ANZIAM J. 58(2017), 417–427
doi:10.1017/S1446181116000353

RARE EVENTS IN THE STOCHASTIC CAMASSA–HOLM EQUATION

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(Received 12 June, 2016; accepted 24 September, 2016; first published online 6 March 2017)

Abstract

We investigate rare or small probability events in the context of large deviations of the stochastic Camassa–Holm equation. By the weak convergence approach and regularization, we get large deviations of the regularized equation. Then, by stochastic equations exponentially equivalent to the corresponding laws, we get large deviations of the stochastic Camassa–Holm equation.

2010 *Mathematics subject classification*: 60H15.

Keywords and phrases: stochastic Camassa–Holm equation, regularization, exponentially equivalent, large deviations.

1. Introduction

We analyse the stochastic Camassa–Holm (CH) equation

$$\begin{cases} du^\epsilon + [u^\epsilon u_x^\epsilon + (1-\partial_x^2)^{-1}\partial_x(u^{\epsilon 2} + \frac{1}{2}u_x^{\epsilon 2})]\,dt = \sqrt{\epsilon}\,dW \\ u^\epsilon(x,0) = \varphi(x), \end{cases} \tag{1.1}$$

where $0 < \epsilon < 1$ is sufficiently small, W is a Wiener process in the Hilbert space H with convolution operator Q and $(1-\partial_x^2)^{-1} f = p * f$, $p = (1/2)e^{-|x|}$ for all $f \in L^2(\mathbb{R})$.

The CH equation was derived by Camassa and Holm [4, 12] as a model of water waves. The well-posedness of (1.1) in $H^s(\mathbb{R})$ for $s > 3/2$ was established by Chen et al. [5]. In this paper, we consider the rare events as described by a large deviation principle (LDP) for the stochastic CH equation (1.1). The LDP is an active and important topic in probability and statistics. Recently, it was found that the weak convergence approach [10] along with stochastic control can be employed to obtain the LDP.

For the special nonlinear terms of (1.1), the LDP for the solution of (1.1) cannot be directly obtained by a weak convergence approach. However, by this method, we can

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get the LDP for u_η^ϵ of the regularized equations

$$\begin{cases} du_\eta^\epsilon + [u_\eta^\epsilon u_{\eta x}^\epsilon + (1-\partial_x^2)^{-1}\partial_x(u_\eta^{\epsilon 2} + \frac{1}{2}u_{\eta x}^{\epsilon 2})]\,dt = \sqrt{\epsilon}\,dW_\eta \\ u_\eta^\epsilon(x,0) = \varphi_\eta(x), \end{cases} \tag{1.2}$$

where $0 < \eta < 1$, $\varphi_\eta = \varphi * \rho_\eta$, $Q_\eta = Q * \rho_\eta$ and ρ_η is the Friedrichs mollifier [11]. Then we prove that the solution of u_η^ϵ is exponentially equivalent to the solution of u^ϵ: that is, for any $\lambda > 0$,

$$\lim_{\epsilon\to 0} \epsilon \ln P\Big(\sup_{t\in[0,T]} \|u_\eta^\epsilon - u^\epsilon\|_{H^s}^2 > \lambda\Big) = -\infty, \tag{1.3}$$

from which it follows that $\{u^\epsilon\}$ satisfies the LDP [8, Theorem 4.2.13].

This paper is organized as follows. In Section 2, some standard definitions and results of the LDP are recalled and then, in Section 3, the main theorems and their proofs are given. The paper concludes with a brief discussion in Section 4.

2. Large deviation principle

Let us first recall some standard definitions and results from the large deviation theory. Let X^ϵ be a family of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in some Polish space E [9].

Definition 2.1. A function $I : E \to [0, +\infty]$ is called a rate function if I is lower semicontinuous. A rate function I is called a good rate function if the level set $\{x \in E : I(x) \le K\}$ is compact for each $K < \infty$.

Definition 2.2. The family $\{X^\epsilon\}$ is said to satisfy the LDP with rate function I if, for each Borel subset A of E,

$$-\inf_{x\in A^o} I(x) \le \liminf_{\epsilon\to 0} \epsilon \ln P\{X^\epsilon \in A\} \le \limsup_{\epsilon\to 0} \epsilon \ln P\{X^\epsilon \in A\} \le -\sup_{x\in\bar{A}} I(x),$$

where A^o and $\bar{A}$ denote the interior and closure of A in E, respectively.

Definition 2.3. The sequence $\{X^\epsilon\}$ is said to satisfy the Laplace principle with rate function I if, for each bounded continuous real-valued function h defined on E

$$\lim_{\epsilon\to 0} \epsilon \ln \mathbb{E}\Big[\exp\Big(-\frac{1}{\epsilon}h(X^\epsilon)\Big)\Big] = -\inf_{x\in E}\{h(x) + I(x)\}.$$

Suppose that $\mathcal{G}^\epsilon : C([0,T]; H) \to E$ is a measurable map and $X^\epsilon = \mathcal{G}^\epsilon(W(\cdot))$. Let $\mathcal{A}$ denote the class of H-valued $\mathcal{F}_t$-predictable processes v which satisfy $\int_0^T \|v(r)\|_H^2\,dr < \infty$ almost surely (a.s.). Let $S_M = \{v \in L^2(0,T;H) \mid \int_0^T \|v(r)\|_H^2\,dr \le M\}$. The set S_M endowed with the weak topology is a Polish space. Define $\mathcal{A}_M = \{v \in \mathcal{A} \mid v(\omega) \in S_M, P\text{-a.s.}\}$.

