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Phase field model with a variable chemical potential

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We study some asymptotic behaviour of phase interfaces with variable chemical potential under the uniform energy bound. The problem is motivated by the Cahn–Hilliard equation, where one has a control of the total energy and chemical potential. We show that the limit interface is an integral varifold with generalized L^p mean curvature. The convergence of interfaces as $\varepsilon \to 0$ is in the Hausdorff distance sense.

1. Introduction

In this paper, we study some asymptotic behaviour of phase interfaces in the van der Waals–Cahn–Hilliard theory of phase transitions. The equation we consider is

$$-\varepsilon \Delta u + \frac{W'(u)}{\varepsilon} = f, \tag{1.1}$$

where $u : U \subset \mathbb{R}^n \to \mathbb{R}$, $n \geqslant 2$, is the normalized density distribution of a two-phase fluid and W is a double well potential with strict local minima at ± 1. The function f is a variable chemical potential field in the two-phase fluid model [22] and $\varepsilon \approx 0$ is a parameter that gives the order of interface thickness.

The associated energy,

$$E_\varepsilon(u) = \int_\Omega \tfrac{1}{2}\varepsilon|\nabla u|^2 + \frac{W(u)}{\varepsilon}, \tag{1.2}$$

and the behaviour of minimizers of this energy with a volume constraint were initially studied in [29, 38] within the framework of Γ-convergence [16] and subsequently generalized by many authors [6, 20, 25, 30, 32, 39]. In this case, the minimizers satisfy equation (1.1) with some constant $f \equiv c$, and the functional $E_\varepsilon(\cdot)$ Γ-converges to the area functional. The limit interface as $\varepsilon \to 0$ is area minimizing, with a given volume constraint. It was also proved [26] that the constant mean curvature of the limit interface is determined by the chemical potential and

$$\sigma = \int_{-1}^{1} \sqrt{\tfrac{1}{2}W(s)}\,\mathrm{d}s.$$

In [23], we studied the behaviour of general critical points of the functional (1.2) with a volume constraint and showed that the interface is close, in the Hausdorff

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distance sense, to a locally constant mean curvature hypersurface when $\varepsilon \approx 0$. This corresponds to studying (1.1) again, with constant $f \equiv c$, with suitable bounds on the energy and c. Note that due to the non-convexity of the functional, there can be solutions that are only locally energy minimizing or even unstable. In [23], we also proved that the interfaces of locally energy-minimizing solutions as $\varepsilon \to 0$ converge to a locally area-minimizing hypersurface.

One motivation to consider (1.1) with variable f, aside from being a natural generalization, comes from the Cahn–Hilliard equation [12], which models various phase separation phenomena in a melted alloy with two stable phases. It is

$$\left.\begin{aligned} u_t = \Delta f \quad \text{on } \Omega \times (0,\infty), \\ f = -\varepsilon \Delta u + \frac{W'(u)}{\varepsilon}, \\ \frac{\partial u}{\partial n} = \frac{\partial f}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \quad \text{on } \Omega. \end{aligned}\right\} \tag{1.3}$$

The Neumann boundary condition reflects the insulation from the outside. It is a fourth-order gradient flow of $E_\varepsilon(\cdot)$ with volume conservation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon(u) = -\int_\Omega |\nabla f|^2 \leqslant 0.$$

The time-scale here is different from the usual setting, where t may be replaced by $t\varepsilon$. For more physical background, derivation of the equation and the related equation such as the Allen–Cahn equation, see, for example, [1, 3, 5, 7, 10–13, 15, 18, 19, 24] and the references therein. Even though it is far from a complete picture, we mention the most relevant references to the present article. With the time-scale under consideration, it is known that the limit problem is the so-called Mullins–Sekerka problem [31]. This was formally derived by Pego [34], and was given a rigorous justification using asymptotic expansions and spectral analysis by Alikakos *et al*. [2]. In the case of radial symmetry and Dirichlet conditions, Stoth [40] proved a global convergence to the limit problem in dimension three. For general domains and solutions, Chen [14] showed that the solutions converge to a weak solution of the limit problem using the notion of varifolds. Here, we take the similar approach to that of Chen in this paper, using varifolds as our working device.

Given a sequence of solutions $\{u^i\}_{i=1}^\infty$ to

$$-\varepsilon_i \Delta u^i + \frac{W'(u^i)}{\varepsilon_i} = f^i,$$

$\varepsilon_i \to 0$, with uniform bounds $\sup_i E_{\varepsilon_i}(u^i) \leqslant E_0$ and $\sup_i \|f^i\|_{W^{1,n}} < \infty$, we show that there exists a subsequence whose interfaces converge, in the Hausdorff distance sense, to a hypersurface with the mean curvature determined by $f^\infty = \lim f^i$, σ and the interface multiplicities. The mean curvature belongs to L^p for any $p < \infty$ with respect to the $(n-1)$-dimensional hypersurface measure of the limit interface. We prove a monotonicity-type formula for the properly scaled energy, which extends the case discussed in [23]. Once this is established, the rectifiability and integrality of the limit varifold follow more or less from the argument in [23]. The proof is technically involved, and we need detailed estimates on the positive part of the so-called discrepancy function $\frac{1}{2}\varepsilon|\nabla u|^2 - W/\varepsilon$ using the Aleksandorov–Bakelman–Pucci (ABP) estimates. The results are applied to the Cahn–Hilliard equation with $n = 2$, showing that there exists a subsequence for a.e. time, with the convergence properties discussed in theorem 2.1, when $\varepsilon \to 0$.

We note that there is an interesting class of unstable solutions with multiple peaks (see [8, 9, 41, 42] and the references therein). They describe nucleation phenomena that are experimentally observed. Their singular limit cannot be captured by our method, since the chemical potential or energy has to blow-up in our scale.

The organization of the paper is as follows. In § 2, we state our assumptions, terminologies and main results. In § 3, we derive our main monotonicity formula for the scaled energy. In § 4, we prove that the limit interface measure is supported on rectifiable sets and that the measure has a.e. integral densities. In the language of varifolds, we show that the limit interface measure is an integral varifold (after dividing by σ), with the generalized mean curvature in L^p for any $p < \infty$. In § 5, we discuss some additional comments and some implications to the Cahn–Hilliard equation.

2. Assumptions and main results

2.1. Hypotheses

We consider the problem with the following assumptions.

Assumption A. The function $W : \mathbb{R} \to [0, \infty)$ is C^3 and $W(\pm 1) = 0$. For some $\gamma \in (-1, 1)$, $W' < 0$ on $(\gamma, 1)$ and $W' > 0$ on $(-1, \gamma)$. For some $\alpha \in (0, 1)$ and $\kappa > 0$, $W''(x) \geqslant \kappa$ for all $|x| \geqslant \alpha$.

Assumption B. $U \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary ∂U. Sequences of $C^3(U)$ functions $\{u^i\}_{i=1}^{\infty}$ and $C^1(U)$ functions $\{f^i\}_{i=1}^{\infty}$ satisfy

$$\varepsilon_i \Delta u^i = \varepsilon_i^{-1} W'(u^i) - f^i \tag{2.1}$$

on U. Here, $\lim_{i\to\infty} \varepsilon_i = 0$ and we assume that there exist c_0, λ_0 and E_0 such that $\sup_U |u^i| \leqslant c_0$,

$$\|f^i\|_{W^{1,n}(U)} = \left(\int_U |f^i|^n + |\nabla f^i|^n\right)^{1/n} \leqslant \lambda_0$$

and

$$\int_U \tfrac{1}{2}\varepsilon_i |\nabla u^i|^2 + \frac{W(u^i)}{\varepsilon_i} \leqslant E_0$$

for all i.

Assumption A requires that W has a standard W shape, with non-degenerate minima at ± 1 and local maximum at γ. The assumptions on u^i and f^i are either motivated by the applications to the Cahn–Hilliard equation (see § 5.2) or regarded simply as a starting point to the problem. The regularity of u is then standard [21]. The requirement of a $W^{1,n}$-norm bound on f^i comes mainly from the technical limit of our approach using the ABP estimates. It is not clear that this can be replaced by a weaker norm bound, such as $W^{1,p}$ for some $p > \frac{1}{2}n$, which we believe

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is ultimately the weakest bound for f^i, to conclude similar convergence results (see § 5.1 for further discussion).

We next discuss a few immediate consequences of the assumptions. Let

$$\Phi(s) = \int_0^s \sqrt{\tfrac{1}{2} W(s)}\, \mathrm{d}s$$

and define new functions

$$w^i = \Phi \circ u^i$$

for each i.

Since $|\nabla w^i| = \sqrt{\frac{1}{2} W(u^i)} |\nabla u^i|$, it follows by the Cauchy–Schwarz inequality that

$$\int_U |\nabla w^i| \leqslant \frac{1}{2} \int_U \tfrac{1}{2} \varepsilon_i |\nabla u^i|^2 + \frac{W(u^i)}{\varepsilon_i} \leqslant \tfrac{1}{2} E_0.$$

We also have $\Phi(-c_0) \leqslant w^i \leqslant \Phi(c_0)$. By the compactness theorem for bounded variation functions, there exists a subsequence, also denoted by $\{w^i\}$, and an a.e. pointwise limit w^∞ such that

$$\lim_{i\to\infty} \int_U |w^i - w^\infty| = 0 \quad \text{and} \quad \int_U |Dw^\infty| \leqslant \liminf_{i\to\infty} \int_U |\nabla w^i|.$$

Here, $|Dw^\infty|$ is the total variation of the vector-valued Radon measure Dw^∞.

Let Φ^{-1} be the inverse of Φ and define

$$u^\infty = \Phi^{-1}(w^\infty).$$

Then $u^i \to u^\infty$ a.e. and, by the Lebesgue dominated convergence theorem,

$$\int_U |u^i - u^\infty| \to 0.$$

Also, by Fatou's lemma and the energy bound, we have

$$\int_U W(u^\infty) = \int_U \lim_{i\to\infty} W(u^i) \leqslant \liminf_{i\to\infty} \int_U W(u^i) = 0.$$

This shows that $u^\infty = \pm 1$ a.e. on U, and the sets $\{u^\infty = \pm 1\}$ have finite perimeter in U, since

$$\|\partial\{u^\infty = 1\}\|(U) = \frac{1}{2} \int_U |Du^\infty| = \frac{1}{\sigma} \int_U |Dw^\infty| \leqslant \frac{E_0}{2\sigma},$$

where we define

$$\sigma = \int_{-1}^1 \sqrt{\tfrac{1}{2} W(s)}\, \mathrm{d}s$$

and where $\|\partial A\|$ denotes the perimeter of A in the measure-theoretic sense (see [17]).

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2.2. The associated varifolds

In this section we associate to each solution of (2.1) a varifold in a natural way. We refer to [4,37] for a comprehensive treatment of varifolds.

Let $G(n, n-1)$ denote the Grassman manifold of unoriented $(n-1)$-dimensional planes in $\mathbb{R}^n$. We say that V is an $(n-1)$-dimensional *varifold* in $U \subset \mathbb{R}^n$ if V is a Radon measure on $G_{n-1}(U) = U \times G(n, n-1)$. Let $V_{n-1}(U)$ denote the set of all $(n-1)$-dimensional varifolds in U. *Convergence in the varifold sense* means convergence in the usual sense of measures. For $V \in V_{n-1}(U)$, we let the *weight* $\|V\|$ be the Radon measure in U defined by

$$\|V\|(A) = V(\{(x, S) \mid x \in A,\ S \in G(n, n-1)\})$$

for each Borel set $A \subset U$. If M is a $(n-1)$-rectifiable subset of U, we define $v(M) \in V_{n-1}(U)$ by

$$v(M)(E) = \mathcal{H}^{n-1}(\{x \in U \mid (x, \operatorname{Tan}^{n-1}(\mathcal{H}^{n-1}\lfloor_M, x)) \in E\})$$

for each Borel set $E \in G_{n-1}(U)$, where $\operatorname{Tan}^{n-1}(\mathcal{H}^{n-1}\lfloor_M, x)$ is the approximate tangent plane to M at x and so exists for $\mathcal{H}^{n-1}$ a.e. $x \in M$.

