THE NON-LINEAR EQUATIONS FOR THE GREEN FUNCTION AND CALCULATION OF THE MAGNETIC FIELD TURBULENT DIFFUSIVITIES AND α-EFFECT

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Abstract. The exact numerical solution of the simplest non-linear equation from the hierarchy of non-linear equations for the averaged Green function shows that such solution allows to calculate the diffusivity and α-effect coefficient with a good accuracy for an arbitrary spectra of turbulence for all values of the characteristic parameter. It is derived also the improved equation describing the evolution of admixture fluctuations in a turbulent medium which takes into account the non-linear equation for the averaged Green function.

1. Turbulent transport coefficients

The process of the transport of some admixture fields (number density $n(\mathbf{r},t)$, magnetic field $\mathbf{B}(\mathbf{r},t)$ etc.) in a given turbulent medium is studied. Usually one is interested in averaged values, e.g. mean number density $\langle n\rangle$ and mean magnetic field $\langle\mathbf{B}\rangle$. From the exact equations for $n = \langle n\rangle + n'$ and $\mathbf{B} = \langle\mathbf{B}\rangle + \mathbf{B}'$ one derives the equations for $\langle n\rangle$ and $\langle\mathbf{B}\rangle$:

$$\left(\partial/\partial t - (D_m + D_T)\nabla^2\right)\langle n\rangle = 0$$

$$\left(\partial/\partial t - (D_m + D_T)\nabla^2\right)\langle\mathbf{B}\rangle = \alpha_T \; rot\,\langle\mathbf{B}\rangle - \beta_T\frac{\partial}{\partial t}\; rot\,\langle\mathbf{B}\rangle \tag{1}$$

which contain the turbulent diffusivity D_T and other transport coefficients α_T and β_T describing the generation of magnetic fields by motions with helicity. D_m is the molecular diffusivity. To calculate the coefficients α_T, D_T and β_T we need the knowledge of the Green function $G(1,2) \equiv G(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)$ of the non-averaged equations $((\partial/\partial t - D_m\nabla\cdot\nabla) \equiv L_0)$:

$$L_0 n = Ln$$

and

$$L_0\mathbf{B} = \hat{L}\mathbf{B} \tag{2}$$

where $Ln = -\nabla(n\mathbf{u})$ and $\hat{L}\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{B} - \mathbf{B}\; div\;\mathbf{u}$. Here $\mathbf{u}(\mathbf{r},t)$ is the turbulent velocity considered as a known stochastic function. We assume that the medium is infinite with a homogeneous, isotropic and stationary ensemble of $\mathbf{u}(\mathbf{r},t)$. The general theory is presented in Silant'ev, 1992, resp. Dolginov and Silant'ev, 1992. It was shown there that the integral equation for $G(1,2)$ may be written in an renormalized form in such a way that the free term coincides with the averaged Green function $\langle G(1,2)\rangle \equiv g(1-2) = g(R,\tau)(\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2, \tau = t_1 - t_2)$. It was derived also the hierarchy of non-linear equations for $\langle G(1,2)\rangle$. The simplest non-linear equation has the form $(dj \equiv d\mathbf{r}_j dt_j)$:

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$$g(1-2) = G_m(1-2) + \int d3 \int d4 G_m(1-3)\left\langle L(3) g(3-4) L(4)\right\rangle g(4-2) \qquad (3)$$

The function G_m is the Green function of the operator L_0 and describes the molecular diffusion. Eq. (3) may be easily solved numerically for the case of incompressible medium. It was shown that,if one uses $g(1-2)$ instead of $G(1,2)$, then the simple formula for D_T has the error $\approx 10\%$ for turbulent spectra with $\xi \gg 1$ and this error monotonically tends to zero with decreasing ξ. Using the simple asymptotic approximation of g it may be written ($\chi = u_0 \tau_0 p/\sqrt{3}$)

$$D_T = \left(\tau_0/3\pi^2\right) \int_0^\infty dp p^4 f(p)(2+\chi)(2+\chi+\chi^2)^{-1} \qquad (4)$$

for the case of incompressible turbulence without helicity. Here $f(p,\tau) = f(p)\exp(-|\tau|/\tau_0)$ describes the two-point correlation of the velocity field

$$\langle \mathbf{u}(\mathbf{r}_1,t_1)\cdot\mathbf{u}(\mathbf{r}_2,t_2)\rangle = \pi^{-2} \int_0^\infty dp p^4 f(p,\tau)\ \exp(i\mathbf{p}\cdot\mathbf{R}). \qquad (5)$$

