}\cap\{(t,x)\in(s_i-t_i,0]\times\mathbb{R}\colon |x+\xi_j(t+t_i)+x_i-\xi_n^0(t+t_i)|>M\},$$

where $n=1,\dots,k$ and $M>0$. We note here that $|x_i|<R+1$ for all $i\in\mathbb{N}$, $|\phi_i(0,0)| = |\psi_i(t_i,y_i)/\Phi(t_i,y_i)| > \delta > 0$. Also, in view of the proof of (3.12) and the assumption (3.17), we can assume that

$$\liminf t_i - s_i > \infty.$$

Without loss of generality we assume that $x_i \to x_0 \in B_{R+1}(0)$, $\lim_{i\to\infty} t_i - s_i = \infty$ (otherwise we take a subsequence).

Step 1. We assert that $\phi_i \to \phi$ locally uniformly, that $\phi(0,0) > \delta$ and that ϕ satisfies

$$\phi_t = \phi_{xx} + f'(w(x+x_0))\phi \quad \text{in } (-\infty, 0]\times\mathbb{R}. \tag{3.20}$$

Let $(t,x)\in B_{t_i,n,j}$, $1\leqslant n\leqslant k$. By (3.9) and (3.18) we have that

$$\begin{aligned}|\phi_i(t,x)| &\leqslant \left|\frac{\psi_i(t+t_i, x+x_i+\xi_j(t+t_i))}{\Phi(t_i, x_i+\xi_j(t_i))}\right| \\ &\leqslant \left|\frac{\Phi(t+t_i, x+x_i+\xi_j(t+t_i))}{\Phi(t_i, x_i+\xi_j(t_i))}\right| \\ &= \frac{e^{\sigma(-(x+x_i+\xi_j(t+t_i))+\xi_{n-1}(t+t_i))} + e^{\sigma(x+x_i+\xi_j(t+t_i)-\xi_{n+1}(t+t_i))}}{e^{\sigma(-(x_i+\xi_j(t_i))+\xi_{j-1}(t_i))} + e^{\sigma(x_i+\xi_j(t_i)-\xi_{j+1}(t_i))}} \\ &\leqslant e^{\sigma(-(x+x_i+\xi_j(t+t_i)-\xi_n(t+t_i))-(\xi_n(t+t_i)-\xi_{n-1}(t+t_i))+x_i+(\xi_j(t_i))-\xi_{j-1}(t_i))} \\ &\quad + e^{\sigma((x+x_i+\xi_j(t+t_i)-\xi_n(t+t_i))+(\xi_n(t+t_i)-\xi_{n+1}(t+t_i))+x_i+(\xi_{j+1}(t_i))-\xi_j(t_i))} \\ &\leqslant C_0(\beta, \|h\|_{L^\infty}, \sup_{1\leqslant j\leqslant k}|\gamma_j|, \sigma, R)\left(\frac{t_i}{t+t_i}\right)^{\sigma/\sqrt{2}} e^{\sigma|x+\xi_j(t+t_i)-\xi_n(t+t_i)|} \\ &\qquad \text{for all } i\in\mathbb{N}, (t,x)\in B_{t_i,n,j},\end{aligned} \tag{3.21}$$

where in the last inequality we have used (2.1).

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Now note here that

$$\bigcup_{i=1}^{\infty} B_{t_i,j,j} = (-\infty, 0] \times \mathbb{R}.$$

Thus, the proof of this step for the assertion is complete.

STEP 2. In this step we prove the following orthogonality condition for ϕ:

$$\int_{\mathbb{R}} \phi(t,x) w'(x+x_0)\,\mathrm{d}x = 0 \quad \text{for all } t \in (-\infty, 0]. \tag{3.22}$$

Let $t \in \bigcap_{i=i_0}^{\infty}(s_i - t_i, 0]$ for some $i_0 \in \mathbb{N}$, and

$$\begin{aligned} x \in B_{t,t_i,n,j} = \{x \in \mathbb{R}\colon & \\ & \tfrac{1}{2}(\xi_n^0(t+t_i) + \xi_{n-1}^0(t+t_i)) - \xi_j(t+t_i) - x_i \leqslant x \\ & \qquad \leqslant \tfrac{1}{2}(\xi_n^0(t+t_i) + \xi_{n+1}^0(t+t_i)) - \xi_j(t+t_i) - x_i\}. \end{aligned}$$

By (3.21) we have that

$$\left| \int_{B_{t,t_i,j,j}} \phi_i(t,x) w'(x+x_i)\,\mathrm{d}x \right| \leqslant C_0 \int_{\mathbb{R}} \mathrm{e}^{-(\sqrt{2}-\sigma)|x|}\,\mathrm{d}x < C. \tag{3.23}$$

Let $n > j$. Then there exists i_0 such that for any $i > i_0$ we have $x > 0$. Also, by (3.21), the assumptions on ξ (see notation 2.1) and the fact that $|x| < R+1$, we obtain

$$\begin{aligned} &\left| \int_{B_{t,t_i,n,j}} \phi_i(t,x) w'(x+x_i)\,\mathrm{d}x \right| \\ &\quad \leqslant C_0 \int_{(\xi_n^0(t+t_i)+\xi_{n-1}^0(t+t_i))/2-\xi_j(t+t_i)-x_i}^{(\xi_n^0(t+t_i)+\xi_{n+1}^0(t+t_i))/2-\xi_j(t+t_i)-x_i} \mathrm{e}^{-\sqrt{2}x+\sigma|x+\xi_j(t+t_i)-\xi_n(t+t_i)|}\,\mathrm{d}x \\ &\quad = C_0 \mathrm{e}^{-\sqrt{2}(\xi_n(t+t_i)-\xi_j(t+t_i))} \int_{(\xi_n^0(t+t_i)+\xi_{n-1}^0(t+t_i))/2-\xi_n(t+t_i)-x_i}^{(\xi_n^0(t+t_i)+\xi_{n+1}^0(t+t_i))/2-\xi_n(t+t_i)-x_i} \mathrm{e}^{-\sqrt{2}y+\sigma|y|}\,\mathrm{d}y. \end{aligned}$$

Now

$$\begin{aligned} &\int_{(\xi_n^0(t+t_i)+\xi_{n-1}^0(t+t_i))/2-\xi_n(t+t_i)-x_i}^{(\xi_n^0(t+t_i)+\xi_{n+1}^0(t+t_i))/2-\xi_n(t+t_i)-x_i} \mathrm{e}^{-\sqrt{2}y+\sigma|y|}\,\mathrm{d}y \\ &\quad = \int_0^{(\xi_n^0(t+t_i)+\xi_{n+1}^0(t+t_i))/2-\xi_n(t+t_i)-x_i} \mathrm{e}^{-\sqrt{2}y+\sigma|y|}\,\mathrm{d}y \\ &\qquad + \int_{(\xi_n^0(t+t_i)+\xi_{n-1}^0(t+t_i))/2-\xi_n(t+t_i)-x_i}^{0} \mathrm{e}^{-\sqrt{2}y+\sigma|y|}\,\mathrm{d}y \\ &\quad \leqslant C(\mathrm{e}^{-((\sqrt{2}-\sigma)/2)(\xi_{n+1}^0(t+t_i)-\xi_n^0(t+t_i))} + \mathrm{e}^{((\sqrt{2}+\sigma)/2)(\xi_n^0(t+t_i)-\xi_{n-1}^0(t+t_i))}) + C_1 \\ &\quad \leqslant C(\mathrm{e}^{-((\sqrt{2}-\sigma)/2\sqrt{2})(\log(-2\sqrt{2}\beta(t+t_i)))} + \mathrm{e}^{((\sqrt{2}+\sigma)/2\sqrt{2})(\log(-2\sqrt{2}\beta(t+t_i)))}) + C_1. \end{aligned}$$

But

$$\mathrm{e}^{-\sqrt{2}(\xi_n(t+t_i)-\xi_j(t+t_i))} \leqslant C\mathrm{e}^{-\sqrt{2}(\xi_n(t+t_i)-\xi_{n-1}(t+t_i))} \leqslant C\mathrm{e}^{-\log(-2\sqrt{2}\beta(t+t_i))}.$$

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Thus, combining all the above, we obtain

$$\left| \int_{B_{t,t_i,n,j}} \phi_i(t,x) w'(x+x_i)\,\mathrm{d}x \right| \leqslant C\mathrm{e}^{-((\sqrt{2}-\sigma)/2\sqrt{2})\log(-2\sqrt{2}\beta(t+t_i))} \xrightarrow{i\to\infty} 0. \quad (3.24)$$

Similarly, the estimate (3.24) is valid if $n < j$. Now note that

$$\begin{aligned} \int_{\mathbb{R}} \phi_i(t,x) w'(x+x_i)\,\mathrm{d}x &= \int_{\mathbb{R}} \frac{\psi_i(t+t_i, x+x_i+\xi_j(t+t_i))}{\Phi(t_i, x_i+\xi_j(t_i))} w'(x+x_i)\,\mathrm{d}x \\ &= \frac{1}{\Phi(t_i, x_i+\xi_j(t_i))} \int_{\mathbb{R}} \psi_i(t+t_i, x) w'(x-\xi_j(t+t_i))\,\mathrm{d}x \\ &= 0. \end{aligned} \quad (3.25)$$

By (3.23)–(3.25) we have that

$$0 = \int_{\mathbb{R}} \phi_i(t,x) w'(x+x_i)\,\mathrm{d}x \to \int_{\mathbb{R}} \phi(t,x) w'(x+x_0)\,\mathrm{d}x$$

and the proof of this assertion follows.

Step 3. In this step we prove the following assertion.