Now we formulate the following sufficient condition for the Laplace principle (equivalently, LDP if E is a Polish space) of X^ϵ as $\epsilon \to 0$.

https://doi.org/10.1017/S1446181116000353 Published online by Cambridge University Press

Assumption 2.4. There exists a measurable map $\mathcal{G}^0 : C([0,T]; H) \to E$ such that the following two conditions hold.

(i) If $\{v_\epsilon \in \mathcal{A}_M \mid \epsilon > 0\}$ as a random variable in S_M converges to $v \in \mathcal{A}_M$ in distribution as $\epsilon \to 0$ for some $0 < M < \infty$, then

$$\mathcal{G}^\epsilon\Big(W(\cdot) + \frac{1}{\epsilon}\int_0^{\cdot} v_\epsilon(r)\,dr\Big) \to \mathcal{G}^0\Big(\int_0^{\cdot} v(r)\,dr\Big)$$

in distribution, as $\epsilon \to 0$.

(ii) For $M < \infty$, the set $K_M = \{\mathcal{G}^0(\int_0^{\cdot} v(r)\,dr) \mid v \in S_M\}$ is a compact subset of E.

For each $g \in E$, define

$$I(g) = \inf_{v \in \{L^2([0,T];H) \mid g = \mathcal{G}^0(\int_0^{\cdot} v(\tau)\,d\tau)\}} \Big\{\frac{1}{2}\int_0^T \|v(r)\|_H^2\,dr\Big\}, \tag{2.1}$$

where the infimum over an empty set is taken as ∞.

The following theorem was proven by Budhiraja and Dupuis [3].

Theorem 2.5. *Let $X^\epsilon = \mathcal{G}^\epsilon(W(\cdot))$. If $\{\mathcal{G}^\epsilon\}$ satisfies Assumption 2.4, then the family $\{X^\epsilon \mid \epsilon > 0\}$ satisfies the Laplace principle in E with the rate function I given by (2.1).*

3. The main results

In this section, we give the LDP for u_η^ϵ and u^ϵ. Denote by $\mathcal{L}^s$ the space of Hilbert–Schmidt operators from H into $H^s(\mathbb{R})$ with the norm $\|Q\|_{\mathcal{L}^s}^2 = \mathrm{tr}(Q^*Q) = \sum_{k\in\mathbb{N}} \|Q^{1/2}e_k\|_s^2$, where $(e_k)_{k\in\mathbb{N}}$ is an orthogonal basis of L^2. Let C be a constant in the rest of the paper, which may alter in different places.

Let $X_T = C([0,T]; H^s)$, $s > 3/2$. It follows that (see [3]) there exists a Borel-measurable function $\mathcal{G}^\epsilon : X_T \to X_T$ such that $u_\eta^\epsilon = \mathcal{G}^\epsilon(W_\eta)$. Set $\tilde{u}_v^\epsilon(\cdot) = \mathcal{G}^\epsilon(W_\eta(\cdot) + (1/\sqrt{\epsilon})\int_0^{\cdot} v_\eta(s)\,ds)$. Then, by Girsanov's theorem [13], $\tilde{u}_v^\epsilon(\cdot)$ is the unique mild solution on $[0,T]$ of the equations

$$\begin{cases} d\tilde{u}_v^\epsilon + [\tilde{u}_v^\epsilon \tilde{u}_{vx}^\epsilon + (1-\partial_x^2)^{-1}\partial_x(\tilde{u}_v^{\epsilon 2} + \frac{1}{2}\tilde{u}_{vx}^{\epsilon 2}) + v_\eta]\,dt = \sqrt{\epsilon}\,dW_\eta \\ \tilde{u}_v^\epsilon(x,0) = \varphi_\eta(x). \end{cases}$$

Let us introduce the skeleton equation [10] associated with (1.2), that is,

$$\begin{cases} d\tilde{u}_v + [\tilde{u}_v \tilde{u}_{vx} + (1-\partial_x^2)^{-1}\partial_x(\tilde{u}_v^2 + \frac{1}{2}\tilde{u}_{vx}^2) + v_\eta]\,dt = 0 \\ \tilde{u}_v(x,0) = \varphi_\eta(x). \end{cases} \tag{3.1}$$

The existence and uniqueness of the solution $\tilde{u}_v$ to (3.1) in $C([0,T]; H^\infty)$ can be obtained (see, for example, [5, Theorem 3.1]).