We associate to each function w^i a varifold V^i defined naturally as follows [14, 23, 24]. By Sard's theorem, $\{w^i = t\} \subset U$ is a C^3 hypersurface for L^1 almost all t. Define $V^i \in V_{n-1}(U)$ by

$$V^i(A) = \int_{-\infty}^{\infty} v(\{w^i = t\})(A)\,\mathrm{d}t$$

for each Borel set $A \subset G_{n-1}(U)$. By the co-area formula [17], we have

$$\|V^i\|(A) = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{w^i = t\} \cap A)\,\mathrm{d}t = \int_A |\nabla w^i|$$

for each Borel set $A \subset U$. One may interpret the varifold V^i as a weighted averaging of the level sets of u^i, which is concentrated around the transition region. The first variation of V^i is given by [33, § 2.1]

$$\delta V^i(g) = \int_U \biggl(\operatorname{div} g - \sum_{j,k=1}^{n} \frac{w^i_{x_j}}{|\nabla w^i|}\frac{w^i_{x_k}}{|\nabla w^i|} g^j_{x_k}\biggr)|\nabla w^i| \tag{2.2}$$

for each $g \in C^1_c(U; \mathbb{R}^n)$.

2.3. Main results

With the above terminology and assumptions A and B, we show the following.

THEOREM 2.1. *Let V^i be the varifold associated with u^i (via w^i), as in §§ 2.1 and 2.2. On passing to a subsequence, we can assume that*

$$f^i \to f^\infty \text{ weakly in } W^{1,n}, \qquad u^i \to u^\infty \text{ a.e.}, \qquad V^i \to V \text{ in the varifold sense.}$$

Moreover, we have the following.

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(1) *For each* $\phi \in C_c(U)$,

$$\|V\|(\phi) = \lim_{i\to\infty}\int \tfrac{1}{2}\phi\varepsilon_i|\nabla u^i|^2 = \lim_{i\to\infty}\int \phi\frac{W(u^i)}{\varepsilon_i} = \lim_{i\to\infty}\int \phi|\nabla w^i|.$$

(2) $\operatorname{supp}\|\partial\{u^\infty = 1\}\| \subset \operatorname{supp}\|V\|$ *and* $\{u^i\}$ *converges locally uniformly to* ± 1 *on* $U \setminus \operatorname{supp}\|V\|$.

(3) *For each* $\tilde{U} \subset\subset U$, $0 < b < 1$, $\{|u^i| \leqslant 1 - b\} \cap \tilde{U}$ *converges to* $\tilde{U} \cap \operatorname{supp}\|V\|$ *in the Hausdorff distance sense.*

(4) $\sigma^{-1}V$ *is an integral varifold. Moreover, the density* $\theta(x) = \sigma N(x)$ *of* V *satisfies*

$$N(x) = \begin{cases} \textit{odd}, & \mathcal{H}^{n-1} \text{ a.e. } x \in M^\infty, \\ \textit{even}, & \mathcal{H}^{n-1} \text{ a.e. } x \in \operatorname{supp}\|V\| \setminus M^\infty, \end{cases}$$

where M^∞ *is the reduced boundary of* $\{u^\infty = 1\}$.

(5) *The generalized mean curvature* H *of* V *is given by*

$$H(x) = \begin{cases} \dfrac{f^\infty(x)}{\theta(x)}\nu^\infty(x), & \mathcal{H}^{n-1} \text{ a.e. } x \in M^\infty, \\ 0, & \mathcal{H}^{n-1} \text{ a.e. } x \in \operatorname{supp}\|V\| \setminus M^\infty, \end{cases}$$

where ν^∞ *is the inward normal for* M^∞.

(6) *The generalized mean curvature* H *belongs to* L^p_{loc} *for any* $p < \infty$ *with respect to* $\|V\|$.

Comments that follow theorem 1 in [23] go with minor changes here as well. Part (1) shows that, in the limit, the energy is equally divided between the two terms of the energy functional (1.2), called the *equipartition of energy*. In fact, our result shows the following: *for any* $\tilde{U} \subset\subset U$ *and for the full sequence* (*not only a subsequence*),

$$\lim_{i\to\infty}\int_{\tilde{U}}\left|\tfrac{1}{2}\varepsilon_i|\nabla u^i|^2 - \frac{W(u^i)}{\varepsilon_i}\right| = 0.$$

This is even without $\frac{1}{2}\varepsilon_i|\nabla u^i|^2\,\mathrm{d}x$ converging to some measure. Part (4) suggests that folding of the interface as $\varepsilon \to 0$ occurs locally as an integer multiple of one-dimensional travelling wave solutions [28], almost everywhere in the measure-theoretic sense.

Without loss of generality, we may assume that $M^\infty \subset \operatorname{supp}\|\partial\{u^\infty = 1\}\|$. We were not able to prove or disprove that $\mathcal{H}^{n-1}(\operatorname{supp}\|\partial\{u^\infty = 1\}\| \setminus M^\infty) = 0$ in general. This is due to the lack of a uniform lower-density estimate for the measure

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$\|\partial\{u^\infty = 1\}\|$ (as opposed to $\|V\|$) at $\mathcal{H}^{n-1}$ a.e. x in the closure of M^∞. On the other hand, if $N(x)$ is odd $\mathcal{H}^{n-1}$ a.e. for $x \in \operatorname{supp}\|V\|$, result (4) shows that

$$\mathcal{H}^{n-1}(\operatorname{supp}\|V\| \setminus M^\infty) = 0 \quad \text{and} \quad \operatorname{supp}\|V\| = \operatorname{supp}\|\partial\{u^\infty = 1\}\|.$$

If $N(x) = 1$ a.e., then this corresponds to 'no energy loss' situation, since

$$\int |Dw^\infty|\phi = \sigma\|\partial\{u^\infty = 1\}\|(\phi) = \|V\|(\phi) = \lim_{i\to\infty} \int |\nabla w^i|\phi$$

for all $\phi \in C_c(U)$. In case f^i are all constants and with no energy loss, the relation between the curvature of the limit interface and the chemical potential is established by Luckhaus and Modica in [26], and we prove here the direct generalization.

It is well known that the support of a rectifiable varifold with L^p mean curvature, $p > n-1$, is locally a $C^{1,\alpha}$ graph on a relatively open dense subset $\mathcal{O}$ [4] for $\alpha = 1-(n-1)/p$. The density function on $\mathcal{O}$ is locally constant and integer valued. If we additionally assume a better bound on f^i, for example, $C^{k,\alpha}$, $0<\alpha<1$, $k \geqslant 0$, standard elliptic estimates [21] show that the support there is $C^{k+2,\alpha}$. On the other hand, we do not know if $\mathcal{H}^{n-1}(\operatorname{supp}\|V\| \setminus \mathcal{O}) = 0$ in general. The density function θ is defined everywhere on the support of $\|V\|$ and is upper-semicontinuous.

If $N = 1$, $\mathcal{H}^{n-1}$ a.e. on $\operatorname{supp}\|V\|$, then the support is locally a $C^{1,\alpha}$ hypersurface of mean curvature given by f^∞/σ, except for a closed set of $\mathcal{H}^{n-1}$ measure zero.

3. Monotonicity formula

In this section, in addition to assumption A, we assume that the functions $u, f : U \to \mathbb{R}$ satisfy assumption B, with u^i, f^i and ε_i replaced by u, f and ε, respectively. We assume that $\tilde{U}$ is open and $\tilde{U} \subset\subset U$.

The main result here is the energy monotonicity formula for

$$E(r,x) = \frac{1}{r^{n-1}} \int_{B_r(x)} \left(\tfrac{1}{2}\varepsilon|\nabla u|^2 + \frac{W(u)}{\varepsilon}\right),$$

given in proposition 3.6. We first derive the identity (3.1), which gives the expression of the radial derivative of $E(r,x)$. The main difficulty in proving the positivity of $(\mathrm{d}/\mathrm{d}r)E(r,x)$ comes from the positive part of the discrepancy function

$$\xi_\varepsilon = \tfrac{1}{2}\varepsilon|\nabla u|^2 - \frac{W(u)}{\varepsilon}.$$

When f is a constant function, we proved in [23] that $\xi_\varepsilon \leqslant c$, independent of ε. There we used a differential inequality satisfied by $\xi_\varepsilon - G(u)$, where G is a suitable modification function, and a maximum-principle-type argument. In this paper, since we only have control of ∇f in the L^n-norm, we use a ABP-type estimate instead. Even though we could not prove the uniform supremum bound, we show $\xi_\varepsilon \leqslant c\varepsilon^{-2/5}$ for all sufficiently small ε (proposition 3.2). This estimate gives us an energy lower bound around the interface for the length-scale of order $\varepsilon^{2/5}$. We then use this lower bound and a covering lemma to obtain the estimate on the integral of ξ_ε (proposition 3.5) satisfied for r larger than the length-scale of order $\varepsilon^{2/5}$. Combined with this estimate, we obtain the monotonicity formula (3.10).

We denote $W - \varepsilon u f$ by $\tilde{W}$.

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LEMMA 3.1. *For $B_r(x) \subset\subset U$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}r}\bigg\{\frac{1}{r^{n-1}}\int_{B_r(x)}\bigg(\tfrac{1}{2}\varepsilon|\nabla u|^2+\frac{\tilde{W}}{\varepsilon}\bigg)\bigg\} = \frac{1}{r^n}\int_{B_r(x)}\bigg(\frac{\tilde{W}}{\varepsilon}-\tfrac{1}{2}\varepsilon|\nabla u|^2\bigg) + \frac{\varepsilon}{r^{n+1}}\int_{\partial B_r(x)}((y-x)\cdot\nabla u)^2 - \frac{1}{r^n}\int_{B_r(x)}((y-x)\cdot\nabla f)u. \tag{3.1}$$

Proof. Multiply both sides of (1.1) by $\nabla u\cdot g$, where $g=(g^1,\dots,g^n)\in C_c^1(U;\mathbb{R}^n)$. Then, after two integrations by parts, we obtain

$$\int_U\bigg(\bigg(\tfrac{1}{2}\varepsilon|\nabla u|^2+\frac{\tilde{W}(u)}{\varepsilon}\bigg)\operatorname{div} g-\varepsilon\sum_{i,j}u_{y_i}u_{y_j}g^j_{y_i}-(\nabla f\cdot g)u\bigg)=0. \tag{3.2}$$

We let $x=0$ by a suitable translation and let $g^j(y)=y_j\rho(|y|)$, where $\rho(|y|)$ is a smooth approximation to the characteristic function χ_{B_r}. Writing $r=|y|$, equation (3.2) becomes

$$\int_U\bigg(\bigg(\tfrac{1}{2}\varepsilon|\nabla u|^2+\frac{\tilde{W}(u)}{\varepsilon}\bigg)(r\rho'+n\rho)-\varepsilon\frac{\rho'}{r}(y\cdot\nabla u)^2-\varepsilon|\nabla u|^2\rho-(\nabla f\cdot y)\rho u\bigg)=0.$$

Letting $\rho\to\chi_{B_r}$ and rearranging terms, we obtain

$$-(n-1)\int_{B_r}\bigg(\tfrac{1}{2}\varepsilon|\nabla u|^2+\frac{\tilde{W}(u)}{\varepsilon}\bigg)+r\int_{\partial B_r}\bigg(\tfrac{1}{2}\varepsilon|\nabla u|^2+\frac{\tilde{W}(u)}{\varepsilon}\bigg) = \int_{B_r}\bigg(\frac{\tilde{W}(u)}{\varepsilon}-\tfrac{1}{2}\varepsilon|\nabla u|^2\bigg)+\frac{\varepsilon}{r}\int_{\partial B_r}(y\cdot\nabla u)^2-\int_{B_r}(y\cdot\nabla f)u.$$

Dividing the above expression by r^n, we obtain (3.1). □

For the moment, we assume the following technical result, which we prove later.