Assertion 3.4. *There exists $C = C(R,\sigma) > 0$, such that*

$$|\phi(t,x)| \leqslant C\mathrm{e}^{-\sigma|x|} \quad \textit{for all } (t,x) \in (-\infty, 0] \times \mathbb{R}. \quad (3.26)$$

Now, note that if $(t,x) \in B_{t_i,n,j}$, by definition of ξ (notation 2.1), we have

$$\mathrm{e}^{\sigma|x+\xi_j(t+t_i)-\xi_n(t+t_i)|} \leqslant C_0\Big(\beta, \|h\|_{L^\infty}, \sup_{1\leqslant j\leqslant k} |\gamma_j|, \sigma, R\Big)\mathrm{e}^{\sigma|x|}.$$

Thus, in view of the proof of (3.21) we have that

$$\left| \frac{g_i(t+t_i, x+x_i+\xi_j(t+t_i))}{\Phi(t_i, x_i+\xi_j(t_i))} \right| \leqslant C \left\| \frac{g_i}{\Phi} \right\|_{L^\infty((s_i,\bar{t}_i)\times\mathbb{R})} \mathrm{e}^{\sigma|x|} \quad \text{for all } i \in \mathbb{N}.$$

In view of the proof of assertion 3.3 we can find i_0 and $M > 0$ by using

$$G(t,x) = M\left(\mathrm{e}^{-\sigma|x|} + \left\| \frac{g_i}{\Phi} \right\|_{L^\infty((s_i,\bar{t}_i)\times\mathbb{R})} \mathrm{e}^{\sigma|x|}\right)$$

as barrier, to prove

$$|\phi_i(t,x)| \leqslant G(t,x) \quad \text{for all } (t,x) \in B^M_{t_i,j,j} \quad \text{for all } i \geqslant i_0.$$

The proof of (3.26) follows if we send $i \to \infty$.

Step 4. Here we prove (3.13).

If we multiply (3.20) by ϕ and integrate with respect to x, by proposition 3.5 we have

$$\begin{aligned} 0 &= \frac{1}{2}\int_{\mathbb{R}} (\phi^2)_t\,\mathrm{d}x + \int_{\mathbb{R}} |\phi_x|^2 - f'(w(x))|\phi|^2\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}} (\phi^2)_t\,\mathrm{d}x + c\int_{\mathbb{R}} |\phi(t,x)|^2\,\mathrm{d}x. \end{aligned}$$

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Setting $a(t) = \int_{\mathbb{R}} |\phi(t,x)|^2 \, \mathrm{d}x$, we have that there exists a c_0 such that

$$a'(t) \leqslant -c_0 a(t) \implies a(t) > a(0)\mathrm{e}^{c_0|t|},$$

which is a contradiction since

$$\|\mathrm{e}^{\sigma|x|}\phi\|_{L^\infty((-\infty,-T_0-\tilde{t}_0)\times\mathbb{R})} < C.$$

□

The following proposition is well known; we give a proof for the reader's convenience.

Proposition 3.5. *Consider the Hilbert space*

$$H = \left\{\zeta \in H^1(\mathbb{R}) \colon \int_{\mathbb{R}} \zeta(x) w'(x) \, \mathrm{d}x = 0\right\}.$$

Then the following inequality is valid:

$$\int_{\mathbb{R}} |\zeta'(x)|^2 - f'(w(x))|\zeta|^2 \, \mathrm{d}x \geqslant c \int_{\mathbb{R}} |\zeta(x)|^2 \, \mathrm{d}x \quad \textit{for all } \zeta \in H \cap L^2(\mathbb{R}). \tag{3.27}$$

Proof. Let $\zeta \in H$ and set $\zeta = w'\phi$. Then

$$\begin{aligned}\int_{\mathbb{R}} |\zeta'(x)|^2 - f'(w(x))|\zeta|^2 \, \mathrm{d}x &= \int_{\mathbb{R}} |w''|^2|\phi|^2 \, \mathrm{d}x + \int_{\mathbb{R}} |w'|^2|\phi'(x)|^2 \\ &\quad + \int_{\mathbb{R}} w''w'(\phi^2)' \, \mathrm{d}x - \int_{\mathbb{R}} f'(w(x))|w'\phi|^2 \, \mathrm{d}x \\ &= \int_{\mathbb{R}} |w'|^2|\phi'(x)|^2 \\ &\geqslant 0.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}} |\zeta'(x)|^2 - f'(w(x))|\zeta|^2 \, \mathrm{d}x = 0 \quad \text{if and only if} \quad \zeta = cw'$$

for some constant c, which implies that $\zeta = 0$.

Now we assume that there exists a sequence $\{\phi_n\}_{n=1}^\infty \in H$ such that

$$\int_{\mathbb{R}} \phi_n^2 \, \mathrm{d}x = 1$$

and

$$\int_{\mathbb{R}} |\phi_n'(x)|^2 - f'(w(x))|\phi_n|^2 \, \mathrm{d}x \leqslant \frac{1}{n}. \tag{3.28}$$

Thus, $\phi_n \rightharpoonup \phi$ in H and $\phi_n \rightharpoonup \phi$ in $L^2(K)$ for any compact subset of $\mathbb{R}$, which implies

$$0 = \int_{\mathbb{R}} \phi_n(x) w' \, \mathrm{d}x \to \int_{\mathbb{R}} \phi w' \, \mathrm{d}x = 0,$$

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$\phi \in H$ and

$$\int_{\mathbb{R}} |\phi'(x)|^2 - f'(w(x))|\phi|^2 \, dx = 0.$$

Thus, $\phi = 0$.

But, by (3.28) we have

$$2 = 2\int_{\mathbb{R}} |\phi|^2 \, dx \leqslant 3\int_{\mathbb{R}} (1-w^2)|\phi|^2 \, dx.$$

This implies that $\phi \neq 0$, which is clearly a contradiction. □

3.2. The problem (3.5) with $g(t,x) = h(t,x) - \sum_{j=1}^{k} c_i(t) w'(x - \xi_j(t))$

In this subsection, we study the following problem:

$$\left.\begin{aligned} \psi_t = \psi_{xx} + f'(z(t,x))\psi + h(t,x) - \sum_{j=1}^{k} c_i(t) w'(x-\xi_j(t)) \quad \text{in } (s,-T_0] \times \mathbb{R}, \\ \psi(s,x) = 0 \quad \text{in } \mathbb{R}, \end{aligned}\right\} \tag{3.29}$$

where $h \in \mathcal{C}_{\Phi}((s,-T) \times \mathbb{R})$ and $c_i(t)$ satisfies the following (nearly diagonal) system

$$\begin{aligned} \sum_{i=1}^{k} c_i(t) \int_{\mathbb{R}} w'(x-\xi_i(t)) w'(x-\xi_j(t)) \, dx \\ = \int_{\mathbb{R}} (\psi_{xx}(t,x) w'(x-\xi_j(t)) + f'(z(t,x)\psi(t,x))) \, dx \\ - \xi_j'(t) \int_{\mathbb{R}} \psi(t,x) w''(x-\xi_j(t)) \, dx \\ + \int_{\mathbb{R}} h(t,x) w'(x-\xi_j(t)) \, dx \quad \text{for all } i = 1,\dots,k, \quad t < -T. \end{aligned} \tag{3.30}$$

We note here that if ψ is a solution of (3.29) and $c_i(t)$ satisfies the above system, then $g(t,x) = h(t,x) - \sum_{j=1}^{k} c_i(t) w'(x-\xi_j(t))$ satisfies (3.6). Thus, in view of the proof of (3.8) we have that ψ satisfies the orthogonality conditions

$$\int_{\mathbb{R}} \psi(t,x) w'(x-\xi_i(t)) \, dx = 0 \quad \text{for all } i = 1,\dots,k, \quad s < t < -T_0.$$

The main result of this subsection is the following.

Lemma 3.6. *Let $h \in \mathcal{C}_{\Phi}((s,-T) \times \mathbb{R})$. Then there exist a uniform constant $T_0 \geqslant T > 0$ and a unique solution T^s of problem (3.29).*

Furthermore, we have that, for all $s < t < -T_0$, T^s satisfies the orthogonality conditions (2.5) and the following estimate:

$$\|T^s\|_{\mathcal{C}_{\Phi}((s,t)\times\mathbb{R})} \leqslant C\|h\|_{\mathcal{C}_{\Phi}((s,t)\times\mathbb{R})}, \tag{3.31}$$

where $C > 0$ is a uniform constant.

To prove the above lemma we need the following result.

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LEMMA 3.7. *Let $T > 0$ be sufficiently large, $h \in \mathcal{C}_\Phi(s, -T)$ and $\psi \in \mathcal{C}_\Phi((s, -T)\times\mathbb{R})$. Then there exist $c_i(t)$, $i = 1, \dots, k$, such that the nearly diagonal system (3.30) holds.*

Furthermore, the following estimates for c_i are valid, for some constant $C > 0$ that does not depend on T, s, t, ψ or f:

$$|c_i(t)| \leqslant C\left(\left(\frac{1}{|t|}\right)^{1+\sigma/2\sqrt{2}}\left\|\frac{\psi}{\Phi}\right\|_{L^\infty((s,-T)\times\mathbb{R})} + \left(\frac{1}{|t|}\right)^{\sigma/\sqrt{2}}\left\|\frac{h}{\Phi}\right\|_{L^\infty((s,-T)\times\mathbb{R})}\right)$$
$$\text{for all } t \in [s, -T] \quad \text{for all } i = 1, \dots, k,$$

$$\left|\frac{c_i(t)w'(x-\xi_i(t))}{\Phi(t,x)}\right| \leqslant C\left(\left(\frac{1}{|t|}\right)^{1-\sigma/2\sqrt{2}}\left\|\frac{\psi}{\Phi}\right\|_{L^\infty((s,-T)\times\mathbb{R})} + \left\|\frac{h}{\Phi}\right\|_{L^\infty((s,-T)\times\mathbb{R})}\right)$$
$$\text{for all } t \in [s, -T] \quad \text{for all } i = 1, \dots, k.$$

Proof. For $i < j$, we have

$$\begin{aligned}
\int_{\mathbb{R}} & w'(x-\xi_i(t))w'(x-\xi_j(t))\,\mathrm{d}x \\
&= \int_{\mathbb{R}} w'(x + (\xi_j(t) - \xi_i(t)))w'(x)\,\mathrm{d}x \\
&= C\int_{\mathbb{R}}\left(\frac{1}{\mathrm{e}^{(\sqrt{2}/2)(x-(\xi_j(t)-\xi_i(t)))} + \mathrm{e}^{(\sqrt{2}/2)(-x+(\xi_j(t)-\xi_i(t)))}}\right)^2 \\
&\qquad \times\left(\frac{1}{\mathrm{e}^{\sqrt{2}x/2} + \mathrm{e}^{-\sqrt{2}x/2}}\right)^2\mathrm{d}x \\
&= C\frac{1}{\mathrm{e}^{\sqrt{2}(\xi_j(t)-\xi_i(t))}}\int_{\mathbb{R}}\left(\frac{1}{\mathrm{e}^{(\sqrt{2}/2)(x-2(\xi_j(t)-\xi_i(t)))} + \mathrm{e}^{-\sqrt{2}x/2}}\right)^2 \\
&\qquad \times\left(\frac{1}{\mathrm{e}^{\sqrt{2}x/2} + \mathrm{e}^{-\sqrt{2}x/2}}\right)^2\mathrm{d}x \\
&= C\frac{1}{\mathrm{e}^{\sqrt{2}(\xi_j(t)-\xi_i(t))}}\int_{\mathbb{R}} F(t,x)\,\mathrm{d}x,
\end{aligned}$$

where

$$F(t,x) = \left(\frac{1}{\mathrm{e}^{(\sqrt{2}/2)(x-2(\xi_j(t)-\xi_i(t)))} + \mathrm{e}^{-\sqrt{2}x/2}}\right)^2\left(\frac{1}{\mathrm{e}^{\sqrt{2}x/2} + \mathrm{e}^{-\sqrt{2}x/2}}\right)^2.$$

Now,

$$\int_{2(\xi_j(t)-\xi_i(t))}^{\infty} F(t,x)\,\mathrm{d}x < C, \qquad \int_{-\infty}^{0} F(t,x)\,\mathrm{d}x < C,$$
$$\int_0^{2(\xi_j(t)-\xi_i(t))} F(t,x)\,\mathrm{d}x \leqslant C((\xi_j(t)-\xi_i(t)) + 1),$$

where the constant $C > 0$ does not depend on t.

