Proceedings of the Edinburgh Mathematical Society (2017) **60**, 231–250
DOI:10.1017/S0013091515000541

EQUIVARIANT MAPS FROM STIEFEL BUNDLES TO VECTOR BUNDLES

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(Received 24 January 2014)

Abstract Let $E \to B$ be a fibre bundle and let $E' \to B$ be a vector bundle. Let G be a compact Lie group acting fibre preservingly and freely on both E and $E' - 0$, where 0 is the zero section of $E' \to B$. Let $f\colon E \to E'$ be a fibre-preserving G-equivariant map and let $Z_f = \{x \in E \mid f(x) = 0\}$ be the zero set of f. In this paper we give a lower bound for the cohomological dimension of the zero set Z_f when a fibre of $E \to B$ is a real Stiefel manifold with a free $\mathbb{Z}/2$-action or a complex Stiefel manifold with a free $\mathbb{S}^1$-action. This generalizes a well-known result of Dold for sphere bundles equipped with free involutions.

Keywords: cohomological dimension; continuous cohomology; equivariant map; Stiefel manifold; projective Stiefel manifold

2010 *Mathematics subject classification:* Primary 55M20; 55M35
Secondary 55R91; 55M10; 55R25

1. Introduction

There are many interesting results regarding continuous maps from spheres to Euclidean spaces. One of them is the classical Borsuk–Ulam theorem. A simple version of the theorem is the following.

Borsuk–Ulam theorem. *If $n \geqslant m$, then for every continuous antipode-preserving map $f\colon \mathbb{S}^n \to \mathbb{R}^m$ there exists a point $x \in \mathbb{S}^n$ such that $f(x) = 0$.*

This simple-looking theorem has a host of extensions and generalizations, and many interesting applications in various areas of mathematics. A comprehensive survey of several topics centred around the Borsuk–Ulam theorem up to 1985 can be seen in [**31**]. A more recent account of various generalizations and applications of the theorem is the recent monograph by Matoušek [**23**].

In this paper we aim to give a generalization of the Borsuk–Ulam theorem in the setting of fibre bundles. This line of thought was initiated by the works of Dold [**7**], Fadell [**11**], Izydorek [**15**], Jaworowski [**17**] and Nakaoka [**25**]. In [**7**], Dold proved the following generalization of the Borsuk–Ulam theorem.

Theorem 1.1. *Let B be a paracompact space. Let $E \to B$ and $E' \to B$ be vector bundles of dimensions n and m, respectively. Let $f\colon S(E) \to E'$ be a fibre-preserving map such that $f(-x) = -f(x)$ for all $x \in S(E)$, where $S(E)$ is the sphere bundle of $E \to B$. If $n > m$ and $Z_f = \{x \in S(E) \mid f(x) = 0\}$, then*

$$\operatorname{cohom.dim}(Z_f) \geqslant \operatorname{cohom.dim}(B) + (n - m - 1).$$

Taking B to be a point yields the classical Borsuk–Ulam theorem. Considering a fibre bundle as a parametrization of the fibre by the base, the above theorem is a parametrized version of the following generalization of the Borsuk–Ulam theorem due to Yang [**33**,**34**].

Theorem 1.2. *Let T be a free involution on $\mathbb{S}^n$ and let $f\colon \mathbb{S}^n \to \mathbb{R}^m$ be a continuous map. If $n \geqslant m$ and $A_f = \{x \in \mathbb{S}^n \mid f(x) = f(T(x))\}$, then*

$$\operatorname{cohom.dim}(A_f) \geqslant (n - m).$$

Dold [**7**] and Nakaoka [**25**] defined certain polynomials associated with fibre bundles with free actions of circle and cyclic groups of prime order. These polynomials were called characteristic polynomials and were used for obtaining parametrized Borsuk–Ulam-type results. Mattos and Santos [**5**] obtained parametrized Borsuk–Ulam theorems for bundles whose fibre has the mod p cohomology algebra (p odd) of a product of two spheres with any free $\mathbb{Z}/p$-action and for bundles whose fibre has the rational cohomology algebra of a product of two spheres with any free $\mathbb{S}^1$-action. Jaworowski obtained parametrized Borsuk–Ulam theorems for lens space bundles in [**19**] and for sphere bundles in [**17**,**18**,**20**]. Singh [**30**] obtained parametrized Borsuk–Ulam theorems for fibre bundles whose fibre has the mod 2 cohomology algebra of a real or a complex projective space with any free involution. In a very recent paper, Mattos *et al.* [**6**] proved results of this kind for fibre bundles whose fibre is a space of type (a, b). Here a space of type (a, b) is a certain product or wedge of spheres and projective spaces depending on the parity of the integers a and b. These spaces were introduced by Toda [**32**] and James [**16**], and have been studied extensively in the context of transformation groups [**8**,**9**,**28**,**29**].