Define $\mathcal{G}^0 : X_T \to X_T$ by

$$\mathcal{G}_0(h) = \begin{cases} \tilde{u}_v & \text{if } h = \int_0^{\cdot} v_\eta(r)\,dr \text{ for some } v \in L^2([0,T]; H^s) \\ 0 & \text{otherwise.} \end{cases}$$

The following lemmas are needed.

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LEMMA 3.1 [2, 5]. *Under the Assumptions 2.4, the following estimates hold for any η satisfying $0 < \eta < 1$ and $s > 0$.*

(i) $\|u_{0\eta}\|^2_{H^q} + \int_0^t \|v_\eta\|^2_{H^q}\,d\tau + \|Q_\eta\|^2_{\mathcal{L}_q} \le c\eta^{(s-q)/2}$ *for any* $q > 0$.
(ii) *If* $q \neq s$, *then* $\mathbb{E}[\|u_{0\eta} - u_0\|^2_{H^q} + \int_0^t \|v_\eta - v\|^2_{H^q}\,d\tau] + \|Q_\eta - Q\|^2_{\mathcal{L}_q} \le c\eta^{(s-q)/2}$.
(iii) *If* $q = s$, *then* $\mathbb{E}[\|u_{0\eta} - u_0\|_{H^q}] + \|Q_\eta - Q\|^2_{\mathcal{L}_q} = o(1)$.

Here c is a constant, independent of η.

LEMMA 3.2. *Let* $s > 3/2$, $\varphi(x) \in H^s$, $k = 0, 1$ *and* $Q \in \mathcal{L}^s$. *Then*

$$\lim_{R\to+\infty} \ln P\Big(\sup_{0\le t\le T} \|\tilde{u}^\epsilon_v\|^2_s > R\Big) = -\infty, \tag{3.2}$$

$$\sup_{t\in[0,T]} \|\tilde{u}_v\|^2_{H^{s+k}} \le C\eta^{-k/2}. \tag{3.3}$$

PROOF. Let $\Phi(x) = \ln(1+x)$, $x > 0$. Then $\Phi'(x) = 1/(1+x)$ and $\Phi''(x) = -1/(1+x)^2$. Define $\tau_R = \inf\{t > 0, \|\tilde{u}^\epsilon_v\|^2_{H^s} \ge R\}$, $R > 0$. Applying Itô's formula [6] to $\Phi(\|\tilde{u}^\epsilon_v(t\wedge\tau_R)\|^2_{H^s})$,

$$\begin{aligned}\Phi(\|\tilde{u}^\epsilon_v(t\wedge\tau_R)\|^2_{H^s}) = \Phi(\|\varphi_\eta\|^2_{H^s}) &+ 2\int_0^{t\wedge\tau_R} \Phi'(\|\tilde{u}^\epsilon_v\|^2_{H^s})(\Lambda^s\tilde{u}^\epsilon_v, \Lambda^s h(\tilde{u}^\epsilon_v, \tilde{u}^\epsilon_{vx}))\,d\tau \\ &+ 2\int_0^{t\wedge\tau_R} \Phi'(\|\tilde{u}^\epsilon_v\|^2_{H^s})(\Lambda^s\tilde{u}^\epsilon_v, \Lambda^s v_\eta)\,d\tau \\ &+ \epsilon\int_0^{t\wedge\tau_R} \Phi''(\|\tilde{u}^\epsilon_v\|^2_{H^s})\|Q_\eta\|^2_{\mathcal{L}^s}\,d\tau \\ &+ 2\sqrt{\epsilon}\int_0^{t\wedge\tau_R} \Phi'(\|\tilde{u}^\epsilon_v\|^2_{H^s})(\Lambda^s\tilde{u}^\epsilon_v, \Lambda^s\,dW_\eta),\end{aligned} \tag{3.4}$$

where

$$h(u, u_x) = uu_x + (1-\partial_x^2)^{-1}\partial_x(u^2 + \tfrac{1}{2}u_x^2). \tag{3.5}$$

Since $\|\tilde{u}^\epsilon_v\|_{L^\infty}, \|\tilde{u}^\epsilon_{vx}\|_{L^\infty} \le C\|\tilde{u}^\epsilon_v\|_{H^s}$ with $s > 3/2$,

$$\begin{aligned}\int_{\mathbb{R}} (\Lambda^s\tilde{u}^\epsilon_v)(\Lambda^s h(\tilde{u}^\epsilon_v, \tilde{u}^\epsilon_{vx}))\,dx &\le C(\|\tilde{u}^\epsilon_{vx}\|_{L^\infty}\|\tilde{u}^\epsilon_v\|^2_{H^s} + \|\tilde{u}^\epsilon_v\|_{H^s}\|\tilde{u}^{\epsilon 2}_v\|_{H^{s-1}}) \\ &\le C(\|\tilde{u}^\epsilon_v\|_{L^\infty} + \|\tilde{u}^\epsilon_{vx}\|_{L^\infty})\|\tilde{u}^\epsilon_v\|^2_{H^s} \le C\|\tilde{u}^\epsilon_v\|^3_{H^s}.\end{aligned} \tag{3.6}$$