Thus, we can easily obtain

$$\int_{\mathbb{R}} w'(x-\xi_i(t))w'(x-\xi_j(t))\,\mathrm{d}x \leqslant C\frac{|\log|t||}{t} \quad \text{for all } i \neq j, \quad i,j=1,\dots,k,$$

where in the above inequality we have used the assumptions on ξ_j (see notation 2.1). Thus, the system is nearly diagonal and we can solve it for sufficiently large T.

In addition, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \Phi(t,x)\,\mathrm{d}x &= \sum_{j=1}^{k}\int_{(\xi_j^0(t)+\xi_{j-1}^0(t))/2}^{(\xi_j^0(t)+\xi_{j+1}^0(t))/2} \exp(\sigma(-x+\xi_{j-1}^0(t))) + \exp(\sigma(x-\xi_{j+1}^0(t)))\,\mathrm{d}x \\ &\leqslant C\left(\frac{1}{|t|}\right)^{\sigma/2\sqrt{2}},\end{aligned} \tag{3.32}$$

where $\xi_0^0=-\infty$, $\xi_{k+1}^0=\infty$ and

$$\begin{aligned}&\left|\int_{\mathbb{R}}(\psi_{xx}+f'(z(t,x))\psi)w'(x-\xi_j(t))\,\mathrm{d}x\right| \\ &\quad = \left|\int_{\mathbb{R}}(f'(w(x-\xi_j(t)))-f'(z(t,x)))\psi(t,x)w'(x-\xi_j(t))\,\mathrm{d}x\right| \\ &\quad = \left|\int_{\mathbb{R}}(f'(w(x))-f'(z(t,x+\xi_j(t))))\psi(t,x+\xi_j(t))w'(x)\,\mathrm{d}x\right| \\ &\quad \leqslant C\left\|\frac{\psi}{\Phi}\right\|_{L^\infty((s,-T)\times\mathbb{R})} \\ &\qquad \times \int_{\mathbb{R}}|(-1)^{j+1}w(x)-z(t,x+\xi_j(t))|\Phi(t,x+\xi_j(t))w'(x)\,\mathrm{d}x \\ &\quad \leqslant C\left\|\frac{\psi}{\Phi}\right\|_{L^\infty((s,-T)\times\mathbb{R})}\frac{1}{|t|}\int_{\mathbb{R}}\Phi(t,x+\xi_j(t))\,\mathrm{d}x \\ &\quad \leqslant C\left\|\frac{\psi}{\Phi}\right\|_{L^\infty((s,-T)\times\mathbb{R})}\left(\frac{1}{|t|}\right)^{1+\sigma/2\sqrt{2}}.\end{aligned} \tag{3.33}$$

In the last inequality we have used the fact that, if $i>j$, then

$$|w(x+\xi_j-\xi_i)+1|w'(x) \leqslant C\frac{1}{\mathrm{e}^{\sqrt{2}(\xi_j(t)-\xi_i(t))}}.$$

Similarly, we have that

$$\left.\begin{aligned}\left|\xi_j'(t)\int_{\mathbb{R}}\psi(t,x)w''(x-\xi_j(t))\,\mathrm{d}x\right| &\leqslant C\left\|\frac{\psi}{\Phi}\right\|_{L^\infty((s,-T)\times\mathbb{R})}\left(\frac{1}{|t|}\right)^{1+\sigma/2\sqrt{2}}, \\ \left|\int_{\mathbb{R}}h(t,x)w'(x-\xi_j(t))\,\mathrm{d}x\right| &\leqslant C\left\|\frac{h}{\Phi}\right\|_{L^\infty((s,-T)\times\mathbb{R})}\left(\frac{1}{|t|}\right)^{\sigma/\sqrt{2}}.\end{aligned}\right\} \tag{3.34}$$

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Thus, by the above inequalities we have

$$|c_i(t)| \leqslant C\left\|\frac{\psi}{\Phi}\right\|_{L^\infty((s,-T)\times\mathbb{R})}\left(\frac{1}{|t|}\right)^{1+\sigma/2\sqrt{2}} \quad \text{for all } i=1,\dots,k.$$

Now if $\frac{1}{2}\xi_{i-1}^0(t)+\xi_i^0(t) \leqslant x \leqslant \frac{1}{2}\xi_{i+1}^0(t)+\xi_i^0(t)$, we have

$$\begin{aligned}
&\left|\frac{c_i(t)w'(x-\xi_i(t))}{\Phi(t,x)}\right| \\
&\quad \leqslant C|t|^{\sigma/\sqrt{2}}\left(\left(\frac{1}{|t|}\right)^{1+\sigma/2\sqrt{2}}\left\|\frac{\psi}{\Phi}\right\|_{L^\infty((s,-T)\times\mathbb{R})}+\left(\frac{1}{|t|}\right)^{\sigma/\sqrt{2}}\left\|\frac{h}{\Phi}\right\|_{L^\infty((s,-T)\times\mathbb{R})}\right) \\
&\quad \leqslant C\left(\left(\frac{1}{|t|}\right)^{1-\sigma/2\sqrt{2}}\left\|\frac{\psi}{\Phi}\right\|_{L^\infty((s,-T)\times\mathbb{R})}+\left\|\frac{h}{\Phi}\right\|_{L^\infty((s,-T)\times\mathbb{R})}\right). \qquad (3.35)
\end{aligned}$$

Combining all above the proof of lemma is complete. □

Proof of lemma 3.6. First, we recall that

$$\begin{aligned}
\sum_{i=1}^k c_i(t)\int_{\mathbb{R}} w'(x-\xi_i(t))w'(x-\xi_j(t))\,\mathrm{d}x
&= \int_{\mathbb{R}} \psi_{xx}(t,x)w'(x-\xi_j(t))+f'(z(t,x)\psi(t,x))\psi\,\mathrm{d}x \\
&\quad -\xi_j'(t)\int_{\mathbb{R}}\psi(t,x)w''(x-\xi_j(t))\,\mathrm{d}x \\
&\quad +\int_{\mathbb{R}} h(t,x)w'(x-\xi_j(t))\,\mathrm{d}x \quad \text{for all } i=1,\dots,k, \quad t<-T. \qquad (3.36)
\end{aligned}$$

We shall prove that there exists a unique solution of problem (3.29) by using a fixed-point argument.

Let

$$X^s = \{\psi\colon \|\psi\|_{\mathcal{C}_\Phi((s,s+1)\times\mathbb{R})}<\infty\}.$$

We consider the operator $A^s\colon X^s\to X^s$ given by

$$A^s(\psi)=T^s(h-C(\psi)),$$

where $T^s(g)$ denotes the solution to (3.5) and $C(\psi)=\sum_{j=1}^k c_i(t)w'(x-\xi_j(t))$. Also by standard parabolic estimates we have

$$\|A^s(\psi)\|_{\mathcal{C}_\Phi((s,s+1)\times\mathbb{R})} \leqslant C_0(\|h-C(\psi)\|_{\mathcal{C}_\Phi((s,s+1)\times\mathbb{R})}) \qquad (3.37)$$

for some uniform constant $C_0>0$. We shall show that the map A^s defines a contraction mapping, and we shall apply the fixed-point theorem to it. Towards this end, set $c=C_0\|h\|_{\mathcal{C}_\Phi((s,-T)\times\mathbb{R})}$ and

$$X_c^s=\{\psi\colon \|\psi\|_{C_\Phi((s,s+1)\times\mathbb{R})}<2c\},$$

where constant C_0 is taken from (3.37) for $C(T,s)=C(s+1,s)$. We note here that, by standard parabolic theory, $C(T,s)=C_0|(-T-s)|$.

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We claim that $A^s(X_c^s) \subset X_c^s$. Indeed, by inequality (3.37) we have

$$\begin{aligned}\|A^s(\psi)\|_{\mathcal{C}_\Phi((s,s+1)\times\mathbb{R})} &\leqslant C_0(\|h-C(\psi)\|_{\mathcal{C}_\Phi((s,s+1)\times\mathbb{R})})\\ &\leqslant C_0(\|h\|_{\mathcal{C}_\Phi((s,-T)\times\mathbb{R})}+\|C(\psi)\|_{\mathcal{C}_\Phi((s,s+1)\times\mathbb{R})})\\ &\leqslant \frac{C_0}{\sqrt{|s+1|}}(\|\psi\|_{\mathcal{C}_\Phi((s,s+1)\times\mathbb{R})})+c\\ &\leqslant c+c,\end{aligned}$$

where in the above inequalities we have used lemma 3.7 and we have chosen $|s|$ to be sufficiently large. Next, we show that A^s defines a contraction map. Indeed, since $C(\psi)$ is linear in ψ we have

$$\begin{aligned}&\|A^s(\psi_1)-A^s(\psi_2)\|_{\mathcal{C}_\Phi((s,s+1)\times\mathbb{R})}\\ &\qquad\leqslant \|C(\psi_1)-C(\psi_2)\|_{\mathcal{C}_\Phi((s,s+1)\times\mathbb{R})}=\|C(\psi_1-\psi_2)\|_{\mathcal{C}_\Phi((s,s+1)\times\mathbb{R})}\\ &\qquad\leqslant \frac{C}{\sqrt{|s+1|}}\|(\psi_1-\psi_2)\|_{\mathcal{C}_\Phi((s,s+1)\times\mathbb{R})}\\ &\qquad\leqslant \tfrac{1}{2}\|(\psi_1-\psi_2)\|_{\mathcal{C}_\Phi((s,s+1)\times\mathbb{R})}.\end{aligned}$$

Combining all the above, by the fixed-point theorem we have that there exists a $\psi^s \in X^s$ such that $A^s(\psi^s) = \psi^s$, meaning that (3.29) has a solution ψ^s for $-T = s+1$.