Formula (4) is valid for turbulence with arbitrary spectrum and has its maximum error $\approx 10\%$ for $\xi \gg 1$. If helicity is absent then D_T is the same for the number density and for the magnetic field. Helicity is described by the correlator

$$\langle \mathbf{u}(\mathbf{r}_1,t_1)\cdot rot\ \mathbf{u}(r_2,t_2)\rangle = -\pi^{-2} \int_0^\infty dp p^4 D(p,\tau)\exp(i\mathbf{p}\cdot\mathbf{R}) \qquad (6)$$

For turbulence with $\xi \gg 1$ we have

$$D_T = \frac{H_0}{2\sqrt{3}\pi^2 u_0} \int_0^\infty dp p^3 \left(\frac{2}{H_0} f(p) + \frac{1}{u_0^2 p^2} D(p)\right)$$

$$\alpha_T = \left(H_0/2\sqrt{3}\ u_0\pi^2\right) \int_0^\infty dp p^3 \left(2H_0^{-1} D(p,0) + u_0^{-2} f(p,0)\right) \qquad (7)$$

$$\beta_T = \left(1/\pi^2 u_0^2\right) \int_0^\infty dp p^2 \left(D(p,0) + f(p,0)\right)$$

where $u_0^2 = \langle \mathbf{u}^2\rangle$ and $H_0 = \langle \mathbf{u}\cdot\ rot\ \mathbf{u}\rangle$. For $\xi \ll 1$ $D_T \approx u_0^2\tau/3$, $\alpha_T \approx H_0\tau_0/3$, $\beta_T \approx H_0\tau^2/3$.
The values $D_T/\left(U_0^2\tau_0/3\right)$, $\alpha_T/(H_0\tau_0/3)$, $\beta_T/\left(H_0\tau_0^2/3\right)$ monotonically decrease with increasing ξ. For large ξ the first two of them decrease as ξ^{-1} and the third as ξ^{-2}.

2. Improvement of equations describing the evolution of admixture fluctuations

The approximate equations which describe the evolution of the two-point correlators $V(R,t_1,t_2) = \langle n(1)n(2)\rangle$ and $T(R,t_1,t_2) = \langle \mathbf{B}(1)\cdot\mathbf{B}(2)\rangle$ firstly have been described by R.C. Bourret (1962). For Fourier transforms of V and T on R these equations are:

$$(\partial/\partial t_1 + p^2 D_m)\,\tilde{V}(p,t_1,t_2) = (2\pi)^{-3}\int d\mathbf{q}\int_0^\infty dt' p^2 q^2 f(|\mathbf{p}-\mathbf{q}|, t_1 - t')$$

$$\cdot(1-\mu^2)\left[\tilde{g}(p,t_2-t')\tilde{V}(q,t_1,t') - \tilde{K}(q,t_1-t')\tilde{V}(p,t',t_2)\right] \equiv \varphi(p,t_1,t_2) \qquad (8)$$

The equation for $\tilde{T}(p,t_1,t_2)$ differs from (8) only by the additional factor $(p^2+q^2-pq\mu)/q^2$ in the first term in brackets. Here $\mathbf{p}\cdot\mathbf{q} = pq\mu$ and $\tilde{g}(p,\tau)$ is the Fourier transform of the mean Green function $\langle G(1,2)\rangle \equiv g(R,\tau)$. Formally the equation for $\tilde{g}(p,\tau)$ may be written in the form:

$$\tilde{g}(p,\tau) = \tilde{G}_m(p,\tau) - (2\pi)^{-3}\int d\mathbf{q}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \tilde{G}_m(p,t_1-t')p^2q^2(1-\mu^2)$$

$$\cdot f(|\mathbf{p}-\mathbf{q}|, t-t')\tilde{K}(q,t-t')\tilde{g}(p,t'-t^2) \qquad (9)$$

The kernel $\tilde{K}(p,\tau)$ is an infinite series of terms depending on $\tilde{g}(p,\tau)$ and correlators of the turbulent velocity. The truncations of this series gives us the hierarchy of non-linear equations mentioned above. The simplest non-linear equation (3) results if $\tilde{K}(p,\tau) = \tilde{g}(p,\tau)$. Because of $\tilde{V}(p,t_2,t_1) = \tilde{V}(p,t_1,t_2)$ and $\tilde{T}(p,t_2,t_1) = \tilde{T}(p,t_1,t_2)$ the action of $(\partial/\partial t_2 + p^2 D_m)$ gives rise to equations with its right hand sides equal to $\varphi(p,t_2,t_1)$. Usually one is interested in the evolution of $\langle n^2(\mathbf{r},t)\rangle$ and $\langle \mathbf{B}^2(\mathbf{r},t)\rangle$ described by $\tilde{V}(p,t,t)$ and $\tilde{T}(p,t,t)$.