Let k and n be integers such that $1 \leqslant k < n$. Then the real Stiefel manifold $V_k(\mathbb{R}^n)$ is the space of orthonormal k-frames in $\mathbb{R}^n$. Similarly, the complex Stiefel manifold $V_k(\mathbb{C}^n)$ is defined as the space of orthonormal k-frames in $\mathbb{C}^n$. Since $V_1(\mathbb{R}^n) = \mathbb{S}^{n-1}$ and $V_1(\mathbb{C}^n) = \mathbb{S}^{2n-1}$, we can view Stiefel manifolds as generalizations of spheres. It is worth mentioning that Stiefel manifolds have been studied extensively in the context of transformation groups. In particular, equivariant maps between Stiefel manifolds and Euclidean spaces have been investigated in [**13**,**14**,**21**,**26**] and many other works. The purpose of this paper is to prove parametrized versions of some of these results and generalize Dold's theorem to fibre bundles whose fibre is either a real Stiefel manifold with a free $\mathbb{Z}/2$-action or a complex Stiefel manifold with a free $\mathbb{S}^1$-action. For simplicity we refer to such bundles as Stiefel bundles. Unlike Dold [**7**], our Stiefel bundles are not necessarily associated with vector bundles.

For real Stiefel bundles we prove the following result.

Theorem 1.3. *Let $V_k(\mathbb{R}^n) \hookrightarrow E \to B$ be a fibre bundle with a fibre-preserving free $\mathbb{Z}/2$-action such that the induced action on each fibre is equivalent to the antipodal action. Suppose that the quotient bundle $X_k(\mathbb{R}^n) \hookrightarrow \bar{E} \to B$ admits a cohomology extension of the fibre with respect to $\mathbb{Z}/2$. Let $\mathbb{R}^m \hookrightarrow E' \to B$ be a vector bundle with a fibre-preserving $\mathbb{Z}/2$-action that is free outside the zero section. Let $f\colon E \to E'$ be a fibre-preserving $\mathbb{Z}/2$-equivariant map and let $Z_f = \{x \in E \mid f(x) = 0\}$. If $(n-k) \geqslant m$, then*

$$\operatorname{cohom.dim}(Z_f) \geqslant \operatorname{cohom.dim}(B) + (n-k-m).$$

We prove the following result for complex Stiefel bundles.

Theorem 1.4. *Let $V_k(\mathbb{C}^n) \hookrightarrow E \to B$ be a fibre bundle with a fibre-preserving free $\mathbb{S}^1$-action such that the induced action on each fibre is equivalent to the standard action. Suppose that the quotient bundle $X_k(\mathbb{C}^n) \hookrightarrow \bar{E} \to B$ admits a cohomology extension of the fibre with respect to $\mathbb{Z}/p$, where p is a prime. Let $\mathbb{R}^m \hookrightarrow E' \to B$ be an even-dimensional vector bundle with a fibre-preserving $\mathbb{S}^1$-action that is free outside the zero section, and let $f\colon E \to E'$ be a fibre-preserving $\mathbb{S}^1$-equivariant map. If $2(n-k) \geqslant m$, then*

$$\operatorname{cohom.dim}(Z_f) \geqslant \operatorname{cohom.dim}(B) + (2n-2k-m+1).$$

The notation in the theorems is explained in §§ 1 and 2. After the above introduction in § 1, we record some useful results in § 2. In § 3 we discuss free actions on Stiefel manifolds and the cohomology structure of their orbit spaces. In § 4 we construct characteristic polynomials for Stiefel bundles following Dold's technique [**7**]. We prove our main theorems in § 5. Finally, in § 6 we give a bound for the cohomological dimension of the coincidence set of a map.