Hence, Young's inequality [15] yields

$$2\Phi'(\|\tilde{u}^\epsilon_v\|^2_{H^s})(\Lambda^s\tilde{u}^\epsilon_v, \Lambda^s h(\tilde{u}^\epsilon_v, \tilde{u}^\epsilon_{vx})) \le C\|\tilde{u}^\epsilon_v\|_{H^s} \le C(1 + \|\tilde{u}^\epsilon_v\|^2_{H^s}). \tag{3.7}$$

By Hölder's and Young's inequalities,

$$\Phi'(\|\tilde{u}^\epsilon_v\|^2_{H^s})\int_{\mathbb{R}} \Lambda^s\tilde{u}^\epsilon_v\Lambda^s v_\eta\,dx \le C(\|v_\eta\|^2_{H^s} + \|\tilde{u}^\epsilon_v\|^2_{H^s}). \tag{3.8}$$

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It follows from (3.4)–(3.8) and Lemma 3.1 that

$$\mathbb{E}\Phi(\|\tilde{u}_v^\epsilon(T\wedge\tau_R)\|_{H^s}^2) \le C + C\int_0^{T\wedge\tau_R} \mathbb{E}\Phi(\|\tilde{u}_v^\epsilon\|_{H^s}^2)\,dt,$$

where $\mathbb{E}\Phi$ is the mathematical expectation of random variable Φ. Applying Gronwall's inequality [14] yields

$$\mathbb{E}\Phi(\|\tilde{u}_v^\epsilon(T\wedge\tau_R)\|_{H^s}^2) \le C.$$

Since $P(\sup_{0\le t\le T}\|\tilde{u}_v^\epsilon\|_s^2 > R)\Phi(R) \le \mathbb{E}\Phi(\|\tilde{u}_v^\epsilon(T\wedge\tau_R)\|_{H^s}^2) \le C$,

$$\ln P\Big(\sup_{0\le t\le T}\|\tilde{u}_v^\epsilon\|_s^2 > R\Big) \le \ln C - \ln(\Phi(R)), \tag{3.9}$$

and making R tend to $+\infty$ in (3.9) proves (3.2).

By multiplying both sides of regularized (3.1) by $\Lambda^s\tilde{u}_v\Lambda^s$, then by integration and estimation similar to the above,

$$\|\tilde{u}_v\|_{H^s}^2 \le \|\varphi_\eta\|_{H^s}^2 + \int_0^t \|v_\eta\|_{H^s}^2\,dr + C\int_0^t \|\tilde{u}_v\|_{H^s}^3\,dr = y(t).$$

Then $dy/dt \le Cy^{3/2}$, which yields $y(t)\le C$, and hence proves (3.3) with $k=0$. Multiplying both sides of regularized (3.1) by $\Lambda^{s+1}\tilde{u}_v\Lambda^{s+1}$ and integrating, and then using (3.3) with $k=0$, gives

$$\begin{aligned}\|\tilde{u}_v\|_{H^{s+1}}^2 &\le \|\varphi_\eta\|_{H^{s+1}}^2 + \int_0^t \|v_\eta\|_{H^{s+1}}^2\,dr + C\int_0^t \|\tilde{u}_v\|_{H^s}\|\tilde{u}_v\|_{H^{s+1}}^2\,dr\\ &\le C\eta^{-1/2} + C\int_0^t \|\tilde{u}_v\|_{H^s}^2\,dr,\end{aligned}$$

which, with Gronwall's inequality, implies (3.3). □

Remark 3.3. We cannot get the secondary moment bound of $\tilde{u}_v^\epsilon$. Fortunately, it is sufficient to prove Theorem 3.4 by the probability bound in (3.2) and the stopping time.

We formulate the Freidlin–Wentzell type estimate [8] for u_η^ϵ.

Theorem 3.4. *The solution set $\{u_\eta^\epsilon\}$ satisfies the LDP in $C([0,T];H^s)$, $s>3/2$ with a good rate function*

$$I(g) = \inf_{\{v\in L^2([0,T];H^s)\,|\,g=\mathcal{G}^0(\int_0^\cdot v_\eta(r)\,dr)\}} \frac{1}{2}\int_0^T \|v_\eta(r)\|_{H^s}^2\,dr.$$

Proof. Suppose that $\{v_\epsilon\}\subset\mathcal{A}_M$ and it converges to $v\in S_M$ in distribution. We prove that $\mathcal{G}^\epsilon(W_\eta(\cdot)+(1/\sqrt{\epsilon})\int_0^\cdot v_{\epsilon\eta}(s)\,ds)$ converges to $\mathcal{G}^0(\int_0^\cdot v_\eta(s)\,ds)$ in distribution, as $\epsilon\to 0$. Let $w_\eta = \tilde{u}_{v_\epsilon}^\epsilon - \tilde{u}_v$. Then w_η satisfies

$$\begin{cases} w_{\eta t} + h(\tilde{u}_{v_\epsilon}^\epsilon, \tilde{u}_{v_\epsilon x}^\epsilon) - h(\tilde{u}_v, \tilde{u}_{vx}) + v_{\epsilon\eta} - v_\eta = \sqrt{\epsilon}\,dW_\eta\\ w_\eta(x,0) = 0,\end{cases}$$

where $h(u,u_x)$ is given in (3.5). For $R>0$, we define a stopping time

$$\tau_R = \inf\{t\in[0,T]; \|\tilde{u}^{\epsilon}_{v_\epsilon}\|^2_{H^s} > R\}.$$

