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The solvability of an elliptic system under a singular boundary condition

J. García-Melián and **J. Sabina de Lis**
Departamento de Análisis Matemático, Universidad de La Laguna,
C/Astrofísico Francisco Sánchez s/n, 32871 La Laguna, Spain
(jjgarmel@ull.es; josabina@ull.es)

R. Letelier-Albornoz*
Departamento de Matemáticas, Universidad de Concepción,
Casilla 160-C, Concepción, Chile

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In this work we are considering both the one-dimensional and the radially symmetric versions of the elliptic system $\Delta u = v^p$, $\Delta v = u^q$ in Ω, where $p, q > 0$, under the boundary condition $u_{|\partial\Omega} = +\infty$, $v_{|\partial\Omega} = +\infty$. It is shown that no positive solutions exist when $pq \leqslant 1$, while we provide a detailed account of the set of (infinitely many) positive solutions if $pq > 1$. The behaviour near the boundary of all solutions is also elucidated, and symmetric solutions (u, v) are completely characterized in terms of their minima $(u(0), v(0))$. Non-symmetric solutions are also deeply studied in the one-dimensional problem.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Boundary blow-up elliptic problems of the form

$$\left.\begin{aligned} \Delta u &= f(u) \quad \text{in } \Omega, \\ u &= +\infty \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

regarding the subjects of existence and uniqueness of positive solutions (sometimes called 'large') together with estimates of their rate of divergence to infinity at $\partial\Omega$, have been the focus of a great number of works. We quote the pioneering papers [5], [33], [32] and [29] concerning Riemannian geometry and Riemann surfaces and [23] and [24], where (1.1) arises in a problem in electrohydrodynamics (see also [26], where stochastic control problems lead to large solutions). A brief account of more recent literature on the problem is provided by [25], [2], [36], [3], [10], [27], [28], [34], [1], [31], [4], [37], [9] and [19].

In the specific case of $f(u) = u^p$, which is closer in some sense to the nonlinearities to be dealt with in this paper, problem (1.1) was considered in [29] for $p = (N + 2)/(N - 2)$, while later generalizations of the form $\Delta u = a(x)u^p$, with a Hölder continuous and positive (up to $\partial\Omega$), were given in [25] ($p \geqslant 3$), [2], [3], [36]

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and [31]. The extension to equations involving the p-Laplacian was first considered in [10]. In [19] the case where a vanishes on $\partial\Omega$ was studied, while a is even allowed to be unbounded near $\partial\Omega$ in [37], [6], [7] and [14] (see an updated account in [15] and the references included there on this issue). It is also worth remarking that [18] gives a study case where problem (1.1) arises in population dynamics. Specifically, $-\Delta u = \lambda(x)u - a(x)u^p$ in Ω, $u = +\infty$ on $\partial\Omega$ with $p > 1$, λ, a Hölder continuous in Ω, $a > 0$ in Ω but $a_{|\partial\Omega} = 0$ (see also [19]).

However, the corresponding problem for elliptic systems, namely

$$\left.\begin{aligned} \Delta u &= f(u,v), && x \in \Omega,\\ \Delta v &= g(u,v), && x \in \Omega,\\ u = v &= +\infty, && x \in \partial\Omega, \end{aligned}\right\} \tag{1.2}$$

has been barely touched on in the literature, even for the more selective classes of nonlinearities f, g. As one might expect, the main handicap in dealing with (1.2) is the lack of results for comparison. Indeed, a natural way to construct solutions in the scalar case (1.1) is to solve the finite datum Dirichlet problem and then let the datum grow to infinity. Comparison principles are then instrumental in showing convergence, say by obtaining suitable estimates. In fact, by means of the method of subsolutions and supersolutions, comparison is a fundamental tool in the cases of (1.2) recently studied in [17] and [15]. In [17], $-f = \lambda u - u^2 + buv$, $-g = \mu v - v^2 + cuv$ with $b, c > 0$, so (1.2) falls in the cooperative regime. Regarding [15], $f = u^r v^p$, $g = u^q v^s$, $p, q, r, s > 0$ and now (1.2) is of competitive type, being the analysis restricted to the parameter range $(r-1)(s-1) - pq \geqslant 0$ (see further comments below). It should be remarked upon here that the research developed in the present paper was the 'starting point' for those works.

On the other hand, problems related to (1.2) arise when studying Lotka–Volterra systems of predator–prey and competitive type, under a zero Dirichlet condition and variable coefficients, some of them vanishing on whole subdomains of Ω (see [8, 11, 12, 30]).

In this paper we concentrate our efforts on the (at first sight) simplest case, where coupling and nonlinearity combine together in (1.2). Namely,

$$\left.\begin{aligned} \Delta u &= v^p, && x \in \Omega,\\ \Delta v &= u^q, && x \in \Omega,\\ u = v &= +\infty, && x \in \partial\Omega, \end{aligned}\right\} \tag{1.3}$$

where $p, q > 0$. As will be seen later, it turns out that the analysis of this kind of system is fairly complicated. Regarding [15], we are dealing here with the complementary case $pq > 1$ (termed there as 'supercritical'), and so no kind of comparison results can be employed. That is why we are restricting ourselves to the study of the one-dimensional version:

$$\left.\begin{aligned} u'' &= v^p, \quad -L < x < L,\\ v'' &= u^q, \quad -L < x < L,\\ u(\pm L) &= v(\pm L) = +\infty, \end{aligned}\right\} \tag{1.4}$$

where $L > 0$, the prime denotes differentiation with respect to x, and we are searching for both symmetric and non-symmetric solutions. We are also analysing

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the radially symmetric version of (1.3) where Ω is the ball $B(0,R)=\{x:|x|<R\}$ and $u(x)=u(r)$, $v(x)=v(r)$, $r=|x|$. In this case (1.2) takes the form

$$\left.\begin{aligned} \frac{d^2u}{dr^2}+\frac{N-1}{r}\frac{du}{dr}&=v^p, \quad 0<r<R,\\ \frac{d^2v}{dr^2}+\frac{N-1}{r}\frac{dv}{dr}&=u^q, \quad 0<r<R,\\ \frac{du}{dr}_{|r=0}=0, \quad u(R)=\infty, &\qquad \frac{dv}{dr}_{|r=0}=0, \quad v(R)=\infty. \end{aligned}\right\} \tag{1.5}$$

The case of a general domain Ω of $\mathbb{R}^N$ will be the subject of future work.

It will be seen below that both problems (1.4), (1.5) admit positive solutions only when $pq>1$, a translation to our setting of the condition $p>1$ which appears in a single equation. But, as a first outstanding difference with respect to the situations considered before, systems (1.4), (1.5) do not exhibit a unique positive solution. More precisely, we can find infinitely many positive symmetric solutions to both problems. We suspect that this non-uniqueness is caused by the fact that our system is of 'competitive type'. Moreover, (1.4) also exhibits non-symmetric solutions and a detailed description of them is provided. It turns out that all solutions share the same asymptotic behaviour at $x=\pm L$.

We next state our first result concerning (1.4). The proof relies on the study of the blow-up property for solutions of the Cauchy problem associated with (1.4), exploiting the scaling invariance of the system.

THEOREM 1.1. *Problem (1.4) admits a positive symmetric solution if and only if*

$$pq>1. \tag{1.6}$$

Moreover, the following properties hold.

(i) Uniqueness. *There exists a unique solution* $(u,v)=(U_1(x),V_1(x))$ *to (1.4) (respectively,* $(U_2(x),V_2(x))$*), positive in* $x\neq 0$*, under the restriction*

$$\inf_{(-L,L)} u = 0 \quad \Big(\textit{respectively,} \quad \inf_{(-L,L)} v=0\Big).$$

(ii) Multiplicity of solutions. *The set of positive symmetric solutions* (u,v) *to (1.4) defines a continuous arc joining* $(U_1(x),V_1(x))$ *to* $(U_2(x),V_2(x))$*. More precisely, the set*

$$\Gamma=\{(\inf u,\inf v)=(u(0),v(0)): \\ (u,v) \textit{ positive, symmetric, solving (1.4)}\}\subset\bar{\mathbb{R}}_+^2$$

is parametrized by $g=(g_1,g_2):[0,1]\to\bar{\mathbb{R}}_+^2$*,* g *continuous,* g_1 *non-decreasing,* g_2 *non-increasing and* $g(0)=(U_1(0),V_1(0))$*,* $g(1)=(U_2(0),V_2(0))$ *(see figure 2).*

(iii) Asymptotic profile. *Every positive symmetric solution* (u,v) *to (1.4) verifies the following asymptotic estimates:*

$$\left.\begin{aligned} u(x)&\sim a d(x)^{-\xi} \quad \textit{as } d(x)\to 0+,\\ v(x)&\sim b d(x)^{-\eta} \quad \textit{as } d(x)\to 0+, \end{aligned}\right\} \tag{1.7}$$

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where $d(x) = \min\{L-x, L+x\}$ *and* $\xi = 2(p+1)/(pq-1)$, $\eta = 2(q+1)/(pq-1)$, *while*

$$a = [\xi(\xi+1)\eta^p(\eta+1)^p]^{1/(pq-1)}, \qquad b = [\eta(\eta+1)\xi^q(\xi+1)^q]^{1/(pq-1)}.$$

As for solutions to (1.5) our main result is the following.

THEOREM 1.2. *The problem*

$$\left.\begin{aligned} \Delta u &= v^p, && x \in B(0,R), \\ \Delta v &= u^q, && x \in B(0,R), \\ u = v &= +\infty, && |x| = R, \end{aligned}\right\} \tag{1.8}$$

admits a radially symmetric positive solution $(u(r), v(r))$ *if and only if*

$$pq > 1.$$

The following properties are also satisfied.

(i) Change of scale. *If* (u,v) *solves (1.8) in* $B(0,R)$, *then for every* $\lambda > 0$, $(u_\lambda(r), v_\lambda(r)) = (\lambda u(\lambda^\theta r), \lambda v^{q+1/p+1}(\lambda^\theta r))$, $\theta = (pq-1)/2(p+1)$ *is a solution of (1.8) in* $B(0, \lambda^{-\theta}R)$.

(ii) Uniqueness. *There exists a unique radially symmetric solution* $(u,v) = (U_1(r), V_1(r))$ *(respectively,* $(U_2(r), V_2(r))$*) to (1.8), positive in* $B(0,R)\setminus\{0\}$ *under the restriction*

$$\inf_{r\in[0,R)} u = 0 \quad \Bigl(\textit{respectively,} \ \inf_{r\in[0,R)} v = 0\Bigr).$$

(iii) Multiplicity. *Problem (1.8) admits infinitely many solutions* $(u(r), v(r))$. *Furthermore, the set*

$$\varGamma_R = \{(u_0, v_0) : u_0 = \inf u,\ v_0 = \inf v,\ (u,v) \ \textit{a radial solution}\} \subset \bar{\mathbb{R}}^2_+$$

is contained in $[0, U_2(0)] \times [0, V_1(0)]$, *while* u_0 *varies non-decreasingly,* v_0 *non-increasingly when* (u_0, v_0) *varies with* $\varGamma_R$.

Our description of the set of positive solutions to (1.4) is completed with the study of its strictly non-symmetric positive solutions. A first result in this direction asserts, among other things, that every symmetric solution generates two one-parametric families of non-symmetric positive solutions to (1.4).

THEOREM 1.3. *A necessary and sufficient condition in order that problem (1.4) possesses a positive solution* (u,v), *regardless of its symmetry, is again*

$$pq > 1.$$

Moreover, such a solution exhibits exactly the same profile at the boundary as a symmetric one. Namely,

$$u(x) \sim ad(x)^{-\xi}, \qquad v(x) \sim bd(x)^{-\eta} \tag{1.9}$$

as $d \to 0+$, where $d(x) = \min\{L - x, L + x\}$, ξ, η are the exponents, and a, b are the coefficients in (1.7).

Furthermore, each symmetric *positive solution (u, v) to (1.4) with*

$$u_0 = \inf u, \qquad v_0 = \inf v$$

gives rise to two non-continuable families of positive solutions satisfying the following properties.

(i) *There exists $\sigma_1^* = \sigma_1^*(u_0, v_0)$, positive and continuous in (u_0, v_0), and a family $\{(\hat{u}(x,\sigma), \hat{v}(x,\sigma))\}_{|\sigma|\leqslant\sigma_1^*}$ of positive solutions to (1.4) such that $(\hat{u}, \hat{v})_{|\sigma=0} = (u, v)$ and*

  (a) $(\hat{u}(x,-\sigma), \hat{v}(x,-\sigma)) = (\hat{u}(-x,\sigma), \hat{v}(-x,\sigma))$ *for each* $|\sigma| \leqslant \sigma_1^*$, $-L < x < L$;

  (b) $\inf \hat{v}(\cdot\,,\sigma) = v_0$ *for* $|\sigma| \leqslant \sigma_1^*$;

  (c) $0 < \inf \hat{u}(\cdot\,,\sigma) < u_0$ *if* $0 < |\sigma| < \sigma_1^*$ *while* $\inf \hat{u}(\cdot\,,\sigma) = 0$ *at* $\sigma = \pm\sigma_1^*$.

  Furthermore, $(\hat{u}(\cdot\,,\sigma), \hat{v}(\cdot\,,\sigma))$ is non-symmetric for $\sigma \neq 0$.

(ii) *Similarly, problem (1.4) exhibits a family $\{(\tilde{u}(x,\sigma), \tilde{v}(x,\sigma))\}_{|\sigma|\leqslant\sigma_2^*}$ of positive solutions where $\sigma_2^* = \sigma_2^*(u_0, v_0)$ is a positive continuous function of (u_0, v_0), such that $(\tilde{u}, \tilde{v})_{|\sigma=0} = (u, v)$ while $(\tilde{u}(\cdot\,,\sigma), \tilde{v}(\cdot\,,\sigma))$ is non-symmetric for $\sigma \neq 0$. In addition,*

  (a) $(\tilde{u}(x,-\sigma), \tilde{v}(x,-\sigma)) = (\tilde{u}(-x,\sigma), \tilde{v}(-x,\sigma))$ *for each* $|\sigma| \leqslant \sigma_2^*$, $-L < x < L$;

  (b) $\inf \tilde{u}(\cdot\,,\sigma) = u_0$ *for* $|\sigma| \leqslant \sigma_2^*$;

  (c) $0 < \inf \tilde{v}(\cdot\,,\sigma) < v_0$ *if* $0 < |\sigma| < \sigma_2^*$ *while* $\inf \hat{v}(\cdot\,,\sigma) = 0$ *at* $\sigma = \pm\sigma_2^*$.

REMARK 1.4. For a non-symmetric solution (u, v) in the family $\{(\hat{u}, \hat{v})\}_{|\sigma|\leqslant\sigma_1^*}$, the parameter σ has the status, after a convenient rescaling, of the derivative of $\hat{u}$ at a certain reference point $x \in (-L, L)$. σ has a similar meaning regarding $(\tilde{u}, \tilde{v})$. See remark 6.5 for precise details and a global bifurcation diagram for non-symmetric positive solutions in the family $(\hat{u}, \hat{v})$.

A second, more ambitious statement ensures the existence of a bidimensional continuum of non-symmetric solutions generated from a symmetric one. An additional result (theorem 1.6) characterizes the set of all possible positive solutions (u, v) to problem (1.4) in terms of their infimums $u_0 = \inf u$, $v_0 = \inf v$ by ascertaining the set $\mathcal{C} \in \mathbb{R}^2$ where their derivatives $(u_x(0), v_x(0))$ must lie. This permits us to find a broad class of non-symmetric solutions exhibiting the property of being a continuous deformation of a symmetric positive one.

THEOREM 1.5. *Let (u, v) be a positive symmetric solution of (1.4) with $u_0 = \inf u$, $v_0 = \inf v$, $\sigma_i^* = \sigma_i^*(u_0, v_0)$, $i = 1, 2$, being the associated values introduced in theorem 1.3. Then there exists a continuous bidimensional and non-continuable family of positive solutions*

$$\{(\tilde{u}(x,\bar{\sigma}), \tilde{v}(x,\bar{\sigma})) : \bar{\sigma} = (\sigma_1, \sigma_2) \in \mathcal{C}_0\}$$

such that the following hold.

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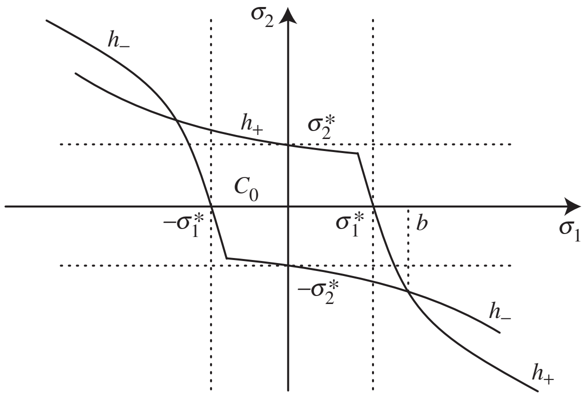

Figure 1. The connected piece, $\mathcal{C}_0$ containing $(0,0)$, of the set of values $\bar\sigma = (\sigma_1, \sigma_2)$ lying between h_- and h_+.

(a) *The $\bar\sigma$-domain $\mathcal{C}_0 \subset \mathbb{R}^2$ is an open bounded-connected set, symmetric with respect to* $(0,0) \in \mathcal{C}_0$, $(\tilde u(\cdot\,,\bar\sigma), \tilde v(\cdot\,,\bar\sigma))_{|\bar\sigma=(0,0)} = (u,v)$, *while*

$$\tilde u(-x,\bar\sigma) = \tilde u(x,-\bar\sigma), \qquad \tilde v(-x,\bar\sigma) = \tilde v(x,-\bar\sigma), \qquad -L < x < L,$$

for all $\bar\sigma \in \mathcal{C}_0$. *Moreover,* $(\tilde u(\cdot\,,\bar\sigma), \tilde v(\cdot\,,\bar\sigma))$ *is non-symmetric if* $\bar\sigma \neq (0,0)$.