We claim that $\psi^s(t,x)$ can be extended to a solution on $(s,-T_0]\times\mathbb{R}$ while still satisfying the orthogonality condition (2.5) and the *a priori* estimate. Towards this end, assume that our solution $\psi^s(t,\cdot)$ exists for $s \leqslant t \leqslant -T$, where $T > T_0$ is the maximal time of the existence. Since ψ^s satisfies the orthogonality condition (2.5), by (3.7) we have

$$\|\psi^s\|_{\mathcal{C}_\Phi((s,-T)\times\mathbb{R})} \leqslant C\|h-C(\psi)\|_{\mathcal{C}_\Phi((s,-T)\times\mathbb{R})}.$$

Thus, if we choose T_0 sufficiently large, we have by lemma 3.7 that

$$\|\psi^s\|_{\mathcal{C}_\Phi((s,-T)\times\mathbb{R})} \leqslant C\|h\|_{\mathcal{C}_\Phi((s,-T)\times\mathbb{R})} \leqslant C\|h\|_{\mathcal{C}_\Phi((s,-T_0)\times\mathbb{R})}.$$

It follows that ψ^s can be extended past time $-T$, unless $T = T_0$. Moreover, (3.31) is satisfied as well and ψ^s also satisfies the orthogonality condition. □

Proof of proposition 3.1. Take a sequence $s_j \to -\infty$ and $\psi_j = \psi^{s_j}$, where ψ^{s_j} is the function (3.29) with $s = s_j$. Then, by (3.7), we can find a subsequence $\{\psi_j\}$ and ψ such that $\psi_j \to \psi$ locally uniformly in $(-\infty,-T_0)\times\mathbb{R}$.

Using (3.7) and standard parabolic theory, we have that ψ is a solution of (3.29) and satisfies (3.4). This concludes the proof. □

4. The nonlinear problem

Going back to the nonlinear problem, function ψ is a solution of (2.4) if and only if $\psi \in C_\Phi((-\infty,-T_0)\times\mathbb{R})$ solves the fixed-point problem

$$\psi = B(\psi), \tag{4.1}$$

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where

$$B(\psi) := A(E(\psi))$$

and A is the operator in proposition 3.1.

Let $T_0 > 1$, we define

$$\varLambda = \Big\{ h \in C^1(-\infty, -T_0] \colon \sup_{t \leqslant -T_0} |h(t)| + \sup_{t \leqslant -T_0} |t||h'(t)| < 1 \Big\}$$

and

$$\|h\|_{\varLambda} = \sup_{t \leqslant -T_0} (|h(t)|) + \sup_{t \leqslant -T_0} (|t||h'(t)|).$$

The main goal in this section is to prove the following proposition.

PROPOSITION 4.1. *Let $\sigma < \sqrt{2}$ and $\nu = (\sqrt{2} - \sigma)/2\sqrt{2}$. There exists a number $T_0 > 0$ depending only on σ such that, for any given functions h in $\varLambda$, there is a solution $\psi = \varPsi(h)$ of (4.1) with respect to $\xi = \xi^0 + h$. The solution ψ satisfies the orthogonality conditions (2.9) and (2.10). Moreover, the following estimate holds:*

$$\|\varPsi(h_1) - \varPsi(h_2)\|_{\mathcal{C}_{\varPhi}((-\infty,-T_0)\times\mathbb{R})} \leqslant \frac{C}{T_0^{\nu}} \|h_1 - h_2\|_{\varLambda}, \tag{4.2}$$

where C is a universal constant.

In order to prove proposition 4.1 we first need to prove some lemmas.

Set

$$X_{T_0} = \Big\{ \psi \colon \|\psi\|_{\mathcal{C}_{\varPhi}((-\infty,-T_0)\times\mathbb{R})} < 2\frac{C_0}{T_0^{\nu}} \Big\}$$

for some fixed constant, C_0.

We denote by $N(\psi, h)$ the function $N(\psi)$ in (3.2) with respect to ψ and $\xi = \xi^0 + h$. In addition, we denote by z_i the corresponding function in (2.3) with respect to $\xi = \xi_i = \xi^0 + h_i$, $i = 1, 2$.

LEMMA 4.2. *Let $h_1, h_2 \in \varLambda$ and $\psi_1, \psi_2 \in X_{T_0}$. Then there exists a constant $C = C(C_0)$ such that*

$$\begin{aligned}\|N(\psi_1, h_1) - N(\psi_2, h_2)\|&_{\mathcal{C}_{\varPhi}((-\infty,-T_0)\times\mathbb{R})} \\ &\leqslant \frac{C}{T_0^{\nu}} (\|\psi_1 - \psi_2\|_{\mathcal{C}_{\varPhi}((-\infty,-T_0)\times\mathbb{R})} + \|h_1 - h_2\|_{\varLambda}).\end{aligned}$$

Proof. First, we shall prove that there exists a constant $C > 0$ that depends only on C_0 such that

$$\|N(\psi_1, h_1) - N(\psi_2, h_1)\|_{\mathcal{C}_{\varPhi}((-\infty,-T_0)\times\mathbb{R})} \leqslant \frac{C}{T_0^{\nu}} \|\psi_1 - \psi_2\|_{\mathcal{C}_{\varPhi}((-\infty,-T_0)\times\mathbb{R})}. \tag{4.3}$$

By straightforward calculation we can easily show that

$$|N(\psi_1, h_1) - N(\psi_2, h_1)| \leqslant \frac{C}{T_0^{\nu}} |\psi_1 - \psi_2| (\varPhi + \varPhi^2),$$

where the constant C depends on C_0 and the proof of (4.3) follows.

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Now we shall prove that

$$\|N(\psi_2,h_1)-N(\psi_2,h_2)\|_{\mathcal{C}_\Phi((-\infty,-T_0)\times\mathbb{R})}\leqslant C\|h_1-h_2\|_\Lambda, \tag{4.4}$$

where the constant C depends on C_0.

By straightforward calculations we have

$$\begin{aligned}|N(\psi_2,h_1)-N(\psi_2,h_2)| &= |-(z_1+\psi_2)^3+z_1^3+3z_1^2\psi_2+(z_2+\psi_2)^3-z_2^3|-3z_2^2\psi_2\\ &\leqslant \frac{C}{T_0^\nu}|h_1-h_2|\Phi^2, \end{aligned}\tag{4.5}$$

which implies (4.4). By using (4.3) and (4.4), the result follows. □

We denote by $E(\psi,h)$ the function $N(\psi)$ in (3.2) with respect to ψ and $\xi=\xi^0+h$.

Lemma 4.3. *Let $h_1,h_2\in\Lambda$. Then there exists constant $C=C(C_0)$ such that*

$$\|E(h_1)-E(h_2)\|_{\mathcal{C}_\Phi((-\infty,-T_0)\times\mathbb{R})}\leqslant \frac{C}{T_0^\nu}\|h_1-h_2\|_\Lambda. \tag{4.6}$$

Proof. Set $\xi=\xi^0+h_1$, $\zeta=\xi^0+h_2$. Let

$$\tfrac12(\xi_j^0(t)+\xi_{j-1}^0(t))\leqslant x\leqslant \tfrac12(\xi_j^0(t)+\xi_{j+1}^0(t)),\quad j=1,\dots,k,$$

with $\xi_0^0=-\infty$ and $\xi_{k+1}^0=\infty$. Note here that there exists $\mu\in[-1,1]$ such that

$$\begin{aligned}|w(x-\xi_{j-1}(t))-w(x-\zeta_{j-1}(t))| &\leqslant C|h_1-h_2\|w'(x-\xi_{j-1}^0(t)+\mu)|\\ &\leqslant C|h_1-h_2\|w'(x-\xi_{j-1}^0(t))|.\end{aligned}$$

Thus, in view of the proof of lemma 2.2 and the above inequality we have

$$\begin{aligned}&\left|f(z_1(t,x))-\sum_{j=1}^k(-1)^{j+1}f(w(x-\xi_j))-f(z_2(t,x))+\sum_{j=1}^k(-1)^{j+1}f(w(x-\zeta_j))\right|\\ &\qquad\leqslant C|h_1-h_2|\,|w'(x-\xi_{j-1}^0(t))|.\end{aligned}$$

Also, we can easily show that

$$\left|\sum_{j=1}^k(-1)^{j+1}w'(x-\xi_j(t))\xi_j'(t)-\sum_{j=1}^k(-1)^{j+1}w'(x-\zeta_j(t))\zeta'(t)\right|\leqslant\frac{C}{t}\|h_1-h_2\|_\Lambda.$$

But, for any

$$\tfrac12(\xi_j^0(t)+\xi_{j-1}^0(t))\leqslant x\leqslant \tfrac12(\xi_j^0(t)+\xi_{j+1}^0(t)),\quad j=1,\dots,k,$$

we have

$$\frac{1}{\Phi}\leqslant C|t|^{\sigma/\sqrt2}\quad\text{and}\quad \frac{1}{\Phi}|w'(x-\xi_{j-1}^0(t))|\leqslant C|t|^{-\nu}.$$

On combining all of the above, we obtain the desired result. □

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LEMMA 4.4. *Let $h_1, h_2 \in \Lambda$, $\psi_1, \psi_2, \psi \in X$. Also let $C(\psi, h, t) = (c_1(t), \dots, c_k(t))$ satisfy*

$$\begin{aligned}\sum_{i=1}^{k} c_i(t) \int_{\mathbb{R}} w'(x-\xi_i(t)) w'(x-\xi_j(t))\,\mathrm{d}x &= \int_{\mathbb{R}} (-f'(w(x-\xi_j)) + f'(z(t,x)))\psi(t,x) w'(x-\xi_j(t))\,\mathrm{d}x \\ &\quad - \xi_j'(t) \int_{\mathbb{R}} \psi(t,x) w''(x-\xi_j(t))\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}} (E(h) + N(\psi,h)) w'(x-\xi_j(t))\,\mathrm{d}x \quad \textit{for all } j = 1,\dots,k, \quad t < -T,\end{aligned}$$

with respect to ψ and $\xi = \xi^0 + h$. Then

$$\begin{aligned}&|C(\psi_1, h_1, t) - C(\psi_2, h_2, t)| \\ &\quad \leqslant \frac{C}{|t|^{1+\sigma/2\sqrt{2}}} \|\psi_1 - \psi_2\|_{\mathcal{C}_\Phi((-\infty,-T_0)\times\mathbb{R})} + \frac{C}{|t|^{\nu+\sigma/\sqrt{2}}} \|h_1 - h_2\|_\Lambda \end{aligned} \tag{4.7}$$

for some positive constant C_0 that depends only on C_0.

