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STOCHASTIC NONZERO-SUM GAMES: A NEW CONNECTION BETWEEN SINGULAR CONTROL AND OPTIMAL STOPPING

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Abstract

In this paper we establish a new connection between a class of two-player nonzero-sum games of optimal stopping and certain two-player nonzero-sum games of singular control. We show that whenever a Nash equilibrium in the game of stopping is attained by hitting times at two separate boundaries, then such boundaries also trigger a Nash equilibrium in the game of singular control. Moreover, a differential link between the players' value functions holds across the two games.

Keywords: Games of singular control; games of optimal stopping; Nash equilibrium; one-dimensional diffusion; Hamilton–Jacobi–Bellman equation; verification theorem

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1. Introduction

Connections between some problems of singular stochastic control (SSC) and questions of optimal stopping (OS) are well known in control theory. In 1966, Bather and Chernoff [5] studied the problem of controlling the motion of a spaceship which must reach a given target within a fixed period of time, and with minimal fuel consumption. This problem of aerospace engineering was modelled in [5] as a SSC problem, and an unexpected link with OS was observed. The value function of the control problem was indeed differentiable in the direction of the controlled state variable, and its derivative coincided with the value function of an OS problem.

The result of Bather and Chernoff was obtained by using mostly tools from analysis. Following this, Karatzas [26], [27], and Karatzas and Shreve [29] employed fully probabilistic methods to perform a systematic study of the connection between SSC and OS for the so-called 'monotone follower problem'. The latter consists of tracking the motion of a stochastic process (a Brownian motion in [26]–[29]) by a nondecreasing control process in order to maximise (minimise) a performance criterion which is concave (convex) in the control variable. Further, a link to OS was shown to hold also for monotone follower problems of finite-fuel type; i.e. where the total variation of the control (the fuel available to the controller) stays bounded (see [18] and [28], and also [4] for dynamic stochastic finite fuel). More recent works provided extensions of the above results to diffusive settings, [6] and [8], to Brownian two-dimensional problems with state constraints [11], to Itô-Lévy dynamics under partial information [36], and to non-Markovian processes [3].

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It was soon realised that these kinds of connection could be established in wider generality with admissible controls which are of bounded variation as functions of time (rather than just monotone). Indeed, under suitable regularity assumptions (including convexity or concavity of the objective functional with respect to the control variable), the value function of a bounded variation control problem is differentiable in the direction of the controlled state variable, and its derivative equals the value function of a two-player zero-sum game of OS (Dynkin game). To the best of the authors' knowledge, this link was first noted in [40] in a problem of controlling a Brownian motion, and then generalised in [7] and [31], and later on also in [22] via optimal switching.

It is important to observe that despite their appearance in numerous settings, connections between SSC and OS are rather 'delicate' and should not be taken for granted, even for monotone follower problems with very simple diffusion processes. Indeed, counterexamples can be found in [14] and [15] where the connection breaks down even if the cost function is arbitrarily smooth and the underlying processes are Ornstein–Uhlenbeck or Brownian motion.

The existing theory on the connection between SSC and OS is well established for *single-agent* optimisation problems. However, the latter are not suitable for the description of more complex systems where strategic interactions between several decision makers play a role. Problems of this kind arise, for instance, in economics and finance when studying productive capacity expansion in an oligopoly [39], the competition for the market-share control [32], or the optimal control of an exchange rate by a central bank (see the introduction of [24] for such an application).

In this paper we establish a new connection between a class of two-player nonzero-sum games of OS (see [16] and the references therein) and certain two-player nonzero-sum games of SSC. These games involve two different underlying (one-dimensional) Itô diffusions. The one featuring in the game of controls will be denoted by $\tilde{X}$, whereas the one featuring in the game of stopping will be denoted by X.

In the game of controls, each player may exert a monotone control to adjust the trajectory of $\tilde{X}$. The first player can only increase the value of $\tilde{X}$ by exerting his/her control, while the second player can decrease the value of $\tilde{X}$ only by exerting his/her control. If player 1 uses a unit of control at time $t > 0$ then he/she must pay $G_1(\tilde{X}_t)$, while at the same time player 2 receives $L_2(\tilde{X}_t)$. A symmetric situation occurs if player 2 exerts control (see Section 2.2). Each player wants to maximise his/her own total expected reward functional.

In the game of stopping, both players observe the dynamics of X and may decide to end the game by choosing a stopping time for X. When the game ends, each player pays a cost according to the following rule: if the ith player stops first, he/she pays G_i; if instead the ith player lets the opponent stop first, he/she pays L_i. Here G_i and L_i are the same functions as in the game of controls and, in general, they depend on the value of X at the random time when the game is ended.

We show that if a Nash equilibrium in the game of stopping is attained by hitting times of two separate thresholds, i.e. the process X is stopped as soon as it leaves an interval (a_*, b_*) on the real line, then the couple of controls that keep $\tilde{X}$ inside $[a_*, b_*]$ with minimal effort (i.e. according to a Skorokhod reflection policy) realises a Nash equilibrium in the game of singular controls. Moreover, we also prove that the value functions of the two players in the game of singular controls can be obtained by suitably integrating their respective ones in the game of OS. The existence of Nash equilibria of the threshold type for the game of stopping holds in a large class of examples as demonstrated in [16]. Here the proof of our main theorem (see Theorem 3.1 below) is based on a verification argument following an educated guess. In order to illustrate an application of our results we present a game of pollution control between a social planner and a firm representative of the productive sector.

Another important result of this paper is a simple explicit construction of Markov-perfect equilibria (i.e. equilibria in which each player dynamically reacts to his/her opponent's decisions) for a class of two-player continuous-time stochastic games of singular control. This is a problem in game theory which has yet to be solved in full generality (see [2, Section 2] and [39]), and here we contribute to further improve results in that direction. We seek for Nash equilibria in the class of control strategies $\mathcal{M}$ which forbids the players to exert simultaneous impulsive controls (i.e. simultaneous jumps of their control variables). On the one hand, this is a convenient choice for technical reasons, but, on the other hand, we also show in Appendix A.1 that it induces no loss of generality in a large class of problems commonly addressed in the literature on SSC.

It is worth emphasising a key difficulty in handling nonzero-sum games. If, for example, player 1 deviates unilaterally from an equilibrium strategy then this has two effects: it worsens player 1's performance, but it also affects player 2's payoff. However, it is impossible to establish *a priori* whether such a deviation benefits or harms player 2. This issue does not arise in single-agent problems and in two-player zero-sum games where the optimisation involves a unique objective functional. From a partial differential equation point of view this is expressed by the fact that our nonzero-sum game of controls is associated to a system of coupled variational inequalities, rather than to a single variational inequality. Thus, there is a fundamental difference between the nature of our results and those already known for certain (single-agent) bounded variation control problems (see, for example, [7] and [40]).

Our work marks a new step towards a global view on the connection between SSC problems and questions of OS by extending the existing results to nonzero-sum, multi-agent optimisation problems. A link between these two classes of optimisation problems is important not only from a purely theoretical point of view but also from a practical point of view. Indeed, as it was pointed out in [29, p. 857] one may hope to 'jump' from one formulation to the other in order to 'pose and solve more favourable problems'. As an example, one may note that questions of existence and uniqueness of optimisers are more tractable in control problems than in stopping ones; on the other hand, a characterisation of optimal control strategies is, in general, a more difficult task than the one of OS rules. Recent contributions to the literature (see, for example, [12], [13], and [20]) have highlighted how the combined approach of SSC and OS is extremely useful to deal with investment/consumption problems for a single representative agent. It is therefore reasonable to expect that our work will increase the mathematical tractability of investment/consumption problems for multiple interacting agents.

The rest of the paper is organised as follows. In Section 2 we introduce the setting, the game of singular controls, and the game of OS. In Section 3 we prove our main result and discuss the assumptions needed. An application to a game of pollution control is considered in Section 4, whereas some proofs and a discussion regarding admissible strategies are collected in Appendix A.

2. Setting

2.1. The underlying diffusions

Denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space equipped with a filtration $\mathsf{F} = (\mathcal{F}_t)_{t\geq 0}$ under the usual hypotheses. Let $\tilde{W} = (\tilde{W}_t)_{t\geq 0}$ be a one-dimensional standard Brownian motionadapted to F, and $(\tilde{X}_t^{\nu,\xi})_{t\geq 0}$ the strong solution (if it exists) to the one-dimensional,

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controlled stochastic differential equation (SDE)

$$\mathrm{d}\tilde{X}_t^{\nu,\xi} = \mu(\tilde{X}_t^{\nu,\xi})\,\mathrm{d}t + \sigma(\tilde{X}_t^{\nu,\xi})\,\mathrm{d}\tilde{W}_t + \mathrm{d}\nu_t - \mathrm{d}\xi_t, \qquad \tilde{X}_0^{\nu,\xi} = x \in \mathcal{I}, \tag{2.1}$$

with $\mathcal{I} := (\underline{x}, \overline{x}) \subseteq \mathbb{R}$ and with μ, σ real-valued functions which we will specify below. Here $(\nu_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ belong to

$$\mathcal{S} := \{\eta\colon (\eta_t(\omega))_{t\geq 0} \text{ left-continuous, adapted, increasing, with } \eta_0 = 0, \mathbb{P}\text{-a.s.}\},$$

where we abbreviate $\mathbb{P}$-almost surely to $\mathbb{P}$-a.s., and we denote

$$\sigma_{\mathcal{I}} := \inf\{t \geq 0\colon \tilde{X}_t^{\nu,\xi} \notin \mathcal{I}\}$$

the first time the controlled process leaves $\mathcal{I}$.

Note that ν and ξ can be expressed as the sum of their continuous part and pure jump part, i.e.

$$\nu_t = \nu_t^c + \sum_{s<t} \Delta\nu_s, \qquad \xi_t = \xi_t^c + \sum_{s<t} \Delta\xi_s,$$

where $\Delta\nu_s := \nu_{s+} - \nu_s$ and $\Delta\xi_s := \xi_{s+} - \xi_s$. Throughout the paper we will consider the process $\tilde{X}^{\nu,\xi}$ killed at $\sigma_{\mathcal{I}}$, and we make the following assumptions on μ and σ.

Assumption 2.1. *The functions μ and σ are in $C^1(\mathcal{I})$ and $\sigma(x) > 0$, $x \in \mathcal{I}$.*

Since μ and σ are locally Lipschitz, for any given $(\nu, \xi) \in \mathcal{S} \times \mathcal{S}$, (2.1) has a unique strong solution (see [37, Theorem V.7] and the text after its proof).

To account for the dependence of $\tilde{X}$ on its initial position, from now on we will write $\tilde{X}^{x,\nu,\xi}$ where appropriate. In the rest of the paper we use the notation $\mathbb{E}_x[f(\tilde{X}_t^{\nu,\xi})] = \mathbb{E}[f(\tilde{X}_t^{x,\nu,\xi})]$ for f Borel-measurable, since $(\tilde{X}, \nu, \xi)$ is Markovian but the initial value of the controls is always 0. Here $\mathbb{E}_x$ is the expectation under the measure $\mathbb{P}_x(\cdot) := \mathbb{P}(\cdot \mid \tilde{X}_0 = x)$ on $(\Omega, \mathcal{F})$. As mentioned in the introduction, (2.1) will be the underlying process in the game of control.

To keep the notation simple and to avoid introducing another filtered probability space, we also assume that the filtered probability space $(\Omega, \mathcal{F}, \mathsf{F}, \mathbb{P})$ is sufficiently rich to allow for the treble $(\Omega, \mathcal{F}, \mathbb{P})$, F, (X, W) to be a weak solution to the SDE

$$\mathrm{d}X_t = (\mu(X_t) + \sigma(X_t)\sigma'(X_t))\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \qquad X_0 = x \in \mathcal{I}, \tag{2.2}$$

where W is another Brownian motion. Note that this requirement does not affect the generality of our results because $\tilde{X}$ and X never feature at the same time in our optimisation problems. In particular, X will appear only in the game of stopping.

Assumption 2.1 guarantees that the above SDE admits a weak solution which is unique in law up to a possible explosion time [30, Chapter 5.5]. Indeed, for every $x \in \mathcal{I}$, there exists $\varepsilon_o > 0$ such that

$$\int_{x-\varepsilon_o}^{x+\varepsilon_o} \frac{1 + |\mu(z)| + |\sigma(z)\sigma'(z)|}{|\sigma(z)|^2}\,\mathrm{d}z < +\infty. \tag{2.3}$$

To account explicitly for the initial condition, we denote by X^x the solution to (2.2) starting from $x \in \mathcal{I}$ at time 0. Due to (2.3), the diffusion X is regular in $\mathcal{I}$, i.e. if $\tau_z := \inf\{t \geq 0\colon X_t^x = z\}$, we have $\mathbb{P}(\tau_z < \infty) > 0$ for every x and z in $\mathcal{I}$ so that the state space cannot be decomposed into smaller sets from which X cannot exit (see [9, Chapter 2]).

We make the following standing assumption.