(b) *There exist continuous decreasing functions* $h_-, h_+ : \mathbb{R} \to \mathbb{R}$ *satisfying*

(i) $h_-(\sigma_1) = -h_+(-\sigma_1)$ *and*

(ii) $h_+(\mp\infty) = \pm\infty$, $h_+(0) = \sigma_2^*$, $h_+(\sigma_1^*) = 0$ *while* $b > \sigma_1^*$ *exists such that* $h_- = h_+$ *at* $\sigma_1 = \pm b$, *being* $h_- < h_+$ *for* $|\sigma_1| < b$.

Moreover, the set $\mathcal{C}_0$ *can be expressed in terms of such functions as (figure 1)*

$$\mathcal{C}_0 = \{\bar\sigma : |\sigma_1| < b,\ h_-(\sigma_1) < \sigma_2 < h_+(\sigma_1)\}. \tag{1.10}$$

(c) *The non-continuable character of the family beyond the boundary* $\partial\mathcal{C}_0$ *of* $\mathcal{C}_0$ *is reflected by the fact that, modulo a suitable scaling, solutions* $(\tilde u, \tilde v)$ *satisfy either* $\inf \tilde u \to 0$ *or* $\inf \tilde v \to 0$ *as* $\bar\sigma \to \bar\sigma_0$ *for every* $\bar\sigma_0 \in \partial\mathcal{C}_0$.

As will be shown later in § 7, the functions h_-, h_+ exclusively depend on u_0, v_0. For the purposes of our next statement we introduce the set

$$\mathcal{C} = \{\bar\sigma : h_-(\sigma_1) < \sigma_2 < h_+(\sigma_1)\}. \tag{1.11}$$

Observe that $\mathcal{C}_0$ is none other than the connected piece of $\mathcal{C}$ to which $(0,0)$ belongs. We are now studying conditions ensuring that some positive (in general, non-symmetric) solutions can be regarded as a continuous perturbation of a symmetric one.

THEOREM 1.6. *Let* u_0, v_0 *be positive. Then the open set* $\mathcal{C}$ *defined in (1.11) is bounded. Furthermore, suppose that* (u,v) *is an arbitrary positive solution to (1.4) in* $(-L,L)$ *with* $u(0) = u_0$, $v(0) = v_0$, $u_x'(0) = u_0'$, $v_x'(0) = v_0'$. *Then the following properties hold.*

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(i) $(u_0', v_0') \in \mathcal{C}$.

(ii) *If* u_0' *and* v_0' *have the same sign, i.e.* $u_0'v_0' \geqslant 0$, *then necessarily* $|u_0'| \leqslant \sigma_1^*$, $|v_0'| \leqslant \sigma_2^*$. *Moreover,* (u, v) *can be continuously deformed to a symmetric positive solution.*

(iii) *If* $u_0'v_0' < 0$ *but* $(u_0', v_0') \in \mathcal{C}_0$, *then* (u, v) *comes from a symmetric positive solution by means of a continuous perturbation.*

REMARK 1.7. As a consequence of the analysis in § 7, it follows that (u_0', v_0') in (ii) belongs to $\mathcal{C}_0$. Theorems 1.5 and 1.6 do not exclude the possible existence of a non-symmetric solution (u, v) that, in the previous terminology, satisfies $u_0'v_0' < 0$ but $(u_0', v_0') \notin \mathcal{C}_0$ (see remark 7.5 (a)). However, the possibility of deforming it to a symmetric solution cannot be ensured in this case.

Finally, it should be pointed out that methods similar to the ones developed here can be used to analyse (1.2) with the nonlinearities $f = \mathrm{e}^u$, $g = \mathrm{e}^v$. The subsequent results and the analysis of broader classes of competitive nonlinearities $f(u, v)$, $g(u, v)$ will be given elsewhere.

The paper is organized as follows: in § 2 we examine the initial-value problem associated with the symmetric solutions to (1.4), providing some preliminary properties. Section 3 is devoted to the statement of the important theorem 3.1, and the proof of some of its consequences. Section 4 consists of the proof of theorem 3.1. Radial solutions to (1.5) are studied in § 5 while a full description of the non-symmetric solutions to (1.4) is contained in § 6.

2. An initial-value problem

In this section we are going to perform a detailed analysis of the positive solutions to the initial-value problem

$$\left.\begin{aligned} u'' &= v^p, \\ v'' &= u^q, \end{aligned}\right\} \qquad \left.\begin{aligned} u(0) &= u_0, \quad u'(0) = 0, \\ v(0) &= v_0, \quad v'(0) = 0, \end{aligned}\right\} \tag{2.1}$$

$p > 0$, $q > 0$, for $u_0, v_0 \geqslant 0$, where such equations are considered to be defined for $u \geqslant 0$, $v \geqslant 0$. We are working with non-negative solutions so it will not be necessary for the moment to extend u^q, v^p to the whole of $\mathbb{R}$. The main features of (2.1) are examined in the following results.

LEMMA 2.1. *Assume that* $u_0, v_0 \geqslant 0$, $(u_0, v_0) \neq (0, 0)$. *Then (2.1) admits a forward solution* (u, v), *defined in a non-continuable interval* $[0, \omega)$, $0 < \omega \leqslant +\infty$, *which is (component-wise) positive, increasing, convex and satisfies*

$$\lim_{x \to \omega-} u = \lim_{x \to \omega-} v = +\infty.$$

REMARK 2.2. (a) The assertions of lemma 2.1 are standard. Notice that full symmetric solutions in the interval $(-\omega, \omega)$ can be obtained by reflection with respect to $x = 0$.

(b) It should be stressed that both components u, v blow up at the same points $x = \pm\omega$ regardless of whether ω is finite or not. That will remain true even if (u, v) is non-symmetric.

LEMMA 2.3 (comparison principle). *Let (u, v), $(\bar{u}, \bar{v})$ be solutions to*

$$\left.\begin{aligned} u'' &= v^p, \\ v'' &= u^q, \end{aligned}\right\} \tag{2.2}$$

with non-negative initial data

$$(u, v, u', v')_{|x=0} = (u_0, v_0, u_0', v_0'), \qquad (\bar{u}, \bar{v}, \bar{u}', \bar{v}')_{|x=0} = (\bar{u}_0, \bar{v}_0, \bar{u}_0', \bar{v}_0')$$

such that $u_0 \leqslant \bar{u}_0$, $v_0 \leqslant \bar{v}_0$, $u_0' \leqslant \bar{u}_0'$, $v_0' \leqslant \bar{v}_0'$ while $(u_0, v_0, u_0', v_0') \neq (\bar{u}_0, \bar{v}_0, \bar{u}_0', \bar{v}_0')$. Then $u < \bar{u}$ and $v < \bar{v}$ in $x > 0$ wherever both solutions are defined.

Proof. Assume, for instance, that $u_0 < \bar{u}_0$, $v_0 \leqslant \bar{v}_0$, with the remaining cases being handled in the same way. Notice that this implies that $u < \bar{u}$ in an interval of the form $[0, \varepsilon)$. If ε is such that $u(\varepsilon) = \bar{u}(\varepsilon)$, then

$$\begin{aligned} v(x) &= v(0) + v'(0)x + \int_0^x \int_0^s u(t)^q \,\mathrm{d}t \,\mathrm{d}s < \bar{v}(0) + \bar{v}'(0)x + \int_0^x \int_0^s \bar{u}(t)^q \,\mathrm{d}t \,\mathrm{d}s \\ &= \bar{v}(x) \end{aligned}$$

when $x \in (0, \varepsilon)$. But also

$$u(\varepsilon) = u(0) + \int_0^\varepsilon \int_0^s v(t)^p \,\mathrm{d}t \,\mathrm{d}s = \bar{u}(0) + \int_0^\varepsilon \int_0^s \bar{v}(t)^q \,\mathrm{d}t \,\mathrm{d}s = \bar{u}(\varepsilon),$$

which implies that $v \equiv \bar{v}$ in $[0, \varepsilon]$, a contradiction. Thus, $u(x) < \bar{u}(x)$, and similarly $v(x) < \bar{v}(x)$. The lemma is proved. □

LEMMA 2.4. *Problem (2.1) has a unique solution (u, v) for every $u_0, v_0 \geqslant 0$, $(u_0, v_0) \neq (0, 0)$, which will be denoted by $(u(\cdot\,, u_0, v_0), v(\cdot\,, u_0, v_0))$. This solution is increasing in u_0 for fixed v_0, and in v_0 for fixed u_0. In the case $(u_0, v_0) = (0, 0)$, $u = 0$, $v = 0$ is the unique non-negative solution if and only if $pq \geqslant 1$.*

Proof. The uniqueness of solutions is a consequence of the standard theory of ordinary differential equations (ODEs) when $u_0, v_0 > 0$, or when $u_0 = 0$ and $q \geqslant 1$ or $v_0 = 0$ and $p \geqslant 1$. Hence we only need to treat the cases $u_0 = 0$, $v_0 > 0$ and $u_0 = v_0 = 0$ (the remaining case $u_0 > 0$, $v_0 = 0$ is similar).

Assume first that $u_0 = 0$, $v_0 > 0$ and let (u, v), $(\bar{u}, \bar{v})$ be two solutions of (2.1). We adapt an argument in [35]. It is easily shown that $u(x), \bar{u}(x) > 0$, $v(x), \bar{v}(x) > v_0$ for $x > 0$. For $\delta > 0$ small, since $(u(x + \delta), v(x + \delta))$ solves (2.2) with positive initial data $(u(\delta), v(\delta), u'(\delta), v'(\delta))$, we obtain by lemma 2.3 that $\bar{u}(x) < u(x + \delta)$, $\bar{v}(x) < v(x + \delta)$. Letting $\delta \to 0+$ we arrive at $\bar{u} \leqslant u$, $\bar{v} \leqslant v$, and a symmetric argument proves the uniqueness.

Now consider the case $u_0 = v_0 = 0$, and assume that $pq \geqslant 1$. As before we only prove uniqueness. For $\delta > 0$ define $|u|_{\infty,\delta} = \sup_{[0,\delta]} u(x)$, $|v|_{\infty,\delta} = \sup_{[0,\delta]} v(x)$. Then, since

$$u(x) = \int_0^x \int_0^s v(t)^p \,\mathrm{d}t \,\mathrm{d}s,$$

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we have $|u|_{\infty,\delta} \leqslant |v|^p_{\infty,\delta}\delta^2/2$, and symmetrically $|v|_{\infty,\delta} \leqslant |u|^q_{\infty,\delta}\delta^2/2$. Combining these two, we arrive at $|u|_{\infty,\delta} \leqslant |u|^{pq}_{\infty,\delta}(\delta^2/2)^{p+1}$, from which $|u|_{\infty,\delta} = 0$ for δ small follows.

The increasing character of the solution with respect to u_0 and v_0 is a direct consequence of lemma 2.3.

Finally, uniqueness of non-negative solutions corresponding to initial conditions $u_0 = v_0 = 0$ fails in the complementary range $0 < pq < 1$. Indeed, the pair

$$(\tilde{u}(x), \tilde{v}(x)) = (ax^{|\xi|}, bx^{|\eta|}),$$

where (cf. theorem 1.1 (iii))

$$|\xi| = 2(p+1)/(1-pq), \qquad |\eta| = 2(q+1)/(1-pq),$$
$$a = \{|\xi|(|\xi|-1)|\eta|^p(|\eta|-1)^p\}^{1/(pq-1)}, \qquad b = \{|\eta|(|\eta|-1)|\xi|^q(|\xi|-1)^q\}^{1/(pq-1)},$$

defines a positive solution to (2.1). The previous ideas also permit us to show that $(\tilde{u}, \tilde{v})$ is the unique positive solution to (2.1). All other possible non-negative solutions are defined by $(u(x), v(x)) = (0, 0)$ for $0 \leqslant x \leqslant \tau$, $(u(x), v(x)) = (\tilde{u}(x-\tau), \tilde{v}(x-\tau))$ if $x \geqslant \tau$, $\tau > 0$ arbitrary (cf. [16]). This concludes the proof. □

Later in § 6 we perturb the initial conditions $(u, v, u', v')_{|x=0}$ of a positive solution (u, v) to (2.2). For our purposes there it is convenient to extend the right-hand sides of (2.2) so that the resulting perturbed solutions can vanish or be negative somewhere. The required uniqueness result for the extended equation to be used in this work is stated in our next theorem. We remark that the proof is more involved than that of lemma 2.4.

THEOREM 2.5. *Suppose that* $pq \geqslant 1$. *Then, for arbitrary initial data*

$$(u_0, v_0, u_0', v_0') \in \mathbb{R}^4,$$

the Cauchy problem

$$\left.\begin{aligned} u'' &= |v|^p, \\ v'' &= |u|^q, \end{aligned}\right\} \qquad \left.\begin{aligned} u(0) &= u_0, \quad u'(0) = u_0', \\ v(0) &= v_0, \quad v'(0) = v_0' \end{aligned}\right\} \tag{2.3}$$

possesses a unique solution (u, v).

Proof. We only need to consider the case $u_0 = 0$ with $0 < q < 1$ and showing local uniqueness. The remaining options correspond either to cases where the standard theory provides local uniqueness or to the symmetric situation $v_0 = 0$ with $0 < p < 1$.

First consider the case $u_0 = 0$, $u_0' \neq 0$, which can be treated according to an idea in [35]. In fact, define $(u, u_1, v, v_1) = (u, u', v, v')$ and perform the change of variable $x \mapsto u$ near zero. This leads to the equivalent four-dimensional system

$$\frac{\mathrm{d}x}{\mathrm{d}u} = u_1^{-1}, \qquad \frac{\mathrm{d}u_1}{\mathrm{d}u} = |v|^p u_1^{-1}, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = v_1 u_1^{-1}, \qquad \frac{\mathrm{d}v_1}{\mathrm{d}u} = |u|^q u_1^{-1},$$

under initial conditions $x(0) = 0$, $u_1(0) = u_0'$, $v(0) = 0$, $v_1(0) = v_0'$, which falls in the scope of the standard uniqueness theory.

Next, let us deal with the case $u_0 = u_0' = 0$. Since $v_0 = v_0' = 0$ implies that (u, v) is trivial (lemma 2.4), we can assume $(v_0, v_0') \neq (0, 0)$. So initially suppose $v_0 \neq 0$. If (u, v), $(\bar{u}, \bar{v})$ solve (2.3), set $y(x) = \bar{u}(x) - u(x)$, $z(x) = \bar{v}(x) - v(x)$. Then, (y, z) satisfies

$$y'' = c(x)z \quad \text{and} \quad z'' = d(x)y,$$

where

$$c(x) = \frac{|\bar{v}|^p - |v|^p}{\bar{v} - v}, \qquad d(x) = \frac{\bar{u}^q - u^q}{\bar{u} - u},$$

and where, under the present assumptions, $p > 1$. Observing that

$$u(x) = (\tfrac{1}{2}|v_0|^p + o(1))x^2, \qquad \bar{u}(x) = (\tfrac{1}{2}|v_0|^p + o(1))x^2,$$

the representation, for small $|x| > 0$,

$$d(x) = q\int_0^1 \frac{\mathrm{d}s}{(u(x) + s(\bar{u}(x) - u(x)))^{1-q}},$$

implies that

$$|x|^{2(1-q)}|d(x)| \leqslant C_1$$

for $|x| \leqslant \delta$, $\delta > 0$ small and certain positive C_1. On the other hand,

$$y(x) = x^2\int_0^1 (1-s)c(sx)z(sx)\,\mathrm{d}s.$$

Since $c(x) = p|v_0|^{p-2}v_0 + o(1)$,

$$\sup_{0<|x|<\delta} \frac{|y(x)|}{x^2} \leqslant C_2|z|_{\infty,\delta},$$

where $C_2 > 0$, $|z|_{\infty,\delta} = \sup_{|x|\leqslant\delta}|z(x)|$. From the identity

$$z(x) = \int_0^x \int_0^t s^2 d(s)\frac{y(s)}{s^2}\,\mathrm{d}s\,\mathrm{d}t,$$

we achieve

$$|z|_{\infty,\delta} \leqslant C_2\delta^2 \sup_{|x|\leqslant\delta} x^2|d(x)|\,|z|_{\infty,\delta},$$

which implies that $z = 0$ in $|x| \leqslant \delta$ if δ is small. Hence $(u, v) = (\bar{u}, \bar{v})$ near zero.

The case $u_0 = u_0' = v_0 = 0$ with $v_0' \neq 0$ is handled in a similar way. In fact,

$$|x|^{(p+2)(1-q)}|d(x)| = O(1), \qquad c(x) = O(|x|^{p-1}),$$

as $x \to 0$. In the same way,

$$\sup_{0<|x|\leqslant\delta} \frac{|y(x)|}{|x|^{p+2}} \leqslant C_3|z'|_{\infty,\delta}.$$

Thus

$$z'(x) = \int_0^x s^{p+2}d(s)\frac{y(s)}{s^{p+2}}\,\mathrm{d}s$$

leads to

$$|z'|_{\infty,\delta} \leqslant C_3\delta \sup_{|x|\leqslant\delta} |x|^{p+2}|d(x)|\,|z'|_{\infty,\delta},$$

and we have $z'=0$ in $|x|\leqslant\delta$. This again implies that $(u,v)=(\bar{u},\bar{v})$ near zero. □

Below we are mainly interested in determining when a positive solution to (2.1) blows up, i.e. $\omega<+\infty$. Before stating a first partial result in that direction let us introduce some notation. For a positive solution (u,v) to (2.1) defined in a non-continuable interval $[0,\omega)$ we fix the notation

$$\omega = T(u_0,v_0).$$

At the moment, $T(u_0,v_0)$ could possibly be $+\infty$ in some circumstances.