PROPOSITION 3.2. *There exist constants c_1 and ε_1, which depend only on c_0, λ_0, W, n and $\operatorname{dist}(\tilde{U},\partial U)$, such that, if $\varepsilon<\varepsilon_1$,*

$$\sup_{\tilde{U}}\bigg(\tfrac{1}{2}\varepsilon|\nabla u|^2-\frac{W}{\varepsilon}\bigg)\leqslant c_1\varepsilon^{-2/5}. \tag{3.3}$$

LEMMA 3.3. *For $B_r(x)\subset\tilde{U}$, there exists c_2, which depends only on c_0, λ_0, n, W and $\operatorname{dist}(\tilde{U},\partial U)$, such that, for any $s<r$ and $\varepsilon<\varepsilon_1$,*

$$E(r,x)-E(s,x)\geqslant -c_2(r+r^{1/2}+r\varepsilon^{-2/5})+\int_s^r\frac{\mathrm{d}\tau}{\tau^n}\int_{B_\tau(x)}\bigg(\frac{W}{\varepsilon}-\tfrac{1}{2}\varepsilon|\nabla u|^2\bigg)^+. \tag{3.4}$$

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Proof. By (3.3) and the Hölder and Sobolev inequalities, we obtain

$$\frac{1}{r^n}\int_{B_r(x)}\left(\tfrac{1}{2}\varepsilon|\nabla u|^2-\frac{\tilde{W}}{\varepsilon}\right)^+ \leqslant \frac{1}{r^n}\int_{B_r(x)}\left\{\left(\tfrac{1}{2}\varepsilon|\nabla u|^2-\frac{W}{\varepsilon}\right)^+ + |uf|\right\}$$
$$\leqslant \omega_n(c_1\varepsilon^{-2/5}+c_0\|f\|_{L^{2n}}r^{-1/2})$$
$$\leqslant \omega_n(c_1\varepsilon^{-2/5}+c_0c(n)\lambda_0 r^{-1/2}).$$

After integrating over $[s,r]$, this gives the bound on the negative part of the first term in (3.1). Also, for the third term in (3.1),

$$\left|\frac{1}{r^n}\int_{B_r(x)}((y-x)\cdot\nabla f)u\right| \leqslant c_0\frac{1}{r^{n-1}}\int_{B_r(x)}|\nabla f| \leqslant c_0c(n)\|\nabla f\|_{L^n} \leqslant c_0c(n)\lambda_0.$$

The difference resulting from W and $\tilde{W}$ may be estimated by

$$\frac{1}{r^{n-1}}\int_{B_r(x)}|uf| \leqslant c_0c(n)\|f\|_{W^{1,n}}r^{1/2} \leqslant c_0c(n)\lambda_0 r^{1/2}.$$

With an appropriate choice of c_2, we obtain (3.4). □

PROPOSITION 3.4. *There exist constants* $0<c_3$, $c_4<1$, *which depend only on* c_0, λ_0, n, W *and* $\operatorname{dist}(\tilde{U},\partial U)$, *such that, if* $B_{\varepsilon^{2/5}}(x)\subset\tilde{U}$, $|u(x)|\leqslant\alpha$ *and* $\varepsilon<\varepsilon_1$, *then*

$$E(r,x)\geqslant c_4 \quad \text{for } \varepsilon\leqslant r\leqslant c_3\varepsilon^{2/5}. \tag{3.5}$$

Proof. Translate x to the origin. By scaling $\tilde{x}=x/\varepsilon$ and $\tilde{u}(\tilde{x})=u(\varepsilon\tilde{x})$, the energy scales as

$$E(\varepsilon,0)=\frac{1}{\varepsilon^{n-1}}\int_{B_\varepsilon}\left(\tfrac{1}{2}\varepsilon|\nabla u|^2+\frac{W}{\varepsilon}\right)\mathrm{d}x=\int_{B_1}(\tfrac{1}{2}|\nabla\tilde{u}|^2+W)\,\mathrm{d}\tilde{x}.$$

In this scale, there exists a constant c_5, which depends only on c_0, λ_0, n, W and $\operatorname{dist}(\tilde{U},\partial U)$, such that $|\nabla\tilde{u}|\leqslant c_5$, due to the standard elliptic L^p estimate [21]. Since $|\tilde{u}(0)|\leqslant\alpha$, there exists some $c_6=c(c_5,W)>0$ such that $W(\tilde{u})\geqslant c_6$ on B_{c_6}. Hence $E(\varepsilon,0)\geqslant\omega_n c_6^{n+1}$. Using (3.4) and setting $s=\varepsilon$, if we restrict r to be less than $c_3\varepsilon^{2/5}$ for sufficiently small $c_3=c(c_2,c_6,n)$ and setting $c_4=\omega_n c_6^{n+1}/2$, we obtain the desired inequality (3.5). □

In the following, we use the energy lower bound (3.5) around the interface. The ball $B_r(x)$ we consider has a radius of order $\varepsilon^{2/5}$ at least, which is larger than $\varepsilon^{4/5}$, where we have the energy lower bound. We cover the interface by a collection of balls with radii $\varepsilon^{4/5}$.

PROPOSITION 3.5. *There exist constants* c_7, c_8, $\varepsilon_2\leqslant\varepsilon_1$, *which depend only on* c_0, λ_0, n, W *and* $\operatorname{dist}(\tilde{U},\partial U)$, *such that, if* $B_r(x)\subset\tilde{U}$, $r\geqslant c_3\varepsilon^{2/5}$ *and* $\varepsilon<\varepsilon_2$, *then*

$$\frac{1}{r^n}\int_{B_r(x)}\left(\tfrac{1}{2}\varepsilon|\nabla u|^2-\frac{W}{\varepsilon}\right)^+ \leqslant c_7E(r,x)+c_8. \tag{3.6}$$

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Proof. We translate x to the origin. We estimate the integral on

$$\mathcal{A} = \{x \in B_r \setminus B_{r-\varepsilon^{4/5}}\},$$
$$\mathcal{B} = \{x \in B_{r-\varepsilon^{4/5}} \,|\, \mathrm{dist}(\{|u| \leqslant \alpha\}, x) < \varepsilon^{4/5}\},$$
$$\mathcal{C} = \{x \in B_{r-\varepsilon^{4/5}} \,|\, \mathrm{dist}(\{|u| \leqslant \alpha\}, x) \geqslant \varepsilon^{4/5}\}.$$

The set $\mathcal{A}$ is the $\varepsilon^{4/5}$-shell of the ball B_r, $\mathcal{B}$ is the $\varepsilon^{4/5}$-neighbourhood of the interface and $\mathcal{C}$ is the complement of the two.

CASE 1 (estimate on $\mathcal{A}$). By (3.3),

$$\frac{1}{r^n}\int_{\mathcal{A}}\left(\tfrac{1}{2}\varepsilon|\nabla u|^2 - \frac{W}{\varepsilon}\right)^+ \leqslant \frac{n\omega_n\varepsilon^{4/5}}{r}c_1\varepsilon^{-2/5} \leqslant \frac{n\omega_n c_1}{c_3}, \tag{3.7}$$

since $r \geqslant c_3\varepsilon^{2/5}$.

CASE 2 (estimate on $\mathcal{B}$). We first estimate $\mathcal{H}^n(\mathcal{B})$. We apply the Vitali covering lemma [37, theorem 3.3] to the family of balls $\{B_{\varepsilon^{4/5}}(x)\}_{x\in\{|u|\leqslant\alpha\}\cap\mathcal{B}}$ (which covers $\mathcal{B}$), so that $\{B_{\varepsilon^{4/5}}(x_i)\}_{i=1}^N$ is a pairwise disjoint subset of the family and so that $\mathcal{B} \subset \bigcup_{i=1}^N B_{5\varepsilon^{4/5}}(x_i)$. Then we have $\mathcal{H}^n(\mathcal{B}) \leqslant \omega_n(5\varepsilon^{4/5})^n N$. On the other hand, by (3.5) and since $\varepsilon < \varepsilon^{4/5} < c_3\varepsilon^{2/5}$ for all sufficiently small ε,

$$c_4(\varepsilon^{4/5})^{n-1} \leqslant \int_{B_{\varepsilon^{4/5}}(x_i)}\left(\tfrac{1}{2}\varepsilon|\nabla u|^2 + \frac{W}{\varepsilon}\right)$$

holds for each $i = 1,\dots,N$. Since they are pairwise disjoint, summing over i, we have

$$Nc_4(\varepsilon^{4/5})^{n-1} \leqslant \int_{B_r}\left(\tfrac{1}{2}\varepsilon|\nabla u|^2 + \frac{W}{\varepsilon}\right).$$

These two estimates show that

$$\mathcal{H}^n(\mathcal{B}) \leqslant \frac{\omega_n 5^n\varepsilon^{4/5}}{c_4}\int_{B_r}\left(\tfrac{1}{2}\varepsilon|\nabla u|^2 + \frac{W}{\varepsilon}\right) = \frac{\omega_n 5^n\varepsilon^{4/5}r^{n-1}E(r,0)}{c_4}.$$

Finally, with (3.3),

$$\frac{1}{r^n}\int_{\mathcal{B}}\left(\tfrac{1}{2}\varepsilon|\nabla u|^2 - \frac{W}{\varepsilon}\right)^+ \leqslant \frac{c_1\varepsilon^{-2/5}\mathcal{H}^n(\mathcal{B})}{r^n} \leqslant \frac{\omega_n 5^n c_1 E(r,0)}{c_3c_4}, \tag{3.8}$$

since $r \geqslant c_3\varepsilon^{2/5}$.