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Assumption 2.2. *The points $\underline{x}$ and $\overline{x}$ are either natural or entrance-not-exit for the diffusion X, hence unattainable. Moreover, $\underline{x}$ and $\overline{x}$ are unattainable for the uncontrolled process $\tilde{X}^{0,0}$.*

For boundary behaviours of diffusions, we refer the reader to [9, p. 15]. Unattainability of $\underline{x}$ and $\overline{x}$ refers to the fact that, for $x \in \mathcal{I}$, the processes X^x and $\tilde{X}^{x,0,0}$ cannot leave the interval $(\underline{x}, \overline{x})$ in finite time, $\mathbb{P}$-a.s. Feller's test for explosion (see, for example, [30, Theorem 5.5.29]) provides necessary and sufficient conditions under which $\underline{x}$ and $\overline{x}$ are unattainable for the diffusions X and $\tilde{X}^{0,0}$. Moreover, specific properties of natural and entrance-not-exit boundaries may be addressed by using the speed measure $m(\mathrm{d}x)$ and the scale function $S(x)$ of the above diffusions (since we are not going to make use of these concepts we simply refer the interested reader to [9, pp. 14–15] for details).

In the next remark we show that if σ' is sufficiently integrable then unattainable boundary points of X are also unattainable for the uncontrolled process $\tilde{X}^{0,0}$.

Remark 2.1. For simplicity, let us assume that $\sigma \in C^2(\mathcal{I})$ so that both (2.1) and (2.2) admit a strong solution. For $x \in \mathcal{I}$, let us define a new measure $\mathbb{Q}_x$ by the Radon–Nikodým derivative

$$Z_t := \frac{\mathrm{d}\mathbb{Q}_x}{\mathrm{d}\mathbb{P}_x}\bigg|_{\mathcal{F}_t} = \exp\bigg\{\int_0^t \sigma'(\tilde{X}_s^{0,0})\,\mathrm{d}\tilde{W}_s - \frac{1}{2}\int_0^t (\sigma')^2(\tilde{X}_s^{0,0})\,\mathrm{d}s\bigg\}, \qquad \mathbb{P}_x\text{-a.s.}$$

which is an exponential martingale under suitable integrability conditions on σ'. Hence, the Girsanov theorem implies that the process $B_t := \tilde{W}_t - \int_0^t \sigma'(\tilde{X}_s^{0,0})\,\mathrm{d}s$ is a standard Brownian motion under $\mathbb{Q}_x$ and it is not difficult to verify that $\mathrm{Law}(\tilde{X}^{0,0} \mid \mathbb{Q}_x) = \mathrm{Law}(X \mid \mathbb{P}_x)$.

It follows, denoting $\sigma_{\mathcal{I}}^0 = \inf\{t > 0\colon \tilde{X}_t^{0,0} \notin \mathcal{I}\}$ and $\tau_{\mathcal{I}} = \inf\{t > 0\colon X_t \notin \mathcal{I}\}$, that $\mathrm{Law}(\sigma_{\mathcal{I}}^0 \mid \mathbb{Q}_x) = \mathrm{Law}(\tau_{\mathcal{I}} \mid \mathbb{P}_x)$. Note also that the measures $\mathbb{Q}_x$ and $\mathbb{P}_x$ are equivalent on $\mathcal{F}_t^{\tilde{W}}$ for all $0 \le t < +\infty$, where $(\mathcal{F}_t^{\tilde{W}})_{t\ge 0}$ is the filtration generated by $\tilde{W}$ (see [30, Chapter 3.5]). In particular, $\{\sigma_{\mathcal{I}}^0 \le t\} \in \mathcal{F}_t^{\tilde{W}}$. Therefore, using the fact that $\underline{x}$ and $\overline{x}$ are unattainable for X, we obtain

$$0 = \mathbb{P}_x(\tau_{\mathcal{I}} \le t) = \mathbb{Q}_x(\sigma_{\mathcal{I}}^0 \le t) \quad\Longrightarrow\quad \mathbb{P}_x(\sigma_{\mathcal{I}}^0 \le t) = 0 \quad \text{for all } t > 0.$$

Hence, $\mathbb{P}_x(\sigma_{\mathcal{I}}^0 < +\infty) = 0$, which proves that $\underline{x}$ and $\overline{x}$ are unattainable for the process $\tilde{X}^{0,0}$ under $\mathbb{P}_x$ for all $x \in \mathcal{I}$.

The infinitesimal generator of the uncontrolled diffusion $\tilde{X}^{x,0,0}$ is denoted by $\mathcal{L}_{\tilde{X}}$ and defined as

$$(\mathcal{L}_{\tilde{X}} f)(x) := \tfrac{1}{2}\sigma^2(x) f''(x) + \mu(x) f'(x), \qquad f \in C^2(\overline{\mathcal{I}}),\ x \in \mathcal{I},$$

whereas the one for X is denoted by $\mathcal{L}_X$ and defined as

$$(\mathcal{L}_X f)(x) := \tfrac{1}{2}\sigma^2(x) f''(x) + (\mu(x) + \sigma(x)\sigma'(x)) f'(x), \qquad f \in C^2(\overline{\mathcal{I}}),\ x \in \mathcal{I}.$$

Letting $r > 0$ be a fixed constant, we have the following assumption.

Assumption 2.3. *Assume that $r > \mu'(x)$ for $x \in \overline{\mathcal{I}}$.*

We denote by ψ and ϕ the fundamental solutions of the ordinary differential equation (see [9, Chapter 2, Section 10])

$$\mathcal{L}_X u(x) - (r - \mu'(x))u(x) = 0, \qquad x \in \mathcal{I}, \tag{2.4}$$

and we recall that they are strictly increasing and decreasing, respectively.

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Finally, we denote by $S'(x)$, $x \in \mathcal{I}$, the density of the scale function of $(X_t)_{t\geq 0}$, and by w the Wronskian

$$w := \frac{\psi'(x)\phi(x) - \phi'(x)\psi(x)}{S'(x)}, \qquad x \in \mathcal{I}, \tag{2.5}$$

which is a positive constant.

Particular attention in this paper is devoted to solutions of (2.1) reflected inside intervals $[a, b] \subset \mathcal{I}$, and we recall here the following result on Skorokhod reflection. Its proof can be found, for example, in [41, Theorem 4.1] (note that μ' and σ' are bounded on $[a, b]$). Below we abbreviate 'such that' to s.t. and $\mathbf{1}_A$ is the indicator function on the event A.

Lemma 2.1. *Let Assumption 2.1 hold. For any $a, b \in \mathcal{I}$ with $a < b$ and any $x \in [a, b]$, there exists a unique couple $(\nu^a, \xi^b) \in \mathcal{S} \times \mathcal{S}$ that solves the Skorokhod reflection problem* $\mathrm{SP}(a, b; x)$ *defined as:*

$$\text{find } (\nu, \xi) \in \mathcal{S} \times \mathcal{S} \text{ s.t.} \begin{cases} \tilde{X}_t^{x,\nu,\xi} \in [a, b], \ \mathbb{P}\text{-a.s.} & \text{for } 0 < t \leq \sigma_{\mathcal{I}}, \\ \displaystyle\int_0^{T\wedge\sigma_{\mathcal{I}}} \mathbf{1}_{\{\tilde{X}_t^{x,\nu,\xi} > a\}} \, \mathrm{d}\nu_t = 0, \ \mathbb{P}\text{-a.s.} & \text{for any } T > 0, \\ \displaystyle\int_0^{T\wedge\sigma_{\mathcal{I}}} \mathbf{1}_{\{\tilde{X}_t^{x,\nu,\xi} < b\}} \, \mathrm{d}\xi_t = 0, \ \mathbb{P}\text{-a.s.} & \text{for any } T > 0. \end{cases} \tag{2.6}$$

It also follows that $\mathrm{supp}\{\mathrm{d}\nu_t^a\} \cap \mathrm{supp}\{\mathrm{d}\xi_t^b\} = \varnothing$.

For future frequent use we also recall the one-sided version of the above result.

Lemma 2.2. *Let Assumption 2.1 hold. For any $a \in \mathcal{I}$, $x \geq a$, and $\xi \in \mathcal{S}$, there exists a unique $\nu^a \in \mathcal{S}$ that solves the Skorokhod reflection problem* $\mathrm{SP}_{a+}^{\xi}(x)$ *defined as:*

$$\text{find } \nu \in \mathcal{S} \text{ s.t.} \begin{cases} \tilde{X}_t^{x,\nu,\xi} \in [a, \overline{x}), \ \mathbb{P}\text{-a.s.} & \text{for } 0 < t \leq \sigma_{\mathcal{I}}, \\ \displaystyle\int_0^{T\wedge\sigma_{\mathcal{I}}} \mathbf{1}_{\{\tilde{X}_t^{x,\nu,\xi} > a\}} \, \mathrm{d}\nu_t = 0, \ \mathbb{P}\text{-a.s.} & \text{for any } T > 0. \end{cases} \tag{2.7}$$

Similarly, for any $b \in \mathcal{I}$, $x \leq b$, and $\nu \in \mathcal{S}$, there exists a unique $\xi^b \in \mathcal{S}$ that solves the Skorokhod reflection problem $\mathrm{SP}_{b-}^{\nu}(x)$ *defined as:*

$$\text{find } \xi \in \mathcal{S} \text{ s.t.} \begin{cases} \tilde{X}_t^{x,\nu,\xi} \in (\underline{x}, b], \ \mathbb{P}\text{-a.s.} & \text{for } 0 < t \leq \sigma_{\mathcal{I}}, \\ \displaystyle\int_0^{T\wedge\sigma_{\mathcal{I}}} \mathbf{1}_{\{\tilde{X}_t^{x,\nu,\xi} < b\}} \, \mathrm{d}\xi_t = 0, \ \mathbb{P}\text{-a.s.} & \text{for any } T > 0. \end{cases} \tag{2.8}$$

The proof of the above lemma is based on a Picard iteration scheme. Although this derivation seems to be standard we could not find a precise reference for our particular setting, and as such we provide a short proof in Appendix A.2.

2.2. The game of controls

We introduce a two-player nonzero-sum game of singular control, where player 1 (respectively, player 2) can influence the dynamics (2.1) by exerting the control ν (respectively, ξ). The game has the following structure: if player 1 uses a unit of control at time $t > 0$, he/she must pay a cost $G_1(\tilde{X}_t^{\nu,\xi})$, while player 2 receives a reward $L_2(\tilde{X}_t^{\nu,\xi})$. A symmetric situation occurs if player 2 exerts control. Both players want to maximise their own expected discounted reward functional Ψ_i defined by

$$\Psi_1(x;\ \nu,\xi) := \mathbb{E}\bigg[\int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt} L_1(\tilde{X}_t^{x,\nu,\xi}) \ominus \mathrm{d}\xi_t - \int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt} G_1(\tilde{X}_t^{x,\nu,\xi}) \oplus \mathrm{d}\nu_t\bigg], \tag{2.9}$$

$$\Psi_2(x;\ \nu,\xi) := \mathbb{E}\bigg[\int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt} L_2(\tilde{X}_t^{x,\nu,\xi}) \oplus \mathrm{d}\nu_t - \int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt} G_2(\tilde{X}_t^{x,\nu,\xi}) \ominus \mathrm{d}\xi_t\bigg], \tag{2.10}$$

where $r > 0$ is the discount rate and the integrals are defined below.

To avoid dealing with controls producing infinite payoffs, we restrict our attention to the couples $(\nu,\xi) \in \mathcal{S} \times \mathcal{S}$ for which

$$\begin{aligned}
&\mathbb{E}\bigg[\int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt} |L_1(\tilde{X}_t^{x,\nu,\xi})| \ominus \mathrm{d}\xi_t + \int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt} |G_1(\tilde{X}_t^{x,\nu,\xi})| \oplus \mathrm{d}\nu_t\bigg] < +\infty,\\
&\mathbb{E}\bigg[\int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt} |L_2(\tilde{X}_t^{x,\nu,\xi})| \oplus \mathrm{d}\nu_t + \int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt} |G_2(\tilde{X}_t^{x,\nu,\xi})| \ominus \mathrm{d}\xi_t\bigg] < +\infty.
\end{aligned} \tag{2.11}$$

We denote the space of such couples by $\mathcal{S}^\circ \times \mathcal{S}^\circ$.

A definition of the integrals with respect to the controls in the presence of state-dependent costs requires some attention because simultaneous jumps of ξ and ν may be difficult to handle. An extended discussion on this matter is provided in Appendix A.1. Here we consider the class of admissible strategies (see Remark 2.2 below)

$$\mathcal{M} := \{(\nu,\xi) \in \mathcal{S}^\circ \times \mathcal{S}^\circ : \mathbb{P}_x(\Delta\nu_t \cdot \Delta\xi_t > 0) = 0 \text{ for all } t \geq 0 \text{ and } x \in \mathcal{I}\}. \tag{2.12}$$

Following [43] (see also [32] and [33] among others), we define the discounted costs of controls by

$$\int_0^T \mathrm{e}^{-rt} g(\tilde{X}_t^{x,\nu,\xi}) \ominus \mathrm{d}\xi_t = \int_0^T \mathrm{e}^{-rt} g(\tilde{X}_t^{x,\nu,\xi})\, \mathrm{d}\xi_t^c + \sum_{t<T} \mathrm{e}^{-rt} \int_0^{\Delta\xi_t} g(\tilde{X}_t^{x,\nu,\xi} - z)\, \mathrm{d}z, \tag{2.13}$$

$$\int_0^T \mathrm{e}^{-rt} g(\tilde{X}_t^{x,\nu,\xi}) \oplus \mathrm{d}\nu_t = \int_0^T \mathrm{e}^{-rt} g(\tilde{X}_t^{x,\nu,\xi})\, \mathrm{d}\nu_t^c + \sum_{t<T} \mathrm{e}^{-rt} \int_0^{\Delta\nu_t} g(\tilde{X}_t^{x,\nu,\xi} + z)\, \mathrm{d}z \tag{2.14}$$

for $T > 0$, $(\nu,\xi) \in \mathcal{M}$, and for any function g such that the integrals are well defined.