In the next statement we have in mind that (as will be shown in §§ 3 and 4) blow-up is only possible in the regime $pq>1$.

LEMMA 2.6. *Assume that $pq>1$ and suppose that a certain solution to (2.1) corresponding to positive initial data (u_0^*,v_0^*) blows up in finite time, i.e. $T(u_0^*,v_0^*)<\infty$. Then, for every positive initial data (u_0,v_0) the solution (u,v) to (2.1) also blows up in finite time. In other words,*

$$T(u_0,v_0)<\infty$$

for every $u_0,v_0>0$. Moreover, the following scaling property holds:

$$T(\lambda u_0,\lambda^{(q+1)/(p+1)}v_0)=\lambda^{-(pq-1)/2(p+1)}T(u_0,v_0),\quad \lambda>0. \tag{2.4}$$

Proof. To begin with, it is easily seen that, regardless of the values of p and q,

$$(u_\lambda,v_\lambda)=(\lambda u(kx),\lambda^{(q+1)/(p+1)}v(kx)),\quad k=\lambda^{(pq-1)/2(p+1)} \tag{2.5}$$

solves (2.1) with data $(\lambda u_0,\lambda^{(q+1)/(p+1)}v_0)$, $\lambda>0$ arbitrary, provided that (u,v) solves (2.1). In this way, $(u(\cdot\,,u_0^*,v_0^*),v(\cdot\,,u_0^*,v_0^*))$ gives rise to a family of solutions blowing up at times

$$x=\lambda^{-(pq-1)/2(p+1)}T(u_0^*,v_0^*).$$

Thus, given any positive solution (u,v) to (2.1) with data (u_0,v_0) it is possible to find λ so small as to have

$$\lambda u_0^*<u_0,\qquad \lambda^{(q+1)/(p+1)}v_0^*<v_0,$$

and lemma 2.3 implies that (u,v) blows up at $x=T(u_0,v_0)$, satisfying

$$0<T(u_0,v_0)\leqslant\lambda^{-(pq-1)/2(p+1)}T(u_0^*,v_0^*).$$

This proves the lemma. □

REMARKS 2.7. (a) As a consequence of the analysis in §§ 3 and 4 it follows that $pq>1$ is a necessary and sufficient condition for the blow-up of all positive solutions to (2.1) including those corresponding to semitrivial data of the form $(u_0^*,0)$ or $(0,v_0^*)$. In fact, in either of the latter cases those solutions remain *positive* for $x>0$, a regime where such analysis remains valid.

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(b) An important consequence of the proof of lemma 2.3 is the fact that $T(\bar{u}_0, \bar{v}_0) \leqslant T(u_0^*, v_0^*)$ provided $u_0^* \leqslant \bar{u}_0$, $v_0^* \leqslant \bar{v}_0$. We are now further proving that $T(\bar{u}_0, \bar{v}_0) < T(u_0^*, v_0^*)$ if $u_0^* < \bar{u}_0$, $v_0^* < \bar{v}_0$. Indeed, if in that case $T(\bar{u}_0, \bar{v}_0) = T(u_0^*, v_0^*) := L$, then $T(u_0, v_0) = L$ in the rectangle $[u_0^*, \bar{u}_0] \times [v_0^*, \bar{v}_0]$. However, the curve

$$\frac{v_0}{v_0^*} = \left(\frac{u_0}{u_0^*}\right)^{(q+1)/(p+1)}$$

goes into the rectangle while (2.4) says that T strictly decreases along it, which is not possible. Thus, the assertion follows.

(c) It is well known from the theory of ODEs that the upper and lower limits of the maximal interval of existence of a solution are only semicontinuous functions of their initial data (see [21]). In our specific case,

$$\liminf_{(u_0', v_0') \to (u_0, v_0)} T(u_0', v_0') \geqslant T(u_0, v_0).$$

However, it is already known that $T(u_0', v_0') \leqslant T(u_0, v_0)$ when $u_0 \leqslant u_0'$, $v_0 \leqslant v_0'$. Therefore,

$$\liminf_{(u_0', v_0') \to (u_0, v_0)} T(u_0', v_0') = T(u_0, v_0)$$

for every (u_0, v_0). Nevertheless, it is pointed out here that the continuity of $T(u_0, v_0)$ will follow from the analysis in § 3 (corollary 3.5).

3. Blow-up for solutions to (2.1): proof of theorem 1.1

Our main purpose in this section is to state a basic result (theorem 3.1 below) that is crucial for elucidating the blow-up property for the non-negative (non-trivial) solutions to (2.1). We are also drawing some important conclusions from it.

Let $(u(x), v(x))$, $0 \leqslant x < \omega$, be any non-continuable solution to (2.1) corresponding to non-negative initial conditions $(u_0, v_0) \neq (0, 0)$ (for the moment $\omega = +\infty$ is still possible). Writing $u_1 = u'(x)$, $v_1 = v'(x)$, where the prime denotes differentiation with respect to x, (u, u_1, v, v_1) solves the equivalent problem:

$$\left.\begin{aligned} u' &= u_1, & u(0) &= u_0, \\ u_1' &= v^p, & u_1(0) &= 0, \\ v' &= v_1, & v(0) &= v_0, \\ v_1' &= u^q, & v_1(0) &= 0. \end{aligned}\right\} \tag{3.1}$$

According to lemma 2.1, u, u_1, v, v_1 and their derivatives up to the second order are all positive. Since (u_0, v_0) is non-trivial and roles played by u and v are symmetric, there is no loss of generality if $v_0 \neq 0$ is assumed below. In particular, we can express $x = x(u_1)$ (the inverse of $x \mapsto u_1$), where $0 \leqslant x < \omega$ as $0 < u_1 < +\infty$. Then $u(u_1) = u(x(u_1))$, $v(u_1) = v(x(u_1))$, $v_1(u_1) = v_1(x(u_1))$ define the solution to the problem

$$\left.\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}u_1} &= \frac{u_1}{v^p}, & u(0) &= u_0,\\ \frac{\mathrm{d}v}{\mathrm{d}u_1} &= \frac{v_1}{v^p}, & v(0) &= v_0,\\ \frac{\mathrm{d}v_1}{\mathrm{d}u_1} &= \frac{u^q}{v^p}, & v_1(0) &= 0. \end{aligned}\right\} \tag{3.2}$$

It also follows from lemma 2.1 that $u(u_1)$, $v(u_1)$, $v_1(u_1)$ all diverge to $+\infty$ as $u_1 \to +\infty$. Next we state our main result in this section. For the sake of clarity, its proof will be postponed until § 4.

THEOREM 3.1. *For $u_0 \geqslant 0$, $v_0 > 0$ let (u, v, v_1) be the solution to (3.2). Then $u(u_1)$, $v(u_1)$, $v_1(u_1)$ satisfy the following asymptotic estimates:*

$$\lim_{u_1\to+\infty} \frac{u}{Au_1^{\alpha}} = 1, \qquad \lim_{u_1\to+\infty} \frac{v}{Bu_1^{\beta}} = 1, \qquad \lim_{u_1\to+\infty} \frac{v_1}{Cu_1^{\gamma}} = 1, \tag{3.3}$$

where the exponents α, β, γ are given by

$$\alpha = \frac{2(p+1)}{pq+2p+1}, \qquad \beta = \frac{2(q+1)}{pq+2p+1}, \qquad \gamma = \frac{pq+2q+1}{pq+2p+1} \tag{3.4}$$

and (A, B, C) is the unique positive solution to

$$\left.\begin{aligned} \alpha AB^p &= 1,\\ \beta B^{p+1} &= C,\\ \gamma B^p C &= A^q. \end{aligned}\right\} \tag{3.5}$$

REMARK 3.2. We point out that $A = (\beta\gamma/\alpha^{(2p+1)/p})^{p/(pq+2p+1)}$, while an explicit expression for B and C will not be strictly required in what follows.

As a consequence of theorem 3.1, we can completely answer the blow-up question for positive solutions to (2.1). This permits us in turn to determine the nature of the set of positive symmetric solutions to (1.4), providing in addition exact asymptotic estimates for their profile near the boundary.

COROLLARY 3.3. *Let $(u, v) \neq (0, 0)$ be any non-negative solution to (2.1) defined in a non-continuable interval $0 \leqslant x < \omega$. Then $\omega < +\infty$, and so blow-up occurs, if and only if $pq > 1$.*

Proof. The solution $(u(x), v(x))$ gives rise to its orbital version

$$(u(u_1), v(u_1), v_1(u_1)),$$

which solves (3.2). Since $u(u_1)$ is increasing with $u \to +\infty$ as $u_1 \to +\infty$, its inverse function $u_1 = U_1(u)$, $U_1(u_0) = 0$, has the same behaviour. Observe in addition that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = U_1(u),$$

thus

$$\omega = \int_{u_0}^{+\infty} \frac{\mathrm{d}s}{U_1(s)}. \tag{3.6}$$

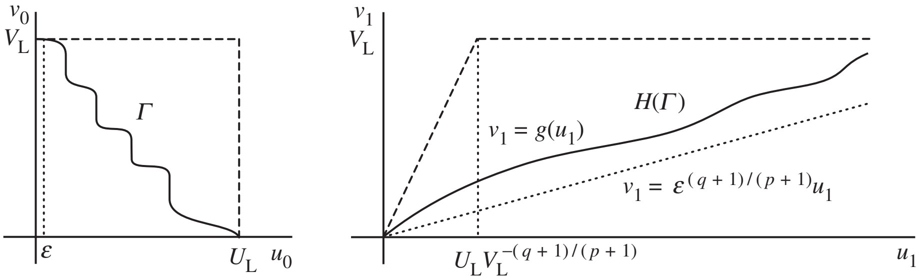

Figure 2. The set Γ in the original variables and after the change $u_1 = u_0^{-(q+1)/(p+1)} v_0$, $v_1 = v_0$.

Therefore, $\omega < +\infty$ amounts to the convergence of the integral. Since, in view of theorem 3.1,

$$U_1(u) \sim \left(\frac{u}{A}\right)^{1/\alpha}, \quad u \to +\infty, \tag{3.7}$$

such convergence is equivalent to $\alpha < 1$, which can be rewritten as $pq > 1$. □

Remark 3.4. The general solution $(U(u_1, u_0, v_0),\ V(u_1, u_0, v_0),\ V_1(u_1, u_0, v_0))$ to (3.2) is as smooth as the equation when observed as a mapping of (u_1, u_0, v_0). The same holds true with the inverse $U_1(u, u_0, v_0)$ of the function $u_1 \mapsto u$. Thus

$$T(u_0, v_0) = \int_{u_0}^{+\infty} \frac{\mathrm{d}s}{U_1(s, u_0, v_0)} = \int_0^{+\infty} \frac{\mathrm{d}s}{U_1(s - u_0, u_0, v_0)},$$

and on this basis the continuity of T with respect to (u_0, v_0) will be proved. We are now stating this fact, and postponing its proof until § 4.

Corollary 3.5. *The function $T(u, v)$ defined in § 2 is continuous.*

An important consequence of the continuity of T and the scale invariance of the equation is that we can completely determine the set of initial data (u_0, v_0) for symmetric solutions to (1.4) in the interval $-L < x < L$.

Corollary 3.6. *Assume that $pq > 1$ and choose $L > 0$ arbitrary. Then the set Γ consisting of all non-negative (u_0, v_0) such that*

$$T(u_0, v_0) = L \tag{3.8}$$

defines a continuous arc $\{(u_0(\sigma), v_0(\sigma)) \in \mathbb{R}_+^2 : 0 \leqslant \sigma \leqslant 1\}$ joining a certain point $(u_0, v_0)_{|\sigma=0} = (0, V_L)$, $V_L > 0$, to $(u_0, v_0)_{|\sigma=1} = (U_L, 0)$, $U_L > 0$, so that $u_0(\sigma)$ is non-decreasing while $v_0(\sigma)$ is non-increasing. In particular, $\Gamma \subset [0, U_L] \times [0, V_L]$ (figure 2).

Proof. By theorem 3.1 and since $pq > 1$, an arbitrary solution to (2.1) with initial conditions $(u_0, 0)$ blows up. The same holds true with the solution corresponding to $(\lambda u_0, 0)$, $\lambda > 0$. Then the scaling property (2.4) provides a unique $\lambda_1 > 0$ such that

$$T(\lambda_1 u_0, 0) = L.$$

We set $U_L = \lambda_1 u_0$. By the same reasoning, for fixed $v_0 > 0$ a positive λ_2 can be found so that, putting $V_L = \lambda_2^{(q+1)/(p+1)} v_0$, the equality

$$T(0, V_L) = L$$

holds. Thus, it follows from lemma 2.3 and remark 2.7 (b) that $T(u_0, v_0) < L$ whenever $u_0 > U_L$ or $v_0 > V_L$, and so $0 \leqslant u_0 \leqslant U_L$, $0 \leqslant v_0 \leqslant V_L$ for every (u_0, v_0) satisfying (3.8). Therefore, Γ is bounded and meets the u_0- and v_0-axes, respectively, at $(U_L, 0)$, $(0, V_L)$.

We now claim that for every $0 < u_0 < U_L$ there is at least a (u_0, v_0) such that $T(u_0, v_0) = L$. Indeed, $T(u_0, 0) > L$ if $u_0 < U_L$, $T(u_0, V_L) \leqslant L$, while in addition $T(u_0, \cdot)$ is continuous in $0 \leqslant v_0 \leqslant V_L$. Hence the claim follows. Unfortunately, it is not possible to ensure the uniqueness of such a solution for each u_0.

To study the form of Γ in the set $Q := [0, U_L] \times [0, V_L]$ it is convenient to introduce the coordinates

$$u_1 = u_0^{-(q+1)/(p+1)} v_0, \qquad v_1 = v_0, \qquad u_0 > 0,$$

and define H as the mapping $(u_0, v_0) \mapsto (u_1, v_1)$. Then the image of $Q_1 = Q \setminus \{u_0 = 0\}$ is

$$H(Q_1) = \{(u_1, v_1) : u_1 > 0,\ 0 \leqslant v_1 \leqslant \min\{U_L^{(q+1)/(p+1)} u_1, V_L\}\}$$

(see figure 2). On the other hand, it is implicit in (2.4) that every curve $u_1 = c$, $c > 0$, contains exactly a unique point (u_0, v_0) with $T(u_0, v_0) = L$. Thus, to every $u_1 > 0$ corresponds a unique solution $(u_1, v_1) \in H(Q_1)$ of the transformed equation

$$T_1(u_1, v_1) = L,$$

where $T_1(u_1, v_1) = T((u_1^{-1} v_1)^{(p+1)/(q+1)}, v_1)$. The continuity of T and a compactness argument then show that the image $H(\Gamma)$ of Γ is the graph of a continuous function $v_1 = g(u_1)$, $u_1 > 0$. Remark 2.7 (b) further implies that g is nondecreasing. Finally, as $(U_L, 0)$ and $(0, V_L)$ are the unique solutions of (3.8) in the u_0- and v_0-axes, it follows from the continuity of T that $\lim_{u_1 \to 0+} g(u_1) = 0$ while $\lim_{u_1 \to +\infty} g(u_1) = V_L$. Thus,

$$\Gamma = H^{-1}\{(u_1, g(u_1)) : u_1 > 0\} \cup \{(U_L, 0), (0, V_L)\}.$$

This proves the lemma. □

A final consequence of theorem 3.1 is the rate of blow-up of solutions to (1.4).

Corollary 3.7. *Assume that $pq > 1$ and let (u, v) be a positive solution to (1.4). Then*

$$\left.\begin{aligned} u(x) &\sim (\xi(\xi+1)\eta^p(\eta+1)^p)^{1/(pq-1)} d(x)^{-\xi} \quad \text{as } x \to \pm L, \\ v(x) &\sim (\eta(\eta+1)\xi^q(\xi+1)^q)^{1/(pq-1)} d(x)^{-\eta} \quad \text{as } x \to \pm L, \end{aligned}\right\} \tag{3.9}$$

where $d(x) = \min\{L - x, L + x\}$, and $\xi = 2(p+1)/(pq-1)$, $\eta = 2(q+1)/(pq-1)$.

Proof. From the proof of corollary 3.3 (replacing ω by L) it follows that

$$\int_{u(x)}^{\infty} \frac{\mathrm{d}s}{U_1(s)} = L - x.$$

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Estimate (3.7) then yields

$$u(x) \sim \left(\frac{\alpha A^{1/\alpha}}{1-\alpha}\right)^{\alpha/(1-\alpha)}(L-x)^{-(\alpha/(1-\alpha))}$$

as $x \to L$, and similarly as $x \to -L$. A little algebra and the use of the expression of A provided in remark 3.2 leads to the first expression in (3.9). The second one is shown in a similar way. This proves the corollary. □

Proof of theorem 1.1. Corollaries 3.3 and 3.6 both imply that $pq > 1$ is a necessary and sufficient condition for the solvability of (1.4) in any domain, together with point (ii) in theorem 1.1. The argument at the beginning of the proof of corollary 3.6 implies the uniqueness assertion in (i). In this case observe that $(U_1(x), V_1(x))$, $(U_2(x), V_2(x))$ are the solutions to (2.1) corresponding to initial data $(U_L, 0)$, $(0, V_L)$, respectively. Finally, the asymptotic profile of the symmetric positive solutions near $x = \pm L$ has been obtained in corollary 3.7. □

4. Proof of theorem 3.1

Proof of theorem 3.1. Let $(u(x), v(x))$ be the solution to (2.1) corresponding to initial conditions $u_0 \geqslant 0$, $v_0 > 0$, while $(u(u_1), v(u_1), v_1(u_1))$, $u_1 \geqslant 0$, stands for the associated (orbital) solution to (3.2). As pointed out earlier, $u(u_1) \to +\infty$, $v(u_1) \to +\infty$, $v_1(u_1) \to +\infty$ as $u_1 \to +\infty$.