CASE 3 (estimate on $\mathcal{C}$). Define a Lipschitz function ϕ as follows:

$$\phi(x) = \min\{1, \varepsilon^{-4/5}\,\mathrm{dist}(\{|x| \geqslant r\} \cup \{|u| \leqslant \alpha\}, x)\}.$$

ϕ is 0 on the set $\{|u| \leqslant \alpha\} \cup \{|x| \geqslant r\}$, 1 on $\mathcal{C}$ and $|\nabla\phi| \leqslant \varepsilon^{-4/5}$. Using this ϕ, we estimate $\frac{1}{2}\varepsilon|\nabla u|^2$ (which is larger than $(\frac{1}{2}\varepsilon|\nabla u|^2 - W/\varepsilon)^+$) on $\mathcal{C}$. Differentiate (1.1) with respect to the kth variable, multiply it by $u_{y_k}\phi^2$ and sum over k. Then

$$\int\sum_{k=1}^n \varepsilon u_{y_k}\Delta u_{y_k}\phi^2 = \int\frac{W''}{\varepsilon}|\nabla u|^2\phi^2 - \nabla f\cdot\nabla u\phi^2.$$

We integrate by parts, and since $W'' \geqslant \kappa$ on $\{|u| \geqslant \alpha\}$,

$$\int \frac{\kappa|\nabla u|^2\phi^2}{\varepsilon} + \varepsilon|\nabla^2 u|^2\phi^2 \leqslant \int |\nabla f||\nabla u|\phi^2 + 2\varepsilon \sum_{k,l=1}^{n} |\phi u_{y_k}\phi_{y_l} u_{y_k y_l}|.$$

By the Cauchy–Schwarz inequality, we then obtain

$$(\text{left-hand side}) \leqslant \int \frac{\varepsilon}{2\kappa}|\nabla f|^2\phi^2 + \frac{\kappa}{2\varepsilon}|\nabla u|^2\phi^2 + \tfrac{1}{2}\varepsilon\phi^2|\nabla^2 u|^2 + 2\varepsilon|\nabla\phi|^2|\nabla u|^2.$$

Relegating two terms to the left-hand side, we have

$$\frac{\kappa}{2\varepsilon}\int |\nabla u|^2\phi^2 \leqslant \int \frac{\varepsilon}{2\kappa}|\nabla f|^2\phi^2 + 2\varepsilon|\nabla\phi|^2|\nabla u|^2.$$

Since $|\nabla\phi| \leqslant \varepsilon^{-4/5}$,

$$\int_{\mathcal{C}} \varepsilon|\nabla u|^2 \leqslant \int_{B_r} \frac{\varepsilon^3|\nabla f|^2}{\kappa^2} + \frac{4\varepsilon^{7/5}|\nabla u|^2}{\kappa}.$$

By Hölder's inequality,

$$\begin{aligned}\frac{1}{r^n}\int_{\mathcal{C}} \tfrac{1}{2}\varepsilon|\nabla u|^2 &\leqslant \frac{\varepsilon^3(\omega_n r^n)^{(n-2)/n}\|\nabla f\|_{L^2}^2}{2\kappa^2 r^n} + \frac{2\varepsilon^{2/5}E(r,0)}{\kappa r} \\ &\leqslant \frac{\lambda_0^2\omega_n^{(n-2)/n}\varepsilon^{11/5}}{2\kappa^2 c_3^2} + \frac{2E(r,0)}{\kappa c_3} \qquad (3.9)\end{aligned}$$

by $r \geqslant c_3\varepsilon^{2/5}$. With appropriate choices of the constants, we obtain (3.6) by (3.7)–(3.9). □

Proposition 3.6. *There exist constants c_9 and r_0, depending only on c_0, λ_0, n, W and* dist$(\tilde{U}, \partial U)$*, with the following property: for any $B_r(x) \subset \tilde{U}$, $\varepsilon < \varepsilon_2$ and $c_3\varepsilon^{2/5} \leqslant s < r \leqslant r_0$,*

$$\mathrm{e}^{c_7 r}E(r,x) - \mathrm{e}^{c_7 s}E(s,x) \geqslant \int_s^r \frac{\mathrm{e}^{c_7\tau}}{\tau^n}\int_{B_\tau}\left(\frac{W}{\varepsilon} - \tfrac{1}{2}\varepsilon|\nabla u|^2\right)^+ \mathrm{d}\tau - c_9 r^{1/2}. \qquad (3.10)$$

Proof. By (3.1) and (3.6), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}r}E(r,x) + \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^n}\int_{B_r(x)} uf\right) \\ &\qquad \geqslant \frac{1}{r^n}\int_{B_r(x)}\left(\frac{W}{\varepsilon} - \tfrac{1}{2}\varepsilon|\nabla u|^2\right)^+ - c_7E(r,x) - c_8 - \frac{1}{r^n}\int_{B_r(x)}(|uf| + r|\nabla f||u|).\end{aligned}$$

The last term is bounded from below by $-c_{10}(n, c_0, \lambda_0)r^{-1/2}$. By multiplying both sides by $\mathrm{e}^{c_7 r}$ and integrating over $[s, r]$, we obtain

$$\begin{aligned}&\mathrm{e}^{c_7 r}E(r,x) - \mathrm{e}^{c_7 s}E(s,x) \\ &\qquad \geqslant \int_s^r \mathrm{e}^{c_7\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\tau^{n-1}}\int_{B_\tau(x)} uf\right)\mathrm{d}\tau + \int_s^r \frac{\mathrm{e}^{c_7\tau}}{\tau^n}\int_{B_\tau}\left(\frac{W}{\varepsilon} - \tfrac{1}{2}\varepsilon|\nabla u|^2\right)^+ \mathrm{d}\tau \\ &\qquad\qquad - 2c_{10}r^{1/2}\mathrm{e}^{c_7 r} - c_8c_7^{-1}(\mathrm{e}^{c_7 r} - \mathrm{e}^{c_7 s}).\end{aligned}$$

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The first term in the right-hand side is estimated from below by $-c(r^{1/2}+r^{3/2})$. Thus, with a suitable restriction on r, we obtain (3.10). □

For the rest of the section, we prove proposition 3.2. First, we need the following lemma, showing that $|u|$ stays within $1+\varepsilon^\eta$ for $\eta<1$. In case f is bounded in the L^∞-norm, we can use the pointwise maximum principle as in [23, proposition 3.2]. Here we need to use integral estimates.

PROPOSITION 3.7. *There exists a constant ε_3, which depends only on λ_0, c_0, n, $0<\eta<1$, $\operatorname{dist}(\tilde U,\partial U)$ and W, such that*

$$\sup_{\tilde U}|u|\leqslant 1+\varepsilon^\eta \tag{3.11}$$

whenever $\varepsilon\leqslant\varepsilon_3$.

Proof. Suppose $B_1\subset\tilde U$. For any $p\geqslant 1$, multiply both sides of (1.1) by $[(u-1)^+]^p\phi^2$, where $\phi\in C_c^\infty(B_1)$, $\phi\geqslant 0$, and integrate by parts. Then we obtain

$$\begin{aligned}-\varepsilon\int p[(u-1)^+]^{p-1}|\nabla u|^2\phi^2+[(u-1)^+]^p2\phi\nabla\phi\cdot\nabla u&\\ =\int\frac{W'}{\varepsilon}[(u-1)^+]^p\phi^2-\int f[(u-1)^+]^p\phi^2.&\end{aligned}$$

For $u\geqslant 1$, $W'(u)\geqslant\kappa(u-1)$ by assumption B. Hence

$$\begin{aligned}\frac{\kappa}{\varepsilon}\int[(u-1)^+]^{p+1}\phi^2&+\int\varepsilon p[(u-1)^+]^{p-1}|\nabla u|^2\phi^2\\ &\leqslant 2\varepsilon\int[(u-1)^+]^p\phi|\nabla\phi||\nabla u|+\int|f|[(u-1)^+]^p\phi^2\\ &\leqslant\tfrac{1}{2}p\varepsilon\int[(u-1)^+]^{p-1}|\nabla u|^2\phi^2+\frac{2\varepsilon}{p}\int[(u-1)^+]^{p+1}|\nabla\phi|^2\\ &\qquad+\frac{\kappa}{2\varepsilon}\int[(u-1)^+]^{p+1}\phi^2+\frac{\varepsilon^pc(p)}{\kappa^p}\int|f|^{p+1}\phi^2,\end{aligned}$$

which shows that

$$\frac{\kappa}{2\varepsilon}\int[(u-1)^+]^{p+1}\phi^2\leqslant\frac{2\varepsilon}{p}\int[(u-1)^+]^{p+1}|\nabla\phi|^2+\frac{\varepsilon^pc(p)}{\kappa^p}\int|f|^{p+1}\phi^2.$$

Since $\|f\|_{L^{p+1}}\leqslant c(n,p)\|f\|_{W^{1,n}}\leqslant c(n,p)\lambda_0$ by the Sobolev inequality for any $p<\infty$, and since $|u|$ is bounded by c_0, we obtain

$$\int_{B_{1-s}}[(u-1)^+]^{p+1}\leqslant c_{12}(s,p,c_0,\lambda_0,\kappa)\varepsilon^{p+1}$$

by iterating the above estimate. To derive a contradiction, assume that $u(x_0)-1\geqslant\varepsilon^\eta$ for some $x_0\in B_{1-s}$. By the gradient estimate, $|\nabla u|\leqslant c_5\varepsilon^{-1}$ (as in the proof of (3.5)). Thus, for $y\in B_{\varepsilon^{1+\eta}/2c_5}(x_0)$,

$$u(y)-1\geqslant u(x_0)-1-\sup|\nabla u|\cdot\frac{\varepsilon^{1+\eta}}{2c_5}\geqslant\tfrac{1}{2}\varepsilon^\eta.$$

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Then

$$c_{12}\varepsilon^{p+1} \geqslant \int_{B_{\varepsilon^{1+\eta}/2c_5}(x_0)} [(u-1)^+]^{p+1} \geqslant \frac{\omega_n \varepsilon^{\eta(p+1)}}{2^{p+1}} \left(\frac{\varepsilon^{1+\eta}}{2c_5}\right)^n.$$

The right-hand side is of order $\varepsilon^{p\eta+\eta(n+1)+n}$. Since $\eta < 1$, for sufficiently large p ($p > (\eta(n+1)+n-1)/(1-\eta)$) and small ε, the inequality cannot hold. This would be a contradiction. Hence we obtain (3.11). $u \geqslant -1-\varepsilon^\eta$ is proved similarly. □

It is convenient to work in the ε-scale, namely, we consider $B_{3d} \subset \tilde{U}$ and consider the rescaled problem

$$-\Delta u + W'(u) = \varepsilon f \quad \text{on } B_{3d/\varepsilon}.$$

Without loss of generality, we choose a suitable unit so that $d=1$. Also, we denote the rescaled discrepancy function by

$$\xi = \tfrac{1}{2}|\nabla u|^2 - W(u).$$

With this scale, we need to prove

$$\sup_{B_{\varepsilon^{-1}}} \xi \leqslant c_1 \varepsilon^{3/5}. \tag{3.12}$$

In the following, we use

$$\xi_G(x) = \tfrac{1}{2}|\nabla u|^2 - W(u) - G(u),$$

where $G : \mathbb{R} > \mathbb{R}$ will be fixed shortly. We first obtain a differential inequality for ξ_G (cf. [14,23,28]). For the reader's convenience, we supply the proof.

LEMMA 3.8. *On* $|\nabla u| > 0$,

$$\begin{aligned}\Delta\xi_G - \frac{2(W'+G')\nabla u}{|\nabla u|^2} \cdot \nabla\xi_G + 2G''\xi_G \\ \geqslant (G')^2 + G'W' - 2G''(W+G) + \varepsilon f(W'+G') - \varepsilon\nabla f \cdot \nabla u.\end{aligned} \tag{3.13}$$

Proof. Compute

$$\begin{aligned}\Delta\xi_G &= |\nabla^2 u|^2 + \sum u_{x_i}(\Delta u)_{x_i} - W''|\nabla u|^2 - W'\Delta u - G''|\nabla u|^2 - G'\Delta u \\ &= |\nabla^2 u|^2 - (W'+G')\Delta u - G''|\nabla u|^2 - \varepsilon\nabla u \cdot \nabla f \quad (\text{by } \Delta u = W' - \varepsilon f) \\ &= |\nabla^2 u|^2 - (W'+G')(W'-\varepsilon f) - 2G''(G+W+\xi_G) - \varepsilon\nabla u\cdot\nabla f,\end{aligned}$$

where the last line is derived by substituting $|\nabla u|^2 = 2(G+W+\xi_G)$. On the other hand,

$$\begin{aligned}|\nabla u|^2|\nabla^2 u|^2 &\geqslant \sum_j \left(\sum_i u_{x_i}u_{x_ix_j}\right)^2 \\ &= \sum_j ((\xi_G)_{x_j} + (W'+G')u_{x_j})^2 \\ &\geqslant 2(W'+G')\nabla u\cdot\nabla\xi_G + (W'+G')^2|\nabla u|^2.\end{aligned}$$

We may then conclude (3.13). □

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Lemma 3.9. *Suppose* $0 < \eta$, $\beta < 1$, $0 \leqslant \iota \leqslant 1$ *satisfy* $2\beta + \iota \geqslant 2\eta$, $0 < s < 1$, $0 < c_{11} < 1$. *Then there exist* $0 < \varepsilon_4 < \varepsilon_3$, $0 < c_{12}$, $c_{13} < \infty$, *depending only on* η, β, ι, s, c_0 λ_0, c_{11}, $\operatorname{dist}(\tilde{U}, \partial U)$, n *and* W, *with the following properties.*

Suppose

(A) $u \in C^3(B_{\varepsilon^{-\beta}})$, $f \in C^1(B_{\varepsilon^{-\beta}})$ *and* $\varepsilon \leqslant \varepsilon_4$ *satisfy*

$$-\Delta u + W'(u) = \varepsilon f$$

on $B_{\varepsilon^{-\beta}}$; *and*

(B) $\sup_{B_{\varepsilon^{-\beta}}} |u| \leqslant 1 + \varepsilon^\eta$, $\sup_{B_{\varepsilon^{-\beta}}} \xi \leqslant c_{11}\varepsilon^\iota$.