Throughout the paper we take functions G_i and L_i satisfying the following assumption.

Assumption 2.4. *Assume that $G_i, L_i : \overline{\mathcal{I}} \to \mathbb{R} \cup \{\pm\infty\}$ with $L_i < G_i$ on $\mathcal{I}$ and with $G_i \in C^1(\mathcal{I})$ and $L_i \in C(\mathcal{I})$. Moreover, the following asymptotic behaviours hold:*

$$\limsup_{x\to\underline{x}} \left|\frac{G_i}{\phi}\right|(x) = 0 \quad \textit{and} \quad \limsup_{x\to\overline{x}} \left|\frac{G_i}{\psi}\right|(x) = 0.$$

Nash equilibria for the game are defined in the following way.

Definition 2.1. For $x \in \mathcal{I}$, we say that a couple $(\nu^*,\xi^*) \in \mathcal{M}$ is a Nash equilibrium if and only if

$$|\Psi_i(x;\ \nu^*,\xi^*)| < +\infty, \qquad i = 1, 2,$$

and

$$\begin{aligned}
\Psi_1(x;\ \nu^*,\xi^*) &\geq \Psi_1(x;\ \nu,\xi^*) \quad \text{for any } \nu \in \mathcal{S} \text{ s.t. } (\nu,\xi^*) \in \mathcal{M},\\
\Psi_2(x;\ \nu^*,\xi^*) &\geq \Psi_2(x;\ \nu^*,\xi) \quad \text{for any } \xi \in \mathcal{S} \text{ s.t. } (\nu^*,\xi) \in \mathcal{M}.
\end{aligned} \tag{2.15}$$

We also say that $V_i(x) := \Psi_i(x;\ \nu^*,\xi^*)$ is the value of the game for the ith player relative to the equilibrium.

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Remark 2.2. In several problems of interest for applications, the functionals (2.9) and (2.10) may be rewritten as the sum of three terms: an integral in time of a state-dependent running profit, plus two integrals with respect to the controls, with constant instantaneous costs (see, for example, [13], [21], and [35] for similar functionals in the case of single-agent optimisation problems). In such cases, the condition in (2.12) relative to jumps of the admissible strategies is not needed. In fact, we show in Appendix A.1 that if at least one player picks a control that reflects the process at a fixed boundary (i.e. solving one of the problems in Lemma 2.2) then the other player has no incentives in picking strategies outside the class $\mathcal{M}$.

Remark 2.3. It is worth noting that, given $a, b \in \mathcal{I}$ with $a < b$, the couple of controls (ν^a, ξ^b) which solves $\mathrm{SP}(a, b; x)$ (see (2.6)) belongs to $\mathcal{M}$. In fact, one can easily check that (ν^a, ξ^b) satisfies (2.11), for example by looking at the proof of [38, Lemma 2.1]. Moreover, by construction, we have $\mathbb{P}_x(\Delta\nu_t^a \cdot \Delta\xi_t^b > 0) = 0$ for all $t \geq 0$.

Remark 2.4. Nash equilibria could, in principle, exist in broader sets than $\mathcal{M}$. However, this fact does not *per se* add useful information. In fact, unless some additional optimality criterion is introduced (for example, maximisation of the total profit of the two players), it is often impossible to rank multiple equilibria according to the players' individual preferences. In this paper we content ourselves with equilibria in $\mathcal{M}$ as these lead to explicit solutions and to the desired connection between OS and SSC.

2.3. The game of stopping

In this section we introduce a two-player nonzero-sum game of stopping where the underlying process is X^x as in (2.2). This is the game which we show is linked to the game of controls introduced in the previous section.

Denote by $\mathcal{T}$ the set of F-stopping times. The ith player chooses $\tau_i \in \mathcal{T}$ with the aim of minimising an expected cost functional $\mathcal{J}_i(\tau_1, \tau_2; x)$, and the game ends at $\tau_1 \wedge \tau_2$. This game has payoffs of immediate stopping given by the functions G_i and L_i appearing in the functionals (2.9) and (2.10) of the game of control. More precisely, we set

$$\begin{aligned}\mathcal{J}_1(\tau_1, \tau_2; x) := \mathbb{E}\bigg[&\exp\bigg(-\int_0^{\tau_1}(r - \mu'(X_s^x))\,\mathrm{d}s\bigg)G_1(X_{\tau_1}^x)\mathbf{1}_{\{\tau_1<\tau_2\}}\\ &+\exp\bigg(-\int_0^{\tau_2}(r - \mu'(X_s^x))\,\mathrm{d}s\bigg)L_1(X_{\tau_2}^x)\mathbf{1}_{\{\tau_1\geq\tau_2\}}\bigg], \end{aligned} \tag{2.16}$$

$$\begin{aligned}\mathcal{J}_2(\tau_1, \tau_2; x) := \mathbb{E}\bigg[&\exp\bigg(-\int_0^{\tau_2}(r - \mu'(X_s^x))\,\mathrm{d}s\bigg)G_2(X_{\tau_2}^x)\mathbf{1}_{\{\tau_2\leq\tau_1\}}\\ &+\exp\bigg(-\int_0^{\tau_1}(r - \mu'(X_s^x))\,\mathrm{d}s\bigg)L_2(X_{\tau_1}^x)\mathbf{1}_{\{\tau_2>\tau_1\}}\bigg]. \end{aligned} \tag{2.17}$$

Here, as in the case of the game of controls, we also introduce the notion of Nash equilibrium.

Definition 2.2. For $x \in \mathcal{I}$, we say that a couple $(\tau_1^*, \tau_2^*) \in \mathcal{T} \times \mathcal{T}$ is a Nash equilibrium if and only if

$$|\mathcal{J}_i(\tau_1^*, \tau_2^*; x)| < +\infty, \qquad i = 1, 2,$$

and

$$\begin{aligned}\mathcal{J}_1(\tau_1^*, \tau_2^*; x) &\leq \mathcal{J}_1(\tau_1, \tau_2^*; x) \quad \text{for all } \tau_1 \in \mathcal{T},\\ \mathcal{J}_2(\tau_1^*, \tau_2^*; x) &\leq \mathcal{J}_2(\tau_1^*, \tau_2; x) \quad \text{for all } \tau_2 \in \mathcal{T}.\end{aligned}$$

We also say that $v_i(x) := \mathcal{J}_i(\tau_1^*, \tau_2^*; x)$ is the value of the game for the ith player relative to the equilibrium.

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Our choice for the game of stopping is motivated by a heuristic argument which is well known in the economic literature on irreversible (partially reversible) investment problems. We briefly illustrate the main ideas below.

In our game of controls, both players are faced with the question of *how* to use their control in order to maximise an expected payoff. This might be interpreted as the problem of two investors who must decide *how to invest* a unit of capital in order to maximise their future expected profits. In the mathematical economics literature (see, for example, [17]), the question is known to be equivalent to the one of *timing* the investment of one unit of capital. The equivalence can be formally explained via an analysis of marginal costs and benefits for each investor.

Here we take the point of view of player 1, but symmetric arguments can be applied to player 2. Given an investment strategy ν, player 1 pays a marginal cost equal to G_1 per unit of investment. However, the upward shift in the controlled dynamics (due to ν) modifies the current level of the state variable, and, therefore, also the player's expected future profit. Such a change in the expected future payoffs, per unit of invested capital, represents the marginal benefit for player 1. As long as the marginal benefit is smaller than the marginal cost, then player 1 should *wait* and do nothing. On the contrary, at times when the marginal benefit equals or exceeds the marginal cost, it is clear that player 1 should invest (at the optimum the marginal benefit is never strictly larger than the marginal cost). In this sense, player 1 is *timing* the decision to incur a (marginal) cost G_1 in exchange for expected future profits. This explains the (random) payoff $G_1(X_{\tau_1})$ in (2.16) and (2.17), while the indicator $\mathbf{1}_{\{\tau_1<\tau_2\}}$ is due to the fact that the previous argument holds until the second player decides to invest. In particular, while player 1 waits for his/her optimal time τ_1 to invest, it may happen that player 2 decides to invest first. This situation produces a marginal cost for player 1 equal to L_1 (which here may be negative or positive), and explains the role of the (random) payoff $L_1(X_{\tau_2})\mathbf{1}_{\{\tau_1\geq\tau_2\}}$ in (2.16) and (2.17).

Since investors try to minimise costs, we are naturally led to consider minimisation of the players' expected discounted marginal costs in (2.16) and (2.17). The specific discount factor adopted here is due to the nature of the underlying controlled diffusion, and it is a technical point which will become clear in the analysis below.

3. The main result

Here we prove the key result of the paper (Theorem 3.1), i.e. a differential link between the value functions v_i, $i = 1, 2$, relative to Nash equilibria in the game of stopping and the value functions V_i, $i = 1, 2$, relative to Nash equilibria in the game of control. The result holds when the equilibrium stopping times for X are hitting times to suitable thresholds so that the related optimally controlled $\tilde{X}$ is reflected at such thresholds.

Theorem 3.1 relies on assumptions regarding the existence of a Nash equilibrium in the game of stopping and suitable properties of the associated values v_1 and v_2. It was shown in [16] that such requirements hold in a broad class of examples, and we will summarise the results of [16] in Proposition 3.1 below for completeness.

For a given connected set $\mathcal{O} \subseteq \mathcal{I}$, in the theorem below we will make use of the Sobolev space $W^{2,\infty}_{\mathrm{loc}}(\mathcal{O})$. This is the space of functions which are twice differentiable in the weak sense on $\mathcal{O}$, and whose weak derivatives up to order 2 are functions in $L^{\infty}_{\mathrm{loc}}(\mathcal{O})$. We will also use the fact that if $u \in W^{2,\infty}_{\mathrm{loc}}(\mathcal{O})$ then $u \in C^1(\mathcal{O})$ by Sobolev embedding; see [10, Chapter 9, Corollary 9.15].

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Theorem 3.1. *Suppose that there exist a_*, b_* with $\underline{x} < a_* < b_* < \overline{x}$ such that the following conditions hold:*

(i) *the stopping times*

$$\tau_1^* := \inf\{t > 0\colon X_t^x \le a_*\}, \qquad \tau_2^* := \inf\{t > 0\colon X_t^x \ge b_*\}$$

form a Nash equilibrium for the game of stopping as in Definition 2.2;

(ii) *the value functions $v_i(x) := \mathcal{J}_i(\tau_1^*, \tau_2^*; x)$, $i = 1, 2$, are such that $v_i \in C(\mathcal{I})$, $i = 1, 2$, with $v_1 \in W^{2,\infty}_{\text{loc}}(\underline{x}, b_*)$ and $v_2 \in W^{2,\infty}_{\text{loc}}(a_*, \overline{x})$;*

(iii) *$v_1 = G_1$ in $(\underline{x}, a_*]$, $v_1 = L_1$ in $[b_*, \overline{x})$, $v_2 = G_2$ in $[b_*, \overline{x})$, and $v_2 = L_2$ in $(\underline{x}, a_*]$. Moreover, they solve the boundary-value problem*

$$(\mathcal{L}_X v_i - (r - \mu')v_i)(x) = 0, \qquad a_* < x < b_*,\ i = 1, 2, \tag{3.1}$$

$$(\mathcal{L}_X v_1 - (r - \mu')v_1)(x) \ge 0, \qquad \underline{x} < x \le a_*, \tag{3.2}$$

$$(\mathcal{L}_X v_2 - (r - \mu')v_2)(x) \ge 0, \qquad b_* \le x < \overline{x}, \tag{3.3}$$

$$v_i \le G_i, \qquad x \in \mathcal{I},\ i = 1, 2. \tag{3.4}$$

Then the strategy profile that prescribes to reflect $\tilde{X}$ at the two barriers a^* and b^* (up to a possible initial jump) forms a Nash equilibrium for the game of control (see Definition 2.1). In particular, for $x \in \mathcal{I}$ and $t \ge 0$, such an equilibrium is realised by the couple of controls

$$\nu_t^* := \mathbf{1}_{\{t>0\}}[(a_* - x)^+ + \nu_t^{a_*}], \qquad \xi_t^* := \mathbf{1}_{\{t>0\}}[(x - b_*)^+ + \xi_t^{b_*}], \tag{3.5}$$

where (ν^{a_*}, ξ^{b_*}) uniquely solves problem $\mathrm{SP}(a_*, b_*; (x \vee a_*) \wedge b_*)$*; see (2.6). Finally, the value functions $V_i(x) = \Psi_i(x; \nu^*, \xi^*)$, $i = 1, 2$, are given by*

$$V_1(x) = \kappa_1 + \int_{a_*}^{x} v_1(z)\,\mathrm{d}z, \qquad V_2(x) = \kappa_2 + \int_{x}^{b_*} v_2(z)\,\mathrm{d}z, \qquad x \in \mathcal{I},$$

with

$$\kappa_1 := \frac{1}{r}\bigg(\frac{\sigma^2}{2}G_1' + \mu G_1\bigg)(a_*), \qquad \kappa_2 := -\frac{1}{r}\bigg(\frac{\sigma^2}{2}G_2' + \mu G_2\bigg)(b_*). \tag{3.6}$$

Proof. The proof is by direct check and it is performed in two steps.