For $1/2 < q < \min\{1, s-1\}$, similar to the estimates in [5, Proposition 3.1],

$$\mathbb{E}\Big[\sup_{t\in[0,T\wedge\tau_R]} \|w_\eta\|^2_{H^q}\Big] \le C(\epsilon+1)\eta^{(s-q)/2}e^{1+\sqrt{R}}. \tag{3.10}$$

By using the estimates of Chen et al. [5, (4.48)–(4.50)], the inequality (3.10), Gronwall's inequality and Itô's formula,

$$\mathbb{E}\Big[\sup_{t\in[0,T\wedge\tau_R]} \|w_\eta\|^2_{H^s}\Big] \le C\Big\{(\epsilon+1)\eta^{(s-q-1)/2} + \int_0^{t\wedge\tau_R} \|v_{\epsilon\eta}-v_\eta\|^2_{H^s}\,dr\Big\}e^{1+\sqrt{R}},$$

from which it follows that

$$\mathbb{E}\Big[\sup_{t\in[0,T\wedge\tau_R]} \|w_\eta\|^2_{H^s}\Big] \to 0 \quad \text{as } \epsilon\to 0. \tag{3.11}$$

Given an arbitrarily small constant $\delta>0$, by Lemma 3.2, one can choose R such that $P(\tau_R\le T)\le\delta/2$. For such R and for all $\lambda>0$, by (3.11), there exists ϵ_0 such that for $\epsilon<\epsilon_0$, $P(\sup_{t\in[0,T]}\|w_\eta(t\wedge\tau_R)\|_{H^s}>\lambda)\le\delta/2$. Therefore

$$P\Big(\sup_{t\in[0,T]}\|w_\eta(t)\|_{H^s}>\lambda\Big) \le P(\tau_R\le T) + P\Big(\sup_{t\in[0,T]}\|w_\eta(t\wedge\tau_R)\|_{H^s}>\lambda\Big)\le\delta.$$

This proves that $\tilde{u}^{\epsilon}_{v_\epsilon}$ converges to $\tilde{u}_v$ in probability in $C([0,T];H^s)$.

By weak compactness of $\tilde{S}_M$, we can select a subsequence of the set $v_{\epsilon\eta}\in\tilde{S}_M$, still denoted in the same way, which converges weakly to a limit $v_\eta\in\tilde{S}_M$. Let $w_\eta=\tilde{u}_{v_\epsilon}-\tilde{u}_v$. Then w satisfies

$$\begin{cases} w_{\eta t} + h(\tilde{u}_{v_\epsilon},\tilde{u}_{v_\epsilon x}) - h(\tilde{u}_v,\tilde{u}_{vx}) + v_{\epsilon\eta} - v_\eta = 0\\ w_\eta(x,0)=0. \end{cases}$$

The rest of the proof can be obtained as (3.10)–(3.11) with $\epsilon=0$, details of which we omit here. This completes the proof of Theorem 3.4. □

The following lemma shows that the probability that the solutions u^ϵ_η, u^ϵ stay outside an energy ball is exponentially small.

LEMMA 3.5. *Let u^ϵ and u^ϵ_η be the solutions of* (1.1) *and* (1.2)*, respectively, and let $k=0,1$. Then*

$$\lim_{R\to\infty}\sup_{0<\epsilon\le1}\epsilon\ln P\Big(\sup_{0\le t\le T}\|u^\epsilon\|^2_{H^s}>R\Big) = -\infty, \tag{3.12}$$

$$\lim_{R\to\infty}\sup_{0<\epsilon\le1}\epsilon\ln P\Big(\sup_{0\le t\le T}\|u^\epsilon_\eta\|^2_{H^{s+k}}>R\eta^{-k/2}\Big) = -\infty. \tag{3.13}$$

PROOF. Define $\psi(\xi)=\int_0^{\xi}1/(1+v^2\ln v)\,dv$, $\Psi_\lambda(\xi)=e^{\lambda\psi(\xi)}$, $\xi>0$. Then

$$\Psi'(\xi)=\frac{\lambda\Psi(\xi)}{1+\xi^2\ln\xi}\quad\text{and}\quad \Psi''(\xi)=\frac{(\lambda^2+\lambda)\Psi(\xi)}{(1+\xi^2\ln\xi)^2}(1-\xi-2\xi\ln\xi).$$