Proof. We omit the proof here, as we can make very similar calculations to those in lemmas 3.7, 4.2 and 4.3. □

Proof of proposition 4.1. (a) We consider the operator $B\colon \mathcal{C}_\Phi((-\infty,-T_0)\times\mathbb{R}) \to \mathcal{C}_\Phi((-\infty,-T_0)\times\mathbb{R})$, where $B(\psi)$ denotes the solution to (4.1). We shall show that the map B defines a contraction mapping and shall apply the fixed-point theorem to it. First, we note by lemma 2.2 and proposition 3.1 that

$$\|B(0)\|_{\mathcal{C}_\Phi((-\infty,-T_0)\times\mathbb{R})} \leqslant \frac{C_0}{T_0^\nu}.$$

By proposition 3.1 and lemma 3.6, we obtain

$$\|B(\psi_1) - B(\psi_2)\|_{\mathcal{C}_\Phi((-\infty,-T_0)\times\mathbb{R})} \leqslant \frac{C}{T_0^\nu} (\|\psi_1 - \psi_2\|_{\mathcal{C}_\Phi((-\infty,-T_0)\times\mathbb{R})}),$$

providing that

$$\|\psi_i\|_{\mathcal{C}_\Phi((-\infty,-T_0)\times\mathbb{R})} \leqslant 2\frac{C_0}{T_0^\nu}.$$

Thus, if we choose T_0 sufficiently large we can apply the fixed-point theorem in

$$X_{T_0} = \left\{ \psi\colon \|\psi\|_{\mathcal{C}_\Phi((-\infty,-T_0)\times\mathbb{R})} < 2\frac{C_0}{T_0^\nu} \right\},$$

to obtain that there exists ψ such that $B(\psi) = \psi$.

(b) For simplicity we set $\psi^1 = \Psi(h_1)$ and $\psi^2 = \Psi(h_2)$. The estimate will be obtained by applying the estimate (3.7). However, because each ψ^i satisfies the orthogonality conditions (2.5) with $\xi(t) = \xi^i(t) := \xi^0(t) + h_i(t)$, the difference $\psi^1 - \psi^2$ does not

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satisfy an exact orthogonality condition. To overcome this technical difficulty, we shall consider instead the difference $Y := \psi^1 - \bar{\psi}^2$, where

$$\bar{\psi}^2 = \psi^2 - \sum_{i=1}^{k} \lambda_i(t) w'(x - \xi_i^1),$$

with

$$\sum_{i=1}^{k} \lambda_i(t) \int_{\mathbb{R}} w'(x - \xi_i^1(t)) w'(x - \xi_j^1(t))\,\mathrm{d}x = \int_{\mathbb{R}} \psi^2(t,x) w'(x - \xi_j^1(t))\,\mathrm{d}x, \qquad j = 1, \dots, k.$$

Clearly, Y satisfies the orthogonality conditions (2.5) with $\xi(t) = \xi^1(t)$. Denote by L_t^i the operator

$$L_t^i \psi^i = \psi_t^i - \psi_{xx}^i + f'(z^i(t,x))\psi^i.$$

By lemmas 4.2–4.4 and the fact that

$$\frac{w'(x - \xi_i^1)}{\Phi} \leqslant C|t|^{\sigma/\sqrt{2}} \quad \text{for all } i = 1, \dots, k,$$

we can easily prove

$$\|Y\|_{\mathcal{C}_\Phi((-\infty,-T_0)\times\mathbb{R})} \leqslant \frac{C}{T_0^\nu}(\|\psi_1 - \psi_2\|_{\mathcal{C}_\Phi((-\infty,-T_0)\times\mathbb{R})} + \|h_1 - h_2\|_\Lambda) + C\left(\sum_{i=1}^{k} \sup_{t\in(-\infty,-T_0)} |t|^{\sigma/\sqrt{2}} |\lambda_i(t)|\right). \quad (4.8)$$

Then, by the orthogonality conditions (2.5) and (3.32), we have

$$\left|\int_{\mathbb{R}} \psi^2(t,x) w'(x - \xi_j^1(t))\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}} \psi^2(t,x)(w'(x - \xi_j^1(t)) - w'(x - \xi_j^2(t)))\,\mathrm{d}x\right| \leqslant \frac{C}{T_0^\nu}|t|^{-\sigma/\sqrt{2}}\|h_1 - h_2\|_\Lambda. \quad (4.9)$$

Now,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \psi^2(t,x) w'(x - \xi_j^1(t))\,\mathrm{d}x\right| = \left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \psi^2(t,x)(w'(x - \xi_j^1(t)) - w'(x - \xi_j^2(t)))\,\mathrm{d}x\right|. \quad (4.10)$$

But,

$$\int_{\mathbb{R}} \psi_t^2(t,x)(w'(x - \xi_j^1(t)) - w'(x - \xi_j^2(t)))\,\mathrm{d}x = -\int_{\mathbb{R}} \psi_{xx}^2(t,x)(w'(x - \xi_j^1(t)) - w'(x - \xi_j^2(t)))\,\mathrm{d}x + \int_{\mathbb{R}} L_t^2\psi^2(w'(x - \xi_j^1(t)) - w'(x - \xi_j^2(t)))\,\mathrm{d}x$$

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$$
\begin{aligned}
&\quad - \int_{\mathbb{R}} f'(z^2(t,x))\psi^2(t,x)(w'(x-\xi_j^1(t)) - w'(x-\xi_j^2(t)))\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \psi^2(t,x)(w'''(x-\xi_j^1(t)) - w'''(x-\xi_j^2(t)))\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}} L_t^2\psi^2(w'(x-\xi_j^1(t)) - w'(x-\xi_j^2(t)))\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}} f'(z^2(t,x))\psi^2(t,x)(w'(x-\xi_j^1(t)) - w'(x-\xi_j^2(t)))\,\mathrm{d}x.
\end{aligned}
$$

By the fixed-point argument in (a) we have that

$$
\left| \int_{\mathbb{R}} \psi_t^2(t,x)(w'(x-\xi_j^1(t)) - w'(x-\xi_j^2(t)))\,\mathrm{d}x \right| \leqslant \frac{C}{T_0^\nu}|t|^{-\sigma/\sqrt{2}}\|h_1-h_2\|_\Lambda. \quad (4.11)
$$

By (4.9)–(4.11) and definitions of λ_i we have that

$$
|\lambda_i(t)| + |\lambda_i'(t)| \leqslant \frac{C}{T_0^\nu}|t|^{-\sigma/\sqrt{2}}\|h_1-h_2\|_\Lambda.
$$

Combining all the above we have that

$$
\|Y\|_{\mathcal{C}_\Phi^0((-\infty,-T_0)\times\mathbb{R})} \leqslant \frac{C}{T_0^\nu}\|\psi_1-\psi_2\|_{\mathcal{C}_\Phi((-\infty,-T_0)\times\mathbb{R})} + C\|h_1-h_2\|_\Lambda.
$$

However,

$$
\begin{aligned}
&\|\psi_1-\psi_2\|_{\mathcal{C}_\Phi((-\infty,-T_0)\times\mathbb{R})} \\
&\qquad \leqslant \|Y\|_{\mathcal{C}_\Phi((-\infty,-T_0)\times\mathbb{R})} + C\biggl(\sum_{i=1}^k \sup_{t\in(-\infty,-T_0)} |t|^{\sigma/\sqrt{2}}|\lambda_i(t)|\biggr) \\
&\qquad \leqslant \frac{C}{T_0^\nu}\|\psi_1-\psi_2\|_{\mathcal{C}_\Phi((-\infty,-T_0)\times\mathbb{R})} + \frac{C}{T_0^\nu}\|h_1-h_2\|_\Lambda,
\end{aligned}
$$

and the proof of inequality (4.2) follows if we choose T_0 to be sufficiently large. □

5. The choice of $\boldsymbol{\xi_i}$

Let T_0 be sufficiently large, let $\frac{\sqrt{2}}{2} < \sigma < \sqrt{2}$ and let $\psi \in \mathcal{C}_\Phi((-\infty,-T_0)\times\mathbb{R})$ be the solution of problem (2.4). We want to find ξ_i such that

$$
\begin{aligned}
0 &= \int_{\mathbb{R}} (-f'(w(x-\xi_j(t))) + f'(z(t,x)))\psi(t,x)w'(x-\xi_j(t))\,\mathrm{d}x \\
&\quad - \xi_j'(t)\int_{\mathbb{R}} \psi(t,x)w''(x-\xi_j(t))\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}} (E+N(\psi))w'(x-\xi_j(t))\,\mathrm{d}x \quad \text{for all } j=1,\dots,k,\ t<-T,
\end{aligned}
$$

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where

$$E = \sum_{j=1}^{k} (-1)^{j+1} w(x - \xi_j(t))\xi_j' + f(z(t,x)) - \sum_{j=1}^{k} (-1)^{j+1} f(w(x - \xi_j(t))),$$
$$N(\psi) = f(\psi(t,x) + z(t,x)) - f(z(t,x)) - f'(z(t,x))\psi.$$

First, we study the error term E. Let $1 < j < k$. Then we have that

$$\int_{\mathbb{R}} \left(f(z(t,x)) - \sum_{i=1}^{k} (-1)^{i+1} f(w(x - \xi_i(t))) \right) w'(x - \xi_j(t))\,\mathrm{d}x$$
$$= \int_{\mathbb{R}} \left(f(z(t, x + \xi_j(t))) - \sum_{i=1}^{k} (-1)^{i+1} f(w(x + \xi_j(t) - \xi_i(t))) \right) w'(x)\,\mathrm{d}x.$$