Step 1. The functions

$$u_1(x) = \kappa_1 + \int_{a_*}^{x} v_1(z)\,\mathrm{d}z, \qquad x \in \mathcal{I},$$

$$u_2(x) = \kappa_2 + \int_{x}^{b_*} v_2(z)\,\mathrm{d}z, \qquad x \in \mathcal{I}, \tag{3.7}$$

with κ_1 and κ_2 as in (3.6), are C^1 on $\mathcal{I}$ (by the continuity of G_i and L_i on $\mathcal{I}$) with $u_1 \in C^2(\underline{x}, b_*)$ since $v_1 \in C^1(\underline{x}, b_*)$, and $u_2 \in C^2(a_*, \overline{x})$ since $v_2 \in C^1(a_*, \overline{x})$. We now show that u_1, u_2 and the boundaries a_*, b_* solve the system of coupled variational problems

$$\begin{cases} (\mathcal{L}_{\tilde{X}} u_1 - r u_1)(x) = 0, & x \in (a_*, b_*), \\ (\mathcal{L}_{\tilde{X}} u_1 - r u_1)(x) \le 0, & x \in (\underline{x}, b_*), \\ u_1'(x) \le G_1(x), & x \in (\underline{x}, b_*), \\ u_1'(x) = G_1(x), & x \in (\underline{x}, a_*), \\ u_1'(x) = L_1(x), & x \in (b_*, \overline{x}), \end{cases} \tag{3.8}$$

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and

$$\begin{cases} (\mathcal{L}_{\tilde{X}} u_2 - r u_2)(x) = 0, & x \in (a_*, b_*), \\ (\mathcal{L}_{\tilde{X}} u_2 - r u_2)(x) \le 0, & x \in (a_*, \overline{x}), \\ u_2'(x) \ge -G_2(x), & x \in (a_*, \overline{x}), \\ u_2'(x) = -G_2(x), & x \in (b_*, \overline{x}), \\ u_2'(x) = -L_2(x), & x \in (\underline{x}, a_*). \end{cases} \tag{3.9}$$

We will give details only about the derivation of (3.9) as the ones for (3.8) are analogous. The last three properties in (3.9) follow by observing that $u_2' = -v_2$, and by using $v_2 = G_2$ in $[b_*, \overline{x})$, $v_2 = L_2$ in $(\underline{x}, a_*]$, and (3.4) (see (iii) in the statement of the theorem). To prove the first equation in (3.9), we use the definition of u_2 (see (3.7)) and explicit calculations to obtain

$$(\mathcal{L}_{\tilde{X}} u_2 - r u_2)(x) = -\frac{\sigma^2(x)}{2} v_2'(x) - \mu(x) v_2(x) - r\kappa_2 - \int_x^{b_*} r v_2(z)\, \mathrm{d}z. \tag{3.10}$$

Then we also use (3.1) to obtain, for $x \in (a_*, b_*)$,

$$\int_x^{b_*} r v_2(z)\, \mathrm{d}z = \int_x^{b_*} (\mathcal{L}_X v_2(z) + \mu'(z) v_2(z))\, \mathrm{d}z. \tag{3.11}$$

Integrating by parts the right-hand side of (3.11), using $v_2(b_*) = G_2(b_*)$ and $v_2'(b_*) = G_2'(b_*)$, and substituting the result back into (3.10), the right-hand side of (3.10) is equal to 0 upon recalling the definition of κ_2 (see (3.6)). Finally, to prove the second equation in (3.9), it is enough to note that, for $x \in [b_*, \overline{x})$,

$$\int_x^{b_*} r v_2(z)\, \mathrm{d}z \ge \int_x^{b_*} (\mathcal{L}_X v_2(z) + \mu'(z) v_2(z))\, \mathrm{d}z,$$

by (3.3) and then argue as before.

Step 2. We now proceed to a verification argument to show that $u_i = V_i$, $i = 1, 2$, and that the strategy profile (3.5) forms a Nash equilibrium. We again provide full details for only u_2 as the proof follows in the same way for u_1.

Recall the dynamics for $\tilde{X}^{\nu,\xi}$ from (2.1), and note that by definition (3.5), the couple of controls (ν^*, ξ^*) solves the Skorokhod reflection problem in $[a_*, b_*]$, up to an initial jump. Moreover, Remark 2.3 guarantees that $(\nu^*, \xi^*) \in \mathcal{M}$.

First we show that $u_2 \ge \Psi_2(x;\, \nu^*, \xi)$ for any admissible ξ. Take $\xi \in \mathcal{S}^\circ$ such that $(\nu^*, \xi) \in \mathcal{M}$. It is important to note that ν^* in (3.5) involves the control ν^{a^*} that solves $\mathrm{SP}^{\xi}_{a^*+}(x \vee a_*)$ of Lemma 2.2 (see (2.6)) for an arbitrary ξ. Recalling that $u_2 \in C^2(a_*, \overline{x})$, we can apply the Itô–Meyer formula, up to a localising sequence of stopping times, to the process $u_2(\tilde{X}^{x,\nu^*,\xi})$ (in particular, we use the fact that $\mathbb{P}_x(\Delta\nu_t^* \cdot \Delta\xi_t > 0) = 0$ for all $t \ge 0$). The integral with respect to the continuous part of the bounded variation process $\nu^* - \xi$ is the difference of the integrals with respect to $\mathrm{d}\nu^{*,c}$ and $\mathrm{d}\xi^c$. For $x \in \mathcal{I}$, we obtain

$$\begin{aligned} u_2(x) &= \mathrm{e}^{-r\theta_y} u_2(\tilde{X}^{x,\nu^*,\xi}_{\theta_y}) - \int_0^{\theta_y} \mathrm{e}^{-rs} (\mathcal{L}_{\tilde{X}} - r) u_2(\tilde{X}^{x,\nu^*,\xi}_s)\, \mathrm{d}s - M_{\theta_y} \\ &\quad - \int_0^{\theta_y} \mathrm{e}^{-rs} u_2'(\tilde{X}^{x,\nu^*,\xi}_s)\, \mathrm{d}\nu^{*,c}_s + \int_0^{\theta_y} \mathrm{e}^{-rs} u_2'(\tilde{X}^{x,\nu^*,\xi}_s)\, \mathrm{d}\xi^c_s \\ &\quad - \sum_{s<\theta_y} \mathrm{e}^{-rs} (u_2(\tilde{X}^{x,\nu^*,\xi}_{s+}) - u_2(\tilde{X}^{x,\nu^*,\xi}_s)), \end{aligned} \tag{3.12}$$

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where M is

$$M_t := \int_0^t \mathrm{e}^{-rs}\sigma(\tilde{X}_s^{x,\nu^*,\xi})u_2'(\tilde{X}_s^{x,\nu^*,\xi})\,\mathrm{d}\tilde{W}_s,$$

and θ_y is the stopping time

$$\theta_y := \inf\{u > 0\colon \tilde{X}_u^{x,\nu^*,0} \geq y\} \quad \text{for } y > b_*.$$

Note that, for any $t \in (0, \theta_y]$, we have $a_* \leq \tilde{X}_t^{x,\nu^*,\xi} \leq \tilde{X}_t^{x,\nu^*,0} \leq y$, hence, the continuity of σ and u_2' imply that $(M_t)_{t\leq\theta_y}$ is a martingale.

Since $(\nu^*, \xi) \in \mathcal{M}$, the process $\tilde{X}^{x,\nu^*,\xi}$ is left-continuous and we have

$$\begin{aligned}&\sum_{s<\theta_y} \mathrm{e}^{-rs}(u_2(\tilde{X}_{s+}^{x,\nu^*,\xi}) - u_2(\tilde{X}_s^{x,\nu^*,\xi}))\\ &\quad= \sum_{s<\theta_y} \mathrm{e}^{-rs}(u_2(\tilde{X}_{s+}^{x,\nu^*,\xi}) - u_2(\tilde{X}_s^{x,\nu^*,\xi}))[\mathbf{1}_{\{\Delta\nu_s^*>0\}} + \mathbf{1}_{\{\Delta\xi_s>0\}}],\end{aligned}$$

where

$$\sum_{s<\theta_y} \mathrm{e}^{-rs}(u_2(\tilde{X}_{s+}^{x,\nu^*,\xi}) - u_2(\tilde{X}_s^{x,\nu^*,\xi}))\mathbf{1}_{\{\Delta\nu_s^*>0\}} = \sum_{s<\theta_y} \mathrm{e}^{-rs} \int_0^{\Delta\nu_s^*} u_2'(\tilde{X}_s^{x,\nu^*,\xi} + z)\,\mathrm{d}z,$$

$$\sum_{s<\theta_y} \mathrm{e}^{-rs}(u_2(\tilde{X}_{s+}^{x,\nu^*,\xi}) - u_2(\tilde{X}_s^{x,\nu^*,\xi}))\mathbf{1}_{\{\Delta\xi_s>0\}} = -\sum_{s<\theta_y} \mathrm{e}^{-rs} \int_0^{\Delta\xi_s} u_2'(\tilde{X}_s^{x,\nu^*,\xi} - z)\,\mathrm{d}z.$$

Hence, (3.12) may be written in a more compact form as (see (2.13) and (2.14))

$$\begin{aligned}u_2(x) &= \mathrm{e}^{-r\theta_y}u_2(\tilde{X}_{\theta_y}^{x,\nu^*,\xi}) - \int_0^{\theta_y} \mathrm{e}^{-rs}(\mathcal{L}_{\tilde{X}} - r)u_2(\tilde{X}_s^{x,\nu^*,\xi})\,\mathrm{d}s - M_{\theta_y}\\ &\quad - \int_0^{\theta_y} \mathrm{e}^{-rs}u_2'(\tilde{X}_s^{x,\nu^*,\xi})_{\oplus}\,\mathrm{d}\nu_s^* + \int_0^{\theta_y} \mathrm{e}^{-rs}u_2'(\tilde{X}_s^{x,\nu^*,\xi})_{\ominus}\,\mathrm{d}\xi_s.\end{aligned}$$

Now, we note that the third and fifth formulae in (3.9) imply that $u_2' \geq -G_2$ on $\mathcal{I}$ and that $u_2'(\tilde{X}_s^{x,\nu^*,\xi}) = -L_2(\tilde{X}_s^{x,\nu^*,\xi})$ for all s in the support of $\mathrm{d}\nu_s^*$ (i.e. for all $s \geq 0$ s.t. $\tilde{X}_s^{x,\nu^*,\xi} \leq a_*$). Moreover, employing the second expression in (3.9) jointly with the fact that $\tilde{X}_s^{x,\nu^*,\xi} \geq a_*$ for $s > 0$, we obtain

$$\begin{aligned}u_2(x) &\geq \mathrm{e}^{-r\theta_y}u_2(\tilde{X}_{\theta_y}^{x,\nu^*,\xi}) - M_{\theta_y} + \int_0^{\theta_y} \mathrm{e}^{-rs}L_2(\tilde{X}_s^{x,\nu^*,\xi})_{\oplus}\,\mathrm{d}\nu_s^*\\ &\quad - \int_0^{\theta_y} \mathrm{e}^{-rs}G_2(\tilde{X}_s^{x,\nu^*,\xi})_{\ominus}\,\mathrm{d}\xi_s.\end{aligned} \tag{3.13}$$

By taking expectations we end up with

$$u_2(x) \geq \mathbb{E}_x\bigg[\mathrm{e}^{-r\theta_y}u_2(\tilde{X}_{\theta_y}^{\nu^*,\xi}) + \int_0^{\theta_y} \mathrm{e}^{-rs}L_2(\tilde{X}_s^{\nu^*,\xi})_{\oplus}\,\mathrm{d}\nu_s^* - \int_0^{\theta_y} \mathrm{e}^{-rs}G_2(\tilde{X}_s^{\nu^*,\xi})_{\ominus}\,\mathrm{d}\xi_s\bigg]. \tag{3.14}$$

We aim at taking limits as $y \to \overline{x}$ in (3.14), and we preliminarily note that $\theta_y \uparrow \sigma_{\mathcal{I}}$ as $y \to \overline{x}$, $\mathbb{P}_x$-a.s.