Define $\tau_R=\inf\{t>0,\|u^\epsilon(t)\|_{H^s}^2\ge R\}$, $R>e$. Let $\xi(t)=\|u^\epsilon(t)\|_{H^s}^2$. Then, by Itô's formula,

$$\begin{aligned}\mathbb{E}\Psi_\lambda(\xi(t\wedge\tau_R))&\le\Psi_\lambda(\xi(0))+C\lambda\mathbb{E}\int_0^{t\wedge\tau_R}\Psi'(\xi(r))\|u^\epsilon\|_{H^s}^3\,dr\\&\quad+C(\lambda^2+\lambda)\sqrt{\epsilon}\|Q\|_{\mathcal{L}^s}^2\mathbb{E}\int_0^{t\wedge\tau_R}\Psi''(\xi(r))\,dr\\&\le C+C(\lambda^2\sqrt{\epsilon}+\lambda\sqrt{\epsilon}+\lambda)\int_0^{t\wedge\tau_R}\mathbb{E}\Psi_\lambda(\xi(t\wedge\tau_R))\,dr,\end{aligned}$$

which, by Gronwall's inequality, implies that

$$\mathbb{E}\Psi_\lambda(\xi(T\wedge\tau_R))\le Ce^{C(\lambda^2\sqrt{\epsilon}+\lambda\sqrt{\epsilon}+\lambda)}.$$

Let $\lambda=1/\epsilon$. Then

$$P\Big(\sup_{0\le t\le T}\|u^\epsilon(t)\|_{H^s}^2>R\Big)\Psi_{1/\epsilon}(R)\le\mathbb{E}\Psi_\lambda(\xi(T\wedge\tau_R))\le Ce^{C(\lambda^2\sqrt{\epsilon}+\lambda\sqrt{\epsilon}+\lambda)},$$

which yields

$$\sup_{0<\epsilon<1}\epsilon\ln P\Big(\sup_{0\le t\le T}\|u^\epsilon(t)\|_{H^s}^2>R\Big)\lesssim 1-\psi(R).\tag{3.14}$$

Note that $\lim_{R\to\infty}\psi(R)=\infty$. By letting R tend to ∞ in (3.14), we prove (3.12).

The proof of (3.13) with $k=0$ is similar to (3.12). Now we prove (3.13) with $k=1$. Define $\tau_R=\inf\{t>0,\|u_\eta^\epsilon(t)\|_{H^s}^2\ge R\}$, $R>0$. Using Itô's formula,

$$\begin{aligned}\|u_\eta^\epsilon(\tau\wedge\tau_R)\|_{H^{s+1}}^2&\le\|\varphi_\eta\|_{H^{s+1}}^2+\epsilon\|Q_\eta\|_{\mathcal{L}^{s+1}}^2+C\int_0^{t\wedge\tau_R}\|u_\eta^\epsilon\|_{H^s}\|u_\eta^\epsilon\|_{H^{s+1}}^2\,d\tau\\&\quad+2\sqrt{\epsilon}\int_0^{t\wedge\tau_R}(\Lambda^{s+1}u_\eta^\epsilon,\Lambda^{s+1}\,dW_\eta),\end{aligned}$$

and the martingale inequality [1, 7] yields

$$\begin{aligned}&4\epsilon\Big\{\mathbb{E}\Big[\sup_{0\le t\le T\wedge\tau_R}\int_0^t(\Lambda^{s+1}u_\eta^\epsilon,\Lambda^{s+1}\,dW_\eta)\Big]^q\Big\}^{2/q}\\&\quad\le Cq\epsilon\Big\{\mathbb{E}\Big[\int_0^T\sup_{0\le\tau\le t\wedge\tau_R}\|u_\eta^\epsilon(\tau)\|_{H^{s+1}}^2\|Q_\eta\|_{\mathcal{L}^{s+1}}^2\,dt\Big]^{q/2}\Big\}^{2/q}\\&\quad\le Cq\epsilon\Big\{\mathbb{E}\Big[\int_0^T\sup_{0\le\tau\le t\wedge\tau_R}\|u_\eta^\epsilon(\tau)\|_{H^{s+1}}^4+\|Q_\eta\|_{\mathcal{L}^{s+1}}^4\,dt\Big]^{q/2}\Big\}^{2/q}\\&\quad\le Cq\epsilon\Big\{\eta^{-1}+\int_0^T\Big(\mathbb{E}\Big[\sup_{0\le\tau\le t\wedge\tau_R}\|u_\eta^\epsilon(\tau)\|_{H^s}^{2q}\Big]\Big)^{2/q}\,dt\Big\}.\end{aligned}\tag{3.15}$$

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Hence

$$\Big(\mathbb{E}\Big[\sup_{0<t<T\wedge\tau_R}\|u_\eta^\epsilon(t)\|_{H^{s+1}}^{2q}\Big]\Big)^{2/q}\le C(1+\epsilon+q\epsilon)\eta^{-1}$$
$$+C(R+q\epsilon)\int_0^T\Big(\mathbb{E}\Big[\sup_{0<r<t\wedge\tau_R}\|u_\eta^\epsilon(r)\|_{H^{s+1}}^{2q}\Big]\Big)^{2/q}\,dt,$$

which implies that

$$\Big(\mathbb{E}\Big[\sup_{0<t<T\wedge\tau_R}\|u_\eta^\epsilon(t)\|_{H^{s+1}}^{2q}\Big]\Big)^{2/q}\le C(1+\epsilon+q\epsilon)\eta^{-1}e^{C(R+q\epsilon)}. \tag{3.16}$$