Let us now introduce the normalization

$$u = X(u_1)u_1^{\alpha}, \qquad v = Y(u_1)u_1^{\beta}, \qquad v_1 = Z(u_1)u_1^{\gamma}, \qquad u_1 > 0, \tag{4.1}$$

where α, β, γ are the exponents introduced in (3.4). Then the coefficients X, Y, Z satisfy the equation

$$u_1\frac{\mathrm{d}X}{\mathrm{d}u_1} = \frac{1}{Y^p} - \alpha X, \qquad u_1\frac{\mathrm{d}Y}{\mathrm{d}u_1} = \frac{Z}{Y^p} - \beta Y, \qquad u_1\frac{\mathrm{d}Z}{\mathrm{d}u_1} = \frac{X^q}{Y^p} - \gamma Z, \qquad u_1 > 0,$$

which, after the change $u_1 = \mathrm{e}^t$, can be written as the autonomous equation

$$\left.\begin{aligned}\frac{\mathrm{d}X}{\mathrm{d}t} &= \frac{1}{Y^p} - \alpha X,\\ \frac{\mathrm{d}Y}{\mathrm{d}t} &= \frac{Z}{Y^p} - \beta Y,\\ \frac{\mathrm{d}Z_1}{\mathrm{d}t} &= \frac{X^q}{Y^p} - \gamma Z,\end{aligned}\right\} \quad t \in \mathbb{R}. \tag{4.2}$$

It should be observed that positive solutions (u, v, v_1) to (3.1) give rise to positive solutions (X, Y, Z) to (4.2), and vice versa. On the other hand, the octant $\mathbb{R}^3_+ = \{X > 0,\ Y > 0,\ Z > 0\}$ is invariant for (4.2). Thus, positive solutions to (4.2) are characterized as those (X, Y, Z) having positive initial data. Finally, notice that $P = (A, B, C)$ (A, B, C given by (3.5)) is the unique positive equilibrium of (4.2).

In order to prove the estimates (3.3), it must then be shown that *every* positive solution (X, Y, Z) to (4.2) is attracted by the equilibrium P, i.e.

$$(X(t), Y(t), Z(t)) \to (A, B, C), \quad t \to +\infty. \tag{4.3}$$

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Thus, a first step in this direction is checking the local stability of P. The linearization of equation (4.2) at $P=(A,B,C)$ has the coefficients matrix

$$\mathcal{A}=\begin{pmatrix} -\alpha & -p\dfrac{\beta\gamma}{\alpha}A^{-(q+1)} & 0 \\ 0 & -\beta(p+1) & \alpha A \\ \alpha q A^{q} & -p\dfrac{\beta\gamma}{\alpha}A^{-1} & -\gamma \end{pmatrix}. \tag{4.4}$$

Its characteristic polynomial is $p(\lambda)=\lambda^3+a_1\lambda^2+a_2\lambda+a_3$, with coefficients given by

$$\left.\begin{aligned} a_1 &= \alpha+\beta(p+1)+\gamma, \\ a_2 &= \alpha\beta(p+1)+\alpha\gamma+\beta\gamma(2p+1), \\ a_3 &= \alpha\beta\gamma(pq+2p+1). \end{aligned}\right\} \tag{4.5}$$

Since a_1, a_3 are positive, the remaining Routh–Hurwitz condition for asymptotic stability $a_1a_2-a_3>0$ reads as

$$\{\alpha+\beta(p+1)+\gamma\}\{\alpha\beta(p+1)+\alpha\gamma+\beta\gamma(2p+1)\}>\alpha\beta\gamma(pq+2p+1),$$

which is equivalent to

$$\begin{aligned} &\{2(p+1)+2(p+1)(q+1)+pq+2q+1\} \\ &\quad\times\{4(p+1)^2(q+1)+2(p+1)(pq+2q+1)+2(2p+1)(q+1)(pq+2q+1)\} \\ &\qquad\qquad > 4(p+1)(q+1)(pq+2p+1)(pq+2q+1), \quad p,q>0. \end{aligned}$$

A bit of computation shows that the last inequality holds since it amounts to the positivity of the sixth-degree symmetric polynomial in p, q, $4p^3q^3+17p^2q^3+\cdots+34p^2+45p+18$ (the omitted remaining eleven terms are positive!) when $p,q>0$. Anyway, the asymptotic stability of P will also be a byproduct of the analysis of the dynamics of (4.2) to be given in a moment.

However, we are proving a stronger property for P: it is a global attractor for all the positive solutions to (4.2). This will be shown in a series of steps. The first one consists of showing that

$$Z=h(X,Y), \quad X>0,\ Y>0,$$

with

$$h(X,Y)=\frac{X^{q+1}}{q+1}+\frac{Y^{p+1}}{p+1},$$

defines a global and exponentially attracting invariant manifold $\mathcal{M}$ to (4.2). In fact, equations in (4.2) can be written as

$$\left[\frac{X^{q+1}}{q+1}\right]'+\alpha X^{q+1}=\frac{X^q}{Y^p}, \qquad \left[\frac{Y^{p+1}}{p+1}\right]'+\beta Y^{p+1}=Z, \qquad Z'+\gamma Z=\frac{X^q}{Y^p}.$$

Observing that $\gamma+1=\alpha(q+1)=\beta(p+1)$, it follows that

$$\left[\mathrm{e}^{(\gamma+1)t}\left(Z-\frac{X^{q+1}}{q+1}+\frac{Y^{p+1}}{p+1}\right)\right]'=0,$$

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which implies that

$$Z(t) = h(X(t), Y(t)) - C_0 \mathrm{e}^{-(\gamma+1)t},$$

where $C_0 = h(X(0), Y(0)) - Z(0)$. This shows the stated property. Notice that P must lie in $\mathcal{M}$, which is also a direct consequence of the equalities

$$\frac{A^{q+1}}{q+1} + \frac{B^{p+1}}{p+1} = \left\{\frac{\gamma}{\alpha(q+1)} + \frac{1}{\beta(p+1)}\right\}C, \qquad \frac{\gamma}{\alpha(q+1)} + \frac{1}{\beta(p+1)} = 1.$$

Due to the attracting character of $\mathcal{M}$, a second step towards getting an insight of the global behaviour of the dynamics of (4.2) is given by studying its restriction to $\mathcal{M}$, which is governed by the equations

$$\left.\begin{aligned} X' &= \frac{1}{Y^p} - \alpha X, \\ Y' &= \frac{1}{q+1}\frac{X^{q+1}}{Y^p} - \frac{\gamma}{p+1}Y. \end{aligned}\right\} \tag{4.6}$$

It can be checked that the following assertions hold:

(a) the first quadrant $\mathbb{R}^2_+ = \{X > 0, Y > 0\}$ is invariant for (4.6),

(b) $P_{\mathcal{M}} = (A, B)$ is its unique positive equilibrium point, and

(c) the divergence

$$\frac{\partial}{\partial X}\left(\frac{1}{Y^p} - \alpha X\right) + \frac{\partial}{\partial Y}\left(\frac{1}{q+1}\frac{X^{q+1}}{Y^p} - \frac{\gamma}{p+1}Y\right)$$

is negative in $X > 0$, $Y > 0$.

This last property ensures that (4.6) cannot exhibit closed orbits in $\mathbb{R}^2_+$.

On the other hand, $P_{\mathcal{M}} = (A, B)$ is asymptotically stable for (4.6). Indeed, the coefficients matrix of the linearized equation around $P_{\mathcal{M}}$ is

$$\mathcal{A}_{\mathcal{M}} = \begin{pmatrix} -\alpha & -p\alpha\dfrac{A}{B} \\ \gamma\left(\dfrac{q+1}{p+1}\right)\dfrac{B}{A} & -\gamma \end{pmatrix},$$

with characteristic polynomial $p_{\mathcal{M}}(\lambda) = \lambda^2 + (\alpha+\gamma)\lambda + 2\gamma$. Therefore, its eigenvalues λ_1, λ_2 both have negative real parts.

It should now be remarked that, due to the invariant character of $\mathcal{M}$, λ_1, λ_2 are also eigenvalues of $\mathcal{A}$, given by (4.4). Thus, the third eigenvalue λ_3 is negative and given by (see (4.5))

$$\lambda_3 = \frac{\det \mathcal{A}}{\lambda_1\lambda_2} = -\frac{\alpha\beta(pq + 2p + 1)}{2}.$$

In particular, this again shows the asymptotic stability of $P = (A, B, C)$.

Finally, due to the asymptotic stability of $P_{\mathcal{M}}$ and the fact that (4.6) does not admit closed orbits, a standard application of Poincaré–Bendixon theorem [21]

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implies that $P_{\mathcal{M}}$ attracts all positive solutions to (4.6), i.e. every positive solution (X, Y) satisfies

$$(X(t), Y(t)) \to (A, B) \quad \text{as } t \to +\infty.$$

To conclude the proof of the global attractiveness of $P = (A, B, C)$ we are combining in what follows both the attractiveness of $\mathcal{M}$ and the global attractiveness of $P_{\mathcal{M}}$ with respect to the positive solutions to the reduced equation (4.6).

We begin with some preliminary technical remarks. For $P_0 = (X_0, Y_0, Z_0)$ set $\phi(t, P_0) = (X(t), Y(t), Z(t))$ the solution to (4.2) with $(X, Y, Z)_{|t=0} = P_0$. Then, performing a local rectification near P of the field associated with (4.2) it is possible to find a small neighbourhood Ω of P, with smooth boundary $\partial\Omega$, such that if $\phi(t_0, P_0) \in \partial\Omega$ for a certain t_0, then $\phi(t, P_0) \in \Omega$ for $t > t_0$. In addition $\phi(t, P_0) \to P$ as $t \to +\infty$. In particular, the field $(X', Y', Z') = (Y^{-p} - \alpha X, ZY^{-p} - \beta Y, X^q Y^{-p} - \gamma Z)$ points inward Ω at every $(X, Y, Z) \in \partial\Omega$.

Put $\Omega_{\mathcal{M}} = \Omega \cap \mathcal{M}$ and for any compact $K \subset \mathbb{R}^2_+$ set $K_{\mathcal{M}} = \{(X, Y, h(X, Y)) : (X, Y) \in K\}$. Since $\phi(t, P_0)$ reaches $\Omega_{\mathcal{M}}$ for every $P_0 \in K_{\mathcal{M}}$, an application (see [22]) of the implicit function theorem leads to the existence of a neighbourhood

$$\mathcal{N} = \{(X, Y, Z) \in \mathbb{R}^3_+ : \operatorname{dist}((X, Y), K) \leqslant \varepsilon,\ |Z - h(X, Y)| \leqslant \delta\}, \quad \varepsilon, \delta > 0,$$

of $K_{\mathcal{M}}$ and of a certain finite time t_K such that $\phi(t, P_0) \in \Omega$ for every $P_0 \in \mathcal{N}$ and $t > t_K$.

A first consequence is the fact that any positive semiorbit $\{\phi(t, P_0) : t \geqslant 0\}$ to (4.2) with $\{(X(t), Y(t)) : t \geqslant 0\} \subset K \subset \mathbb{R}^2_+$, for a certain compact K, satisfies $\phi(t, P_0) \to P$ as $t \to +\infty$. Indeed, since $\mathcal{M}$ is attracting, then $\phi(t, P_0) \in \mathcal{N}$ for large t.

As a second conclusion, if a positive semiorbit $\{\phi(t, P_0) : t \geqslant 0\}$ has

$$(X(t_n), Y(t_n)) \in K$$

for a certain compact $K \subset \mathbb{R}^2_+$ and a sequence $t_n \to +\infty$, then again $\phi(t, P_0) \to P$ as $t \to +\infty$.

Now, let us choose any positive solution (X, Y, Z) to (4.2) coming, via (4.1), from a solution to the initial-value problem (3.1). We claim that

$$Z(t) < h(X(t), Y(t)) \tag{4.7}$$

for $t < 0$ and $|t|$ large (a proof is delayed to remarks 4.1 below). As $\mathcal{M}$ is invariant this means that (4.7) holds for any t and so

$$Z(t) = h(X(t), Y(t)) - C_0 \mathrm{e}^{-(\gamma+1)t}, \quad t \in \mathbb{R},$$

with $C_0 > 0$. Therefore, the component $(X(t), Y(t))$ of any positive solution to (4.2) satisfies the following non-autonomous equation:

$$\left.\begin{aligned} X' &= \frac{1}{Y^p} - \alpha X, \\ Y' &= \frac{1}{Y^p}\left(\frac{X^{q+1}}{q+1} - C_0 \mathrm{e}^{-(\gamma+1)t} - \frac{\gamma}{p+1} Y^{p+1}\right). \end{aligned}\right\} \tag{4.8}$$

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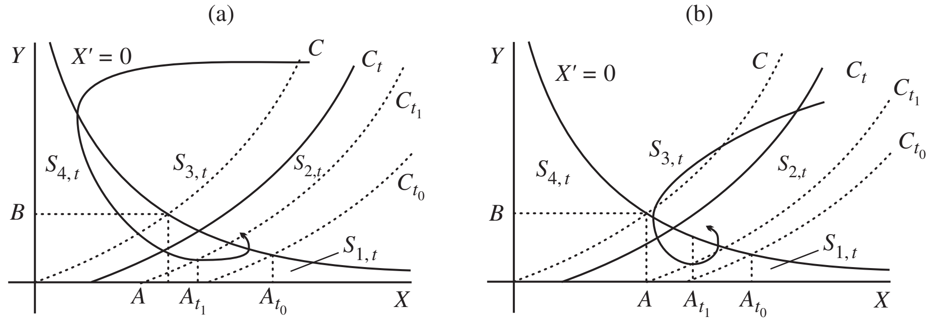

Figure 3. Options (a) (left) and (b) (right).

For immediate use we are setting the t-variable sectors

$$S_{1,t} = \{X' > 0,\ Y' > 0\}, \qquad S_{2,t} = \{X' < 0,\ Y' > 0\},$$
$$S_{3,t} = \{X' < 0,\ Y' < 0\}, \qquad S_{4,t} = \{X' > 0,\ Y' < 0\},$$

denoting by C_t the mobile null cline $Y' = 0$, namely,

$$Y = \left(\frac{p+1}{\gamma}\right)^{1/p+1} \left(\frac{X^{q+1}}{q+1} - C_0 \mathrm{e}^{-(\gamma+1)t}\right)_+^{1/p+1}, \quad X > 0,$$

while (A_t, B_t) will denote the unique positive point where $X' = 0$, $Y' = 0$ at time t. Observe that $(A_t, B_t) \to (A, B)$ while the curve $C_t \to C$ as $t \to +\infty$, where C is the positive null cline $Y' = 0$ corresponding to (4.6), i.e.

$$\frac{X^{q+1}}{q+1} - \left(\frac{\gamma}{p} + 1\right) Y^{p+1} = 0.$$

Our immediate and final objective will be to show that $(X(t), Y(t)) \to (A, B)$ as $t \to +\infty$, whatever the positive solution (X, Y, Z) to (4.2) is. In what follows, several different possibilities will be considered, all of them leading to this conclusion. Our first remark is that

$$(X(t), Y(t)) \in S_{3,t} \quad \text{or} \quad (X(t), Y(t)) \in S_{4,t} \tag{4.9}$$

for $t < 0$ and $|t|$ large, provided, respectively, that the initial conditions (u_0, v_0) satisfy $u_0 > 0$ or $u_0 = 0$ ($v_0 > 0$ in both cases; see remarks 4.1 below). We are assuming the former option in what follows since, in any case, the reasoning is unaffected by this initial assumption.

Next observe that if a solution (X, Y) to (4.8) satisfies $(X(t), Y(t)) \in S_{i,t}$ for t greater than a certain t_0 and some $1 \leqslant i \leqslant 4$, then (X, Y) becomes bounded and by monotonicity $(X(t), Y(t)) \to (A, B)$ as $t \to +\infty$. Since $Z = h(X, Y)$, $(X, Y, Z) \to P$ as $t \to +\infty$.

Thus, the remaining option is that $(X(t), Y(t))$ leaves every $S_{i,t}$ after a finite time t and even in this case we are showing that $(X, Y) \to (A, B)$ as $t \to +\infty$. Since (X, Y) starts at $S_{3,t}$ and we are supposing that it leaves $S_{3,t}$ at a finite time, then only the following options are possible:

(a) (X, Y) exits $S_{3,t}$ after reaching $X' = 0$ at the component $0 < X \leqslant A$;

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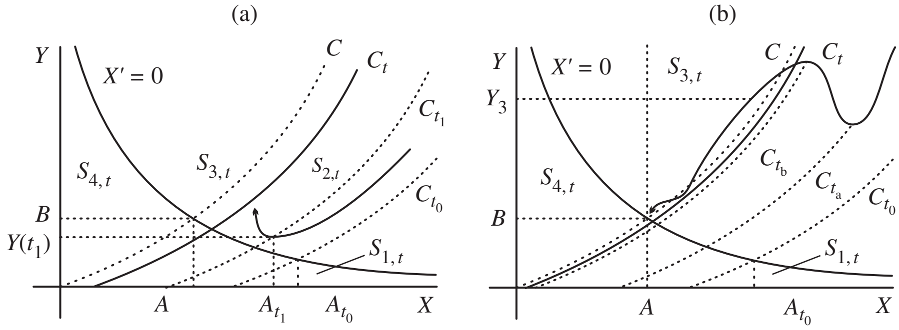

Figure 4. A transition $S_{3,t}$ to $S_{2,t}$ (a) and a possible behaviour in case (i) after transitions $S_{3,t}$ to $S_{2,t}$ and $S_{2,t}$ to $S_{3,t}$ at, respectively, $t = t_a$, $t = t_b$ (b).