(i) By (3.7) it is easy to see that

$$\begin{aligned}
|u_2(\tilde{X}^{\nu^*,\xi}_{\theta_y})| &\le \kappa_2 + \int_{a_*}^{b_*} |v_2(z)|\,\mathrm{d}z + \int_{b_*}^{b_*\vee \tilde{X}^{\nu^*,\xi}_{\theta_y}} |G_2(z)|\,\mathrm{d}z \\
&\le C_2 + \int_{b_*}^{b_*\vee \tilde{X}^{\nu^*,0}_{\theta_y}} |G_2(z)|\,\mathrm{d}z \\
&\le C_2 + \int_{b_*}^{y} |G_2(z)|\,\mathrm{d}z
\end{aligned}$$

for some $C_2 > 0$, where we have used $v_2 = G_2$ on $[b_*, \overline{x})$ and $\tilde{X}^{\nu^*,\xi}_{\theta_y} \le \tilde{X}^{\nu^*,0}_{\theta_y} \le y$, $\mathbb{P}_x$-a.s. Hence, we have

$$\mathbb{E}_x[\mathrm{e}^{-r\theta_y} u_2(\tilde{X}^{\nu^*,\xi}_{\theta_y})] \ge -\mathbb{E}_x[\mathrm{e}^{-r\theta_y}]\bigg(C_2 + \int_{b_*}^{y} |G_2(z)|\,\mathrm{d}z\bigg). \tag{3.15}$$

Using Assumption 2.2, Lemma A.2 guarantees

$$\limsup_{y\uparrow\overline{x}} \mathbb{E}_x[\mathrm{e}^{-r\theta_y}]\bigg(C_2 + \int_{b_*}^{y} |G_2(z)|\,\mathrm{d}z\bigg) \le 0$$

so that (3.15) yields

$$\liminf_{y\uparrow\overline{x}} \mathbb{E}_x[\mathrm{e}^{-r\theta_y} u_2(\tilde{X}^{\nu^*,\xi}_{\theta_y})] \ge 0.$$

(ii) Recall the integrability conditions (2.11) in the definition of $\mathcal{M}$. Then, using the fact that $\theta_y \uparrow \sigma_{\mathcal{I}}$ as $y \uparrow \infty$, and applying the dominated convergence theorem, we obtain

$$\begin{aligned}
&\lim_{y\to\overline{x}} \mathbb{E}_x\bigg[\int_0^{\theta_y} \mathrm{e}^{-rs} L_2(\tilde{X}^{\nu^*,\xi}_s)\oplus \mathrm{d}\nu^*_s - \int_0^{\theta_y} \mathrm{e}^{-rs} G_2(\tilde{X}^{\nu^*,\xi}_s)\ominus \mathrm{d}\xi_s\bigg] \\
&\quad = \mathbb{E}_x\bigg[\int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rs} L_2(\tilde{X}^{\nu^*,\xi}_s)\oplus \mathrm{d}\nu^*_s - \int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rs} G_2(\tilde{X}^{\nu^*,\xi}_s)\ominus \mathrm{d}\xi_s\bigg].
\end{aligned}$$

Finally, we combine items (i) and (ii) and take limits in (3.14) as $y \to \overline{x}$ to obtain

$$u_2(x) \ge \mathbb{E}_x\bigg[\int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rs} L_2(\tilde{X}^{\nu^*,\xi}_s)\oplus \mathrm{d}\nu^*_s - \int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rs} G_2(\tilde{X}^{\nu^*,\xi}_s)\ominus \mathrm{d}\xi_s\bigg].$$

Hence, $u_2(x) \ge \Psi_2(x;\ \nu^*, \xi)$ for any $\xi \in \mathcal{S}$ such that $(\nu^*, \xi) \in \mathcal{M}$.

Now, repeating the steps above with $\xi = \xi^*$, the inequalities in (3.13) and (3.14) become strict equalities due to the fact that $\tilde{X}^{x,\nu^*,\xi^*}_t \in [a_*, b_*]$ for all $t > 0$ and $u_2'(\tilde{X}^{x,\nu^*,\xi^*}_t) = -G_2(\tilde{X}^{x,\nu^*,\xi^*}_t)$ on $\operatorname{supp}\{\mathrm{d}\xi^*_t\}$. Moreover, the process $u_2(\tilde{X}^{\nu^*,\xi^*})$ is bounded so that passing to the limit as $y \to \overline{x}$ yields

$$\lim_{y\uparrow\overline{x}} \mathbb{E}_x[\mathrm{e}^{-r\theta_y} u_2(\tilde{X}^{\nu^*,\xi^*}_{\theta_y})] = 0$$

by dominated convergence and Assumption 2.2. Hence, $u_2(x) = \Psi(x;\ \nu^*, \xi^*) = V_2(x)$. □

Remark 3.1. From the game-theoretic point of view, Nash equilibria of Theorem 3.1 above are Markov perfect [34] (also called Nash equilibria in closed-loop strategies), i.e. equilibria in which players' actions depend only on the 'payoff-relevant' state variable $\tilde{X}$. Our result provides a simple construction of closed-loop Nash equilibria for specific continuous-time stochastic games of singular control. Since this problem has yet to be solved in game theory in its full generality (see the discussion in [2, Section 2] and [39]), our work contributes to fill this gap.

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3.1. On the assumptions in Theorem 3.1

In this section we give sufficient conditions under which a_* and b_* as in Theorem 3.1 exist. Moreover, in Remark 3.2 we provide algebraic equations for a_* and b_* which can be solved at least numerically. Recall ϕ and ψ, i.e. the fundamental decreasing and increasing solutions to (2.4), and recall that $r > \mu'(x)$ for $x \in \overline{\mathcal{I}}$ by Assumption 2.3. We need the following set of functions.

Definition 3.1. Let $\mathcal{A}$ be the class of real-valued functions $H \in C^2(\mathcal{I})$ such that

$$\limsup_{x\to\underline{x}} \left|\frac{H}{\phi}\right|(x) = 0, \qquad \limsup_{x\to\overline{x}} \left|\frac{H}{\psi}\right|(x) = 0, \tag{3.16}$$

$$\mathbb{E}_x\left[\int_0^{\sigma_\mathcal{I}} \exp\left(-\int_0^t (r-\mu'(X_s))\,\mathrm{d}s\right)|h(X_t)|\,\mathrm{d}t\right] < \infty \tag{3.17}$$

for all $x \in \mathcal{I}$, and with $h(x) := (\mathcal{L}_X H - (r-\mu')H)(x)$. We denote by $\mathcal{A}_1$ (respectively, $\mathcal{A}_2$) the set of all $H \in \mathcal{A}$ such that h is strictly positive (respectively, negative) on $(\underline{x}, x_h)$ and strictly negative (respectively, positive) on $(x_h, \overline{x})$ for some $x_h \in \mathcal{I}$ with $\liminf_{x\to\underline{x}} h(x) > 0$ (respectively, $\limsup_{x\to\underline{x}} h(x) < 0$) and $\limsup_{x\to\overline{x}} h(x) < 0$ (respectively, $\liminf_{x\to\overline{x}} h(x) > 0$).

We also need the following assumption, which will hold in the rest of this section.

Assumption 3.1. *For $i = 1, 2$, it holds that $G_i \in \mathcal{A}_i$ and*

$$\limsup_{x\to\underline{x}} \left|\frac{L_i}{\phi}\right|(x) < +\infty, \qquad \limsup_{x\to\overline{x}} \left|\frac{L_i}{\psi}\right|(x) < +\infty. \tag{3.18}$$

Moreover, letting $\hat{x}_1$ and $\hat{x}_2$ in $\mathcal{I}$ be such that

$$\begin{aligned}\{x : (\mathcal{L}_X G_1 - (r-\mu')G_1)(x) > 0\} &= (\underline{x}, \hat{x}_1),\\ \{x : (\mathcal{L}_X G_2 - (r-\mu')G_2)(x) > 0\} &= (\hat{x}_2, \overline{x}),\end{aligned} \tag{3.19}$$

we assume that $\hat{x}_1 < \hat{x}_2$.

The above condition $\hat{x}_1 < \hat{x}_2$ implies that, for any value of the process X, at least one player has a running benefit from waiting (see [16, Section 1]).

The proof of the next proposition can be found in Appendix A.2. In its statement, we denote

$$\vartheta_i(x) := \frac{G_i'(x)\phi(x) - G_i(x)\phi'(x)}{wS'(x)}, \qquad i = 1, 2,$$

with $w > 0$ as in (2.5). We also remark that the proposition holds under all the standing assumptions made so far in the paper (i.e. Assumptions 2.1–3.1). For the convenience of the reader, we also recall that $\underline{x}$ and $\overline{x}$ are natural for X if the process cannot start from $\underline{x}$ and $\overline{x}$ and, moreover, when started in $(\underline{x}, \overline{x})$ cannot reach $\underline{x}$ or $\overline{x}$ in finite time. On the other hand, $\underline{x}$ is entrance-not-exit if the process can be started from $\underline{x}$, but if started from $x > \underline{x}$ it cannot reach $\underline{x}$ in finite time. We refer the reader to [9, pp. 14–15] for further details.

Proposition 3.1. *Either of the following conditions are sufficient for the existence of a_* and b_* fulfilling (i)–(iii) of Theorem 3.1:*

(i) *$\underline{x}$ and $\overline{x}$ are natural boundaries for $(X_t)_{t\geq 0}$;*

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(ii) $\underline{x}$ *is an entrance-not-exit boundary and* $\overline{x}$ *is a natural boundary for* $(X_t)_{t\geq 0}$*; moreover, the following hold:*

- $\vartheta_1(\underline{x}+) := \lim_{x\downarrow\underline{x}} \vartheta_1(x) < (L_1/\psi)(x_2^{\infty})$*, where* x_2^{∞} *uniquely solves* $\vartheta_2(x) = (G_2/\psi)(x)$ *in* $(\hat{x}_2, \overline{x})$*;*
- $\sup\{x > \underline{x} : L_1(x) = \vartheta_1(\underline{x}+)\psi(x)\} \leq \hat{x}_2$*;*
- $\lim_{x\uparrow\overline{x}}(L_1/\phi)(x) > -\infty$.

Remark 3.2. An important byproduct of our connection between nonzero-sum games of control and nonzero-sum games of stopping is that the equilibrium thresholds a_* and b_* of Theorem 3.1 are a solution to a system of algebraic equations which can be computed at least numerically. In the terminology of singular-control theory, these equations correspond to the smooth-fit conditions $V_1''(a_*+) = G_1'(a_*)$ and $V_2''(b_*-) = -G_2'(b_*)$, and were obtained via a geometric constructive approach in [16] (see Theorem 3.2 therein). We recall the system here for completeness:

$$\frac{G_1}{\phi}(a_*) - \frac{L_1}{\phi}(b_*) - \vartheta_1(a_*)\left(\frac{\psi}{\phi}(a_*) - \frac{\psi}{\phi}(b_*)\right) = 0,$$
$$\frac{G_2}{\phi}(b_*) - \frac{L_2}{\phi}(a_*) - \vartheta_1(b_*)\left(\frac{\psi}{\phi}(b_*) - \frac{\psi}{\phi}(a_*)\right) = 0, \qquad (3.20)$$

where $a_* < \hat{x}_1$ and $b_* > \hat{x}_2$.

Uniqueness of the solution to (3.20) was discussed in [16, Theorem 3.8].

4. A game of pollution control

In order to understand the nature of our Assumptions 2.4 and 3.1, and illustrate an application of our results, we present here a game version of a pollution control problem.

A social planner wants to keep the level of pollution low while the productive sector of the economy (modelled as a single representative firm) wants to increase the production capacity. If we assume that the pollution level is proportional to the firm's production capacity (see, for example, [25] and [42]), then the problem translates into a game of capacity expansion. Indeed, the representative firm aims at maximising profits by investing to increase the production level, whereas the social planner aims at keeping the pollution level under control through environmental regulations which effectively cap the maximum production rate.

For the production capacity, we consider a controlled geometric Brownian motion (as, for example, in [12], [13], and [21])

$$\mathrm{d}\tilde{X}_t^{\nu,\xi} = \hat{\mu}\tilde{X}_t^{\nu,\xi}\,\mathrm{d}t + \hat{\sigma}\tilde{X}_t^{\nu,\xi}\,\mathrm{d}\tilde{W}_t + \mathrm{d}\nu_t - \mathrm{d}\xi_t, \qquad \tilde{X}_0^{\nu,\xi} = x \in \mathbb{R}_+,$$

for some $\hat{\mu} \in \mathbb{R}$ and $\hat{\sigma} > 0$. The firm has a running operating profit $\pi(x)$, which is C^1 and strictly concave, and a positive cost per unit of investment $\alpha_1(x)$. The social planner has an instantaneous utility function $u(x)$, which is C^1, decreasing, and strictly concave. Notice indeed that the social planner's utility decreases with increasing pollution levels. Moreover, if the pollution is high, the marginal benefit from decreasing it is large, whereas if the pollution is low, a further contraction of the economy has very little or no benefit. Since imposing a reduction of production might also have some negative impact on social welfare (for example, it might cause an increase in the level of unemployment), we introduce a positive 'cost' (in terms of the expected total utility) associated to the social planner's policies and we denote it

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by $\alpha_2(x)$. For simplicity here, we assume that $\alpha_i(x) \equiv \alpha_i > 0$, $i = 1, 2$, and the objective functionals for the firm, denoted by Ψ_1, and the social planner, denoted by Ψ_2, are given by

$$\begin{aligned}\Psi_1(x;\ \nu, \xi) &:= \mathbb{E}_x\bigg[\int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt}\pi(\tilde{X}_t^{\nu,\xi})\,\mathrm{d}t - \alpha_1 \int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt}\,\mathrm{d}\nu_t\bigg],\\ \Psi_2(x;\ \nu, \xi) &:= \mathbb{E}_x\bigg[\int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt}u(\tilde{X}_t^{\nu,\xi})\,\mathrm{d}t - \alpha_2 \int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt}\,\mathrm{d}\xi_t\bigg].\end{aligned} \tag{4.1}$$

Both players want to maximise their respective functional by picking admissible strategies from $\mathcal{M}$. As explained in Lemma A.1 below, in this context there is no loss of generality for our scope in considering $\mathcal{M}$ rather than $\mathcal{S}^\circ \times \mathcal{S}^\circ$.