2. Some preliminaries

All spaces under consideration will be paracompact Hausdorff spaces and the cohomology used will be the Čech cohomology. The Čech cohomology theory satisfies the continuity property in the sense that if a cohomology class vanishes on a closed set, then it also vanishes on a neighbourhood of that set. We refer the reader to [**10**, Chapter X] for details on Čech cohomology. Throughout the paper the cyclic group $\mathbb{Z}/p$ of order p (p any prime) will be used as a coefficient group in cohomology.

A space X is said to be of finite covering dimension if there is some integer n such that for every open covering $\mathcal{U}$ of X there is an open refinement $\mathcal{V}$ of $\mathcal{U}$ that has order at most $n+1$. The covering dimension of X, denoted by $\operatorname{cov.dim}(X)$, is defined as the smallest n for which this statement holds. The cohomological dimension, denoted $\operatorname{cohom.dim}(X, A)$, of a paracompact Hausdorff space X with respect to an abelian group A is the largest positive integer n such that $H^n(X, Y; A) \neq 0$ for some closed subspace Y of X. We refer the reader to [**24**] for basic results on dimension theory. We will also use the following well-known result of Quillen [**27**].

Theorem 2.1 (Quillen [27, Proposition A.11]). *Let G be a compact Lie group acting on a paracompact Hausdorff space X and let X/G be the orbit space. If A is any abelian group, then*

$$\text{cohom.}\dim(X/G, A) \leqslant \text{cohom.}\dim(X, A).$$

Let G be a compact Lie group acting continuously and freely on a space X. Then

$$X \to X/G$$

is a principal G-bundle. Let

$$G \hookrightarrow E_G \to B_G$$

be the universal principal G-bundle, where B_G is the classifying space of the group G. Then we can take a classifying map

$$X/G \to B_G$$

for the principal G-bundle $X \to X/G$. The group G acts diagonally on $X \times E_G$ with orbit space

$$X_G = (X \times E_G)/G.$$

The projection $X \times E_G \to E_G$ is G-equivariant and gives a fibration

$$X \hookrightarrow X_G \to B_G,$$

called the Borel fibration [**3**, Chapter IV]. The Leray–Serre spectral sequence $\{E_r^{*,*}, d_r\}$ associated with the Borel fibration converges to $H^*(X_G; \mathbb{Z}/2)$ as an algebra, with

$$E_2^{k,l} = H^k(B_G; \mathcal{H}^l(X; \mathbb{Z}/2)),$$

the cohomology of the base B_G with locally constant coefficients $\mathcal{H}^l(X; \mathbb{Z}/2)$ twisted by a canonical action of $\pi_1(B_G)$ (see [**22**]).

We recall that $B_{\mathbb{Z}/2} = \mathbb{R}P^\infty$ and

$$H^*(B_{\mathbb{Z}/2}; \mathbb{Z}/2) \cong \mathbb{Z}/2[s],$$

where s is a homogeneous element of degree 1. Similarly, $B_{\mathbb{S}^1} = \mathbb{C}P^\infty$ and, for any prime p,

$$H^*(B_{\mathbb{S}^1}; \mathbb{Z}/p) \cong \mathbb{Z}/p[t],$$

where t is a homogeneous element of degree 2.

According to Bredon [**4**, p. 372], a fibre bundle $X \hookrightarrow E \to B$ is said to admit a cohomology extension of the fibre with respect to $\mathbb{Z}/p$ if the inclusion of a typical fibre $X \hookrightarrow E$ induces surjection in the cohomology

$$H^*(E, \mathbb{Z}/p) \twoheadrightarrow H^*(X, \mathbb{Z}/p).$$

Clearly, any trivial bundle admits a cohomology extension of the fibre. Also, any projective space bundle admits a cohomology extension of the fibre (see § 4). Assuming that a fibre bundle admits a cohomology extension of the fibre, we can use the well-known Leray–Hirsch theorem [**4**, p. 372, Theorem 1.4] in the proofs of our theorems.