(b) (X, Y) leaves $S_{3,t}$ at some time t after crossing $X' = 0$ at $A < X \leqslant A_t$ (figure 3);

(c) (X, Y) leaves $S_{3,t}$ after meeting C_t into the region $X' \leqslant 0$.

In case (a), (X, Y) strictly enters $S_{4,t}$ and then exits this region at finite time after reaching the mobile null cline C_t at a first time $t = t_1$. It should be remarked that $(X(t_1), Y(t_1)) \neq (A_{t_1}, B_{t_1})$. Indeed, if on the contrary equality holds, then $X'(t_1) = Y'(t_1) = X''(t_1) = 0$ (where the prime denotes differentiation with respect to t), while

$$Y''(t_1) = B_{t_1}^{-p} C_0(\gamma + 1)\mathrm{e}^{-(\gamma+1)t_1} > 0, \qquad X'''(t_1) = -pB_{t_1}^{-p-1} Y''(t_1) < 0.$$

Since $X(t) = X'''(t_1)(t - t_1)^3/6 + o((t - t_1)^3)$ this means that $X' < 0$ for $t \neq t_1$ where $Y' < 0$ for $t < t_1$ and $Y' > 0$ for $t > t_1$ ($0 < |t - t_1|$ small). Thus the only way of reaching (A_t, B_t) at $t = t_1$ is having $X' < 0, Y' < 0$ (i.e. $(X(t), Y(t)) \in S_{3,t}$) for $t < t_1$, $|t - t_1|$ small, and this is not the situation in the present case. Therefore, (X, Y) enters $S_{1,t}$ after $t = t_1$ and finally reaches $S_{2,t}$ at finite time. Exactly in the same way, the solution (X, Y) also reaches $S_{2,t}$ in case (b) after crossing $\{X' = 0,\ A \leqslant X \leqslant A_{t_0}\}$, $S_{4,t}$ and $S_{1,t}$, respectively.

In case (c) define $t_1 = \sup\{t : (X(\tau), Y(\tau)) \in \bar{S}_{3,t}$ for each $\tau \leqslant t\}$ and assume that $t_1 < +\infty$ since the opposite implies that $(X, Y) \to (A, B)$ as $t \to +\infty$. Then there exists $t_2 > t_1$ such that $(X, Y)_{|t=t_2} \in S_{2,t_2}$. Once (X, Y) reaches $S_{2,t}$ at $t = t_2$, the only possible options are the following.

(i) $(X(t), Y(t)) \in \overline{S_{2,t} \cup S_{3,t}} = \{X' \leqslant 0\}$ for $t \geqslant t_2$ but not discarding infinitely many transitions between $S_{2,t}$ and $S_{3,t}$ as $t \to +\infty$ (due to analyticity, only a finite number of transitions is possible for bounded t!).

(ii) $(X(t), Y(t)) \in S_{3,t}$ at some $t = t_3$, $t_3 > t_2$ and (X, Y) satisfies option (a) in $t \geqslant t_3$.

(iii) $(X(t), Y(t)) \in S_{3,t}$ for some $t = t_3'$, $t_3' > t_2$ and (X, Y) satisfies option (b) in $t \geqslant t_3'$.

In case (i) we claim that (X, Y) remains bounded for $t \geqslant t_2$. In fact, we have on one hand that $Y \geqslant (\alpha X)^{-1/p}$ (the only possible contact of (X, Y) with the

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null cline $X' = 0$ is the point (A_t, B_t)). On the other hand, $A < X(t) < X(t_2)$ for $t > t_2$ since $X(t) = A$ at a finite time t implies $Y(t) > B$ ($Y(t) = B$ leads to option (a)) and this, in turn, forces (X, Y) to enter S_3 and reach $X' = 0$ at a finite later time (option (a)) which is not the present case. In addition, either $\gamma Y^{p+1}/p + 1 \leqslant X^{q+1}/q + 1$ for $t \geqslant t_2$, in which case (X, Y) remains bounded, or $\gamma Y^{p+1}(t_3)/p + 1 > X^{q+1}(t_3)/q + 1$ at a later time $t_3 > t_2$ (figure 4). Since $Y'(t_3) < 0$ (and so $Y(t) < Y_3 := Y(t_3)$ for $t > t_3$, t close to t_3), Y will never again reach the value Y_3 because $Y(t)$ should be decreasing when approaching Y_3 from below. Thus $Y(t) \leqslant Y_3$ for $t \geqslant t_3$ and then (X, Y) also remains bounded. As pointed out above, this entails $(X, Y) \to (A, B)$ as $t \to +\infty$.

In both options (ii) and (iii), (X, Y) returns to $S_{2,t}$ in finite time. Thus, (X, Y) may either perform a finite number of returns to $S_{2,t}$ following (ii) or (iii) until it chooses option (i) (and then (X, Y) asymptotically converges to (A, B)) or (X, Y) returns infinitely many times to $S_{2,t}$ by means of either (ii) or (iii). However, in this last assumption, (X, Y) meets the compact segment $K_1 = \{(X, Y) : 0 \leqslant X \leqslant A,\ Y = B\}$ (see remarks 4.1 below) every time it crosses $X' = 0$ in (ii), while (X, Y) meets the compact arc $K_2 = \{(X, Y) : A \leqslant X \leqslant A_{t_0},\ Y^p = \alpha X\}$ every time it crosses $X' = 0$ in option (iii). In both cases, (X, Y) passes through the compact $K_1 \cup K_2$ infinitely many times as $t \to +\infty$ and, as also remarked before, this entails that $(X, Y) \to (A, B)$ as $t \to +\infty$. This completes the proof of theorem 3.1. □

Proof of corollary 3.5. Suppose with no loss of generality that $v_0 > 0$ and assume that $(u_{0n}, v_{0n}) \to (u_0, v_0)$. To prove the continuity of T write (corollary 3.3)

$$T(u_{0n}, v_{0n}) = \left\{ \int_0^{\delta} + \int_{\delta}^{\bar{u}} + \int_{\bar{u}}^{+\infty} \right\} \frac{\mathrm{d}s}{U_1(s - u_{0n}, u_{0n}, v_{0n})}, \tag{4.10}$$

with $0 < \delta < \bar{u}$ to be chosen now. Regarding the first integral, the continuous dependence of the solution $(u, v, v_1) = (U(u_1, u_0, v_0), V(u_1, u_0, v_0), V_1(u_1, u_0, v_0))$ to (3.2) on (u_0, v_0), the fact that $v_0 > 0$ and so the v_{0n} are bounded away from zero, and

$$\tfrac{1}{2} U_1^2 = \int_{u_{0n}}^{u} V(U_1(s, u_{0n}, v_{0n}), u_{0n}, v_{0n})^p \, \mathrm{d}s$$

imply that positive constants δ, k_1 and k_2, not depending on n, can be found so that

$$2k_1(u - u_{0n}) \leqslant U_1^2(u, u_{0n}, v_{0n}) \leqslant 2k_2(u - u_{0n})$$

for all $|u - u_{0n}| \leqslant \delta$ and $n \in \mathbb{N}$. Lebesgue's dominated convergence theorem then implies the convergence of the first integral in (4.10) to the corresponding one with u_0, v_0 replacing u_{0n}, v_{0n} in the integrand. The convergence of the second integral follows from the uniform convergence of $U_1(u, u_{0n}, v_{0n})$ to $U_1(u, u_0, v_0)$ in any bounded interval $J \subset R^+$. As for the third integral we deal with $U(u_1, u_{0n}, v_{0n})$ instead of the inverse U_1 and write

$$U(u_1, u_{0n}, v_{0n}) = X(t, u_{0n}, v_{0n}) u_1^{\alpha}, \quad t = \log u_1.$$

We now have that

$$\begin{aligned}(X(\cdot\,, u_{0n}, v_{0n}), Y(\cdot\,, u_{0n}, v_{0n}), Z(\cdot\,, u_{0n}, v_{0n}))& \\ \to (X(\cdot\,, u_0, v_0), Y(\cdot\,, u_0, v_0), Z(\cdot\,, u_0, v_0))&\end{aligned}$$

in bounded intervals, in particular, for any fixed $t = t_1$. Since the last solution lies in the neighbourhood Ω quoted in the proof of theorem 3.1 for $t \geqslant \bar{t} > t_1$, the same holds with the solutions for n large and $t \geqslant \bar{t}$ (by reducing $\bar{t}$ a little bit if necessary). We will have, in particular, that $X(t, u_{0n}, v_{0n}) \leqslant A + \eta$, $\eta > 0$ small, for $t \geqslant \bar{t}$ and so

$$0 < U(u_1, u_{0n}, v_{0n}) \leqslant (A + \eta) u_1^{\alpha}, \quad u_1 \geqslant \mathrm{e}^{\bar{t}},$$

and, after an appropriate choice of $\bar{u}$, we will have

$$\frac{1}{U_1(u, u_{0n}, v_{0n})} \leqslant \left(\frac{A+\eta}{u}\right)^{1/\alpha}, \quad u \geqslant \bar{u},$$

for n large. This enables us to introduce the limit in the last integral and the continuity of T follows.

Remarks 4.1. (a) The power X^{q+1} in equation (4.6) could be replaced by $|X|^{q+1}$ and all semiorbits to this equation starting at the positive semiaxis $X = 0$, $Y > 0$ are still attracted by the point (A, B). This fact allows us to include the extreme $(X, Y) = (0, B)$ in the construction of the compact K_1 for the purposes of the proof of theorem 3.1.

(b) We now prove claims (4.7) and (4.9). First assume that u_0 and v_0 are both positive. By using the u_1 variable instead of t we observe that $X \simeq u_0 u_1^{-\alpha}$, $Y \simeq v_0 u_1^{-\beta}$ and $Z \simeq u_0^q v_0^{-p} u_1^{1-\gamma}$ as $u_1 \to 0+$ $(t \to -\infty)$. Then observe that

$$u_1 X' \simeq v_0^{-p} u_1^{p\beta} - \alpha u_0 u_1^{-\alpha} \to -\infty, \qquad u_1 Y' \simeq \{v_0^{-2p} u_0^q u_1^2 - \beta v_0\} u_1^{-\beta} \to -\infty,$$

as $u_1 \to 0+$. This gives the first of (4.9). As for (4.7) observe that

$$\{X^{q+1}/(q+1) + Y^{p+1}/(p+1)\} \simeq \{u_0^{q+1}/(q+1) + v_0^{p+1}/(p+1)\} u_1^{-(\gamma+1)},$$

which is greater than Z since $Z = O(u_1^{-(\gamma-1)})$ as $u_1 \to 0+$. Thus (4.7) holds.

Next assume that $u_0 = 0$, $v_0 > 0$. Now $u \simeq (v_0^{-p}/2) u_1^2$ while

$$v_1 \simeq (v_0^{-p(q+1)}/\{2^q(2q+1)\}) u_1^{2q+1} \quad \text{as } u_1 \to 0+.$$

Hence

$$u_1 X' \simeq \tfrac{1}{2} v_0^{-p} (2 - \alpha) u_1^{2-\alpha} > 0, \quad u_1 Y' \simeq \left[\frac{v_0^{-p(q+2)}}{2^q(2q+1)} u_1^{2q+2} - \beta v_0\right] u_1^{-\beta} \to -\infty,$$

as $u_1 \to 0+$ (observe that $0 < \alpha < 2$), which gives the second of (4.9). Finally, observe that

$$Z \simeq \frac{v_0^{-p(q+1)}}{2^q(2q+1)} u_1^{2q+1-\gamma},$$

while still $\{X^{q+1}/(q+1) + Y^{p+1}/(p+1)\} = O(u_1^{-(\gamma+1)})$ as $u_1 \to 0+$ and (4.7) holds again.

5. Radial solutions

For the study of radial solutions to problem (1.8) we require some preliminary properties concerning the initial-value problem:

$$\left.\begin{aligned} u_{rr} + \frac{N-1}{r}u_r = v^p,\\ v_{rr} + \frac{N-1}{r}v_r = u^q,\\ u(0)=u_0, \qquad u'(0)=0, \qquad v(0)=v_0, \qquad v'(0)=0,\end{aligned}\right\} \tag{5.1}$$

$p, q > 0$, u_0, v_0 non-negative. The first fact to be quoted is that such a problem admits a non-continuable solution $(u(r), v(r))$ defined in an interval $[0, \omega)$, $0 < \omega \leqslant +\infty$ provided $(u_0, v_0) \neq (0, 0)$. In addition, both u and v are positive and increasing while $\lim_{r\to\omega} u = \lim_{r\to\omega} v = +\infty$. To show this we only need to solve (5.1) in a small interval $[0, \delta]$. Then the so-constructed local solution can be continued according to standard ODE theory and the remaining assertions easily follow. Thus, to find a local solution observe that (5.1) can be equivalently written as

$$u(r) = u_0 + \int_0^r \int_0^\rho \left(\frac{s}{\rho}\right)^{N-1} v(s)^p \, \mathrm{d}s \, \mathrm{d}\rho := \mathcal{T}_1(u, v),$$
$$v(r) = v_0 + \int_0^r \int_0^\rho \left(\frac{s}{\rho}\right)^{N-1} u(s)^q \, \mathrm{d}s \, \mathrm{d}\rho := \mathcal{T}_2(u, v).$$

By choosing $\delta > 0$ small so that $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2)$ maps

$$\{(u, v) \in (C[0, \delta])^2 : |u - u_0|_{\infty,\delta},\ |v - v_0|_{\infty,\delta} \leqslant \delta\}$$

continuously into itself, Schauder's fixed-point theorem provides a solution in $[0, \delta]$. In the regular cases where both u_0 and v_0 are positive or either $p \geqslant 1$ if $v_0 = 0$ $(u_0 > 0)$ or $q \geqslant 1$ if $u_0 = 0$ $(v_0 > 0)$, Banach's fixed point can be used instead to get a unique solution in $[0, \delta]$.

On the other hand, it follows from the fixed-point equation

$$(u, v) = (\mathcal{T}_1(u, v), \mathcal{T}_2(u, v))$$

that solutions $(u(r), v(r))$, $(\tilde{u}(r), \tilde{v}(r))$ corresponding to initial data $(\tilde{u}_0, \tilde{v}_0) \neq (u_0, v_0)$ satisfy $u(r) < \tilde{u}(r)$, $v(r) < \tilde{v}(r)$, provided that $u_0 \leqslant \tilde{u}_0$, $v_0 \leqslant \tilde{v}_0$.

As another remark, for each $u_0, v_0 \geqslant 0$, (5.1) admits a unique solution (u, v) provided $pq > 1$. We only need to check this in the cases $(u_0, v_0) = (0, 0)$ or when $u_0 = 0$, $v_0 > 0$ and $0 < q < 1$ (the complementary case $v_0 = 0$, $u_0 > 0$ with $0 < p < 1$ being identically handled). The proof of the former case is exactly that for the same case in lemma 2.4. As for the latter case assume that (u, v), $(\tilde{u}, \tilde{v})$ solve (5.1) and set $(u_\delta(r), v_\delta(r)) = (u(r + \delta), v(r + \delta))$, $\delta > 0$ small. Then both $\tilde{u}(r) < u_\delta(r)$ and $\tilde{v}(r) < v_\delta(r)$ for $0 \leqslant r \leqslant c$ and a certain $c > 0$. However, in the radial case one finds that $\{r^{N-1}(u_\delta - \tilde{u})'\}' \geqslant 0$ and $\{r^{N-1}(u_\delta - \tilde{u})'\}' \geqslant 0$ in $[0, c]$. Since $u_\delta'(0)$, $v_\delta'(0) > 0$, we conclude that $\tilde{u}(c) < u_\delta(c)$, $\tilde{v}(c) < v_\delta(c)$ and by the same token, those strict inequalities propagate to the common domains of definition of $(\tilde{u}, \tilde{v})$ and (u_δ, v_δ). We get $\tilde{u} \leqslant u$, $\tilde{v} \leqslant v$ by letting $\delta \to 0+$. The reverse inequality follows in the same way and the proof of uniqueness is concluded.

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For immediate use, $T_\rho(u_0, v_0)$ will designate the right extreme ω of the maximal domain of existence of the solution to (5.1).

Proof of theorem 1.2. First we will prove that $pq > 1$ is a necessary and sufficient condition in order that every non-trivial non-negative solution to (5.1) blows up at a finite r, i.e. $T_\rho(u_0, v_0) < +\infty$ for every $u_0, v_0 \geqslant 0$, $(u_0, v_0) \neq (0, 0)$.

For the necessity assume that $(u(r), v(r))$ is positive and solves (5.1) with $pq \leqslant 1$. Since both u and v are increasing,

$$u_{rr} \leqslant u_{rr} + \frac{N-1}{r} u_r = v^p,$$

and similarly $v_{rr} \leqslant u^q$, which implies that $u(r) \leqslant \bar{u}(r)$, $v(r) \leqslant \bar{v}(r)$, $(\bar{u}, \bar{v})$ being the solution of the one-dimensional problem (2.1) with the same initial conditions as (u, v). The comparison gives $T(u_0, v_0) \leqslant T_\rho(u_0, v_0)$ while $T(u_0, v_0) = +\infty$ if $pq \leqslant 1$. Hence (u, v) does not blow up at a finite r.