From (2) one easily derives the exact relation

$$\frac{d}{dt}\langle n^2(\mathbf{r},t)\rangle = -2D_m(2\pi)^{-3}\int d\mathbf{p} p^2 \tilde{V}(p,t,t) \qquad (10)$$

It shows that the full intensity of the scalar field fluctuations decreases only due to molecular diffusion. Hence this relation plays to a certain extent the role of an energy conservation law. The integral term in (8) is the difference of two large terms in comparison with the molecular dissipation term. It is important that relation (10) can be derived also from (8). This needs the choice $\tilde{K}(p,\tau) = \tilde{g}(p,\tau)$, i.e. the approximate equation (8) is more accurate if one uses $\tilde{g}(p,\tau)$ as a solution of the non-linear equation (3). This consideration shows that any new possible equation for V and T must be consistent with some new equation for $\langle G(1,2)\rangle$. It seems therefore that in the case of magnetic field diffusion the choice $\tilde{K}(p,\tau) = \tilde{g}(p,\tau)$ is also more accurate then any another one.

Kasantsev (1967) studied equation (8) for short-correlated turbulence $f(p,\tau) = \tau_0 f(p)\delta(\tau)$. He proved that for this turbulence (8) is an exact equation and that the exact equation for $\langle G(1,2)\rangle$ is the usual diffusion equation (1) with the diffusivity $D_m + u_0^2\tau/3$. Further, Kasantsev et al. (1983) used (8) with $\tilde{K}(p,\tau) = \tilde{G}_m(p,\tau)$ in order to study the small-scale magnetic dynamo for an arbitrary correlator $f(p,\tau)$. As we have seen the choice $\tilde{K}(p,\tau) = \tilde{G}_m(p,\tau$ is not satisfactory and may give the results far from reality. The use of a short-correlated turbulence model is also unsatisfactory for the consideration of phenomena in small-scale space intervals (so-called inertial interval of turbulence). The analysis of the hierarchy of non-linear equations shows that in the inertial interval of turbulence the Green function has the form $\tilde{g}(p,\tau) \approx exp(-u_0 p\tau/\sqrt{3})$, i.e. it describes the averaged inertial motion of the basis gas. Kasantsev (1967) and Kasantsev et al. (1983) used the Green function $\tilde{g}(p,\tau) = exp(-(D_m + D_T)p^2\tau)$, which leads from (8) to the Schrödinger type equation for the calculation of the increments of the small-scale magnetic dynamo. If one uses the more approciate non-linear equation (3) it may be derived an improved equation which is not of Schrödinger type. It seems that a short-correlated model of turbulence gives over-estimated increments of the magnetic dynamo. Even for $t = 0$ (the beginning of the evolution) this model gives rise to an increasing derivative

$$\frac{d}{dt}\left\langle B^2(\boldsymbol{\tau},t)\right\rangle\Big|_{t=0} = -\int \frac{d\mathbf{p}}{(2\pi)^3}\tilde{T}(p,0,0)\left[2D_m p^2 - \frac{2\tau_0}{3\pi^2}\int_0^\infty dq q^6 f(q)\right] > 0 \quad (11)$$

instead of the true expression which follows directly from (8)

$$\frac{d}{dt}\left\langle B^2(\boldsymbol{\tau},t)\right\rangle\Big|_{t=0} = -\int \frac{d\mathbf{p}}{(2\pi)^3} 2D_m p^2 \tilde{T}(p,0,0) < 0 \quad (12)$$

It should be noted that the exact relation (10) is valid automatically for short-correlated models of turbulence because any kernel $\tilde{K}(p,\tau)$ and the Green function obey the relation $\tilde{K}(p,0) = 1$, $\tilde{g}(p,0) = 1$. Even this feature of a short-correlated turbulence shows its artificial character.

Our considerations result in the criticism that the problem of a small-scale magnetic dynamo needs further investigation.

References

Silant'ev, N.A.: 1992, *JETP* **101**, 1216.
Dolginov, A.Z., Silant'ev, N.A.: 1992, *Geophys. Astrophys. Fluid Dynamics* **63**, 139.
Bourret, R.C.: 1962, *Nuovo Cim.* **26**
Kasantsev, A.P.: 1967, *JETP* **63**, 1806.
Kasantsev, A.P., Ruzmaikin, A.A., Sokolov, D.D.: 1983, *JETP* **88**, 487.