Then

$$\sup_{B_{(1-s)\varepsilon^{-\beta}}} \xi \leqslant c_{12}\{\varepsilon^{1-\beta}(\|\nabla f\|_{L^n(B_{\varepsilon^{-\beta}})} + \|f\|_{L^n(B_{\varepsilon^{-\beta}} \cap \{|f| \geqslant c_{13}\varepsilon^{\eta-1}\})}) + \varepsilon^\eta\}. \quad (3.14)$$

Proof. We choose $\tilde{\phi} \in C^\infty(B_1)$ such that

$$\tilde{\phi}(x) = \begin{cases} 1 & \text{on } B_1 \setminus B_{1-s/2}, \\ 0 & \text{on } B_{1-s}, \end{cases}$$

$0 \leqslant \tilde{\phi} \leqslant 1$, $|\nabla\tilde{\phi}|, |\nabla^2\tilde{\phi}|, |\nabla\tilde{\phi}|/\sqrt{\tilde{\phi}} \leqslant c_s$, where we denote generic positive constants depending only on s and n as c_s. For simplicity, we use the same notation c_s for such constants. Then define

$$\phi(x) = c_{11}\varepsilon^\iota\tilde{\phi}(\varepsilon^\beta x)$$

for $x \in B_{\varepsilon^{-\beta}}$. With this definition and by the properties of $\tilde{\phi}$, we have

$$|\nabla\phi| \leqslant c_{11}c_s\varepsilon^{\beta+\iota}, \qquad |\nabla^2\phi| \leqslant c_{11}c_s\varepsilon^{2\beta+\iota}, \qquad |\nabla\phi|/\sqrt{\phi} \leqslant c_{11}^{1/2}c_s\varepsilon^{\beta+\iota/2}. \quad (3.15)$$

Let $G(u) = c_{14}\varepsilon^\eta(1 - \frac{1}{8}(u-\gamma)^2)$, where $c_{14} \geqslant 1$ will be determined later, and define

$$\tilde{\xi} = \xi - G - \phi \quad (= \xi_G - \phi).$$

By restricting ε to depend only on γ and by assumption (B), we have $G > 0$. We later use $G'W' \geqslant 0$ on $u \in [-1, 1]$ as well as $G'' < 0$ for any u. On $B_{\varepsilon^{-\beta}} \setminus B_{(1-s/2)\varepsilon^{-\beta}}$, $\phi = c_{11}\varepsilon^\iota$, hence by assumption (B), $\tilde{\xi} \leqslant 0$ there. Also, $\tilde{\xi} = \xi - G$ on $B_{(1-s)\varepsilon^{-\beta}}$, since $\phi = 0$ there. In particular, if $\tilde{\xi} \leqslant 0$ on $B_{\varepsilon^{-\beta}}$, then $\xi \leqslant G \leqslant c_{14}\varepsilon^\eta$ on $B_{(1-s)\varepsilon^{-\beta}}$. In the following, we denote

$$M = \sup_{B_{\varepsilon^{-\beta}}} \tilde{\xi}^+$$

and assume that $M > 0$ (or else we are done). We next apply the ABP maximum principle to $\tilde{\xi}^+$ on $B_{\varepsilon^{-\beta}}$. Note that $\tilde{\xi}^+ = 0$ on $\partial B_{\varepsilon^{-\beta}}$. Let Γ^+ be the upper contact set of the graph of $\tilde{\xi}^+$. Then, with the diameter of the domain being $2\varepsilon^{-\beta}$, we have [21, lemma 9.3]

$$\frac{M}{2\varepsilon^{-\beta}} \leqslant \frac{1}{n\omega_n^{1/n}}\left(\int_{\Gamma^+} (-\Delta\tilde{\xi})^n\right)^{1/n}.$$

Note that $\tilde{\xi}^+ = \tilde{\xi}$ and $\Delta\tilde{\xi} \leqslant 0$ on Γ^+, and the slope of the support hyperplane is equal to the gradient of $\tilde{\xi}$. Since such hyperplanes have slope less than $M\varepsilon^\beta/s$, we have

$$|\nabla\tilde{\xi}| \leqslant \frac{M\varepsilon^\beta}{s} \quad \text{on } \Gamma^+. \tag{3.16}$$

Also, since $\tilde{\xi}^+ = 0$ on $B_{\varepsilon^{-\beta}} \setminus B_{(1-s/2)\varepsilon^{-\beta}}$, the height of the contact points of the graph of $\tilde{\xi}^+$ and the supporting hyperplanes have a lower bound,

$$\tilde{\xi} = \tilde{\xi}^+ \geqslant \tfrac{1}{2}sM \quad \text{on } \Gamma^+. \tag{3.17}$$

In the following, we carefully estimate $-\Delta\tilde{\xi}$ from above. In doing so, we need to consider three cases, and define three subsets of $\Gamma^+ \cap B_{\varepsilon^{-\beta}}$ by

$$\begin{aligned}
\mathcal{A} &= \{x \mid |u(x)| > 1\},\\
\mathcal{B} &= \{x \mid \tfrac{1}{2}(\gamma+1) < |u(x)| \leqslant 1\},\\
\mathcal{C} &= \{x \mid |u(x)| \leqslant \tfrac{1}{2}(\gamma+1)\}.
\end{aligned}$$

Note that they are mutually disjoint sets by the definition. By (3.13), we have

$$\begin{aligned}
-\Delta\tilde{\xi} &= -\Delta\xi_G + \Delta\phi\\
&\leqslant \bigg\{-\frac{2(W'+G')\nabla u}{|\nabla u|^2}\cdot\nabla\xi_G\bigg\}\\
&\qquad + \{2G''\xi_G - (G')^2\} + \{-G'W' + 2G''(W+G)\}\\
&\qquad + \{-\varepsilon f(W'+G') + \Delta\phi\} + \{\varepsilon\nabla u\cdot\nabla f\}\\
&\equiv \{\mathrm{I}\} + \{\mathrm{II}\} + \{\mathrm{III}\} + \{\mathrm{IV}\} + \{\mathrm{V}\}.
\end{aligned}$$

(I) Since $\xi_G = \tilde{\xi} + \phi$, using

$$|\nabla u| = \sqrt{2}\sqrt{\tilde{\xi} + W + G + \phi} \geqslant \sqrt{\tilde{\xi}} + \sqrt{\phi},$$

we have

$$\begin{aligned}
\mathrm{I} &\leqslant \frac{2}{|\nabla u|}(|W'| + c_{14}\varepsilon^\eta)|\nabla\tilde{\xi} + \nabla\phi|\\
&\leqslant 2(|W'| + c_{14}\varepsilon^\eta)\bigg(\frac{|\nabla\tilde{\xi}|}{\sqrt{\tilde{\xi}}} + \frac{|\nabla\phi|}{\sqrt{\phi}}\bigg)\\
&\leqslant 2(|W'| + c_{14}\varepsilon^\eta)\bigg(\sqrt{\frac{2M}{s^3}}\varepsilon^\beta + c_{11}^{1/2}c_s\varepsilon^{\beta+\iota/2}\bigg)\\
&\leqslant (|W'| + c_{14}\varepsilon^\eta)c_s c_{11}^{1/2}\varepsilon^{\beta+\iota/2}.
\end{aligned}$$

The last two lines are by (3.15)–(3.17), as well as $M \leqslant c_{11}\varepsilon^\iota$. By separating into three cases, we find

$$\mathrm{I} \leqslant c_s c_{11}^{1/2}\varepsilon^{\beta+\iota/2} \times \begin{cases} (\sup|W''|\varepsilon^\eta + c_{14}\varepsilon^\eta) & \text{on } \mathcal{A},\\ (|W'| + c_{14}\varepsilon^\eta) & \text{on } \mathcal{B},\\ (\sup|W'| + c_{14}\varepsilon^\eta) & \text{on } \mathcal{C}. \end{cases}$$

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(II) Since $\xi_G = \tilde{\xi} + \phi \geqslant 0$ on Γ^+ and $G'' < 0$, we have

$$\mathrm{II} \leqslant 0.$$

(III) By the choice of G, both terms are non-positive on $\mathcal{B}$ and $\mathcal{C}$. Moreover, since $G'' = -\frac{1}{4}c_{14}\varepsilon^\eta$ and $G \geqslant \frac{1}{2}c_{14}\varepsilon^\eta$ for $|u| \leqslant 1 + \varepsilon^\eta$ for suitably small ε, we have

$$\mathrm{III} \leqslant -\tfrac{1}{8}(c_{14}\varepsilon^\eta)^2 \begin{cases} +c_{14}\varepsilon^{2\eta}\sup|W''| & \text{on } \mathcal{A}, \\ -\frac{1}{8}c_{14}\varepsilon^\eta(1-\gamma)|W'| & \text{on } \mathcal{B}, \\ -\frac{1}{4}c_{14}\varepsilon^\eta \min_{|u|\leqslant(\gamma+1)/2} W(u) & \text{on } \mathcal{C}. \end{cases}$$

The first term comes from $G''G$. Note that $\min_{|u|\leqslant(\gamma+1)/2} W(u) > 0$ is a strictly positive constant.

(IV) By (3.11) and (3.15),

$$\begin{aligned} \mathrm{IV} &\leqslant |f|\varepsilon(|W'| + c_{14}\varepsilon^\eta) + c_{11}c_s\varepsilon^{2\beta+\iota} \\ &\leqslant c_{11}c_s\varepsilon^{2\beta+\iota} + \begin{cases} (\varepsilon^{1+\eta}\sup|W''| + \varepsilon^{1+\eta}c_{14})|f| & \text{on } \mathcal{A}, \\ (\varepsilon|W'| + c_{14}\varepsilon^{1+\eta})|f| & \text{on } \mathcal{B}, \\ (\varepsilon\sup|W'| + c_{14}\varepsilon^{1+\eta})|f| & \text{on } \mathcal{C}. \end{cases} \end{aligned}$$

Next we sum the four terms on each set and evaluate them from above. Note that the terms in III are 'good terms', giving the necessary negative contributions.