By (3.13), for any $M>0$, there exists a constant R such that, for any $\epsilon\in(0,1]$,

$$P\Big(\sup_{t\in[0,T]}\|u_\eta^\epsilon\|_{H^s}^2>R\Big)<e^{-M/\epsilon}. \tag{3.17}$$

For such R, take $q=2/\epsilon$. Then, by (3.16),

$$\sup_{0<\epsilon<1}\epsilon\ln P\Big(\sup_{t\in[0,T\wedge\tau_R]}\|u_\eta^\epsilon\|_{H^{s+1}}^2>R_1\eta^{-1/2}\Big)\le\sup_{0<\epsilon<1}\ln\Bigg(\frac{\big(\mathbb{E}\big[\sup_{t\in[0,T\wedge\tau_R]}\|u_\eta^\epsilon\|_{H^{s+1}}^{2q}\big]\big)^{2/q}}{R_1^2\eta^{-1}}\Bigg)$$
$$\le C(R+2)+\ln(3C)-2\ln(R_1)\to-\infty,$$

as $R_1\to\infty$. Hence there exists R_1 such that

$$P\Big(\sup_{t\in[0,T\wedge\tau_R]}\|u_\eta^\epsilon\|_{H^{s+1}}^2>R_1\eta^{-1/2}\Big)<e^{-M/\epsilon}. \tag{3.18}$$

By (3.17) and (3.18),

$$\begin{aligned}
&P\Big(\sup_{t\in[0,T]}\|u_\eta^\epsilon\|_{H^{s+1}}^2>R_1\eta^{-1/2}\Big)\\
&\quad\le P\Big(\sup_{t\in[0,T]}\|u_\eta^\epsilon\|_{H^{s+1}}^2>R_1\eta^{-1/2},\ \sup_{t\in[0,T]}\|u_\eta^\epsilon\|_{H^s}^2\le R\Big)+P\Big(\sup_{t\in[0,T]}\|u_\eta^\epsilon\|_{H^s}^2>R\Big)\\
&\quad\le P\Big(\sup_{t\in[0,T\wedge\tau_R]}\|u_\eta^\epsilon\|_{H^{s+1}}^2>R_1\eta^{-1/2}\Big)+P\Big(\sup_{t\in[0,T]}\|u_\eta^\epsilon\|_{H^s}^2>R\Big)<2e^{-M/\epsilon},
\end{aligned}$$

from which (3.13) is obtained. This completes the proof. □

Now we present the main result of the paper, as follows.

THEOREM 3.6. *The solution set* $\{u^\epsilon\}$ *satisfies the LDP in* $C([0,T];H^s)$, $s>3/2$ *with a good rate function*

$$I(g)=\inf_{\{v\in L^2([0,T];H^s)|g(t)=\mathcal{G}^0(\int_0^\cdot v(\tau)\,d\tau)\}}\frac{1}{2}\int_0^T\|v\|_{H^s}^2\,dt.$$

https://doi.org/10.1017/S1446181116000353 Published online by Cambridge University Press

PROOF. By Theorem 3.4 and [8, Theorem 4.2.13], we just need to prove that (1.3) holds. For $R > 0$, we define the stopping time

$$\tau_R^1 = \inf\{t \mid \|u_\eta^\epsilon\|_{H^{s+1}}^2 > R\eta^{-1/2} \quad \text{or} \quad \|u_\eta^\epsilon\|_{H^s}^2 + \|u^\epsilon\|_{H^s}^2 > R\}.$$

Then

$$\begin{aligned} P\Big(&\sup_{0\le t\le T} \|u_\eta^\epsilon - u^\epsilon\|_{H^s}^2 > \lambda, \sup_{0\le t\le T} (\|u_\eta^\epsilon\|_{H^s}^2 + \|u^\epsilon\|_{H^s}^2) \le R, \sup_{0\le t\le T} \|u_\eta^\epsilon\|_{H^{s+1}}^2 \le R\Big) \\ &\le P\Big(\sup_{0\le t\le \tau_R^1} \|u_\eta^\epsilon - u^\epsilon\|_{H^s}^2 > \lambda\Big). \end{aligned}$$

Let $w = u_\eta^\epsilon - u^\epsilon$. Then w satisfies the equations

$$\begin{cases} w_t + h(u_\eta^\epsilon, u_{\eta x}^\epsilon) - h(u^\epsilon, u_x^\epsilon) = \sqrt{\epsilon}\, d(W_\eta - W) \\ w_\eta(x,0) = w_{\eta 0} = \varphi_\eta - \varphi. \end{cases}$$

