THE WIENER-PITT PHENOMENON ON THE HALF-LINE

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It has been well known for many years (**2**) that if $F_\mu(t)$ is the Fourier-Stieltjes transform of a bounded measure μ on the real line R, which is bounded away from zero, it does *not* follow that $[F_\mu(t)]^{-1}$ is also the Fourier-Stieltjes transform of a measure. It seems of interest (as was remarked, in conversation, by J. D. Weston) to consider measures on the half-line $R^+ = [0, \infty[$, instead of on R. The Fourier-Stieltjes transform is now replaced by the Laplace-Stieltjes transform

$$L_\mu(\zeta) = \int_0^\infty e^{-x\zeta}\, d\mu(x) \quad (\mathscr{R}\zeta \geqq 0),$$

and the problem is: if μ is a bounded measure, and

$$| L_\mu(\zeta) | \geqq k > 0 \quad (\mathscr{R}\zeta \geqq 0),$$

is it true that $[L_\mu(\xi)]^{-1}$ is the Laplace-Stieltjes transform of a measure also? The answer, as will be shown below, is negative; the Wiener-Pitt phenomenon occurs. One may of course ask (and to some extent answer) a similar question in the case of a general semigroup. Some extensions (for example, to the positive quadrant in R^2, and similar situations) are immediate; we do not attempt to discuss the general problem here.

The occurrence of the phenomenon in the case of R^+ follows quite easily from results already known for R. Let $M(R)$ be the Banach algebra of bounded measures on R, and let $\lambda \in M(R)$ satisfy

(i) $\| \lambda \| = 1$;
(ii) the support of λ is contained in $[-1, 1]$;
(iii) $F_\lambda(t)$ is real for all t;
(iv) the spectrum of λ contains i ($i^2 = -1$).

It is clear that (i) and (iii) together imply that $-1 \leqq F_\lambda(t) \leqq 1$ for all t. The existence of such measures has been established in general locally compact abelian groups (see (**1**) or (**4**)). A simple example on the real line may be obtained (**3**) by writing λ_n for the measure with mass $\frac{1}{2}$ at each of $\pm 1/n!$, and taking λ to be the infinite convolution product $\lambda_2 * \lambda_3 * \lambda_4 \ldots$. Let μ be the measure $\delta_3 * (\delta_0 - \lambda^2)$, that is, the measure obtained by translating $(\delta_0 - \lambda^2)$ through a distance $+3$ (we write δ_x for the measure with mass 1 at x). Then μ may be regarded as a measure on either R or R^+; its support is contained in $(1, 5)$.

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Let m be a homomorphism of $M(R)$ on to the complex field such that $m(\lambda^2) = -1$; let $m(\delta_3) = e^{i\theta}$, and consider the measure

$$\nu = 2\delta_0 - e^{-i\theta}\mu.$$

Then

$$m(\nu) = 2 - e^{-i\theta}e^{i\theta}(1+1) = 0,$$

and so ν has no inverse in $M(R)$ and, *a fortiori*, no inverse in $M(R^+)$.

On the other hand, $L_\nu(\zeta)$ is bounded away from zero in the right half plane. Writing $\zeta = \xi + i\eta$ we have in general, for $\xi \geqq 0$,

$$\begin{aligned} |L_\nu(\zeta) - 2| &= \left| \int_0^\infty e^{-x\zeta} d\mu(x) \right| \\ &= \left| \int_1^5 e^{-x\zeta} d\mu(x) \right| \\ &\leqq |e^{-\zeta}| \, \| \mu \| \\ &= 2e^{-\xi}, \end{aligned}$$

so that $L_\nu(\zeta)$ is bounded away from zero in every half plane $\xi \geqq \xi_0 > 0$. Also, for $\xi = 0$,

$$L_\nu(i\eta) = F_\nu(\eta) = 2 - e^{-i\theta}e^{-3i\eta}(1 - F_\lambda^2(\eta)).$$

Since $0 \leqq F_\lambda^2(\eta) \leqq 1$, it follows that $|L_\nu(i\eta)| \geqq 1$. Finally, $L_\nu(\zeta)$ is continuous in ξ, uniformly in η, at $\xi = 0$ (and indeed everywhere). Taking $\xi > 0$ we have

$$\begin{aligned} |L_\nu(i\eta) - L_\nu(\xi + i\eta)| &= \left| \int_0^\infty (1 - e^{-x\xi}) e^{-ix\eta} d\nu(x) \right| \\ &= \left| \int_0^5 (1 - e^{-x\xi}) e^{-ix\eta} d\nu(x) \right| \\ &\leqq \| \nu \| \sup_{0 \leqq x \leqq 5} (1 - e^{-x\xi}) \\ &= 4(1 - e^{-5\xi}). \end{aligned}$$

From this, the required result follows immediately.

Since if $[L_\nu(\xi)]^{-1}$ were of the form $L_\sigma(\zeta)$ for some bounded measure σ, it would follow that $\sigma = \nu^{-1}$, it is clear that the Wiener-Pitt phenomenon occurs.

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