($\mathrm{I} + \cdots + \mathrm{IV}$ on $\mathcal{A}$)

$$\begin{aligned} \mathrm{I} + \cdots + \mathrm{IV} \leqslant c_{14}^2\varepsilon^{2\eta}\biggl(&\frac{c_s c_{11}^{1/2}}{c_{14}^2}\sup|W''|\varepsilon^{\beta+\iota/2-\eta} + \frac{c_s c_{11}^{1/2}}{c_{14}}\varepsilon^{\beta+\iota/2-\eta} \\ &- \frac{1}{8} + \frac{\sup|W''|}{c_{14}} + \frac{c_{11}c_s}{c_{14}^2}\varepsilon^{2\beta+\iota-2\eta}\biggr) \\ &+ (\varepsilon^{1+\eta}\sup|W''| + \varepsilon^{1+\eta}c_{14})|f|. \end{aligned}$$

Since $2\eta \leqslant 2\beta + \iota$, we may restrict c_{14} large, depending only on c_{11}, c_s, W, so that

$$\begin{aligned} \mathrm{I} + \cdots + \mathrm{IV} &\leqslant -\tfrac{1}{16}c_{14}^2\varepsilon^{2\eta} + (\varepsilon^{1+\eta}\sup|W''| + \varepsilon^{1+\eta}c_{14})|f| \\ &\leqslant c_{15}\varepsilon^{1+\eta}|f| \cdot \chi_{\{|f|\geqslant c_{16}\varepsilon^{\eta-1}\}} \end{aligned} \tag{3.18}$$

for suitable choices of c_{15}, $c_{16} = c(c_{11}, c_s, W) > 0$. Here, χ_A denotes the characteristic function of A.

($\mathrm{I} + \cdots + \mathrm{IV}$ on $\mathcal{B}$)

$$\begin{aligned} \mathrm{I} + \cdots + \mathrm{IV} \leqslant |W'|c_{14}\varepsilon^\eta&\biggl(\frac{c_s c_{11}^{1/2}}{c_{14}}\varepsilon^{\beta+\iota/2-\eta} - \tfrac{1}{8}(1-\gamma)\biggr) \\ &+ (c_{14}\varepsilon^\eta)^2\biggl(\frac{c_s c_{11}^{1/2}}{c_{14}}\varepsilon^{\beta+\iota/2-\eta} - \frac{1}{8} + \frac{c_{11}c_s}{c_{14}^2}\varepsilon^{2\beta+\iota-2\eta}\biggr) \\ &+ (\varepsilon|W'| + c_4\varepsilon^{1+\eta})|f|. \end{aligned}$$

Again with $2\eta \leqslant 2\beta + \iota$ and restricting c_{14} large, we have

$$\begin{aligned} \mathrm{I} + \cdots + \mathrm{IV} &\leqslant -\tfrac{1}{16}(1-\gamma)c_{14}\varepsilon^{\eta}|W'| - \tfrac{1}{16}c_{14}^2\varepsilon^{2\eta} + (\varepsilon|W'| + c_{14}\varepsilon^{1+\eta})|f| \\ &\leqslant c_{15}\varepsilon|f| \cdot \chi_{\{|f| \geqslant c_{16}\varepsilon^{\eta-1}\}}, \end{aligned} \tag{3.19}$$

where c_{15}, c_{16} are chosen appropriately again.

($\mathrm{I} + \cdots + \mathrm{IV}$ on $\mathcal{C}$)

$$\begin{aligned} \mathrm{I} + \cdots + \mathrm{IV} \leqslant c_{14}\varepsilon^{\eta}\biggl(&\frac{c_s c_{11}^{1/2} \sup|W'|}{c_{14}}\varepsilon^{\beta+\iota/2-\eta} + c_s c_{11}^{1/2}\varepsilon^{\beta+\iota/2} \\ &- \frac{1}{4}\min_{|u|\leqslant(\gamma+1)/2} W(u) + \frac{c_{11}c_s}{c_{14}}\varepsilon^{2\beta+\iota-\eta}\biggr) \\ &+ (\varepsilon\sup|W'| + c_{14}\varepsilon^{1+\eta})|f|. \end{aligned}$$

Using $0 < \eta \leqslant \beta + \frac{1}{2}\iota$ and restricting ε small and c_{14} large, depending only on η, W, c_{11} and c_s, we have

$$\begin{aligned} \mathrm{I} + \cdots + \mathrm{IV} &\leqslant -\tfrac{1}{8}c_{14}\varepsilon^{\eta}\min_{|u|\leqslant(\gamma+1)/2} W(u) + (\varepsilon\sup|W'| + c_{14}\varepsilon^{1+\eta})|f| \\ &\leqslant c_{15}\varepsilon|f| \cdot \chi_{\{|f| \geqslant c_{16}\varepsilon^{\eta-1}\}} \end{aligned} \tag{3.20}$$

for suitable choices of c_{15}, c_{16}.

($\mathrm{I} + \cdots + \mathrm{V}$) Combining (3.18)–(3.20) and $|\nabla u| \leqslant c_5$, we have

$$\mathrm{I} + \cdots + \mathrm{V} \leqslant c_5\varepsilon|\nabla f| + c_{15}\varepsilon|f| \cdot \chi_{\{|f| \geqslant c_{16}\varepsilon^{\eta-1}\}}$$

on Γ^+. Thus we have

$$\biggl(\int_{\Gamma^+}(-\Delta\tilde{\xi})^n\biggr)^{1/n} \leqslant c_5\varepsilon\|\nabla f\|_{L^n(\Gamma^+)} + c_{15}\varepsilon\|f\|_{L^n(\Gamma^+\cap\{|f|\geqslant c_{16}\varepsilon^{\eta-1}\})}.$$

Since

$$\sup_{B_{\varepsilon^{-\beta}(1-s)}}\tilde{\xi} \leqslant M \leqslant \frac{2\varepsilon^{-\beta}}{n\omega_n^{1/n}}\biggl(\int_{\Gamma^+}(-\Delta\tilde{\xi})^n\biggr)^{1/n},$$

$\tilde{\xi} = \xi - G$ and $G \leqslant c_{14}\varepsilon^{\eta}$, we have the desired estimate by suitably choosing c_{12} and c_{13}. □

Here we give a proof for (3.12). By the Sobolev inequality (applied to the original scale), for any $p < \infty$, we have

$$\biggl(\varepsilon^n\int_{B_{(1-s)\varepsilon^{-1}}}|f|^p\biggr)^{1/p} \leqslant c(p,n,s)\lambda_0.$$

In particular, for any $t \geqslant 1$,

$$\begin{aligned} \biggl(\int_{B_{(1-s)\varepsilon^{-1}}\cap\{|f|\geqslant t\}}|f|^n\biggr)^{1/n} &\leqslant t^{1-p/n}\biggl(\int_{B_{(1-s)\varepsilon^{-1}}}|f|^p\biggr)^{1/n} \\ &\leqslant t^{1-p/n}\varepsilon^{-1/p}c(p,n,s)\lambda_0. \end{aligned}$$

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Thus, if $t = \varepsilon^{\eta-1}$, with $0 < \eta$, with an appropriately large p, the left-hand side of (3.14) is bounded in terms of λ_0 uniformly for all small ε. Now we use (3.14) with $\eta = \beta = \frac{1}{2}$ and $\iota = 0$ to any ball $B_{\varepsilon^{-1/2}}(x) \subset B_{\varepsilon^{-1}}$. Condition (B) is satisfied for a suitable constant c_{11}, and $2\beta + \iota \geqslant 2\eta$ is satisfied. Thus we have

$$\sup_{B_{(1-s)\varepsilon^{-1}}} \xi \leqslant c_{12}(\varepsilon^{1/2} + \varepsilon^{1/2}\|\nabla f\|_{L^n} + \varepsilon^{1/2}\|f\|_{L^n(B_{\varepsilon^{-1}}\cap\{|f|\geqslant c_{13}\varepsilon^{-1/2}\})}).$$

Next we apply lemma 3.9 again, with $\eta = \frac{3}{5}$, $\beta = \frac{2}{5}$, $\iota = \frac{1}{2}$ and the new c_{11}. With this choice, $2\eta \leqslant 2\beta + \iota$ is satisfied. Since $1 - \beta = \eta = \frac{3}{5}$, we obtain (3.12).

REMARK 3.10. The exponent $\frac{3}{5}$ in (3.12) is simply a convenient choice for us, but the argument works just as well, as long as the exponent is strictly larger than $\frac{1}{2}$. If we assume that $f \in W^{1,p}$ for p sufficiently larger than n (for example, $p \geqslant 2n$), then we can prove a better estimate with the exponent equal to 1. As far as we can see, however, we could not obtain such estimate for the case $p = n$.

4. Rectifiability and integrality of the limit varifold

PROPOSITION 4.1. *There exist constants $0 < D_1 \leqslant D_2 < \infty$ and $r_0 > 0$, which depend only on λ_0, c_0, E_0, $\operatorname{dist}(\tilde{U}, \partial U)$ and W, such that*

$$D_1 r^{n-1} \leqslant \mu(B_r(x)) \leqslant D_2 r^{n-1}$$

for all $0 < r < r_0$, $x \in \operatorname{supp}\mu \cap \tilde{U}$ and $B_r(x) \subset \tilde{U}$.

Proof. The existence of D_2 is immediate from (3.10).

To establish the lower bound, let $x \in \operatorname{supp}\mu \cap \tilde{U}$.

CLAIM. *On passing to a subsequence, there exist $x_i \in \tilde{U}$ such that $u^i(x_i) \in [-\alpha, \alpha]$ and $x_i \to x$ as $i \to \infty$.*

Proof of the claim. Suppose the converse. Then there exists some $s > 0$ such that $B_s(x) \subset \tilde{U}$ and $B_s(x) \cap \{|u^i| \leqslant \alpha\} = \emptyset$ for all sufficiently large i. For each such i, either $u^i > \alpha$ on $B_s(x)$ or $u^i < -\alpha$ on $B_s(x)$. If $u^i > \alpha$, by using the argument in proposition 3.7, one shows that

$$u^i \in [1 - \varepsilon_i^{3/4}, 1 + \varepsilon_i^{3/4}] \quad \text{on } B_{s/2}(x)$$

for all sufficiently large i. Similarly, if $u^i < -\alpha$, then $u^i \in [-1 - \varepsilon_i^{3/4}, -1 + \varepsilon_i^{3/4}]$ on $B_{s/2}(x)$. This implies that $W(u^i) = O(\varepsilon_i^{3/2})$ and thus $W(u^i)/\varepsilon_i \to 0$ uniformly on $B_{s/2}(x)$ as $i \to \infty$.