For $1/2 < p < \min\{s-1, 1\}$, by Itô's formula, similarly to (3.15)–(3.16),

$$\Big(\mathbb{E}\Big[\sup_{t\in[0,T\wedge\tau_R^1]} \|w\|_{H^p}^{2q}\Big]\Big)^{2/q} \le C(1+\epsilon+p\epsilon)\eta^{(s-p)/2} e^{C(R+p\epsilon)}.$$

Then, similarly to the proof of (3.13),

$$\lim_{R\to\infty} \sup_{0<\epsilon\le 1} \epsilon \ln P\Big(\sup_{0\le t\le T} \|w\|_{H^p}^2 > R\eta^{(s-p)/2}\Big) = -\infty. \tag{3.19}$$

For $R > 0$, we define the stopping time

$$\tau_R^2 = \inf\{t \mid \|w\|_{H^p}^2 > R\eta^{(s-p)/2}\}.$$

Let $\tau_R = \tau_R^1 \wedge \tau_R^2$. Similarly to (3.15), applying Itô's formula to $\|w(t\wedge\tau_R)\|_{H^s}^2$ yields

$$\begin{aligned} \Big(\mathbb{E}\Big[\sup_{0\le t\le T} \|w\|_{H^s}^{2q}\Big]\Big)^{2/q} &\le \|w_{\eta 0}\|_{H^s}^4 + C(q+\epsilon)\epsilon\|Q_\eta - Q\|_{\mathcal{L}^s}^4 + C\eta^{s-p-1} \\ &\quad + C(R+q\epsilon)\int_0^T \Big(\mathbb{E}\Big[\sup_{0\le t\le T} \|w\|_{H^s}^{2q}\Big]\Big)^{2/q} dt. \end{aligned}$$

Then Gronwall's inequality implies that

$$\Big(\mathbb{E}\Big[\sup_{0\le\tau\le T\wedge\tau_R} \|w\|_{H^s}^2\Big]^q\Big)^{2/q} \le [\|w_{\eta 0}\|_{H^s}^4 + C(q+\epsilon)\epsilon\|Q_\eta - Q\|_{\mathcal{L}^s}^4 + C\eta^{s-p-1}]e^{C(R+q\epsilon)T}. \tag{3.20}$$

By Lemma 3.5 and (3.19), for any $M > 0$, there exists R such that

$$\sup_{0<\epsilon\le 1} \epsilon \ln P\Big(\sup_{0\le t\le T} \|v\|_{H^s}^2 > R\Big) \le -M \tag{3.21}$$

with $v = u^\epsilon, u_\eta^\epsilon, u_\eta^\epsilon$ or w.

For such R, taking $q = 2/\epsilon$ in (3.20),

$$\begin{aligned} A &= \epsilon \ln P\Big(\sup_{0\le t\le T} \|w\|_{H^s}^2 > \lambda, \sup_{0\le t\le T} (\|u^\epsilon\|_{H^s}^2 + \|u_\eta^\epsilon\|_{H^s}^2) \le R, \sup_{0\le t\le T} \|u_\eta^\epsilon\|_{H^{s+1}}^2 \le R\eta^{-1/2}, \\ &\qquad \sup_{0\le t\le T} \|w\|_{H^p}^2 \le R\eta^{(s-p)/2}\Big) \\ &\le \epsilon \ln P\Big(\sup_{0\le t\le T\wedge\tau_R} \|w\|_{H^s}^2 > \lambda\Big) \\ &\le \epsilon \ln \Big(\lambda^{-q}\mathbb{E}\Big(\sup_{0\le t\le T} \|w\|_{H^s}^{2q}\Big)\Big) \\ &\le -2\ln\lambda + C(R+2) + \ln(\|w_{\eta 0}\|_{H^s}^4 + C(q+\epsilon)\epsilon\|Q_\eta - Q\|_{\mathcal{L}^s}^4 + C\eta^{s-p-1}). \end{aligned} \tag{3.22}$$

Taking $\eta = \epsilon$, it follows from (3.22) that $\lim_{\epsilon\to 0} A = -\infty$. Thus, there exists ϵ_0 such that, for any ϵ satisfying $0 < \epsilon \le \epsilon_0$,

$$A \le -M. \tag{3.23}$$

From (3.21) and (3.23), it follows that there exists a constant ϵ_0 such that, for any $\epsilon \in (0, \epsilon_0]$, $P(\sup_{0\le t\le T} \|u_\eta^\epsilon - u^\epsilon\|_{H^s}^2 > \lambda) \le 5e^{-M/\epsilon}$. Since M is arbitrary, the proof is now complete. □

4. Conclusion

Usually, the LDP of the stochastic evolution equation can be shown by a weak convergence. However, it cannot be used to get the LDP of the stochastic CH equation (1.1). In this paper, we first consider the corresponding regularized equation, then we obtain the LDP for the stochastic equation, exponentially equivalent to the corresponding laws. This opens up a new approach to getting the LDP for the stochastic shallow water equations.

Acknowledgements

This work is partially supported by China NSF Grant Nos. 11401532, 11501511, 11671359, Zhejiang Provincial NSF of China under Grant No. LQ14A010015, LQ15A010012.

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