As for the sufficiency suppose on the contrary that $T_\rho(u_0, v_0) = +\infty$ for the solution (u, v) to (5.1). Observe that the radial equations are again invariant under the scale change (2.5):

$$u_\lambda(r) = \lambda u(kr), \quad v_\lambda(r) = \lambda^{(q+1)/(p+1)} v(kr), \quad k = \lambda^{(pq-1)/(2(p+1))},$$

with $\lambda > 0$, and so T_ρ satisfies the scaling relation (2.4). This means that we can assume that both u_0 and v_0 are as large as desired.

On the other hand, observe that (5.1) can be written as (where the prime denotes differentiation with respect to r)

$$\left.\begin{aligned} r^{N-1}(r^{N-1}u')' &= r^{2(N-1)}v^p, \qquad u(0) = u_0, \qquad u'(0) = 0,\\ r^{N-1}(r^{N-1}v')' &= r^{2(N-1)}u^q, \qquad v(0) = v_0, \qquad v'(0) = 0,\end{aligned}\right\} \tag{5.2}$$

which, after the change [13]

$$\rho = \begin{cases} \dfrac{1}{N-2}\left\{1 - \dfrac{1}{r^{N-2}}\right\} & N \geqslant 3, \\ \log r & N = 2, \end{cases}$$

takes the form (where the superposed dot denotes differentiation with respect to ρ)

$$\left.\begin{aligned} \ddot{u} &= g(\rho)v^p, \qquad u(-\infty) = u_0, \qquad \dot{u}(-\infty) = 0,\\ \ddot{v} &= g(\rho)u^q, \qquad v(-\infty) = v_0, \qquad \dot{v}(-\infty) = 0,\end{aligned}\right\} \tag{5.3}$$

where $g(\rho) = r^{2(N-1)}$. Assume that $N \geqslant 3$ below (the case $N = 2$ is slightly simpler). Observe that $-\infty < \rho < c_N := 1/(N-2)$ in that case and the solution $(u(\rho), v(\rho))$ is defined in $(-\infty, c_N)$.

Fix $\rho_0 \in (-\infty, c_N)$. By using the scaling ideas in § 2 it is possible to choose $u_0, v_0 > 0$ large enough so that the solution $(\tilde{u}(\rho), \tilde{v}(\rho))$ to the auxiliary problem

$$\left.\begin{aligned} \ddot{u} &= g(\rho_0)v^p, \qquad \rho \geqslant \rho_0,\\ \ddot{v} &= g(\rho_0)u^q, \qquad \rho \geqslant \rho_0,\\ u(\rho_0) = u_0, \qquad \dot{u}(\rho_0) &= 0, \qquad v(\rho_0) = v_0, \qquad \dot{v}(\rho_0) = 0,\end{aligned}\right\} \tag{5.4}$$

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blows up at a time ρ_ω such that $\rho_0 < \rho_\omega < c_N$. Since $g(\rho)$ is increasing, $u(\rho_0) > u_0$, $v(\rho_0) > v_0$ and $\dot{u}(\rho_0)$, $\dot{v}(\rho_0)$ are positive, we conclude that $u(\rho) > \tilde{u}(\rho)$ and $v(\rho) > \tilde{v}(\rho)$ for $\rho \geqslant \rho_0$. Thus (u, v) must blow up before ρ_ω, that is, at a finite r. This is just the opposite of what was assumed, the sufficiency being thus proved.

To conclude the proof of theorem 1.2 observe that the scaling relation (2.4) implies the uniqueness assertion in (i). Giving to $U_R, V_R > 0$ the same status as that given to the values U_L, V_L introduced in corollary 3.8, solutions $(U_1(r), V_1(r))$ and $(U_2(r), V_2(r))$ then correspond, respectively, to initial data $(u_0, v_0) = (U_R, 0)$, $(0, V_R)$ in (5.1). The proof of (ii) follows from the fact that T_ρ decreases if both u_0 and v_0 increase. However, it should be observed that since the continuity of T_ρ is not available, the corresponding continuity of $\mathit{\Gamma}_R$ cannot be ensured for the moment. □

6. Non-symmetric solutions: the one-dimensional case

Let us begin with some preliminary remarks. Regarding symmetry, suppose that $(u(x), v(x))$ is non-trivial, non-negative (i.e. both u and v non-negative) and solves (1.4). It follows that u, v are strictly convex and so two unique $x_{\min}, y_{\min} \in (-L, L)$ exist such that $u(x_{\min}) = \inf u$ and $v(y_{\min}) = \inf v$. It can then be checked that (u, v) is symmetric if and only if $x_{\min} = y_{\min}$. In such a case, $x_{\min} = y_{\min} = 0$. Thus $x_{\min} \neq y_{\min}$ provides a test for non-symmetry.

A second remark concerns the extended problem (2.3). In the range $pq > 1$, denote by $(u(x), u_1(x), v(x), v_1(x))$ the non-continuable solution to the initial-value problem (see theorem 2.5)

$$\left.\begin{aligned} u' &= u_1, & u(0) &= u_0, \\ u_1' &= |v|^p, & u_1(0) &= u_0', \\ v' &= v_1, & v(0) &= v_0, \\ v_1' &= |u|^q, & v_1(0) &= v_0' \end{aligned}\right\} \tag{6.1}$$

(where a prime denotes differentiation with respect to x) defined in (ω_1, ω_2) with (u_0, u_0', v_0, v_0') arbitrary (especially regarding sign). Suppose that u, v become positive with u', v' positive (respectively, negative) at some $x = x_1$. It then follows from the proof of theorem 3.1 that ω_2 (ω_1) is finite and hence such a solution blows up there. On the contrary, no solution to (6.1) can exhibit this behaviour when $pq \leqslant 1$.

A further feature also has to do with characterizing the finiteness of ω_i, $i = 1, 2$. Notice that the group $u_1 v_1 - h(u, v)$, $h(u, v) = |u|^q u/(q+1) + |v|^p v/(p+1)$, remains constant on solutions of (6.1).

Lemma 6.1. *Suppose that $pq > 1$ and let (u, v) be a non-continuable solution to (2.3) defined on (ω_1, ω_2). If*

$$u'v' - h(u, v) \neq 0 \tag{6.2}$$

at some x, then both *ω_1 and ω_2 are finite. Moreover, this property is equivalent to (6.2) if (u, v) is non-negative and non-trivial.*

Proof. It is sufficient to simply study the behaviour at ω_2 (otherwise, perform the change $x \mapsto -x$). Two alternative options will be considered in turn: (a) u', v' negative in (ω_1, ω_2), and (b) there exists x_0 where either $u' \geqslant 0$ or $v' \geqslant 0$.

Case (a) is not compatible with (6.2) since fixing x_1 we get by convexity $u'(x_1) < u'(x) < 0$, $v'(x_1) < v'(x) < 0$ together with $u(x_1) + u'(x_1)(x - x_1) < u(x) < u(x_1)$, $v(x_1)+v'(x_1)(x-x_1) < v(x) < v(x_1)$ for $x_1 < x < \omega_2$. Thus $\omega_2 = \infty$. The existence of $\lim u'$, $\lim v'$ as $x \to \infty$ and the identities

$$u'(x) = u'(x_1) + \int_{x_1}^{x} |v|^p, \qquad v'(x) = v'(x_1) + \int_{x_1}^{x} |u|^q$$

imply the convergence of

$$\int_{x_1}^{\infty} |v|^p, \qquad \int_{x_1}^{\infty} |u|^q$$

and hence $u \to 0$, $v \to 0$ as $x \to \infty$. Therefore, $\lim u' = \lim v' = 0$ as $x \to \infty$, $u'v' - h(u, v) \to 0$ as $x \to \infty$ and (6.2) cannot hold, as we wanted to prove.

As for (b), suppose, for instance, that $u'(x_0) \geqslant 0$. Then $u' > 0$ for $x_0 < x < \omega_2$ since due to (6.2) (u, v) is non-trivial. If in addition $v' \geqslant 0$ at some $x_2 \geqslant x_0$, then u', v' are positive in $x > x_2$ and by convexity one finds that both u and v must become non-negative at some $x > x_2$. In view of the preliminary remark, $\omega_2 < \infty$. The opposite option, $v' < 0$ for $x_0 \leqslant x < \omega_2$, cannot occur. In fact, one has that v remains bounded for x bounded and, by means of (6.1), the same happens to u, u', u''. In conclusion, $\omega_2 = \infty$ and as before $\int_{x_0}^{\infty} |u|^q < \infty$ implying $u \to 0$ as $x \to \infty$. However, strict convexity of u is not compatible with that behaviour.

We finally show that (6.2) is necessary to have both ω_1 and ω_2 finite in case of non-negative solutions. In fact, if $(u, v) \neq (0, 0)$ violates (6.2) somewhere, then $u'v' = h(u, v)$ for all x and $u'v'$ is positive. It can be assumed without loss of generality that both u' and v' are negative in (ω_1, ω_2) and hence, as shown in (a), $\omega_2 = \infty$ (observe that ω_1 must be finite in this case). □

Remarks 6.2. (a) The necessity of (6.2) for the finiteness of ω_1, ω_2 fails since symmetric two-signed solutions to (2.3) can be constructed so that $u'v' = h(u, v)$ for all x while ω_1 and ω_2 are finite.

(b) The functions

$$u(x) = \frac{a}{x^\xi}, \quad v(x) = \frac{b}{x^\eta}, \quad x > 0,$$

with ξ, η the exponents and a, b the constants in (1.7), provide an explicit example of a solution behaving as in the last part of the proof of the lemma. By the way, they define the 'prototype' positive solution to problem (1.4) in an unbounded interval, say $(0, \infty)$.

The proof of theorem 1.3 is a consequence of the next perturbation result.

Theorem 6.3. *Suppose that $pq > 1$ and $u_0 > 0$, $v_0 \geqslant 0$. Consider the problem*

$$\left.\begin{aligned} u'' &= |v|^p, \\ u(0) &= u_0, \\ u'(0) &= \sigma, \end{aligned}\right\} \qquad \left.\begin{aligned} v'' &= |u|^q, \\ v(0) &= v_0, \\ v'(0) &= 0, \end{aligned}\right\} \tag{6.3}$$

which can be regarded as a perturbation of problem (2.3) controlled by the parameter

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σ. There then exist $\sigma_1^*>0$ and continuous functions $\omega^+,\omega^-:[-\sigma_1^*,\sigma_1^*]\to\mathbb{R}_+$, ω^+ non-increasing, ω^- non-decreasing such that

(i) *the non-continuable solution $(u(x,\sigma),v(x,\sigma))$ to (6.3) is positive for $x\neq 0$ if and only if $|\sigma|<\sigma_1^*$ and its definition domain in that range for σ is $(-\omega^-(\sigma),\omega^+(\sigma))$;*

(ii) *$(u(x,-\sigma),v(x,-\sigma))=(u(-x,\sigma),v(-x,\sigma))$ for all $\sigma\in\mathbb{R}$;*

(iii) *if $x_{\min}(\sigma)$ is defined as $u(x_{\min}(\sigma),\sigma)=\inf u(\cdot\,,\sigma):=h(\sigma)$, then $x_{\min}$ and h are continuous in $|\sigma|\leqslant\sigma_1^*$, C^1 in $\sigma\neq 0$, $x_{\min}(-\sigma)=-x_{\min}(\sigma)$, $h(-\sigma)=h(\sigma)$, $h(\pm\sigma_1^*)=0$, while both $x_{\min}$ and h decrease in $0\leqslant\sigma\leqslant\sigma_1^*$; moreover, $\inf v(\cdot\,,\sigma)=v_0$ for all σ;*

(iv) *$(u(\cdot\,,\sigma),v(\cdot\,,\sigma))$ blows up at $x=-\omega^-,\omega^+$ for every $|\sigma|\leqslant\sigma_1^*$; moreover,*

$$u(x,\sigma)\sim\frac{a}{d(x)^{\xi}},\qquad v(x,\sigma)\sim\frac{b}{d(x)^{\eta}},$$

as $d(x)=\min\{\omega^+-x,x+\omega^-\}\to 0+$, where a, b, ξ and η are the coefficients and exponents involved in theorems 1.1 and 1.3;

(v) *as a function of (u_0,v_0) in $u_0,v_0\geqslant 0$, σ_1^* is continuous and separately increasing in u_0 and v_0; moreover, σ_1^* vanishes at $u_0=0$.*

Proof. Since (ii) follows from uniqueness (theorem 2.5) and replacing x by $-x$ in (6.3), only the case $\sigma\geqslant 0$ needs to be studied. In addition, as condition (6.2) in lemma 6.1 holds (check at $x=0$), the non-continuable solution (u,v) to (6.3) has both ω_1 and ω_2 finite, regardless of whether $\sigma\in\mathbb{R}$. This fact and continuous dependence on σ assure the positivity of the number

$$\sigma_1^*=\sup\{\sigma\geqslant 0:\inf u(\cdot\,,\sigma)>0\},\tag{6.4}$$

where we remark that, whenever possible, the dependence of (u,v) on σ will not be written below. By convexity one finds $x_{\min}<0$ for $\sigma>0$, while the implicit function theorem implies that $x_{\min}(\sigma)$ is C^1 in $\sigma>0$.

On the other hand, it follows by comparison (lemma 2.3) that both $u(x,\sigma)$ and $v(x,\sigma)$ decrease (respectively, increase) when σ increases in each $x<0$ ($x>0$) provided $\inf u\geqslant 0$. We next use this fact to show that $\sigma_1^*<\infty$. Otherwise, for a fixed $x_0<0$ we have $u>0$ in $x_0\leqslant x\leqslant 0$ for all $\sigma>0$. The estimate $v_0<v(x,\sigma)<v(x,0)$ in $(x_0,0]$ together with

$$u_x(x_0,\sigma)=\sigma-\int_{x_0}^{0}|v|^p$$

imply that $(u_x)_{|x=x_0}\sim\sigma$ as $\sigma\to\infty$. That is not possible since $u_0>-u_x(x_0,\sigma)x_0$. Thus, $\sigma_1^*<\infty$. Remark that lemma 6.1 implies that $x_{\min}>-\infty$ at $\sigma=\sigma_1^*$. Furthermore, σ_1^* is the unique $\sigma>0$ such that $\inf u=0$. In fact, for $\sigma=\sigma_1^*$,

$$-x_{\min}=\frac{1}{\sigma}\biggl\{u_0+\int_{x_{\min}}^{0}\int_{t}^{0}v^p\biggr\}.$$

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If $\inf u = 0$ for $\sigma > \sigma_1^*$, then $x_{\min}(\sigma) = x_{\min}(\sigma_1^*)$, while both $u(\cdot\,,\sigma)$ and $u(\cdot\,,\sigma_1^*)$ are positive in $(x_{\min}(\sigma_1^*), 0]$. That contradicts the fact that the above expression for $x_{\min}$ decreases with σ.

It is clear that $h(\sigma) = \inf u(\cdot\,,\sigma)$ decreases in $0 \leqslant \sigma \leqslant \sigma_1^*$, while for $0 \leqslant \sigma_1 < \sigma_2 \leqslant \sigma_1^*$ the option $x_{\min}(\sigma_2) \geqslant x_{\min}(\sigma_1)$ cannot happen since

$$\sigma_2 = \int_{x_{\min}(\sigma_2)}^{0} v(\cdot\,,\sigma_2) \leqslant \int_{x_{\min}(\sigma_1)}^{0} v(\cdot\,,\sigma_2) < \int_{x_{\min}(\sigma_1)}^{0} v(\cdot\,,\sigma_1) = \sigma_1,$$

contrary to the initial assumption. Thus $x_{\min}$ decreases in $[0, \sigma_1^*]$.

The remaining assertions in (i)–(iv) can be proved by setting $\omega^- = -\omega_1$, $\omega^+ = \omega_2$ and observing that, from the proof of theorem 3.1, blow-up rates in (iv) are common for all positive solutions (u, v) existing only up to some finite value $x = \omega$. As in corollary 3.5, these estimates yield the continuity of $\omega^{\pm}$.