For $\frac{1}{2}\varepsilon_i|\nabla u^i|^2$, by using the argument in proposition 3.5 (estimate on $\mathcal{C}$), one shows that

$$\int_{B_{s/2}(x)} \varepsilon_i |\nabla u^i|^2 \leqslant O(\varepsilon_i^2) \to 0$$

as $i \to \infty$. Hence we may conclude that $\mu(B_{s/2}(x)) = 0$, which is a contradiction to $x \in \operatorname{supp}\mu$. This ends the proof of the claim. □

For any $x \in \tilde{U} \cap \operatorname{supp} \mu$, $B_r(x) \subset \tilde{U}$ and $r \leqslant r_0$, propositions 3.4 and 3.6 show that

$$\begin{aligned}\frac{1}{r^{n-1}}\mu(B_r(x)) &\geqslant \lim_{i\to\infty} \frac{1}{r^{n-1}} \int_{B_{r/2}(x_i)} \tfrac{1}{2}\varepsilon_i|\nabla u^i|^2 + \frac{W(u^i)}{\varepsilon_i} \\ &\geqslant \liminf 2^{1-n} \mathrm{e}^{c_7 r/2}\{-c_9 r^{1/2} + \mathrm{e}^{-c_7 c_3 \varepsilon_i^{2/5}} E(c_3 \varepsilon_i^{2/5}, x_i)\} \\ &\geqslant 2^{1-n} \mathrm{e}^{-c_7 r/2}\{-c_9 r^{1/2} + \mathrm{e}^{-c_7 c_3} c_4\}.\end{aligned}$$

Thus, by suitably restricting r, we prove the existence of D_1. □

PROPOSITION 4.2. *Either $u^i \to +1$ or $u^i \to -1$ uniformly on each compact subset of $U \setminus \operatorname{supp} \|V\|$. In particular,* $\operatorname{supp} \|\partial\{u^\infty = 1\}\| \subset \operatorname{supp} \|V\|$. *The terms $\frac{1}{2}\varepsilon_i|\nabla u^i|^2$ and $\varepsilon_i^{-1} W(u^i)$ converge uniformly to zero on compact subsets of $U \setminus \operatorname{supp} \|V\|$.*

Proof. This follows immediately from the argument for the previous proposition. □

Let

$$\xi^i = \tfrac{1}{2}\varepsilon_i|\nabla u^i|^2 - \frac{W(u^i)}{\varepsilon_i}$$

and define (passing to a subsequence if necessary) the measure $|\xi|$ on U by

$$|\xi|(\phi) = \lim_{i\to\infty} \int |\xi^i|\phi$$

for non-negative $\phi \in C_c(U)$. Thus $|\xi|$ is the measure theoretic limit of the absolute values of the discrepancy functions.

PROPOSITION 4.3. *$|\xi|$ is the zero measure and so $\xi^i \to 0$ in $L^1_{\mathrm{loc}}(U)$. Moreover, both $\frac{1}{2}\varepsilon_i|\nabla u^i|^2 - |\nabla w^i|$ and $(1/\varepsilon_i)W(u^i) - |\nabla w^i|$ also converge to zero in $L^1_{\mathrm{loc}}(U)$.*

Proof. First we claim that

$$\liminf_{r\to 0} \frac{1}{r^{n-1}}|\xi|(B_r(x)) = 0 \tag{4.1}$$

for all $x \in \operatorname{supp}|\xi| \cap \tilde{U}$. Otherwise, there would exist $x \in \operatorname{supp}|\xi| \cap \tilde{U}$, $R > 0$ and $b > 0$ such that $R \leqslant r_0$ and $|\xi|(B_r(x)) \geqslant br^{n-1}$ for all $0 < r \leqslant R$. Define

$$\begin{aligned} r_1 &= \min\{b(8D_2c_7 + c_8)^{-1}, R\}, \\ r_2 &= r_1 \min\{\exp[-4b^{-1}(c_9 r_1^{1/2} + 4D_2 \exp[c_7 r_1])], \tfrac{1}{2}\}.\end{aligned}$$

By proposition 4.1 and the definition of $|\xi|$, we may choose a large enough i such that $c_3\varepsilon_i^{2/5} \leqslant r_2$ and

$$\frac{1}{\tau^{n-1}} \int_{B_\tau(x)} \tfrac{1}{2}\varepsilon_i|\nabla u^i|^2 + \frac{W(u^i)}{\varepsilon_i} \leqslant 2D_2, \qquad \frac{1}{\tau^{n-1}} \int_{B_\tau(x)} |\xi^i| \geqslant \tfrac{1}{2}b$$

for all $r_2 \leqslant \tau \leqslant r_1$. By proposition 3.5 and the definition of r_1,

$$\begin{aligned}\frac{1}{\tau^{n-1}}\int_{B_\tau(x)}\left(\frac{W(u^i)}{\varepsilon_i}-\tfrac{1}{2}\varepsilon_i|\nabla u^i|^2\right)^+ &\geqslant \frac{1}{\tau^{n-1}}\int_{B_\tau(x)}|\xi^i|-(c_7E(u^i,\tau,x)+c_8)\tau\\ &\geqslant \tfrac{1}{2}b-(c_7 2D_2+c_8)r_1\\ &\geqslant \tfrac{1}{4}b\end{aligned}$$

for all $r_2 \leqslant \tau \leqslant r_1$. By proposition 3.6,

$$\begin{aligned}2D_2 &\geqslant \frac{1}{r_1^{n-1}}\int_{B_{r_1}(x)}\tfrac{1}{2}\varepsilon_i|\nabla u^i|^2+\frac{W(u^i)}{\varepsilon_i}\\ &\geqslant \mathrm{e}^{-c_7r_1}\left(\tfrac{1}{4}b\int_{r_2}^{r_1}\frac{\mathrm{d}\tau}{\tau}-c_9r_1^{1/2}\right)\\ &= \mathrm{e}^{-c_7r_1}\left(\tfrac{1}{4}b\ln\left(\frac{r_1}{r_2}\right)-c_9r_1^{1/2}\right)\\ &\geqslant 4D_2,\end{aligned}$$

which is a contradiction, so we have proved (4.1).

Combined with proposition 4.1 and $\operatorname{supp}|\xi| \subset \operatorname{supp}\mu$, we have

$$\liminf_{r\to 0}\frac{|\xi|(B_r(x))}{\mu(B_r(x))}=0$$

for all $x \in \operatorname{supp}|\xi|$. A standard result in measure theory then shows that $|\xi| = 0$.

It follows that $\xi^i > 0$ in $L^1_{\mathrm{loc}}(U)$.

By completing the square and using $2|\nabla w^i| = \sqrt{2W(u^i)}|\nabla u^i|$, we see that

$$\begin{aligned}\left|\tfrac{1}{2}\varepsilon_i|\nabla u^i|^2+\frac{W(u^i)}{\varepsilon_i}-2|\nabla w^i|\right| &= \left(\sqrt{\tfrac{1}{2}\varepsilon_i}|\nabla u^i|-\sqrt{\frac{W(u^i)}{\varepsilon_i}}\right)^2\\ &\leqslant \left|\tfrac{1}{2}\varepsilon_i|\nabla u^i|^2-\frac{W(u^i)}{\varepsilon_i}\right|\\ &= |\xi^i|.\end{aligned}$$

This implies the remaining claims in the proposition. ☐

Proposition 4.4. *The limit varifold V satisfies $\|V\| = \frac{1}{2}\mu$ and is rectifiable. The first variation of V is given by*

$$\delta V(g)=\frac{1}{2}\int_U u^\infty \operatorname{div}(f^\infty g)=-\int_{M^\infty} f^\infty g\cdot\nu^\infty\,\mathrm{d}\mathcal{H}^{n-1}$$

for any $g \in C^1_c(U;\mathbb{R}^n)$, where $M^\infty \subset \operatorname{supp}\|V\|$ is the reduced boundary of $\{u^\infty = 1\}$ and f^∞ on M^∞ is the trace of $f^\infty \in W^{1,n}(U)$. The generalized mean curvature vector H is given by

$$H(x)=\begin{cases}\dfrac{f^\infty(x)}{\theta(x)}\nu^\infty(x), & \mathcal{H}^{n-1} \text{ a.e. } x\in M^\infty,\\ 0, & \mathcal{H}^{n-1} \text{ a.e. } x\in \operatorname{supp}\|V\|\setminus M^\infty,\end{cases}$$

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where θ is the density function for $\|V\|$. Moreover,

$$f^\infty \lfloor M^\infty \in L^p_{\mathrm{loc}}(U, \mathcal{H}^{n-1}) \quad \textit{for any } 1 \leqslant p < \infty.$$

Proof. Since $\|V\| = \lim \|V^i\|$ and $\|V^i\| = |\nabla w^i| \, \mathrm{d}\mathcal{L}^n$, it follows from proposition 4.3 and the definition of μ that $\frac{1}{2}\mu = \|V\|$. Next, we rearrange terms in (3.2) and, using the fact that

$$\frac{u^i_{x_j}}{|\nabla u^i|} = \frac{w^i_{x_j}}{|\nabla w^i|},$$

we have

$$\int \biggl(\operatorname{div} g - \sum_{j,k} \frac{w^i_{x_j}}{|\nabla w^i|} \frac{w^i_{x_k}}{|\nabla w^i|} g^j_{x_k} \biggr) \varepsilon_i |\nabla u^i|^2 = \int \biggl\{ \biggl(\tfrac{1}{2}\varepsilon_i |\nabla u^i|^2 - \frac{W(u^i)}{\varepsilon_i} + u^i f^i \biggr) \operatorname{div} g + u^i g \cdot \nabla f^i \biggr\}$$

for any $g \in C^1_c(U)$. Since $\varepsilon_i |\nabla u^i|^2 - 2|\nabla w^i|$ and ξ^i converge to 0, u^i converges to u^∞ strongly in L^p for any $1 < p < \infty$ and f^i converges weakly in $W^{1,n}$ to f^∞, we have

$$\delta V(g) = \frac{1}{2} \int u^\infty (f^\infty \operatorname{div} g + g \cdot \nabla f^\infty).$$

Here we used (2.2), and note that V^i converges to V in the sense of varifold, hence $\delta V^i(g) \to \delta V(g)$. To justify the integration by parts for $f^\infty g$, we use the following theorem due to Meyers and Ziemer [27] and the idea to use it for the similar purpose is due to Schätzle [35, theorem 1.3].

THEOREM 4.5 (cf. theorem 5.12.4 of [43]). *Let μ be a positive Radon measure on $\mathbb{R}^n$ satisfying*

$$K(\mu) \equiv \sup_{\substack{x \in \mathbb{R}^n, \\ r > 0}} \frac{1}{r^{n-1}} \mu(B_r(x)) < \infty.$$

Then

$$\biggl| \int_{\mathbb{R}^n} \phi \, \mathrm{d}\mu \biggr| \leqslant c(n) K(\mu) \int_{\mathbb{R}^n} |\nabla \phi| \, \mathrm{d}\mathcal{L}^n$$

for all $\phi \in C^1_c(\mathbb{R}^n)$.

Since

$$\mathcal{H}^{n-1} \lfloor M^\infty \leqslant \frac{\omega_{n-1}}{2D_1} \|V\| \quad \text{and} \quad \|V\|(B_r(x)) \leqslant \tfrac{1}{2} D_2 r^{n-1},$$

we have

$$K(\mathcal{H}^{n-1} \lfloor M^\infty \cap \tilde{U}) \leqslant \tfrac{1}{4} D_2 D_1^{-1} \omega_{n-1}.$$

Thus

$$\biggl| \int_{M^\infty} \phi \, \mathrm{d}\mathcal{H}^{n-1} \biggr| \leqslant c(n) D_2 D_1^{-1} \int_U |\nabla \phi|$$

for all $\phi \in C^1_c(\tilde{U})$. Using this and smoothly approximating f^∞, we obtain

$$-\int_{M^\infty} f^\infty g \cdot \nu \, \mathrm{d}\mathcal{H}^{n-1} = \frac{1}{2} \int_U \operatorname{div}(f^\infty g) u^\infty$$

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for all $g \in C_c^1(\tilde{U};\mathbb{R}^n)$. Moreover, for any $1 \leqslant p < \infty$,

$$\int_{M^\infty \cap \tilde{U}} |f^\infty|^p \,\mathrm{d}\mathcal{H}^{n-1} \leqslant c(n) D_2 D_1^{-1} \int_U p|f^\infty|^{p-1}|\nabla f^\infty| \leqslant c(n,p) D_2 D_1^{-1} \|f^\infty\|_{W^{1,n}(U)}^p < \infty,$$

by the Sobolev inequality. Since

$$|\delta V(g)| \leqslant \sup |g| \int_{M^\infty \cap \tilde{U}} |f^\infty| \,\mathrm{d}\mathcal{H}^{n-1}$$

for all $g \in C_c^1(\tilde{U};\mathbb{R}^n)$, $\|\delta V(g)\|$ defines a Radon measure on U. By the lower-density estimate bound of $\|V\|$ and Allard's rectifiability theorem [4, theorem 5.5.(1)], we can conclude that V is rectifiable. It also follows that $\|\delta V\|$ is absolutely continuous with respect to $\mathcal{H}^{n-1} \lfloor M^\infty$, and hence with respect to $\|V\|$ as well. Thus

$$\delta V(g) = -\int g \cdot H \,\mathrm{d}\|V\| = -\int_{\operatorname{supp}\|V\|} g \cdot H\theta \,\mathrm{d}\mathcal{H}^{n-1}.$$

The expression for H follows from the second expression for $\delta V(g)$ in the statement of the proposition. □

Note that this proves that $H \in L^p_{\mathrm{loc}}(U;\mathbb{R}^n)$ with respect to $\|V\|$ as well for any $1 \leqslant p < \infty$. By the standard theory for a varifold with its mean curvature in L^p, $p \geqslant n-1$, the density function θ is well defined everywhere on $\operatorname{supp}\|V\|$ and upper-semicontinuous on U. This fact also follows directly from proposition 3.6.