The game with functionals (4.1) will be tackled directly with the same methods developed in the previous sections. Indeed, the additional running cost terms require only a minor tweak to our method. Motivated by the analysis of the previous sections, we look at the game of stopping where two players want to minimise the cost functionals

$$\begin{aligned}\widehat{\mathcal{J}}_1(\tau_1, \tau_2;\ x) &:= \mathbb{E}_x\bigg[\mathrm{e}^{-(r-\hat{\mu})\tau_1}\alpha_1 \mathbf{1}_{\{\tau_1<\tau_2\}} + \int_0^{\tau_1\wedge\tau_2} \mathrm{e}^{-(r-\hat{\mu})t}\pi'(X_t)\,\mathrm{d}t\bigg],\\ \widehat{\mathcal{J}}_2(\tau_1, \tau_2;\ x) &:= \mathbb{E}_x\bigg[\mathrm{e}^{-(r-\hat{\mu})\tau_2}\alpha_2 \mathbf{1}_{\{\tau_2\le\tau_1\}} - \int_0^{\tau_1\wedge\tau_2} \mathrm{e}^{-(r-\hat{\mu})t}u'(X_t)\,\mathrm{d}t\bigg],\end{aligned} \tag{4.2}$$

where the underlying process solves

$$\mathrm{d}X_t = (\hat{\mu} + \hat{\sigma}^2)X_t\,\mathrm{d}t + \hat{\sigma}X_t\,\mathrm{d}W_t \quad \text{for } t > 0, \qquad X_0 = x > 0.$$

Theorem 3.1 holds in this setting and links the game of control (4.1) to the game of stopping (4.2). In particular, in the statement of Theorem 3.1 we should now refer to the games in (4.1) and (4.2) and replace (3.1)–(3.4) by

$$\begin{aligned}(\mathcal{L}_X v_1 - (r-\hat{\mu})v_1)(x) &= -\pi'(x), \qquad a_* < x < b_*,\ i = 1, 2,\\ (\mathcal{L}_X v_2 - (r-\hat{\mu})v_2)(x) &= u'(x), \qquad a_* < x < b_*,\ i = 1, 2,\\ (\mathcal{L}_X v_1 - (r-\hat{\mu})v_1)(x) &\ge -\pi'(x), \qquad \underline{x} < x \le a_*,\\ (\mathcal{L}_X v_2 - (r-\hat{\mu})v_2)(x) &\ge u'(x), \qquad b_* \le x < \overline{x},\\ v_i(x) &\le \alpha_i, \qquad x \in \mathcal{I},\ i = 1, 2.\end{aligned}$$

Moreover, the constants κ_i are adjusted as follows:

$$\kappa_1 := \frac{1}{r}(\hat{\mu}\alpha_1 + \pi)(a_*), \qquad \kappa_2 := -\frac{1}{r}(\hat{\mu}\alpha_2 - u)(b_*).$$

Everything else remains the same, including the proof of the theorem, which can be repeated by following the exact same steps.

We now discuss sufficient conditions under which the game of stopping (4.2) admits a Nash equilibrium. In order to refer directly to the results for the stopping game from Section 3.1, it is convenient to rewrite (4.2) in the form of (2.16) and (2.17).

Here $\mathcal{I} = \mathbb{R}_+$ since X is a geometric Brownian motion. For $r > \hat{\mu}$, we define functions Π' and U' via the ordinary differential equations

$$(\mathcal{L}_X - (r-\hat{\mu}))\Pi'(x) = -\pi'(x), \qquad (\mathcal{L}_X - (r-\hat{\mu}))U'(x) = u'(x), \tag{4.3}$$

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and by imposing growth conditions at 0 and ∞. In particular, letting $\tau_n := \inf\{t \geq 0\colon X_t \notin (1/n, n)\}$, we require that

$$\lim_{n\to\infty} \mathbb{E}_x[\mathrm{e}^{-(r-\hat{\mu})\tau_n}\Pi'(X_{\tau_n})] = \lim_{n\to\infty} \mathbb{E}_x[\mathrm{e}^{-(r-\hat{\mu})\tau_n}U'(X_{\tau_n})] = 0. \tag{4.4}$$

A specific choice for π and u is discussed below, but for now we observe that, by the Dynkin formula and (4.3), we obtain

$$\mathbb{E}_x\bigg[\int_0^{\tau\wedge\tau_n} \mathrm{e}^{-(r-\hat{\mu})t}\pi'(X_t)\,\mathrm{d}t\bigg] = \Pi'(x) - \mathbb{E}_x[\mathrm{e}^{-(r-\hat{\mu})(\tau\wedge\tau_n)}\Pi'(X_{\tau\wedge\tau_n})],$$
$$\mathbb{E}_x\bigg[\int_0^{\tau\wedge\tau_n} \mathrm{e}^{-(r-\hat{\mu})t}u'(X_t)\,\mathrm{d}t\bigg] = \mathbb{E}_x[\mathrm{e}^{-(r-\hat{\mu})(\tau\wedge\tau_n)}U'(X_{\tau\wedge\tau_n})] - U'(x).$$

Letting $n \to \infty$ in the above expressions, using (4.4), and substituting the result back into (4.2), we obtain the original formulation for $\mathcal{J}_1$ and $\mathcal{J}_2$ (see (2.16) and (2.17)) by setting

$$G_1(x) = \alpha_1 - \Pi'(x), \quad G_2(x) = \alpha_2 - U'(x), \qquad L_1(x) = -\Pi'(x), \quad L_2(x) = -U'(x).$$

It remains only to verify that it is possible to choose π and u such that Assumption 3.1 and condition (4.4) hold. Hence, we can apply Proposition 3.1.

We now set

$$\zeta_1(x) := (\mathcal{L}_X G_1 - (r-\hat{\mu})G_1)(x) = \pi'(x) - (r-\hat{\mu})\alpha_1,$$
$$\zeta_2(x) := (\mathcal{L}_X G_2 - (r-\hat{\mu})G_2)(x) = -u'(x) - (r-\hat{\mu})\alpha_2$$

and we note that ζ_1 is decreasing by concavity of π whereas ζ_2 is increasing by concavity of u. For instance, assuming the Inada conditions

$$\lim_{x\to\infty}\pi'(x) = 0, \qquad \lim_{x\to 0}\pi'(x) = +\infty, \qquad \lim_{x\to\infty}u'(x) = -\infty, \qquad \lim_{x\to 0}u'(x) = 0,$$

it follows that (3.19) holds for some $\hat{x}_i$, $i = 1, 2$, which depend on the specific choice of π and u.

Let us now consider the case of $\pi(x) = x^\lambda$ and $u(x) = -x^\delta$, where $\lambda \in (0, 1)$ and $\delta > 1$. For sufficiently large $r > \hat{\mu}$, we can guarantee (3.17) and (4.4). Moreover, denoting by γ_1 (respectively, γ_2) the positive (respectively, negative) root of the second-order equation $\frac{1}{2}\hat{\sigma}^2\gamma(\gamma-1) + (\hat{\mu}+\hat{\sigma}^2)\gamma - (r-\hat{\mu}) = 0$, conditions (3.16) on G_1 and G_2 are satisfied if $\lambda > \max\{0, \gamma_2+1\}$ and $1 < \delta < 1+\gamma_1$. Clearly, (3.18) holds by the same arguments. Finally, we have

$$\hat{x}_1 = \bigg(\frac{r\alpha_1}{\lambda}\bigg)^{-1/(1-\lambda)}, \qquad \hat{x}_2 = \bigg(\frac{r\alpha_2}{\delta}\bigg)^{1/(\delta-1)}$$

so that a suitable choice of α_1 and α_2 ensures that $\hat{x}_1 < \hat{x}_2$.

Appendix A.

A.1. Cost integrals and the set of strategies $\mathcal{M}$

It is well known in the SSC literature that state-dependent instantaneous costs of control give rise to questions concerning the definition of integrals representing the cumulative cost of exercising control.

In [43], Zhu provided a definition consistent with the classical verification argument used in SSC for the solution to a Hamilton–Jacobi–Bellman equation derived by the dynamic

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programming principle. This definition has been adopted in several other papers concerning explicit solutions of SSC problems (see, for example, [32] and [33]), and this is also the one that we use in our (2.13) and (2.14). Another, perhaps more natural, possibility is instead to define the integral as a Riemann–Stieltjes integral as, for example, in Alvarez [1].

Despite this formal difference, it is remarkable that the two definitions for the cost of exercising control lead essentially to the same optimal strategies for problems of the monotone follower type. In particular, it is possible to obtain Zhu's integral from the Riemann–Stieltjes integral by taking the limit as $n \to \infty$ of a sequence of controls that, at a given time t, make n instantaneous jumps of length h/n for a fixed $h > 0$. The optimality of this behaviour is illustrated, for example, by Alvarez [1, Corollary 1], and it is often referred to as a 'chattering policy'. The inconvenience with this approach is that the control obtained in the limit is not admissible in our $\mathcal{S}$, and, therefore, optimisers can be obtained only in a larger class.

Zhu's integral has proved to work very well in problems with monotone controls (representing, for instance, irreversible investments) or with controls of bounded variation (representing, for instance, partially reversible investment policies). In particular, the latter are often chosen in such a way that the controller's decision to invest/disinvest reflects the minimal decomposition of the control process (see, for example, [13], [19], and [23]). In other words, investment and disinvestment do not occur at the same time, and this assumption is often justified by conditions on the absence of arbitrage opportunities.

Here instead we have agents who use their controls independently, and it is unclear why *a priori* they should decide not to contrast each other's moves by acting simultaneously. To elaborate more on this point and understand our choice of the set $\mathcal{M}$, it is convenient to look at particular cases of our problem.

In some instances, it is interesting to include in our functionals (2.9) and (2.10) a state-dependent running cost π_i and use constant marginal costs/rewards of control α_i, β_i (see our example in Section 4 or the problems studied in [13], [21], or [35]). The corresponding functionals are

$$\begin{aligned}\widehat{\Psi}_1(x;\ \nu,\xi) &:= \mathbb{E}\bigg[\int_0^{\sigma_\mathcal{I}} \mathrm{e}^{-rt}\pi_1(\tilde{X}_t^{x,\nu,\xi})\,\mathrm{d}t + \int_0^{\sigma_\mathcal{I}} \mathrm{e}^{-rt}\alpha_1\,\mathrm{d}\xi_t - \int_0^{\sigma_\mathcal{I}} \mathrm{e}^{-rt}\beta_1\,\mathrm{d}\nu_t\bigg],\\ \widehat{\Psi}_2(x;\ \nu,\xi) &:= \mathbb{E}\bigg[\int_0^{\sigma_\mathcal{I}} \mathrm{e}^{-rt}\pi_2(\tilde{X}_t^{x,\nu,\xi})\,\mathrm{d}t + \int_0^{\sigma_\mathcal{I}} \mathrm{e}^{-rt}\alpha_2\,\mathrm{d}\nu_t - \int_0^{\sigma_\mathcal{I}} \mathrm{e}^{-rt}\beta_2\,\mathrm{d}\xi_t\bigg].\end{aligned} \tag{A.1}$$

In these cases, the integrals with respect to the controls are simply understood as Riemann–Stieltjes integrals. For $\alpha_i < \beta_i$, we prove that, if one of the two players opts for a control that reflects the process at a threshold then the other player's best response avoids simultaneous jumps of the controls. The condition $\alpha_i < \beta_i$ is the analogue in this context of the absence of arbitrage in, for example, [13], [21], and [35]. The result is illustrated in the next lemma.

Lemma A.1. *Consider the game with functionals (A.1). Let $a, b \in \mathcal{I}$, recall Lemma 2.2, and assume that $\alpha_i < \beta_i$, $i = 1, 2$. If player 1 (respectively, player 2) chooses*

$$\tilde{\nu}_t^a := \mathbf{1}_{\{t>0\}}[(a-x)^+ + \nu_t^a] \tag{A.2}$$

(respectively, $\tilde{\xi}^b := \mathbf{1}_{\{t>0\}}[(x-b)^+ + \xi_t^b]$), where ν^a solves $\mathrm{SP}_{a+}^{\xi}(x \vee a)$ (respectively, ξ^b solves $\mathrm{SP}_{b-}^{\nu}(x \wedge b)$), then the best reply

$$\hat{\xi}_a := \arg\max \widehat{\Psi}_2(x;\ \tilde{\nu}^a, \xi)$$

is such that $(\tilde{\nu}^a, \hat{\xi}_a) \in \mathcal{M}$ (respectively, $(\hat{\nu}_b, \tilde{\xi}^b) \in \mathcal{M}$ with $\hat{\nu}_b := \arg\max \widehat{\Psi}_1(x; \nu, \tilde{\xi}^b)$).