For simplicity we assume that i is even. Set

$$g = \sum_{i=1}^{j-2} (-1)^{i+1} (w(x + \xi_j(t) - \xi_i(t)) - 1) + \sum_{i=j+2}^{k} (-1)^{i+1} (w(x + \xi_j(t) - \xi_i(t)) + 1),$$
$$g_1 = w(x + \xi_j - \xi_{j-1}) - 1,$$
$$g_2 = w(x + \xi_j - \xi_{j+1}) + 1.$$

Using the fact that $\int_{\mathbb{R}} f(w(x))w'(x)\,\mathrm{d}x = 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} f(z(t, x - \xi_j(t))w'(x))\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}} (g + g_1 - w(x) + g_2)(1 - (g + g_1 + g_2 - w(x))^2)w'(x)\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}} (g_1 + g_2 - 3w^2(x)g_1 - 3w^2(x)g_2 + 3w(x)g_1^2 + 3w(x)g_2^2 - g_1^3 - g_2^3)w'(x)\,\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}} F_0(t,x)w'(x)\,\mathrm{d}x,
\end{aligned} \tag{5.1}$$

where

$$F_0(t,x) = O(g) + O(g_1 g_2).$$

We note that

$$\int_{\mathbb{R}} |g| w'(x)\,\mathrm{d}x \leqslant C \sum_{\substack{i=1,\\ i \neq j-1, j, j+1}}^{k} \exp(-\sigma|\xi_i(t) - \xi_j(t)|),$$
$$\int_{\mathbb{R}} |g_1 g_2| w'(x)\,\mathrm{d}x \leqslant C \exp(-\sqrt{2}|\xi_{j+1}(t) - \xi_{j-1}(t)|).$$

Let

$$F_1(t,x) = \sum_{\substack{i=1,\\ i \neq j-1, j, j+1}}^{k} f(w(x + \xi_i(t) - \xi_j(t))).$$

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Then

$$\begin{aligned}\int_{\mathbb{R}}\bigg(\sum_{i=1}^{k}(-1)^{j+1}f(w(x+\xi_j(t)-\xi_i(t)))\bigg)w'(x)\,\mathrm{d}x\\ &= \int_{\mathbb{R}}(f(g_1+1)+f(g_2-1))w'(x)\,\mathrm{d}x+\int_{\mathbb{R}}F_1(t,x)w'(x)\,\mathrm{d}x\\ &= \int_{\mathbb{R}}(-2g_1-3g_1^2-g_1^3-2g_2+3g_2^2-g_2^3)w'(x)\,\mathrm{d}x+\int_{\mathbb{R}}F_1(t,x)w'(x)\,\mathrm{d}x.\end{aligned} \tag{5.2}$$

In addition, we have that

$$\int_{\mathbb{R}}|F_1(t,x)|w'(x)\,\mathrm{d}x \leqslant C\sum_{\substack{i=1,\\ i\neq j-1,j,j+1}}^{k}\exp(-\sigma|\xi_i(t)-\xi_j(t)|).$$

By (5.1), (5.2) we have

$$\begin{aligned}&\int_{\mathbb{R}}\bigg(f(z(t,x-\xi_j(t)))-\sum_{i=1}^{k}(-1)^{i+1}f(w(x+\xi_j(t)-\xi_i(t)))\bigg)w'(x)\,\mathrm{d}x\\ &\quad = 3\int_{\mathbb{R}}(g_1+g_2)(1-w^2(x))w'(x)\,\mathrm{d}x+3\int_{\mathbb{R}}g_1^2(1+w(x))w'(x)\,\mathrm{d}x\\ &\qquad +3\int_{\mathbb{R}}g_2^2(w(x)-1)w'(x)\,\mathrm{d}x+\int_{\mathbb{R}}F_0(t,x)w'(x)\,\mathrm{d}x-\int_{\mathbb{R}}F_1(t,x)w'(x)\,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}}g_1(1-w^2(x))w'(x)\,\mathrm{d}x\\ &\quad=\int_{\mathbb{R}}\frac{-2\mathrm{e}^{-\sqrt{2}(x+\xi_j-\xi_{j-1})/2}}{\mathrm{e}^{(\sqrt{2}/2)(x+\xi_j-\xi_{j-1})}+\mathrm{e}^{-\sqrt{2}(x+\xi_j-\xi_{j-1})/2}}(1-w^2(x))w'(x)\,\mathrm{d}x\\ &\quad=-2\mathrm{e}^{-\sqrt{2}(\xi_j-\xi_{j-1})}\int_{\mathbb{R}}\frac{1}{\mathrm{e}^{\sqrt{2}x}+\mathrm{e}^{-\sqrt{2}(\xi_j-\xi_{j-1})}}(1-w^2(x))w'(x)\,\mathrm{d}x\\ &\quad=-2\mathrm{e}^{-\sqrt{2}(\xi_j-\xi_{j-1})}\int_{\mathbb{R}}\mathrm{e}^{-\sqrt{2}x}(1-w^2(x))w'(x)\,\mathrm{d}x\\ &\qquad-2\mathrm{e}^{-\sqrt{2}(\xi_j-\xi_{j-1})}\int_{\mathbb{R}}\bigg(\frac{1}{\mathrm{e}^{\sqrt{2}x}+\mathrm{e}^{-\sqrt{2}(\xi_j-\xi_{j-1})}}-\mathrm{e}^{-\sqrt{2}x}\bigg)(1-w^2(x))w'(x)\,\mathrm{d}x\\ &\quad=-2\mathrm{e}^{-\sqrt{2}(\xi_j-\xi_{j-1})}\bigg(\int_{\mathbb{R}}\mathrm{e}^{-\sqrt{2}x}(1-w^2(x))w'(x)\,\mathrm{d}x\\ &\qquad\qquad+\int_{\mathbb{R}}F_2(t,x)(1-w^2(x))w'(x)\,\mathrm{d}x\bigg).\end{aligned}$$

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Now,

$$\begin{aligned}&\left|\int_{\mathbb{R}} F_2(t,x)(1-w^2(x))w'(x)\,\mathrm{d}x\right|\\&\qquad = \mathrm{e}^{-\sqrt{2}(\xi_j-\xi_{j-1})}\int_{\mathbb{R}} \frac{1}{\mathrm{e}^{\sqrt{2}x}(\mathrm{e}^{\sqrt{2}x}+\mathrm{e}^{-\sqrt{2}(\xi_j-\xi_{j-1})})}(1-w^2(x))w'(x)\,\mathrm{d}x\\&\qquad \leqslant C(\xi_j-\xi_{j-1})\mathrm{e}^{-\sqrt{2}(\xi_j-\xi_{j-1})}.\end{aligned}$$

Similarly, for g_2 we have

$$\begin{aligned}&\int_{\mathbb{R}} g_2(1-w^2(x))w'(x)\,\mathrm{d}x\\&\quad = 2\mathrm{e}^{-\sqrt{2}(\xi_j-\xi_{j-1})}\\&\qquad \times\left(\int_{\mathbb{R}} \mathrm{e}^{-\sqrt{2}x}(1-w^2(x))w'(x)\,\mathrm{d}x+\int_{\mathbb{R}} F_3(t,x)(1-w^2(x))w'(x)\,\mathrm{d}x\right),\end{aligned}$$

where

$$\left|\int_{\mathbb{R}} F_3(t,x)(1-w^2(x))w'(x)\,\mathrm{d}x\right| \leqslant C(\xi_{j+1}-\xi_j)\mathrm{e}^{-\sqrt{2}(\xi_{j+1}-\xi_j)}.$$

Now,

$$\begin{aligned}&\int_{\mathbb{R}} g_1^2(1+w(x))w'(x)\,\mathrm{d}x\\&\qquad\qquad \leqslant C\mathrm{e}^{-2\sqrt{2}(\xi_j-\xi_{j-1})}\int_{\mathbb{R}} \frac{1}{\mathrm{e}^{2\sqrt{2}x}+\mathrm{e}^{-2\sqrt{2}(\xi_j-\xi_{j-1})}}(1+w(x))w'(x)\,\mathrm{d}x.\end{aligned}$$

But,

$$\begin{aligned}&\int_{\mathbb{R}} \frac{1}{\mathrm{e}^{2\sqrt{2}x}+\mathrm{e}^{-2\sqrt{2}(\xi_j-\xi_{j-1})}}(1+w(x))w'(x)\,\mathrm{d}x\\&\qquad = \int_{-\infty}^{-\xi_j-\xi_{j-1}} \frac{1}{\mathrm{e}^{2\sqrt{2}x}+\mathrm{e}^{-2\sqrt{2}(\xi_j-\xi_{j-1})}}(1+w(x))w'(x)\,\mathrm{d}x\\&\qquad\quad + \int_{-\xi_j-\xi_{j-1}}^{0} \frac{1}{\mathrm{e}^{2\sqrt{2}x}+\mathrm{e}^{-2\sqrt{2}(\xi_j-\xi_{j-1})}}(1+w(x))w'(x)\,\mathrm{d}x\\&\qquad\quad + \int_{0}^{\infty} \frac{1}{\mathrm{e}^{2\sqrt{2}x}+\mathrm{e}^{-2\sqrt{2}(\xi_j-\xi_{j-1})}}(1+w(x))w'(x)\,\mathrm{d}x\\&\qquad \leqslant C((\xi_j-\xi_{j-1})+1).\end{aligned}$$

Thus, we have

$$\int_{\mathbb{R}} g_1^2(1+w(x))w'(x)\,\mathrm{d}x \leqslant C(\xi_j-\xi_{j-1})\mathrm{e}^{-2\sqrt{2}(\xi_j-\xi_{j-1})}.$$

Similarly,

$$\int_{\mathbb{R}} g_2^2(1-w(x))w'(x)\,\mathrm{d}x \leqslant C(\xi_{j+1}-\xi_j)\mathrm{e}^{-2\sqrt{2}(\xi_{j+1}-\xi_j)}.$$

By assumptions on ψ we have

$$
\begin{aligned}
\int_{\mathbb{R}} &|N(\psi)| w'(x-\xi_j(t))\,\mathrm{d}x \\
&\leqslant C\int_{\mathbb{R}} \Phi^2(t,x) w'(x-\xi_j(t))\,\mathrm{d}x = C\int_{\mathbb{R}} \Phi^2(t,x+\xi_j(t)) w'(x)\,\mathrm{d}x \\
&\leqslant C\sum_{i=1}^{k}\int_{(\xi_i^0(t)+\xi_{i-1}^0(t))/2-\xi_j(t)}^{(\xi_i^0(t)+\xi_{i+1}^0(t))/2-\xi_j(t)} (\exp(2\sigma(-x-\xi_j(t)+\xi_{i-1}^0)) \\
&\qquad\qquad + \exp(2\sigma(x+\xi_j(t)-\xi_{i+1}^0))) w'(x)\,\mathrm{d}x.
\end{aligned}
$$