Next, we prove that $\theta(x) = N\sigma$ for some positive integer $\mathcal{H}^{n-1}$ a.e. on $\operatorname{supp}\|V\|$. With modifications, the line of proof is very similar to that of [23, § 5], so we point out the difference so that the reader may follow the proof. There are two points that must be dealt with: the first is that we only have $f \in W^{1,n}$, so that f is 'almost bounded' but not quite, and the second is that the discrepancy function has an estimate that is not as good as the case in [23], where we had $\xi \leqslant c\varepsilon$. These points can be resolved by replacing the pointwise estimates in [23] by suitable integral estimates.

The first proposition shows that there is only small energy uniformly in ε in the region $\{u \approx \pm 1\}$.

PROPOSITION 4.6 (cf. proposition 5.1 of [23]). *Assume that assumption B is true, with u, ε, f and B_3, and suppose $s > 0$. Then there exist positive constants b and ε_5, depending only on λ_0, c_0, E_0, W and s, such that*

$$\int_{B_1 \cap \{|u| \geqslant 1-b\}} \frac{W(u)}{\varepsilon} \leqslant s$$

whenever $\varepsilon \leqslant \varepsilon_5$.

Proof. We use two lemmas [23, lemmas 5.2, 5.3]. For the first lemma, one only need to use the ABP estimate instead of the pointwise estimate. Since it is a straightforward modification, we omit the proof. The rest of the proof goes through with minor modifications of constants. □

The second proposition deals with 'cutting' the varifold horizontally into stacked single-layered interfaces. Define $T : \mathbb{R}^n > \mathbb{R}^{n-1}$ by $T(x) = (x_1, \dots, x_{n-1})$. Also define $\nu = (\nu_1, \dots, \nu_n) = \nabla u/|\nabla u|$ whenever $|\nabla u| \neq 0$ and $\nu = 0$ when $|\nabla u| = 0$. We additionally define

$$e_\varepsilon = \tfrac{1}{2}\varepsilon|\nabla u|^2 + \frac{W(u)}{\varepsilon}, \qquad \xi_\varepsilon = \tfrac{1}{2}\varepsilon|\nabla u|^2 - \frac{W(u)}{\varepsilon}.$$

In [23, proposition 5.5], one needs to change hypothesis (2).

PROPOSITION 4.7. *Corresponding to each R, E_0, s and N such that $0 < R < \infty$, $0 < E_0 < \infty$, $0 < s < 1$ and N is a positive integer, there exists $\eta > 0$ with the following property.*

Assume the following.

(1) *$Y \subset \mathbb{R}^n$ has no more than $N+1$ elements, $T(y) = 0$ for all $y \in Y$, $a > 0$, $|y - z| > 3a$ for all $y, z \in Y$ and* $\operatorname{diam} Y \leqslant \eta R$.

(2) *On $\{x \in \mathbb{R}^n \mid \operatorname{dist}(x, Y) < R\}$, u satisfies (1.2) with $\|f\|_{W^{1,n}} \leqslant \eta$, $|u| \leqslant 2$ and*
$$\int_a^R \frac{\mathrm{d}r}{r^n} \int_{B_r(x)} \left(\tfrac{1}{2}\varepsilon|\nabla u^2| - \frac{W(u)}{\varepsilon}\right)^+ \leqslant \eta R \quad \text{for each } x \in Y.$$

(3) *For each $y \in Y$ and $a \leqslant r \leqslant R$,*
$$\int_{B_r(y)} |\xi_\varepsilon| + (1 - (\nu_n)^2)\varepsilon|\nabla u|^2 \, \mathrm{d}y \leqslant \eta r^{n-1},$$
$$\int_{B_r(y)} \varepsilon|\nabla u|^2 \leqslant E_0 r^{n-1}.$$

Then we have
$$\sum_{y \in Y} \frac{1}{a^{n-1}} \int_{B_a(y)} e_\varepsilon \leqslant s + \frac{1+s}{R^{n-1}} \int_{\{x \mid \operatorname{dist}(Y,x) < R\}} e_\varepsilon.$$

For the proof, one also modifies hypothesis (4) in [23, lemma 5.4], just like above hypothesis (2). With this, the proof of lemma 5.4 goes through with minor changes, and hence the above proposition follows.

The third proposition deals with the ε scale. Here, one needs a pointwise estimate on ξ_ε, but $\xi_\varepsilon \leqslant c_1\varepsilon^{-2/5}$ is sufficient.

PROPOSITION 4.8. *Given $0 < s < 1$ and $0 < b < 1$, there exist $0 < \eta < 1$ and $1 < L < \infty$, depending on W, with the following property. Assume $0 < \varepsilon < 1$ and u satisfies (1.2) and $\xi_\varepsilon \leqslant \eta\varepsilon^{-1}$ on $B_{4\varepsilon L}(0)$, with $\|f\|_{W^{1,n}} \leqslant \eta$, $|u(0)| \leqslant 1 - b$ and*
$$\int_{B_{4\varepsilon L(0)}} (|\xi_\varepsilon| + (1 - (\nu_n)^2)\varepsilon|\nabla u|^2) \leqslant \eta(4\varepsilon L)^{n-1}.$$
Then we have $T^{-1}(0) \cap \{x \in B_{3L\varepsilon}(0) \mid u(x) = u(0)\} = \{0\}$ and
$$\left|\frac{1}{\omega_{n-1}(L\varepsilon)^{n-1}} \int_{B_{L\varepsilon}(0)} e_\varepsilon - 2\sigma\right| \leqslant s.$$

We only point out that the place $\xi_\varepsilon \leqslant \eta$ was used in the proof of [23, propostion 5.6] is where we wanted to conclude that $|u| \leqslant 1 - \tilde{b}$ on $B_{4L}(0)$ by suitably restricting η. But, in fact, this can be done by having only $\xi_\varepsilon \leqslant \eta\varepsilon^{-1}$. Also, instead of a C^2 estimate for u (or z), we only need a $C^{1,\beta}$ estimate for some $\beta > 0$, which is available even if $f \in W^{1,n}$.

To end the proof of theorem 2.1, we point out that the main difference from that of [23] is condition (2) of proposition 4.7. For this, note that the constants c_1, c_7 and c_8 in propositions 3.2 and 3.5 of this paper scale like r, hence they are small constants in the blowup argument. It is not hard, then, to verify proposition 4.7 (2) above using propositions 3.2 and 3.5 for a rescaled sequence of solutions. The argument in [23] then shows that the density of the limit varifold $\|V\|$ is $N\sigma$, where N is an integer, $\mathcal{H}^{n-1}$ a.e. on the support of $\|V\|$.

5. Concluding remarks and applications

5.1. On the Sobolev norm of chemical potential

In this paper we consider the situation where we control the $W^{1,n}$-norm of chemical potential f as $\varepsilon > 0$. We encounter a serious difficulty in relaxing the control when we estimate the supremum bound on $\frac{1}{2}\varepsilon|\nabla u|^2 - W/\varepsilon$. As noted in the beginning, it is conjectured that a control of $W^{1,p}$ for some $p > \frac{1}{2}n$ should be sufficient to obtain our result, with the limit mean curvature belonging to L^q space for a smaller q. This follows from the following heuristic argument. If the interface M is a C^1 hypersurface and if H belongs to $W^{1,p}(U)$ for some $p > \frac{1}{2}n$, the trace of H on M belongs to $L^q(M)$ for $q > n-1$. If the mean curvature of M is given by H, the well-known result on the regularity of integral varifold [4] shows that the monotonicity formula for the scaled energy ($(n-1)$-dimensional area) holds for M. Since f roughly corresponds to the mean curvature field of the interface, one expects that f being in $W^{1,p}(U)$ for some $p > \frac{1}{2}n$ may be sufficient to obtain a monotonicity-type formula, which may also prove all the subsequent rectifiability and integrality of the limit varifold. Quite relevant to this point are recent articles by Schätzle [35, 36], where he studied the convergence of integral varifolds with their mean curvature given by Sobolev functions in $W^{1,p}(U)$, $p > \frac{1}{2}n$. His results also strongly suggest that the multiplicity of the limit varifold in this paper is $\mathcal{H}^{n-1}$ a.e. σ, namely, no folding, on $M^\infty \cap \{f^\infty \neq 0\}$. Geometrically, if there is an odd number folding on $M^\infty \cap \{f^\infty \neq 0\}$, it implies that at least one of the interfaces has to bend in a wrong direction as they converge, which seems unlikely. We would like to resolve these points in the future.

5.2. Implications to the Cahn–Hilliard equation

Consider a sequence of smooth initial data u_0^i, $i = 1, \dots,$ and ε_i, with

$$\limsup E_{\varepsilon_i}(u_0^i) < \infty, \qquad \varepsilon_i \to 0.$$

Suppose also that there exists a constant $m_0 \in (-|U|, |U|)$ such that

$$\int_U u_0^i = m_0 \quad \text{for all } i.$$

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Let u^i be the solution to the Cahn–Hilliard equation (1.3) with the initial data. As noted, f^i corresponding to u^i satisfies

$$E_{\varepsilon_i}(u_0^i) = E_{\varepsilon_i}(u^i(\cdot,t)) + \int_0^t \int_U |\nabla f^i|^2$$

for all $t \geqslant 0$. Moreover, Chen [14, lemma 3.4] proved that

$$\|f^i(\cdot,t)\|_{L^2(U)} \leqslant C(E_{\varepsilon_i}(u^i) + \|\nabla f^i(\cdot,t)\|_{L^2(U)})$$

holds for all t and i for all ε_i small. Thus

$$\int_0^t \|f^i(\cdot,t)\|^2_{H^1(U)} \leqslant C$$

for all large i. For L^1 a.e. t, we have $\liminf \|f^i(\cdot,t)\|_{H^1(U)} < \infty$ by Fatou's lemma. We now restrict our attention to $n = 2$ and such t. With a suitable growth condition on W (such as $W \geqslant |u|^k$, $k > 2$, for all large u) and the Neumann boundary condition, one may show that there exists a constant $c = c(U, \|f^i\|_{H^1(U)})$ such that $\sup |u^i| \leqslant c$. Thus, for $n = 2$, assumptions A and B are satisfied for a subsequence on this time slice. Unfortunately, even though we may conclude that there exists a subsequence for a.e. time slice for which we may apply our result, the choice of the subsequence may differ for each t. Note that the bound on $\|f^i(\cdot,t)\|_{H^1(U)}$ implies that the time derivative of the total energy stays finite, not allowing a violent jump of mass there.

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