Regarding the continuity of $\sigma_1^*(u_0, v_0)$ it first follows from lemma 2.3 that σ_1^* increases separately in u_0, v_0. Thus, if $P_n = (u_{0n}, v_{0n}) \to P_0 = (u_0, v_0)$, then $\sigma_{1n}^* := \sigma_1^*(P_n)$ remains bounded. We are showing that any limit point of σ_{1n}^* must be σ_1^*, hence $\sigma_1^*(P_n) \to \sigma_1^*$ as desired. Assume that for a certain subsequence $\sigma_{1n'}^* \to \sigma_\infty$. Redefining n' as n set $(u_n(x), v_n(x))$ the solution to (6.3) corresponding to initial data u_{0n}, v_{0n} with $\sigma = \sigma_{1n}^*$, $(u(x), v(x))$ the solution corresponding to $\sigma = \sigma_\infty$ with non-continuable interval (ω_1, ω_2). Supposing that $u_0 > 0$, both ω_1 and ω_2 are finite (lemma 6.1) and so $u' < 0$ at some $b < 0$. Since $(u_n, v_n) \to (u, v)$ in $C^2[b, 0]$ [21], $x_n^* := x_{\min}(\sigma_{1n}^*) \geqslant b$ for large n and has a limit point $x^* \in [b, 0]$. One finds that $u \geqslant 0$ together with $u_{|x=x^*} = 0$. Therefore, $\inf u(\cdot\,,\sigma_\infty) = 0$, which necessarily implies that $\sigma_\infty = \sigma_1^*$. The proof is easily adapted to achieve the case $u_0 = 0$. □

Remark 6.4. Theorem 6.3 admits the corresponding version in which $v'(0)$ varies instead of $u'(0)$, which is kept at zero. More precisely, consider the full σ_1, σ_2 perturbation problem,

$$\left.\begin{aligned} u'' &= |v|^p, \\ u(0) &= u_0, \\ u'(0) &= \sigma_1, \end{aligned}\right\} \qquad \left.\begin{aligned} v'' &= |u|^q, \\ v(0) &= v_0, \\ v'(0) &= \sigma_2, \end{aligned}\right\} \tag{6.5}$$

whose unique (theorem 2.5) non-continuable solution $(u(x, \bar{\sigma}), v(x, \bar{\sigma}))$, $\bar{\sigma} = (\sigma_1, \sigma_2)$, is defined in $(\omega_1, \omega_2) = (-\omega^-(\bar{\sigma}), \omega^+(\bar{\sigma}))$. By setting $\sigma_1 = 0$ and assuming $u_0 \geqslant 0$, $v_0 > 0$, one shows the existence of $\sigma_2^* > 0$ such that $(u(x, (0, \sigma_2)), v(x, (0, \sigma_2)))$ is non-negative if and only if $|\sigma| \leqslant \sigma_2^*$ with $\inf v(\cdot\,, (0, \pm\sigma_2^*)) = 0$, $\omega^{\pm}(0, \sigma_2)$ behaving as $\omega^{\pm}(\sigma)$ and the non-negative solutions blowing-up at $x = \pm\omega^{\pm}(0, \sigma_2)$ according to the same rates as those of theorem 6.3 (iv).

We will treat the σ_1, σ_2 perturbation problem (6.5) in more detail in a moment. We proceed first to the proof of theorem 1.3.

Proof of theorem 1.3. We have already shown the preliminary facts of the theorem. Thus we only need to deal with case (i). In order to construct the family $(\hat{u}, \hat{v})$, consider $(u(\cdot\,,\sigma), v(\cdot\,,\sigma))$ as in theorem 6.3 and define (check the alternative

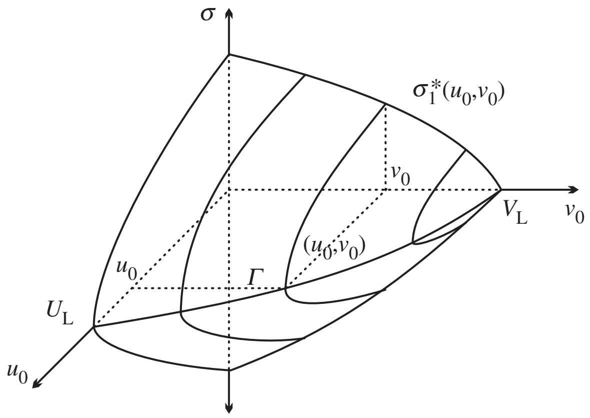

Figure 5. Bifurcation surface for non-symmetric solutions to (1.4) in the family $(\hat{u},\hat{v})$. Symmetric solutions correspond to points $(u_0,v_0,0)$ at curve Γ. The surface is symmetric with respect to the u_0,v_0 plane.

notation of remark 6.4)

$$\left.\begin{aligned} m &= \tfrac{1}{2}(\omega^+(\sigma,0)-\omega^-(\sigma,0)),\\ l &= \tfrac{1}{2}(\omega^+(\sigma,0)+\omega^-(\sigma,0)),\\ \lambda &= (l/L)^{1/\theta}, \end{aligned}\right\} \tag{6.6}$$

where $\theta=(pq-1)/2(p+1)$. The family of positive solutions to (1.4) we are searching for can be constructed by using the scaling properties in lemma 2.6. Namely,

$$(\hat{u}(x,\sigma),\hat{v}(x,\sigma)) = (\lambda u(\lambda^\theta x+m,(\sigma,0)),\lambda^{(q+1)/(p+1)}v(\lambda^\theta x+m,(\sigma,0))). \tag{6.7}$$

with $|\sigma|\leqslant\sigma_1^*$. Observe that $(\hat{u},\hat{v})$ is non-symmetric for $\sigma\neq 0$ since $x_{\min}\neq y_{\min}$ in that case.

According to remark 6.4 the proof of (ii) is identical. □

REMARK 6.5. A bifurcation surface for the family $(\hat{u},\hat{v})$ of non-symmetric solutions with respect to the generating symmetric solutions is shown in figure 5 (a corresponding diagram for $(\tilde{u},\tilde{v})$ is entirely similar). The parameters involved are u_0, v_0 corresponding to the minima in symmetric solutions and σ. Observe that for $|\sigma|\leqslant\sigma_1^*$ parameter σ has the value $\sigma=\lambda^{-\theta-1}\hat{u}'_{|x=-m\lambda^{-\theta}}$ in the case of the family $(\hat{u},\hat{v})$ (for $(\tilde{u},\tilde{v})$ the corresponding parameter is $\sigma=\lambda^{-\theta-(q+1)/p+1}\tilde{v}'_{|x=-m\lambda^{-\theta}}$ with $|\sigma|\leqslant\sigma_2^*$).

7. Proofs of theorems 1.5 and 1.6

We begin with a basic result. Its proof is a consequence of lemmas 2.3 and 6.1 and will therefore be omitted.

LEMMA 7.1. *Let $(u(\cdot,\bar{\sigma}),v(\cdot,\bar{\sigma}))$, $\bar{\sigma}=(\sigma_1,\sigma_2)$, be the solution of the problem (6.5) with domain of existence (ω_1,ω_2). Assume that both u_0 and v_0 are positive while*

$$\inf u(\cdot,\bar{\sigma})_{|\bar{\sigma}=\bar{\sigma}_0}\geqslant 0, \qquad \inf v(\cdot,\bar{\sigma})_{|\bar{\sigma}=\bar{\sigma}_0}\geqslant 0, \tag{7.1}$$

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for a certain $\bar\sigma_0 = (\sigma_{0,1}, \sigma_{0,2}) \in \{\sigma_1 \geqslant 0, \sigma_2 \geqslant 0\}$, $\bar\sigma_0 \neq 0$. *Then,* $\inf u(\cdot\,, \bar\sigma) > 0$, $\inf v(\cdot\,, \bar\sigma) > 0$ *and both* ω_1 *and* ω_2 *are finite for every* $0 \leqslant \bar\sigma \leqslant \bar\sigma_0$, *i.e.* $0 \leqslant \sigma_1 \leqslant \sigma_{01}$, $0 \leqslant \sigma_2 \leqslant \sigma_{02}$, *and* $\bar\sigma \neq \bar\sigma_0$. *In particular, if* $(u(\cdot\,, \bar\sigma), v(\cdot\,, \bar\sigma))$ *is positive with* $\sigma_i \geqslant 0$, $i = 1, 2$, *then necessarily* $\sigma_i < \sigma_i^*$ *for* $i = 1, 2$.

Remark 7.2. Observe that the extreme case $\inf u = \inf v = 0$ at $\bar\sigma = \bar\sigma_0$ is allowed in (7.1), being in that case $\omega_1 = -\infty$.

The proof of theorem 1.6 (ii) can now be given since an arbitrary positive solution (u, v) to (1.4) with $u'(0)v'(0) \geqslant 0$ satisfies the conditions of lemma 7.1 with $\bar\sigma_0 = (u'(0), v'(0))$, after changing x by $-x$, if necessary. In fact, notice that lemma 7.1 ensures that the solution $(u(\cdot\,, \bar\sigma), v(\cdot\,, \bar\sigma))$ to (6.5) is positive and has $\omega_1 = -\omega^-(\bar\sigma)$, $\omega_2 = \omega^+(\bar\sigma)$ finite for $0 \leqslant \bar\sigma \leqslant \bar\sigma_0$. The finiteness of both ω^- and ω^+ together with estimates (1.9) permit showing the continuity of the functions $\omega^\pm$ in $0 \leqslant \bar\sigma \leqslant \bar\sigma_0$ as in the proof of corollary 3.5. By replacing $(u(\cdot\,, (\sigma, 0)), v(\cdot\,, (\sigma, 0)))$ with $(u(\cdot\,, \bar\sigma), v(\cdot\,, \bar\sigma))$ in (6.7), also substituting $\omega^\pm(\sigma, 0)$ with $\omega^\pm(\bar\sigma)$ in (6.6), $0 \leqslant \bar\sigma \leqslant \bar\sigma_0$, we obtain a continuous bidimensional family of positive solutions connecting (u, v) with the symmetric solution attained when $\bar\sigma = 0$, as desired. □

The proofs of theorem 1.5 and the remaining part of theorem 1.6 require a study of the following function,

$$g(\sigma_1) = \sup\{\sigma_2 : \inf v(\cdot\,, (\sigma_1, \sigma_2')) > 0 \text{ for } 0 \leqslant \sigma_2' \leqslant \sigma_2\}, \tag{7.2}$$

and of its dual version,

$$f(\sigma_2) = \sup\{\sigma_1 : \inf u(\cdot\,, (\sigma_1', \sigma_2)) > 0 \text{ for } 0 \leqslant \sigma_1' \leqslant \sigma_1\}, \tag{7.3}$$

where $(u(\cdot\,, \bar\sigma), v(\cdot\,, \bar\sigma))$ stands for the solution of (6.5).

Both the 'supremum' and the restriction $0 \leqslant \sigma_2' \leqslant \sigma_2$ can be replaced in the definition of g by 'infimum' and the range $\sigma_2 \leqslant \sigma_2' \leqslant 0$, respectively (this is nothing else but the effect of the change $x \mapsto -x$ in problem (1.4)). Then what we find is just the symmetric function $g_-(\sigma_1) := -g(-\sigma_1)$ of g. The symmetric function $f_-(\sigma_2) := -f(-\sigma_2)$ has exactly the same meaning.

The main features exhibited by g are collected in the next lemma.

Lemma 7.3. *The function* g *is well defined and positive in* $\mathbb{R}$. *In addition,*

(a) g *is decreasing and continuous in* $\sigma_1 \leqslant 0$ *with* $g(0) = \sigma_2^*$. *Moreover,*

$$\inf v(\cdot\,, \bar\sigma) < 0,$$

for $\sigma_1 \leqslant 0$ *and* $\sigma_2 > g(\sigma_1)$ *and hence (6.5) does not admit positive solutions for* $\sigma_1 \leqslant 0$ *and* $\sigma_2 \geqslant g(\sigma_1)$;

(b) $\lim_{\sigma_1 \to -\infty} g(\sigma_1) = +\infty$ *and, more precisely,*

$$g(\sigma_1) \sim \frac{1}{q+1}[(q+2)v_0]^{(q+1)/(q+2)}(-\sigma_1)^{q/(q+2)} \tag{7.4}$$

as $\sigma_1 \to -\infty$;

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(c) *$g \in L^\infty_{\rm loc}[0,\infty)$ while g is continuous and decreasing in a neighbourhood of any value $\sigma_1 \geqslant 0$ such that $\inf u(\cdot\,,\bar\sigma) > 0$ at $\bar\sigma = (\sigma_1, g(\sigma_1))$. In particular, g is continuous and decreasing in $(-\infty,\delta]$ for some positive $0<\delta<\sigma_1^*$.*

Proof. By theorem 6.3, we have $g(0) = \sigma_2^*$ (see remark 6.4). Let us show the finiteness of g for $\sigma_1 \neq 0$. If $\sigma_1 > 0$ and there exists $\bar\sigma_n = (\sigma_1, \sigma_{2,n})$ with $\sigma_{2,n} \to \infty$ and $\inf v(\cdot\,,\bar\sigma_n) > 0$, then lemma 7.1 implies $\inf u(\cdot\,,\bar\sigma_n) < 0$ for large n. In particular, $u(\cdot\,,\bar\sigma_n)$ will be defined in $[-u_0/\sigma_1, 0]$ since, by convexity, $x_{\min} < -u_0/\sigma_1$ for large n, $x_{\min}$ being the point where $u(\cdot\,,\bar\sigma_n)$ attains the infimum. In addition, $u(x,\bar\sigma_n) \leqslant u_0$ in that interval. If we now take $\hat x \in [-u_0/\sigma_1, 0]$, then

$$0 < \inf v(\cdot\,,\bar\sigma_n) \leqslant v(\hat x,\bar\sigma_n) \leqslant v_0 + \sigma_{2,n}\hat x + \tfrac{1}{2}\hat x^2 u_0^q,$$

and so

$$(-\hat x)\sigma_{2,n} \leqslant v_0 + \tfrac{1}{2}\hat x^2 u_0^q,$$

which is not compatible with the divergence of $\sigma_{2,n}$. We have shown, in particular, that for fixed $\sigma_1 > 0$, the set of those $\sigma_2 > 0$ such that $\inf v(\cdot\,,\bar\sigma) > 0$ is bounded above with a bound that can be chosen independent of σ_1 for σ_1 varying in small intervals in $[0,\infty)$. Hence $g \in L^\infty_{\rm loc}(0,\infty)$ (it will even be shown below that g is continuous in an interval containing $\sigma_1 = 0$).

As for $\sigma_1 < 0$, $u(x,\bar\sigma) \geqslant u_0$ wherever it is defined in $x \leqslant 0$. Lemma 2.3 implies that $u(x,\bar\sigma) < u(x,(\sigma_1,0))$ for each $-\omega^-(\sigma_1,0) < x < 0$ and for every $\sigma_2 > 0$ such that $\inf v(\cdot\,,\bar\sigma) > 0$. Fixing now $\tilde x \in (-\omega^-(\sigma_1,0),0)$, such numbers σ_2 satisfy

$$(-\tilde x)\sigma_2 \leqslant v_0 + \int_0^{\tilde x}\int_0^t u(s,(\sigma_1,0))\,\mathrm{d}s\,\mathrm{d}t,$$

and so they are bounded above. Thus $g(\sigma_1)$ must be finite. Notice also that in this case, lemma 2.3 provides that g decreases with σ_1 while necessarily $\inf v(\cdot\,,\bar\sigma) < 0$ for $\sigma_2 > g(\sigma_1)$.

To complete the proof of (a) let us show the continuity of g in $\sigma_1 \leqslant 0$. First, observe that for $\sigma_1 \leqslant 0$ both limits $g(\sigma_1-) \geqslant g(\sigma_1+) \geqslant \sigma_2^*$ are finite. Since (6.2) holds at $\bar\sigma = (\sigma_1, g(\sigma_1\pm))$, lemma 6.1 assures that both extremes $\pm\omega^\pm(\bar\sigma)$ of the existence interval of the solution $(u(\cdot\,,\bar\sigma), v(\cdot\,,\bar\sigma))$ to (6.5) corresponding to $\bar\sigma = (\sigma_1, g(\sigma_1\pm))$ are finite. This is crucial to enable us to conclude that, after a careful application of the continuous dependence results in [21], $\inf v(\cdot\,,\bar\sigma) = 0$ at $\bar\sigma = (\sigma_1, g(\sigma_1\pm))$. Since $\sigma_1 \leqslant 0$, $g(\sigma_1)$ is just the unique value of σ_2 such that $\inf v = 0$. Hence $g(\sigma_1\pm) = g(\sigma_1)$ as desired.

Let us prove the continuity assertion in (c) and so assume that $\inf u(\cdot\,,\bar\sigma) > 0$ at $\bar\sigma = (\sigma_1, g(\sigma_1))$. As shown in the proof of lemma 6.1, the positivity of $\inf u$ implies also the finiteness of the extremes $\pm\omega^\pm(\bar\sigma)$ of the interval of existence for the solution (u,v) to (6.5) corresponding to $\bar\sigma = (\sigma_1, g(\sigma_1))$. Continuous dependence provides the existence of a small $\eta > 0$ such that $\inf u(\cdot\,,\bar\sigma') > 0$ for every $\bar\sigma'$ satisfying $|\sigma_i' - \sigma_i| \leqslant \eta$, $i = 1,2$. Given $\varepsilon > 0$, $0 < \varepsilon_1 < \max\{\varepsilon,\eta\}$ and $0 < \delta < \eta$ can be found, with the help of lemma 2.3, such that $\inf v(\cdot\,,\bar\sigma') > 0$ in $\sigma_2' = g(\sigma_1) - \varepsilon_1$, $\inf v(\cdot\,,\bar\sigma') < 0$ in $\sigma_2' = g(\sigma_1) + \varepsilon_1$ for each σ_1' such that $|\sigma_1' - \sigma_1| \leqslant \delta$. Therefore, $g(\sigma_1) - \varepsilon < g(\sigma_1') < g(\sigma_1) + \varepsilon$ in the last interval. Moreover, lemma 7.1 directly implies that g decreases there. This concludes the proof of (c).

We finally show the asymptotic estimate (7.4) for g. To this proposal we are equivalently analysing the behaviour as $\sigma_1 \to \infty$ of the symmetric function g_- of g (see the remarks after (7.3)):

$$g_-(\sigma_1) = \inf\{\sigma_2 < 0 : \inf v(\cdot\,, \bar\sigma) > 0\}.$$

For $\bar\sigma = (\sigma_1, g_-(\sigma_1))$ define $y_{\min}$ as the point where $v(x, \bar\sigma)$ attains its minimum. Notice that $g_-(\sigma_1) < 0$ implies that $y_{\min} > 0$. We first prove that

$$\lim_{\sigma_1 \to \infty} g_-(\sigma_1) = -\infty.$$

If, on the contrary, $\inf_{\sigma_1 \geqslant 0} g_- = -l > -\infty$, we find by convexity that $v(x, \bar\sigma)$, $\bar\sigma = (\sigma_1, g_-(\sigma_1))$, is defined in $[0, v_0/l]$ wherein $v \leqslant v_0$. By using that $u(x, \bar\sigma) \geqslant u_0 + \sigma_1 x$ and fixing $\tilde x \in [0, v_0/l]$ we arrive at

$$v_0 \geqslant v(\tilde x, \bar\sigma) \geqslant v_0 - l\tilde x + \int_0^{\tilde x} \int_0^t (u_0 + \sigma_1 s)^q \, \mathrm{d}s \, \mathrm{d}t \to \infty,$$

as $\sigma_1 \to \infty$. Since this is not possible, it follows that $l = \infty$.