Now, note that

$$
\int_{(\xi_j^0(t)+\xi_{j-1}^0(t))/2-\xi_j(t)}^{(\xi_j^0(t)+\xi_{j+1}^0(t))/2-\xi_j(t)} \mathrm{e}^{2\sigma(x+\xi_j(t)-\xi_{j+1}^0)} w'(x)\,\mathrm{d}x \leqslant C\exp(-\sigma-\tfrac{\sqrt{2}}{2})(\xi_{j+1}-\xi_j(t)).
$$

Thus, we can easily prove that

$$
\int_{\mathbb{R}} |N(\psi)| w'(x-\xi_j(t))\,\mathrm{d}x \leqslant C\sum_{i=1,\, i\neq j}^{k} \mathrm{e}^{(-\sigma-\sqrt{2})|\xi_i(t)-\xi_j(t)|}.
$$

Also, we have

$$
\sum_{i=1}^{k}(-1)^{j+1}\xi_i'\int_{\mathbb{R}} w'(x-\xi_i(t))w'(x-\xi_j(t))\,\mathrm{d}x = -\xi_j'(t)\int_{\mathbb{R}} |w'(x)|^2\,\mathrm{d}x + F_4(t),
$$

where

$$
|F_4(t)| \leqslant C\sum_{i=1,\, i\neq j}^{k} |\xi_i'|\mathrm{e}^{-\sigma|\xi_i-\xi_j|}.
$$

Finally,

$$
\begin{aligned}
\left|\xi_j'(t)\int_{\mathbb{R}}\psi(t,x)w''(x-\xi_j(t))\,\mathrm{d}x\right| &\leqslant C|\xi_j'(t)|\int_{\mathbb{R}}\Phi(t,x+\xi_j(t))w''(x)\,\mathrm{d}x \\
&\leqslant C|\xi_j'(t)|(\exp(-\tfrac{1}{2}\sigma-\sqrt{2})|\xi_{j+1}(t)-\xi_j(t)| \\
&\qquad + \exp(-\tfrac{1}{2}\sigma-\sqrt{2})|\xi_{j-1}(t)-\xi_j(t)|).
\end{aligned}
$$

Similarly, for $j=1,\dots,k$, we obtain the respective ordinary differential equation for $\xi=(\xi_1,\dots,\xi_k)$:

$$
\begin{aligned}
\frac{1}{\beta}\xi_j' - \exp(-\sqrt{2}(\xi_{j+1}-\xi_j)) + \exp(-\sqrt{2}(\xi_j-\xi_{j-1})) = F_i(\xi',\xi),& \\
j=1,2,\dots,k,\ t\in(0,-T_0],& \qquad (5.3)
\end{aligned}
$$

with $\xi_{k+1}=\infty$ and $\xi_0=-\infty$.

We recall here our assumption $T_0>1$ and we define

$$
\Lambda = \Big\{h\in C^1(-\infty,-T_0]\colon \sup_{t\leqslant -T_0}|h(t)| + \sup_{t\leqslant -T_0}|t||h'(t)| < 1\Big\}
$$

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and

$$\|h\|_{\Lambda} = \sup_{t \leqslant -T_0} (|h(t)|) + \sup_{t \leqslant -T_0} (|t| |h'(t)|).$$

We set

$$\bar{F}(h', h) = F(\xi', \xi),$$

where $\xi = \xi^0 + h$. Determining the above, and using lemmas 4.9–4.11 and (4.2), we prove the following result.

Proposition 5.1. *Let* $\frac{\sqrt{2}}{2} < \sigma < \sqrt{2}$ *and* $h, h_1, h_2 \in \Lambda$. *Then there exists a constant* $C = C(\sigma)$ *such that*

$$|\bar{F}(h', h)| \leqslant \frac{C}{|t|^{1/2+\sigma/\sqrt{2}}}$$

and

$$|\bar{F}(h'_1, h_2) - \bar{F}(h'_1, h_2)| \leqslant \frac{C}{|t|^{1/2+\sigma/\sqrt{2}}} \|h_1 - h_2\|_{\Lambda}.$$

In the remainder of this section we shall study system (5.3) using some ideas from [6].

5.1. Choice of ξ^0

Let $k \geqslant 4$ be an even number. First, we want to find a solution of the problem

$$\frac{1}{\beta}\xi'_j - \exp(-\sqrt{2}(\xi_{j+1} - \xi_j)) + \exp(-\sqrt{2}(\xi_j - \xi_{j-1})) = 0, \quad j = 1, 2, \ldots, k,\ t \in (0, -T_0], \tag{5.4}$$

with $\xi_{k+1} = \infty$ and $\xi_0 = -\infty$. We set

$$R_l(\xi) := -\exp(-\sqrt{2}(\xi_{j+1} - \xi_j)) + \exp(-\sqrt{2}(\xi_j - \xi_{j-1}))$$

and

$$\boldsymbol{R}(\xi) = \begin{bmatrix} R_1(\xi) \\ \vdots \\ R_k(\xi) \end{bmatrix}.$$

We want to solve the system $\xi' + \beta \boldsymbol{R}(\xi) = 0$. To do so we find first a convenient representation of the operator $\boldsymbol{R}(\xi)$. Let us consider the auxiliary variables

$$\boldsymbol{v} := \begin{bmatrix} \bar{\boldsymbol{v}} \\ v_k \end{bmatrix}, \qquad \bar{\boldsymbol{v}} = \begin{bmatrix} v_1 \\ \vdots \\ v_{k-1} \end{bmatrix},$$

defined in terms of ξ as

$$v_l = \xi_{l+1} - \xi_l \quad \text{with } l = 1, \ldots, k-1, \qquad v_k = \sum_{l=1}^{k} \xi_l,$$

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and define the operators

$$\boldsymbol{S}(\boldsymbol{v}) := \begin{bmatrix} \bar{\boldsymbol{S}}(\bar{\boldsymbol{v}}) \\ 0 \end{bmatrix}, \qquad \bar{\boldsymbol{S}}(\bar{\boldsymbol{v}}) = \begin{bmatrix} S_1(\bar{\boldsymbol{v}}_1) \\ \vdots \\ S_{k-1}(\bar{\boldsymbol{v}}_1) \end{bmatrix},$$

where

$$S_l(\bar{\boldsymbol{v}}) : R_{l+1}(\xi) - R_l(\xi) = \begin{cases} 2\mathrm{e}^{-\sqrt{2}v_1} - \mathrm{e}^{\sqrt{2}v_2} & \text{if } l = 1, \\ -\mathrm{e}^{\sqrt{2}v_{l-1}} + 2\mathrm{e}^{-\sqrt{2}v_l} - \mathrm{e}^{\sqrt{2}v_{l-1}} & \text{if } 2 \leqslant l \leqslant k-2, \\ 2\mathrm{e}^{-\sqrt{2}v_k} - \mathrm{e}^{\sqrt{2}v_{k-1}} & \text{if } l = k-1. \end{cases}$$

Then the operators $\boldsymbol{R}$ and $\boldsymbol{S}$ are correlated through the formula

$$\boldsymbol{S}(\boldsymbol{v}) = \boldsymbol{B}\boldsymbol{R}(\boldsymbol{B}^{-1}\boldsymbol{v}),$$

where $\boldsymbol{B}$ is the constant, invertible $k \times k$ matrix

$$\boldsymbol{B} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 \\ 1 & \cdots & 1 & 1 & 1 \end{bmatrix}.$$

Then, through the relation $\xi = \boldsymbol{B}^{-1}\boldsymbol{v}$, the system $\xi' + \beta\boldsymbol{R}(\xi) = 0$ is equivalent to $\boldsymbol{v}' + \beta\boldsymbol{S}(\boldsymbol{v}) = 0$, which decouples into

$$\bar{v} + \beta\bar{\boldsymbol{S}}(\bar{\boldsymbol{v}}) = 0, \qquad v_k' = 0,$$

where

$$\bar{\boldsymbol{S}}(\bar{\boldsymbol{v}}) = \boldsymbol{C} \begin{bmatrix} \mathrm{e}^{-\sqrt{2}v_1} \\ \vdots \\ \mathrm{e}^{-\sqrt{2}v_{k-1}} \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & -1 & 2 \end{bmatrix}. \tag{5.5}$$

We choose simply $v_k = 0$ and look for a solution $\boldsymbol{v}^0(t) = (\bar{\boldsymbol{v}}(t)^0, 0)$ of the system, where $\bar{\boldsymbol{v}}^0(t)$ has the form

$$\bar{v}_l^0(t) = \tfrac{1}{\sqrt{2}} \log(-2\sqrt{2}\beta t) + b_l \tag{5.6}$$

for constants b_l to be determined.

Substituting this expression into the system, we find the following equations for the numbers b_l:

$$\boldsymbol{C} \begin{bmatrix} \mathrm{e}^{-\sqrt{2}b_1} \\ \vdots \\ \mathrm{e}^{-\sqrt{2}b_{k-1}} \end{bmatrix} = \frac{1}{\beta} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

We compute

$$b_l = -\frac{1}{\sqrt{2}} \log\left(\frac{1}{2\beta}(k-l)l\right), \quad l = 1, \dots, k-1,$$

explicitly. Now, we note that $b_l = b_{k-l}$ for $l = 1, \dots, k-1$. Thus, by (5.4) we have that

$$\xi_{k-j+1} = -\xi_j, \quad j \leqslant \tfrac{1}{2}k,$$

and

$$\xi_j = \frac{1}{\sqrt{2}}\left(j - \frac{k+1}{2}\right) \log(-2\sqrt{2}\beta t) + \gamma_j,$$

where

$$-\gamma_j = \gamma_{k-j+1} = \frac{1}{2}\sum_{i=j}^{k-j} b_i \quad \text{for } j \leqslant \tfrac{1}{2}k.$$

5.2. Solution of problem (5.3)

We keep the notation of the previous subsection, and we write problem (5.3) in the form

$$\xi' + \beta \boldsymbol{R}(\xi) = \boldsymbol{F}(\xi', \xi) \quad \text{in } (-\infty, -T_0].$$

Let $\xi^0 = (\xi_1^0, \dots, \xi_k^0)^{\mathrm{T}}$, where

$$\xi_j^0 = \frac{1}{\sqrt{2}}\left(j - \frac{k+1}{2}\right) \log(-2\sqrt{2}\beta t) + \gamma_j.$$