Proof. Let $x, a \in \mathcal{I}$ and $\xi \in \mathcal{S}^{\circ}$ (recall (2.11)) and consider ν^a solving $\mathrm{SP}^{\xi}_{a+}(x \vee a)$ (see (2.7)). We want to perform a pathwise comparison of the cost functional for player 2 under two different controls. In particular, we fix $\omega \in \Omega$ and assume that there exists (a stopping time) $t_0 = t_0(\omega) > 0$ such that $(\Delta\tilde{\nu}^a_{t_0} \cdot \Delta\xi_{t_0})(\omega) > 0$. With no loss of generality, we may assume that $\tilde{X}^{x,\tilde{\nu}^a,\xi}_{t_0}(\omega) > a$ and that the downward jump $\Delta\xi_{t_0}$ is trying to push the process below a, i.e.

$$\Delta\xi_{t_0}(\omega) > [\tilde{X}^{x,\tilde{\nu}^a,\xi}_{t_0} - a](\omega). \tag{A.3}$$

This push causes the immediate reaction of the control $\tilde{\nu}^a$ and, therefore, a simultaneous jump of the two controls. The case in which $\tilde{X}^{x,\tilde{\nu}^a,\xi}_{t_0}(\omega) \le a$ can be dealt with in the same way up to trivial changes.

We denote by ξ^0 a control in $\mathcal{S}^{\circ}$ such that

$$\xi^0_t(\omega) = \begin{cases} \xi_t(\omega), & t \le t_0, \\ \xi_t(\omega) - [\Delta\xi_{t_0} - (\tilde{X}^{x,\tilde{\nu}^a,\xi}_{t_0} - a)](\omega), & t > t_0. \end{cases}$$

In particular, $\xi^0(\omega)$ is the same as $\xi(\omega)$ but the jump size at $t_0(\omega)$ is reduced so that the process is not pushed below a. For ν^a solving $\mathrm{SP}^{\xi^0}_{a+}(x \vee a)$, the jump at t_0 is not triggered. Therefore, $\tilde{X}^{x,\tilde{\nu}^a,\xi^0}_{t_0+}(\omega) = a$ due only to the downward push given by ξ^0. Now we observe that the (random) Borel measure $\mathrm{d}\tilde{\nu}^a$, induced by $\tilde{\nu}^a$ in response to ξ, differs from the measure $\mathrm{d}\tilde{\nu}^a$, induced by $\tilde{\nu}^a$ in response to ξ^0, only for a mass at t_0 (which is needed to compensate for the jump of ξ). Moreover, since ν^a solves $\mathrm{SP}^{\xi}_{a+}(x \vee a)$ for any ξ then $\tilde{X}^{x,\tilde{\nu}^a,\xi}_t(\omega) = \tilde{X}^{x,\tilde{\nu}^a,\xi^0}_t(\omega)$ for all $t > 0$, since $\tilde{X}^{x,\tilde{\nu}^a,\xi}_{t_0+}(\omega) = \tilde{X}^{x,\tilde{\nu}^a,\xi^0}_{t_0+}(\omega) = a$ and nothing else has changed for $t \ne t_0$.

It is now easy to see that the couple $(\tilde{\nu}^a, \xi)$ requires an additional cost for player 2 compared to the couple $(\tilde{\nu}^a, \xi^0)$ and, therefore, cannot be optimal. For the sake of clarity here, we denote by $\nu^{a,\xi}$ the solution to $\mathrm{SP}^{\xi}_{a+}(x \vee a)$ and by ν^{a,ξ^0} the solution to $\mathrm{SP}^{\xi^0}_{a+}(x \vee a)$, and also we set $\tilde{\nu}^{a,\xi}$ and $\tilde{\nu}^{a,\xi^0}$ as in (A.2).

So we obtain

$$\begin{aligned} &\int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt}\pi_2(\tilde{X}^{x,\tilde{\nu}^{a,\xi},\xi}_t)\,\mathrm{d}t + \int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt}\alpha_2\,\mathrm{d}\tilde{\nu}^{a,\xi}_t - \int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt}\beta_2\,\mathrm{d}\xi_t \\ &\quad = \int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt}\pi_2(\tilde{X}^{x,\tilde{\nu}^{a,\xi^0},\xi^0}_t)\,\mathrm{d}t + \int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt}\alpha_2\,\mathrm{d}\tilde{\nu}^{a,\xi^0}_t - \int_0^{\sigma_{\mathcal{I}}} \mathrm{e}^{-rt}\beta_2\,\mathrm{d}\xi^0_t \\ &\qquad + \mathrm{e}^{-rt_0}(\alpha_2 - \beta_2)[\Delta\xi_{t_0} - (X^{x,\tilde{\nu}^{a,\xi},\xi}_{t_0} - a)], \end{aligned}$$

and the last term is negative as $\alpha_2 < \beta_2$ and by (A.3). Since the above argument can be repeated for any simultaneous jump of $\tilde{\nu}^a$ and ξ, and any $\omega \in \Omega$, the proof is complete. □

The point of the above lemma is that if the costs of control are constant then a simple condition for the absence of arbitrage opportunities implies that if one player picks a reflecting strategy then the other one will pick a control such that $(\nu, \xi) \in \mathcal{M}$. Therefore, under such assumptions, the equilibria constructed in Theorem 3.1 are also equilibria in the larger class $\mathcal{S}^{\circ} \times \mathcal{S}^{\circ}$.

A.2. Auxiliary results

We recall here the fundamental solutions ϕ and ψ of (2.4), and recall also that $\underline{x}$ and $\overline{x}$ are unattainable for X of (2.2) and for the uncontrolled diffusion $\tilde{X}^{0,0}$ of (2.1) (see Assumption 2.2).

Lemma A.2. *Let $a_* \in \mathcal{I}$ be arbitrary but fixed. Set*

$$\nu^*_t := \mathbf{1}_{\{t>0\}}[(a_* - x)^+ + \nu^{a_*}_t]$$

with ν^{a_*} solving the Skorokhod reflection problem $\mathrm{SP}^0_{a_*+}(x \vee a_*)$ of Lemma 2.2. For $y \in (a_*, \overline{x})$, set $\theta_y := \inf\{t > 0 \colon \tilde{X}^{\nu_*,0}_t \geq y\}$ and

$$q(x, y) := \mathbb{E}_x[\mathrm{e}^{-r\theta_y}], \qquad x \in \mathcal{I}.$$

Then, for $i = 1, 2$, we have

$$\lim_{y \uparrow \overline{x}} q(x, y)\bigg(1 + \int_{a_*}^{y} |G_i(z)| \,\mathrm{d}z\bigg) = 0. \tag{A.4}$$

Similarly, let $b_* \in \mathcal{I}$ be arbitrary but fixed. Set

$$\xi^*_t := \mathbf{1}_{\{t>0\}}[(x - b_*)^+ + \xi^{b_*}_t]$$

with ξ^{b_*} the solution to the Skorokhod reflection problem $\mathrm{SP}^0_{b_*-}(x \wedge b_*)$ of Lemma 2.2. For $y \in (\underline{x}, b_*)$, set $\eta_y := \inf\{t > 0 \colon \tilde{X}^{0,\xi_*}_t \leq y\}$ and

$$p(x, y) := \mathbb{E}_x[\mathrm{e}^{-r\eta_y}], \qquad x \in \mathcal{I}.$$

Then, for $i = 1, 2$, we have

$$\lim_{y \downarrow \underline{x}} p(x, y)\bigg(1 + \int_{b_*}^{y} |G_i(z)| \,\mathrm{d}z\bigg) = 0.$$

Proof. We provide a full proof only for the first claim as the one for the second claim follows by similar arguments. The existence of a solution to $\mathrm{SP}^0_{a_*+}(x \vee a_*)$ can be found in [41, Theorem 4.1] for coefficients μ, σ in (2.1) which are uniformly Lipschitz-continuous. The relaxation to locally Lipschitz-continuous coefficients (Assumption 2.1) follows by standard arguments as the ones used in the proof of Lemma 2.2 below.

We note that

$$\tilde{X}^{x,\nu^*,0}_t = \tilde{X}^{x \vee a_*, \nu^{a_*}, 0}_t, \qquad t > 0,$$

and, therefore, θ_y is equal to $\tilde{\theta}_y := \inf\{t > 0 \colon \tilde{X}^{x\vee a_*, \nu^{a_*},0}_t \geq y\}$ and $q(x, y) = q(a_*, y)$ for $x \leq a_*$. Functionals involving $\tilde{\theta}_y$ have well-known analytical properties, and from now on we will make no distinction between θ_y and $\tilde{\theta}_y$.

For $x \geq y$, we have $q(x, y) = 1$, whereas in Lemma 2.1 and Corollary 2.2 of [38], it was shown that the function $q(\cdot, y)$ solves

$$(\mathcal{L}_{\tilde{X}} - r)q(x, y) = 0, \qquad x \in (a_*, y), \tag{A.5}$$

with boundary conditions

$$q(y-, y) := \lim_{x \uparrow y} q(x, y) = 1, \qquad q_x(a_*+, y) := \lim_{x \downarrow a_*} q_x(x, y) = 0.$$

In particular, we refer to the condition at a_* as the reflecting boundary condition.

Since $q(\cdot, y)$ solves (A.5), it may then be written as

$$q(x, y) = A(y)\tilde{\psi}(x) + B(y)\tilde{\phi}(x), \qquad x \in (a_*, y),$$

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where $\tilde{\psi}$ and $\tilde{\phi}$ denote the fundamental increasing and decreasing solutions, respectively, of $(\mathcal{L}_{\tilde{X}} - r)u = 0$ on $\mathcal{I}$. By imposing the reflecting boundary condition, we obtain

$$B(y) = -A(y)\frac{\tilde{\psi}'(a_*)}{\tilde{\phi}'(a_*)},$$

which substituting back into the expression for q yields

$$q(x, y) = A(y)\bigg(\tilde{\psi}(x) - \frac{\tilde{\psi}'(a_*)}{\tilde{\phi}'(a_*)}\tilde{\phi}(x)\bigg). \tag{A.6}$$

Now, imposing the boundary condition at y, we also obtain

$$A(y) = \bigg(\tilde{\psi}(y) - \frac{\tilde{\psi}'(a_*)}{\tilde{\phi}'(a_*)}\tilde{\phi}(y)\bigg)^{-1}. \tag{A.7}$$

Note that $-\tilde{\psi}'(a_*)/\tilde{\phi}'(a_*) > 0$, thus implying $A(y), B(y) > 0$ and $q(x, y) > 0$, as expected. Since the sample paths of $\tilde{X}^{\nu^*,0}$ are continuous for all $t > 0$ then $y \mapsto q(x, y)$ must be strictly decreasing. Hence,

$$q_y(x, y) = A'(y)\bigg(\tilde{\psi}(x) - \frac{\tilde{\psi}'(a_*)}{\tilde{\phi}'(a_*)}\tilde{\phi}(x)\bigg) < 0,$$

which implies $A'(y) < 0$ since the term in brackets is positive. From (A.7) and direct computation, we obtain

$$A'(y) = -\frac{1}{(A(y))^2}\bigg(\tilde{\psi}'(y) - \frac{\tilde{\psi}'(a_*)}{\tilde{\phi}'(a_*)}\tilde{\phi}'(y)\bigg),$$

and $A'(y) < 0$ implies that

$$\bigg(\tilde{\psi}'(y) - \frac{\tilde{\psi}'(a_*)}{\tilde{\phi}'(a_*)}\tilde{\phi}'(y)\bigg) > 0. \tag{A.8}$$

The latter inequality is important in order to prove (A.4).

The assumed regularity of μ and σ (see Assumption 2.1) implies that $\tilde{\psi}'$ solves $\mathcal{L}_X u(x) - (r - \mu'(x))u(x) = 0$ in $\mathcal{I}$ (see (2.4)), and it can therefore be written as a linear combination of the fundamental increasing and decreasing functions ψ and ϕ, i.e.

$$\tilde{\psi}'(x) = \alpha\psi(x) + \beta\phi(x) \quad \text{for some } \alpha, \beta \in \mathbb{R}. \tag{A.9}$$

Analogously,

$$\tilde{\phi}'(x) = \gamma\psi(x) + \delta\phi(x). \tag{A.10}$$

Moreover, since $\tilde{\psi}' > 0$ and $\tilde{\phi}' < 0$ in $\mathcal{I}$, and $\underline{x}$ and $\overline{x}$ are unattainable for X, then it must be that $\alpha, \beta \geq 0$ and $\gamma, \delta \leq 0$ (since $\psi(x)/\phi(x) \to \infty$ as $x \to \overline{x}$ and $\psi(x)/\phi(x) \to 0$ as $x \to \underline{x}$). Noting that $y > a_*$ was arbitrary, inequality (A.8) becomes

$$\bigg(\alpha - \gamma\frac{\tilde{\psi}'(a_*)}{\tilde{\phi}'(a_*)}\bigg)\psi(y) + \bigg(\beta - \delta\frac{\tilde{\psi}'(a_*)}{\tilde{\phi}'(a_*)}\bigg)\phi(y) > 0, \qquad y > a_*. \tag{A.11}$$

We aim to show that $\alpha > 0$ and we can do it by considering two cases separately.

Case 1. Assume that $\gamma < 0$. Since the second term in (A.11) can be made arbitrarily small by letting $y \to \overline{x}$ then it must be $\alpha > \gamma\tilde{\psi}'(a_*)/\tilde{\phi}'(a_*) > 0$.