On the other hand, v, v_x vanish at $x = y_{\min}$. Thus, the following identities hold:

$$v_0 = \int_0^{y_{\min}} s u^q \, \mathrm{d}s, \qquad -g_-(\sigma_1) = \int_0^{y_{\min}} s u^q \, \mathrm{d}s, \tag{7.5}$$

where we have used the positivity of $u = u(x, \bar\sigma)$ in $x \geqslant 0$. Indeed, $u(x, \bar\sigma) \geqslant u_0 + \sigma_1 x$ in $x \geqslant 0$. By using the first equality in (7.5) we achieve

$$v_0 \geqslant \frac{\sigma_1^q}{q+2} y_{\min}^{q+2}.$$

This means that $y_{\min} \to 0+$ as $\sigma_1 \to \infty$. We recall that $0 \leqslant v(x, \bar\sigma) \leqslant v_0$ in $[0, y_{\min}]$ and conclude, by employing the second equality in (7.5),

$$\int_0^{y_{\min}} (u_0 + \sigma_1 s)^q \, \mathrm{d}s \leqslant -g_-(\sigma_1) \leqslant \int_0^{y_{\min}} (u_0 + \sigma_1 s + \tfrac{1}{2} v_0^p s^2)^q \, \mathrm{d}s. \tag{7.6}$$

Hence,

$$-g_-(\sigma_1) \leqslant y_{\min} \int_0^1 (u_0 + \sigma_1 y_{\min} t + \tfrac{1}{2} v_0^p y_{\min}^2 t^2)^q \, \mathrm{d}t.$$

We immediately achieve that $\sigma_1 y_{\min} \to \infty$ as $\sigma_1 \to \infty$ since $\sigma_1 y_{\min} = O(1)$ on a possible sequence $\sigma_{1,n} \to \infty$ together with the previous estimate imply $-g_-(\sigma_1) = O(1)$ on that sequence, which is not possible.

The first equality in (7.5) leads to

$$\int_0^{y_{\min}} s(u_0 + \sigma_1 s)^q \, \mathrm{d}s \leqslant v_0 \leqslant \int_0^{y_{\min}} s(u_0 + \sigma_1 s + \tfrac{1}{2} v_0^p s^2)^q \, \mathrm{d}s,$$

and so

$$\frac{(\sigma_1 y_{\min})^q}{q+2} \leqslant v_0 y_{\min}^{-2} \leqslant \int_0^1 t(u_0 + \sigma_1 y_{\min} t + \tfrac{1}{2} v_0^p y_{\min} t^2)^q \, \mathrm{d}t.$$

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Since

$$\int_0^1 t(u_0+\sigma_1 y_{\min}t+\tfrac{1}{2}v_0^p y_{\min}t^2)^q\,\mathrm{d}t \sim \frac{(\sigma_1 y_{\min})^q}{q+2}, \quad \sigma_1\to\infty,$$

then

$$\frac{1}{y_{\min}} \sim \left\{\frac{\sigma_1^q}{(q+2)v_0}\right\}^{1/(q+2)}, \quad \sigma_1\to\infty. \tag{7.7}$$

Now, after multiplying (7.6) by σ_1, we get

$$\sigma_1 y_{\min}\int_0^1 (u_0+\sigma_1 y_{\min}t)^q\,\mathrm{d}t \leqslant -\sigma_1 g_-(\sigma_1)$$
$$\leqslant \sigma_1 y_{\min}\int_0^1 (u_0+\sigma_1 y_{\min}t+\tfrac{1}{2}v_0^p y_{\min}^2 t^2)^q\,\mathrm{d}t.$$

Hence,

$$\sigma_1 g_-(\sigma_1) \sim -\frac{(\sigma_1 y_{\min})^{q+1}}{q+1}, \quad \sigma_1\to\infty,$$

which, together with (7.7), leads to the desired estimate (7.4). This finishes the proof of the lemma. □

REMARK 7.4. The dual function f of g defined in (7.3) satisfies, when conveniently transposed, the properties of g in lemma 7.3. In particular, $f(\sigma_2)$ is continuous and decreasing in the interval $-\infty<\sigma_2\leqslant\delta_1$ for certain positive $\delta_1<\sigma_2^*$, $f(0)=\sigma_1^*$ while $f(\sigma_2)\to\infty$ as $\sigma_2\to-\infty$ with the asymptotic behaviour,

$$f(\sigma_2)\sim C_p(-\sigma_2)^{p/(p+2)}, \quad \sigma_2\to-\infty, \tag{7.8}$$

where $C_p=[(p+2)u_0]^{(p+1)/(p+2)}/(p+1)$. More importantly, for $\sigma_2\leqslant 0$, problem (6.5) does not admit positive solutions for $\sigma_1\geqslant f(\sigma_2)$. Finally, f will be decreasing and continuous in a certain neighbourhood of any σ_2 where $\inf v(\cdot\,,(f(\sigma_2),\sigma_2))>0$.

Let us now complete the proof of theorem 1.6 and show theorem 1.5. Define the function,

$$h_+(\sigma_1)=\begin{cases} g(\sigma_1), & \sigma_1\leqslant 0,\\ g\wedge g_1(\sigma_1), & 0<\sigma_1<\sigma_1^*,\\ f^{-1}(\sigma_1), & \sigma_1\geqslant\sigma_1^*,\end{cases}$$

where $g\wedge g_1(\sigma_1)=\min\{g(\sigma_1),g_1(\sigma_1)\}$ and the function g_1 is defined in $0<\sigma_1<\sigma_1^*$ as

$$g_1(\sigma_1)=\sup\{\sigma_2 : \inf u(\cdot\,,(\sigma_1,\sigma_2'))>0 \text{ for } 0\leqslant\sigma_2'\leqslant\sigma_2\}.$$

Note that g_1 may be infinite at some σ_1.

We claim that h_+ is continuous and decreasing. In fact, lemma 7.1 directly implies that $g\wedge g_1$ decreases in the interval $(0,\sigma_1^*)$. On the other hand, $g\wedge g_1(\sigma_1)=g(\sigma_1)$ for $0<\sigma_1<\delta$ (see lemma 7.3), while $g\wedge g_1(\sigma_1)=f^{-1}(\sigma_1)$ for $\sigma_1^*-\delta_2<\sigma_1<\sigma_1^*$ and a certain $\delta_2>0$ small enough (see remark 7.4). Moreover, lemma 7.3 (c) ensures that $g\wedge g_1=g$ in a neighbourhood of any σ_1 where $g(\sigma_1)<g_1(\sigma_1)$. Thus $g\wedge g_1$ is continuous in that neighbourhood. The corresponding assertion holds true for those σ_1 where $g_1(\sigma_1)<g(\sigma_1)$ being $g\wedge g_1=f^{-1}$ in a certain neighbourhood

of σ_1 (remark 7.4). To complete the proof of the claim we show the continuity of $g \wedge g_1$. Indeed, assume on the contrary that $g \wedge g_1(\sigma_1-) > g \wedge g_1(\sigma_1+)$ and define $\sigma_2' = g \wedge g_1(\sigma_1-)$, $\sigma_2'' = g \wedge g_1(\sigma_1+)$, with $\bar{\sigma}' = (\sigma_1, \sigma_2')$, $\bar{\sigma}'' = (\sigma_1, \sigma_2'')$. A careful use of continuous dependence of the solutions to (6.5) with respect to $\bar{\sigma}$ permits us to ensure that $\inf u(\cdot, \bar{\sigma}') \geqslant 0$, $\inf v(\cdot, \bar{\sigma}') \geqslant 0$. Lemma 7.1 then implies that $\inf u(\cdot, \bar{\sigma}'') > 0$, $\inf v(\cdot, \bar{\sigma}'') > 0$. However, that is not possible, again due to continuous dependence.

Observe that the function h_+ satisfies all the properties stated in theorem 1.5 (b), with the exception of the existence of the value $\sigma_1 = b$ (see further details below). Moreover, by its own definition no positive solutions to (6.5) are possible for $0 < \sigma_1 < \sigma_1^*$ provided $\sigma_2 \geqslant g \wedge g_1(\sigma_1)$. Hence, that problem cannot exhibit positive solutions for $\sigma_2 \geqslant h_+(\sigma_1)$ and by introducing the symmetric function $h_-(\sigma_1) := -h_+(-\sigma_1)$, exactly the same assertion holds true for $\sigma_2 \leqslant h_-(\sigma_1)$. In other words, the set (see (1.11)),

$$\mathcal{C} = \{\bar{\sigma} : h_-(\sigma_1) < \sigma_2 < h_+(\sigma_1)\},$$

characterizes the existence of positive solutions to (6.5) regarding the values of $\bar{\sigma}$. This provides the proof of theorem 1.6 (i).

We further show the boundedness of $\mathcal{C}$. Since $\mathcal{C} \cap \{\sigma_1 \geqslant 0, \sigma_2 \geqslant 0\} \subset [0, \sigma_1^*] \times [0, \sigma_2^*]$, due to the symmetry of $\mathcal{C}$ it is sufficient to show that $\mathcal{C} \cap \{\sigma_1 \geqslant 0, \sigma_2 \leqslant 0\}$ is bounded. In fact, observe that $h_+ = f^{-1}$ for $\sigma_1 \geqslant \sigma_1^*$ and so (see remark 7.4)

$$h_+(\sigma_1) \sim -C_p^{-(p+2)/p} \sigma_1^{(p+2)/p}, \quad \sigma_1 \to \infty.$$

Since $h_-(\sigma_1) = g_-(\sigma_1)$ for $\sigma_1 \geqslant 0$ and $g_-(\sigma_1) \sim -C_q \sigma_1^{q/(q+2)}$ as $\sigma_1 \to \infty$ (C_q is the coefficient in (7.4)), we conclude that $h_+(\sigma_1) < h_-(\sigma_1)$ for large σ_1. Thus $\mathcal{C}$ is bounded. On the other hand, observe that $h_- < -\sigma_2^*$ in $(0, \sigma_1^*]$ while $h_+ > 0$ in $(0, \sigma_1^*)$ (figure 1). This means that a first value $b > \sigma_1^*$ must exist such that $h_-(b) = h_+(b)$.

Next consider the set $\mathcal{C}_0 = \{\bar{\sigma} : h_-(\sigma_1) < \sigma_2 < h_+(\sigma_1), |\sigma_1| < b\}$ (see (1.10)). By construction, both extremes $\pm\omega^{\pm}(\bar{\sigma})$ of the maximal interval of existence of the solution (u, v) to (6.5) (a solution which is, in addition, positive) are finite and, by the reasons already explained in the proof of theorem 1.6(ii), they vary continuously as $\bar{\sigma} \in \mathcal{C}_0$. Therefore, the scale change (6.7),

$$(\tilde{u}(x, \bar{\sigma}), \tilde{v}(x, \bar{\sigma})) = (\lambda u(\lambda^{\theta} x + m, \bar{\sigma}), \lambda^{(q+1)/(p+1)} v(\lambda^{\theta} x + m, \bar{\sigma})), \tag{7.9}$$

with $\omega^{\pm}(\bar{\sigma})$ replacing the values of $\omega^{\pm}(\sigma, 0)$ in (6.6), defines a continuous bidimensional family of positive solutions to (1.4) which produces a symmetric solution exclusively at $\bar{\sigma} = 0$. This concludes the proofs of theorem 1.5 and theorem 1.6 (iii).

Remarks 7.5. (a) The open set $\mathcal{C}$ could possibly exhibit a connected piece $\mathcal{C}_1$ different from $\mathcal{C}_0$ in $\{\sigma_1 < 0, \sigma_2 < 0\}$ if h_- and h_+ coincide in values σ_1 greater than b. In this case and regarding assertion (iii) of theorem 1.6, it is unclear if a positive solution (u, v) to (1.4) with derivatives $(u_x'(0), v_x'(0)) \in \mathcal{C}_1$ could be deformed, keeping its sign, to produce a symmetric solution.

(b) When $\bar{\sigma} \to \partial\mathcal{C}_0$, different types of non-negative solutions (u, v) to (1.4) are obtained in the limit, one of whose components (or even both of them) vanishes at a unique point of $(-L, L)$. In the case of $\partial\mathcal{C}_0 \cap \{\sigma_1 \geqslant 0, \sigma_2 \leqslant 0\}$, solutions to (6.5)

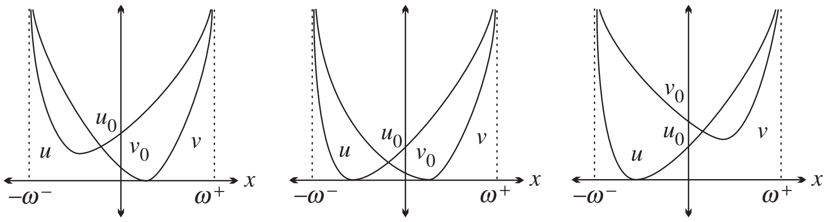

Figure 6. Solutions $(u(\cdot\,,\bar\sigma), v(\cdot\,,\bar\sigma))$ of (6.5) for $\bar\sigma \in \partial\mathcal{C}_0 \cap \{\sigma_1 \geqslant 0\}$ corresponding to $\sigma_1\sigma_2 < 0$. The second configuration corresponds to the values $\sigma_1 = b$, $\sigma_2 = h_+(b)$.

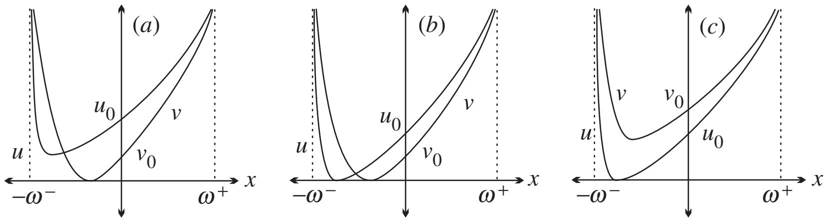

Figure 7. Non-negative solutions associated with values $\bar\sigma \in \partial\mathcal{C}_0 \cap \{\sigma_1 \geqslant 0\}$ in the case where $\sigma_1\sigma_2 > 0$. Occurrence of solutions as in the middle configuration require some additional information.

lead, by means of the scaling (7.9), to solutions (u, v) to (1.4) such that $\inf u > 0$ and $\inf v = 0$ for $\sigma_2 = g_-(\sigma_1)$, $0 \leqslant \sigma_1 < b$, $\inf u = \inf v = 0$ at $\sigma_2 = g_-(b) = f^{-1}(b)$, while $\inf u = 0$ and $\inf v > 0$ for $\sigma_2 = f^{-1}(\sigma_1)$, $\sigma_1^* \leqslant \sigma_1 < b$ (figure 6). The situation in the case $\partial\mathcal{C}_0 \cap \{\sigma_1 \geqslant 0, \sigma_2 \geqslant 0\}$ is slightly different. In the arc $\sigma_2 = h_+(\sigma_1)$, $0 \leqslant \sigma_1 \leqslant \sigma_1^*$, solutions $(u(\cdot\,,\bar\sigma), v(\cdot\,,\bar\sigma))$ of (6.5) exhibit two kind of features. In the first one, $\inf u > 0$ and $\inf v = 0$ (as is the case for $\sigma_1 \sim 0$) or $\inf u = 0$ and $\inf v > 0$ (which happens at least near σ_1^*). Both behaviours are observed in the solutions to (1.4) obtained after the change (7.9). See the first and third parts of figure 7. A second one, due to the existence of at least some σ_1 such that the solution $(u(\cdot\,,\bar\sigma), v(\cdot\,,\bar\sigma))$ to (6.5), satisfies $\inf u = \inf v = 0$. However, a solution to (1.4) with the same behaviour is now generated via (7.9) only when $-\omega^-(\bar\sigma) > -\infty$ with $\bar\sigma = (\sigma_1, h_+(\sigma_1))$ (the case in figure 7*b*). According to lemma 6.1, this case is characterized by the fact that $\sigma_1\sigma_2 < h(u_0, v_0)$ at $\sigma_2 = h_+(\sigma_1)$. Indeed, we can obtain conditions on u_0, v_0 and parameters p, q ensuring that $\sigma_1 h_+(\sigma_1) < h(u_0, v_0)$ for all $\sigma_1 \in [0, \sigma_1^*]$ and so all possible solutions (u, v) to (6.5) in the second kind generate solutions to (1.4) (details are omitted for the sake of brevity). If, on the contrary, $\sigma_1 h_+(\sigma_1) = h(u_0, v_0)$, then $\omega^-(\bar\sigma_0) = \infty$ at $\bar\sigma_0 = (\sigma_1, h_+(\sigma_1))$ and the change (7.9) plainly has no sense. Moreover, it is unclear which is the limit profile in $x \in (-L, L)$ of the solutions $(\tilde u(x, \bar\sigma), \tilde v(x, \bar\sigma))$ to (1.4) obtained in (7.9) when $\bar\sigma \in \mathcal{C}_0$ and $\bar\sigma \to \bar\sigma_0$.

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