We look for a solution of the form $\xi = \xi^0 + h$. Thus, h satisfies

$$\begin{aligned} h' + \beta D_\xi \boldsymbol{R}(\xi^0)h &= \boldsymbol{F}(\xi^{0\prime} + h', \xi^0 + h) + \beta D_\xi \boldsymbol{R}(\xi^0)h - \beta \boldsymbol{R}(\xi^0) \\ &= \boldsymbol{E}(h', h) \quad \text{in } (-\infty, -T_0]. \end{aligned} \tag{5.7}$$

By proposition 5.1, we have

$$\left.\begin{aligned} |\boldsymbol{E}(0,0)| &\leqslant C\left(\frac{1}{|t|}\right)^{1/2+\sigma/\sqrt{2}}, \\ |\boldsymbol{E}(h_1', h_1) - \boldsymbol{E}(h_1', h_1)| &\leqslant C\left(\frac{1}{t}\right)^{1/2+\sigma/\sqrt{2}} |h_1 - h_2| + C\left(\frac{1}{t}\right)^{1/2+\sigma/\sqrt{2}} |h_1' - h_2'|. \end{aligned}\right\} \tag{5.8}$$

Also, restricting ourselves to symmetric ξ, h then satisfies the symmetry condition

$$h_{k-j+1} = -h_j, \quad j \leqslant \tfrac{1}{2}k.$$

In addition, this implies that the solution ψ is even with respect to x, and thus we have that

$$E_{k-j+1} = E_j, \quad j \leqslant \tfrac{1}{2}k. \tag{5.9}$$

Set

$$v^0 = \boldsymbol{B}\xi^0 \quad \text{and} \quad p = \boldsymbol{B}h.$$

Then we have that $\boldsymbol{E}(h', h) = \boldsymbol{E}(\boldsymbol{B}^{-1}h', \boldsymbol{B}^{-1}h) = \boldsymbol{E}(p', p)$, and by

$$\boldsymbol{S}(\boldsymbol{v}) = \boldsymbol{B}\boldsymbol{R}(\boldsymbol{B}^{-1}\boldsymbol{v})$$

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we have that

$$\boldsymbol{S}(\boldsymbol{v^0}) = \boldsymbol{B}\boldsymbol{R}(\xi^0)\boldsymbol{B}^{-1}.$$

Thus, (5.7) is equivalent to

$$p' + \beta D_v \boldsymbol{S}(\boldsymbol{v^0})p = \boldsymbol{B}\boldsymbol{E}(p', p) := \boldsymbol{L}(p', p) \quad \text{in } (-\infty, -T_0]. \tag{5.10}$$

By (5.9) we have that $\boldsymbol{L}_k = 0$. Thus, writing $p = (\bar{p}, p_k)$ and $\boldsymbol{L} = (\bar{\boldsymbol{L}}, L_k)$, the latter system decouples as

$$\bar{p}' + \beta D_{\bar{v}} \bar{\boldsymbol{S}}(\bar{\boldsymbol{v}}^0) = \bar{\boldsymbol{L}}(\bar{p}', \bar{p}) \quad \text{in } (-\infty, -T_0], \qquad p_k' = 0, \tag{5.11}$$

where we have simply chosen $p_k = 0$.

Now, by (5.6) we have

$$\begin{aligned} D_{\bar{v}} \bar{\boldsymbol{S}}(\bar{\boldsymbol{v}}^0) &= -\sqrt{2}\boldsymbol{C} \begin{bmatrix} \mathrm{e}^{-\sqrt{2}v_1} & 0 & \cdots & 0 \\ 0 & \mathrm{e}^{-\sqrt{2}v_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{e}^{-\sqrt{2}v_{k-1}} \end{bmatrix} \\ &= \frac{1}{2\beta t}\boldsymbol{C} \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{k-1} \end{bmatrix}, \end{aligned}$$

where $a_l = (1/2\beta)(k - l)l$, $l = 1, \ldots, k - 1$, and the matrix $\boldsymbol{C}$ is given in (5.5). $\boldsymbol{C}$ is symmetric and positive definite. Indeed, a straightforward computation yields that its eigenvalues are given explicitly by

$$1, \quad \frac{1}{2}, \quad \ldots, \quad \frac{k-1}{k}.$$

We consider the symmetric, positive definite square-root matrix of $\boldsymbol{C}$ and denote it by $\boldsymbol{C}^{1/2}$. Then, setting

$$\bar{p} = \boldsymbol{C}^{1/2}w, \qquad Q(w', w) = \boldsymbol{C}^{-1/2}\bar{\boldsymbol{L}}(\boldsymbol{C}^{1/2}w', \boldsymbol{C}^{1/2}w),$$

we see that (5.11) becomes

$$w' + \frac{1}{2t}\boldsymbol{A}w = Q(w', w), \tag{5.12}$$

where

$$\boldsymbol{A} = \boldsymbol{C}^{1/2} \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{k-1} \end{bmatrix} \boldsymbol{C}^{1/2}.$$

In particular, $\boldsymbol{A}$ has positive eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$. Let there be an orthogonal matrix $\boldsymbol{\Lambda}$ such that $\boldsymbol{D} = \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{\Lambda}$, where $\boldsymbol{D}$ is the diagonal matrix such that $A_{ii} = \lambda_i$,

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$i = 1, \dots, k-1$. Now, setting

$$\omega = \boldsymbol{\Lambda}^{\mathrm{T}} w, \qquad \boldsymbol{\Gamma}(\omega', \omega) = \boldsymbol{\Lambda}^{\mathrm{T}} Q(\boldsymbol{\Lambda}\omega', \boldsymbol{\Lambda}\omega),$$

(5.12) becomes equivalent to

$$\omega' + \frac{1}{2t}\boldsymbol{D}\omega = \boldsymbol{\Gamma}(\omega', \omega) \quad \text{in } (-\infty, -T_0). \tag{5.13}$$

We shall solve (5.13) by using the fixed-point theorem in a suitable space with initial data $w(T_0) = 0$. If ω is a solution of problem (5.13) with initial data, it takes the following form:

$$\omega_i(t) = -\frac{1}{(-t)^{\sqrt{\lambda_i}}} \int_t^{-t_0} (-s)^{\sqrt{\lambda_i}} \Gamma_i(\omega', \omega)\, \mathrm{d}s. \tag{5.14}$$

Let $A(\omega)$ be a solution of (5.14). Then $\boldsymbol{\Gamma}$ satisfies the same estimates in (5.8) and we have

$$|A(0)| \leqslant C_1 \left(\frac{1}{T_0}\right)^{\sigma/\sqrt{2}-1/2}. \tag{5.15}$$

Similarly,

$$|t|\,|A(0)'| \leqslant C_2 \left(\frac{1}{T_0}\right)^{\sigma/\sqrt{2}-1/2} \tag{5.16}$$

if we choose $t_0 > 1$. Thus, we consider the space

$$X = \{h \in C^1(-\infty, -t_0] \colon \|h\|_\Lambda \leqslant 2c_0\},$$

where $c_0 = C_1 + C_2$ are the constants in (5.15) and (5.16). Thus,

$$|A(h_1) - A(h_2)| \leqslant C \left(\frac{1}{T_0}\right)^{\sigma/\sqrt{2}-1/2} \|h_1 - h_2\|_\Lambda,$$

$$|t| |A'(h_1) - A'(h_2)| \leqslant C \|h_1 - h_2\|_\Lambda \left(\frac{1}{T_0}\right)^{\sigma/\sqrt{2}-1/2},$$

and we have

$$\|A(h_1) - A(h_2)\|_\Lambda \leqslant C(\sigma) \left(\frac{1}{T_0}\right)^{\sigma/\sqrt{2}-1/2}.$$

The result follows by the fixed-point theorem if we choose T_0 sufficiently large. □

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References

1 S. M. Allen and J. W. Cahn. A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta Metall.* **27** (1979), 1084–1095.

2 J. Carr and R. L. Pego. Metastable patterns in solutions of $u_t = \varepsilon^2 u_{xx} - f(u)$. *Commun. Pure Appl. Math.* **42** (1989), 523–576.

3 J. Carr and R. L. Pego. Invariant manifolds for metastable patterns in $u_t = \varepsilon^2 u_{xx} - f(u)$. *Proc. R. Soc. Edinb.* A **116** (1990), 133–160.

4 X. Chen, J.-S. Guo and H. Ninomiya. Entire solutions of reaction–diffusion equations with balanced bistable nonlinearities. *Proc. R. Soc. Edinb.* A **136** (2006), 1207–1237.

5 X. Chen, J.-S. Guo, F. Hamel, H. Ninomiya and J. Roquejoffre. Traveling waves with paraboloid like interfaces for balanced bistable dynamics. *Annales Inst. H. Poincaré Analyse Non Linéaire* **24** (2007), 369–393.

6 P. Daskalopoulos, M. del Pino and N. Sesum. Type II ancient compact solutions for the Yamabe flow. Preprint, 2012. (Available at https://arXiv.org/abs/1209.5479.)

7 M. del Pino, M. Kowalczyk and J. Wei. Traveling waves with multiple and nonconvex fronts for a bistable semilinear parabolic equation. *Commun. Pure Appl. Math.* **66** (2013), 481–547.

8 G. Fusco and J. K. Hale. Slow-motion manifolds, dormant instability, and singular perturbations. *J. Dynam. Diff. Eqns* **1** (1989), 75–94.

9 F. Hamel and N. Nadirashvili. Travelling fronts and entire solutions of the Fisher–KPP equation in $\mathbb{R}^N$. *Arch. Ration. Mech. Analysis* **157** (2001), 91–163.

10 F. Hamel, R. Monneau and J.-M. Roquejoffre. Stability of travelling waves in a model for conical flames in two space dimensions. *Annales Scient. Éc. Norm. Sup.* **37** (2004), 469–506.

11 H. Matano, M. Nara and M. Taniguchi. Stability of planar waves in the Allen–Cahn equation. *Commun. PDEs* **34** (2009), 976–1002.

12 H. Ninomiya and M. Taniguchi. Existence and global stability of traveling curved fronts in the Allen–Cahn equations. *J. Diff. Eqns* **213** (2005), 204–233.

13 M. Taniguchi. Multi-dimensional traveling fronts in bistable reaction–diffusion equations. *Discrete Contin. Dynam. Syst.* **32** (2012), 1011–1046.