Case 2. Assume that $\gamma = 0$. If $\alpha = 0$ then the first term on the left-hand side of (A.11) is 0 and by using (A.9) and (A.10), we obtain from (A.11),

$$0 < \left(\beta - \delta\frac{\tilde{\psi}'(a_*)}{\tilde{\phi}'(a_*)}\right)\phi(y) = \left(\beta - \delta\frac{\beta\phi(a_*)}{\delta\phi(a_*)}\right)\phi(y) = 0,$$

hence a contradiction. So it must be $\alpha > 0$.

Finally, for fixed $x \in (\underline{x}, y)$, there exists a constant $C = C(a_*, x) > 0$ such that (A.6) and (A.7) lead to

$$0 \le q(x, y)\left(1 + \int_{a_*}^{y} |G_i(z)|\,\mathrm{d}z\right) \le \frac{C}{\tilde{\psi}(y)}\left(1 + \int_{a_*}^{y} |G_i(z)|\,\mathrm{d}z\right). \tag{A.12}$$

Now letting $y \to \overline{x}$, we have $\tilde{\psi}(y) \to \infty$ as $\overline{x}$ is unattainable for $\tilde{X}^{0,0}$. We have two possibilities:

- $\int_{a_*}^{\overline{x}} |G_i(z)|\,\mathrm{d}z < +\infty$ and, therefore, (A.4) holds trivially from (A.12);
- $\int_{a_*}^{\overline{x}} |G_i(z)|\,\mathrm{d}z = +\infty$ so that by using L'Hôpital's rule in (A.12), (A.9), and Assumption 2.4, we obtain

$$\lim_{y\to\overline{x}} \frac{1}{\tilde{\psi}(y)} \int_{a_*}^{y} |G_i(z)|\,\mathrm{d}z = \lim_{y\to\overline{x}} \frac{|G_i(y)|}{\alpha\psi(y)} = 0.$$

This completes the proof. □

Proof of Lemma 2.2. We provide a short proof of the existence of a unique solution to the Skorokhod reflection problem $\mathrm{SP}^{\xi}_{a+}(x)$.

Note that the drift and diffusion coefficients in (2.1) are locally Lipschitz-continuous due to Assumption 2.1. So we first prove the result for Lipschitz coefficients, and then extend it to locally Lipschitz ones. Note that here we are not assuming sublinear growth of μ and σ but we rely on the nonattainability of $\underline{x}$ and $\overline{x}$ for the uncontrolled process $\tilde{X}^{0,0}$. Existence of a unique solution to problem $\mathrm{SP}^{\nu}_{b-}(x)$ can be shown by analogous arguments. For simplicity, from now on we just write SP^{ξ}_{a+} and omit the dependence on x.

Step 1: Lipschitz coefficients. Here we assume that $\mu, \sigma \in \mathrm{Lip}(\mathcal{I})$ with a constant smaller than $L > 0$. Let $a \in \mathcal{I}$, $x \ge a$, $\xi \in \mathcal{S}$, and consider the sequence of processes defined recursively by $X^{[0]}_t = x$, $\nu^{[0]}_t = 0$, and

$$X^{[k+1]}_t = x + \int_0^t \mu(X^{[k]}_u)\,\mathrm{d}u + \int_0^t \sigma(X^{[k]}_u)\,\mathrm{d}W_u + \nu^{[k+1]}_t - \xi_t, \tag{A.13}$$

$$\nu^{[k+1]}_t = \sup_{0\le s\le t}\left[a - x - \int_0^s \mu(X^{[k]}_u)\,\mathrm{d}u - \int_0^s \sigma(X^{[k]}_u)\,\mathrm{d}W_u + \xi_s\right] \tag{A.14}$$

for any $k \ge 0$ and $t \ge 0$. Note that at any step the process $X^{[k+1]}$ is kept above the level a by the process $\nu^{[k+1]}$ with minimal effort, i.e. according to a Skorokhod reflection at a. The Lipschitz continuity of μ and σ allows us to obtain, from (A.13), the estimate

$$\mathbb{E}_x\left[\sup_{0\le s\le t} |X^{[k+1]}_s - X^{[k]}_s|^2\right] \le C\mathbb{E}_x\left[\int_0^t |X^{[k]}_s - X^{[k-1]}_s|^2\,\mathrm{d}s\right] \tag{A.15}$$

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for $k \geq 1$ and for some positive $C := C(x, a, L)$. Since, for $k = 0$, we have $\mathbb{E}_x[\sup_{0\leq s\leq t} |X_s^{[1]} - x|^2] \leq Rt$ for some $R := R(x, a, L) > 0$, then an induction argument together with (A.15) yield

$$\mathbb{E}_x\Big[\sup_{0\leq s\leq t} |X_s^{[k+1]} - X_s^{[k]}|^2\Big] \leq \frac{(R_0 t)^{k+1}}{(k+1)!}, \qquad k \geq 0, \tag{A.16}$$

for some other positive $R_0 := R_0(x, a, L)$. Analogously,

$$\mathbb{E}_x\Big[\sup_{0\leq s\leq t} |\nu_s^{[k+1]} - \nu_s^{[k]}|^2\Big] \leq \frac{(R_1 t)^{k+1}}{(k+1)!}, \qquad k \geq 0, \tag{A.17}$$

with $R_1 := R_1(x, a, L) > 0$.

Thanks to (A.16) and (A.17) we can now proceed with an argument often used in SDE theory for the proof of the existence of strong solutions (see, for example, the proof of [30, Chapter 5, Theorem 2.9]), i.e. we use the Chebyshev inequality and the Borel–Cantelli lemma to find that $(X^{[k+1]}, \nu^{[k+1]})_{k\geq 0}$ converges a.s., locally uniformly in time, as $k \uparrow \infty$. We denote this limit by $(\tilde{X}^{\nu^a,\xi}, \nu^a)$. By the Lipschitz continuity of μ and σ, and the same arguments as above, we also find that the sequences $(\int_0^t \mu(X_u^{[k]})\,\mathrm{d}u)_{k\geq 0}$ and $(\int_0^t \sigma(X_u^{[k]})\,\mathrm{d}W_u)_{k\geq 0}$ converge a.s., locally uniformly in time. Then we have a.s. (up to a possible subsequence)

$$\begin{aligned}
\nu_t^a &= \lim_{k\uparrow\infty} \nu_t^{[k+1]} \\
&= \lim_{k\uparrow\infty} \sup_{0\leq s\leq t}\Big[a - x - \int_0^s \mu(X_u^{[k]})\,\mathrm{d}u - \int_0^s \sigma(X_u^{[k]})\,\mathrm{d}W_u + \xi_s\Big] \\
&= \sup_{0\leq s\leq t}\Big[a - x - \int_0^s \mu(\tilde{X}_u^{\nu^a,\xi})\,\mathrm{d}u - \int_0^s \sigma(\tilde{X}_u^{\nu^a,\xi})\,\mathrm{d}W_u + \xi_s\Big].
\end{aligned}$$

It thus follows that $(\tilde{X}^{\nu^a,\xi}, \nu^a)$ solve SP_{a+}^{ξ}. Finally, uniqueness can be proved as, for example, in the proof of [41, Theorem 4.1].

Step 2: locally Lipschitz coefficients. Here we assume that μ and σ are as in Assumption 2.1. Let $x_n \uparrow \overline{x}$ and define

$$\mu_n(x) = \mu(x)\mathbf{1}_{\{x\leq x_n\}} + \mu(x_n)\mathbf{1}_{\{x>x_n\}}, \qquad \sigma_n(x) = \sigma(x)\mathbf{1}_{\{x\leq x_n\}} + \sigma(x_n)\mathbf{1}_{\{x>x_n\}}.$$

For each n, we denote by $\mathrm{SP}_{a+}^{\xi(n)}$ the Skorokhod problem SP_{a+}^{ξ} but for the dynamics

$$\mathrm{d}X_t = \mu_n(X_t)\,\mathrm{d}t + \sigma_n(X_t)\,\mathrm{d}W_t + \mathrm{d}\nu_t - \mathrm{d}\xi_t$$

rather than for (2.1).

Since, for each n, we have μ_n and σ_n uniformly Lipschitz on $[a, \overline{x})$, then step 1 guarantees that there exists a unique $(X^{(n)}, \nu^{(n)})$ that solves $\mathrm{SP}_{a+}^{\xi(n)}$. We denote $\tau_n := \inf\{t > 0\colon X_t^{(n)} \geq x_n\}$ and, for all $t \leq \tau_n$, we have

$$\begin{aligned}
X_t^{(n)} &= x + \int_0^t \mu_n(X_u^{(n)})\,\mathrm{d}u + \int_0^t \sigma_n(X_u^{(n)})\,\mathrm{d}W_u + \nu_t^{(n)} - \xi_t \\
&= x + \int_0^t \mu(X_u^{(n)})\,\mathrm{d}u + \int_0^t \sigma(X_u^{(n)})\,\mathrm{d}W_u + \nu_t^{(n)} - \xi_t
\end{aligned}$$

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and

$$\nu_t^{(n)} = \sup_{0\le s\le t}\left[a - x - \int_0^s \mu_n(X_u^{(n)})\,\mathrm{d}u - \int_0^s \sigma_n(X_u^{(n)})\,\mathrm{d}W_u + \xi_s\right]$$
$$= \sup_{0\le s\le t}\left[a - x - \int_0^s \mu(X_u^{(n)})\,\mathrm{d}u - \int_0^s \sigma(X_u^{(n)})\,\mathrm{d}W_u + \xi_s\right].$$

Since the coefficients above do not depend on n, by construction the process $(X_t^{(n)}, \nu_t^{(n)})$ also solves SP_{a+}^{ξ} for $t \le \tau_n$. Uniqueness of the solution for $\mathrm{SP}_{a+}^{\xi(n)}$ implies that $(X_t^{(n)}, \nu_t^{(n)})$ is also the solution to $\mathrm{SP}_{a+}^{\xi(m)}$ for $t \le \tau_m$, for each $m \le n$, and, therefore, the unique solution to SP_{a}^{ξ} up to the stopping time τ_n.

Fix an arbitrary $T > 0$. For all $\omega \in \{\tau_n > T\}$ and all $t \le T$, we can define $(\tilde{X}_t^{\nu^a,\xi}, \nu_t^a) := (X_t^{(n)}, \nu_t^{(n)})$ so that the couple $(\tilde{X}_t^{\nu^a,\xi}, \nu_t^a)$ is the unique solution to SP_{a+}^{ξ} for $t \le T$. It remains to show that $\lim_{n\to\infty}\mathbb{P}(\tau_n > T) = 1$ so that we have constructed a unique solution to SP_{a+}^{ξ} for almost every $\omega \in \Omega$ up to time T.

Let us consider first the $\xi \equiv 0$ case. From Lemma A.2, it follows that $\mathbb{E}_x[\mathrm{e}^{-r\theta_{x_n}}] \to 0$ as $n \to \infty$ with $\theta_{x_n} = \inf\{t > 0 \colon \tilde{X}_t^{\nu^a,0} \ge x_n\}$, and, therefore, $\theta_{x_n} \to \infty$ $\mathbb{P}_x$-a.s. Hence,

$$\lim_{n\to\infty}\mathbb{P}(\tau_n > T) = \lim_{n\to\infty}\mathbb{P}(\theta_{x_n} > T) = 1,$$

since $\tau_n = \theta_{x_n}$, $\mathbb{P}$-a.s. To conclude, it suffices to note that $\tilde{X}_t^{\nu^a,\xi} \le \tilde{X}_t^{\nu^a,0}$ for all $t > 0$ and arbitrary $\xi \in \mathcal{S}$. Then $\overline{x}$ is also unattainable for $\tilde{X}^{\nu^a,\xi}$. □

Proof of Proposition 3.1. The proofs can be found in [16]; here we provide precise references to the relevant results in each case. In particular, we must note that Appendix A.3 of [16] addresses the specific setting of the state-dependent discount factor $r - \mu'(x)$ that appears in our stopping functionals (2.16) and (2.17).

(i) This follows from Theorem 3.2 (and Appendix A.3) of [16].

(ii) This follows from Proposition 3.12 (and Appendix A.3) of [16]. For the sake of completeness, we note that in order to prove that x_2^{∞} uniquely solves $\vartheta_2(x) = (G_2/\psi)(x)$ in $(\hat{x}_2, \overline{x})$ it is useful to change variables. Defining $y = (\psi/\phi)(x) =: F(x)$, where F is strictly increasing, and introducing $\hat{G}_2(y) := [(G_2/\phi) \circ F^{-1}](y)$, $y > 0$, it follows from simple algebra (see [16, Appendix A.1]) that $\vartheta_2(x) = (G_2/\psi)(x)$ is equivalent to $\hat{G}_2'(y)y = \hat{G}_2(y)$. It was shown in [16, Lemma 3.6] that the latter equation has a unique root y_2^{∞} in the interval $(\hat{y}_2, \infty)$ with $\hat{y}_2 := F(\hat{x}_2)$ and $\infty = \lim_{x\uparrow\overline{x}} F(x)$. Therefore, $x_2^{\infty} = F^{-1}(y_2^{\infty})$ solves the initial problem in $(\hat{x}_2, \overline{x})$. □

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