3. Free actions on Stiefel manifolds and their quotients

For $1 \leqslant k < n$ let $V_k(\mathbb{R}^n)$ denote the real Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$. It is a compact connected Hausdorff smooth manifold with

$$\dim(V_k(\mathbb{R}^n)) = kn - \frac{k(k+1)}{2}.$$

Similarly, the complex Stiefel manifold $V_k(\mathbb{C}^n)$ defined as the space of orthonormal k-frames in $\mathbb{C}^n$ is a compact connected Hausdorff smooth manifold with

$$\dim(V_k(\mathbb{C}^n)) = 2nk - k^2.$$

The real Stiefel manifold $V_k(\mathbb{R}^n)$ admits the antipodal involution given by

$$(x_1, \ldots, x_k) \mapsto (-x_1, \ldots, -x_k) \quad \text{for } (x_1, \ldots, x_k) \in V_k(\mathbb{R}^n).$$

The quotient space, denoted by $X_k(\mathbb{R}^n)$, is known as a real projective Stiefel manifold. Similarly, the complex Stiefel manifold $V_k(\mathbb{C}^n)$ admits the standard free $\mathbb{S}^1$-action given by

$$(\zeta, (z_1, \ldots, z_k)) \mapsto (\zeta z_1, \ldots, \zeta z_k) \quad \text{for } \zeta \in \mathbb{S}^1 \text{ and } (z_1, \ldots, z_k) \in V_k(\mathbb{C}^n).$$

The quotient of $V_k(\mathbb{C}^n)$ by this action is known as a complex projective Stiefel manifold, which we denote by $X_k(\mathbb{C}^n)$. Note that $X_1(\mathbb{R}^n) = \mathbb{R}P^{n-1}$ and $X_1(\mathbb{C}^n) = \mathbb{C}P^{n-1}$. Hence, projective Stiefel manifolds can be thought of as generalizations of projective spaces.

Projective Stiefel manifolds are important objects of study in topology and their various invariants have been investigated in detail. One of their important properties is that they classify sections of multiples of line bundles, which is useful in the immersion problem of real projective spaces into Euclidean spaces.

Let p be a prime. Let $V(v_1, \ldots, v_l)$ denote any commutative and associative algebra with unit over $\mathbb{Z}/p$ generated by the set $\{v_1, \ldots, v_l\}$ such that $v_j^2 = v_{2j}$ if $2j \leqslant l$ and $v_j^2 = 0$ otherwise, and such that the set of square-free monomials $\{v_1^{\varepsilon_1} \cdots v_l^{\varepsilon_l} \mid \varepsilon_j \in \{0, 1\}\}$ form an additive basis of $V(v_1, \ldots, v_l)$. Note that if p is an odd prime and all the generators are of odd degree, then $V(v_1, \ldots, v_l)$ is the exterior algebra over $\mathbb{Z}/p$ generated by $\{v_1, \ldots, v_l\}$. Let the notation $V(v_1, \ldots, v_{j-1}, \widehat{v_j}, v_{j+1}, \ldots, v_l)$ mean that v_j is not included in the generating set.

For integers $1 \leqslant k \leqslant n$, define

$$N(p, n, k) = \min\left\{ j \;\middle|\; n - k + 1 \leqslant j \leqslant n \text{ and } \binom{n}{j} \not\equiv 0 \mod p \right\}.$$

It is clear that $n - k + 1 \leqslant N(p, n, k) \leqslant n$ and

$$N(p, n, n) \leqslant N(p, n, n-1) \leqslant \cdots \leqslant N(p, n, 2) \leqslant N(p, n, 1) = n.$$

This number has been computed in many cases in a very recent paper by Petrović and Prvulović [**26**]. Using Lucas's formula, they proved that $N(p, n, k) = n$ for each k if and only if $n = p^r$ for some r. In a similar way, they proved that $N(p,n,k) = n-k+1$ for each k if and only if $n = p^r - 1$ for some r.

Gitler and Handel [**12**] used the Leray–Serre spectral sequence associated with the fibration

$$V_k(\mathbb{R}^n) \hookrightarrow X_k(\mathbb{R}^n) \to \mathbb{R}P^\infty$$

to compute the mod 2 cohomology algebra of $X_k(\mathbb{R}^n)$. Following a similar approach, Astey *et al*. [**1**] used the fibration

$$V_k(\mathbb{C}^n) \hookrightarrow X_k(\mathbb{C}^n) \to \mathbb{C}P^\infty$$

to compute the mod p cohomology algebra of $X_k(\mathbb{C}^n)$. Their results are recorded in the following theorems, which will be crucial for our proofs.

Theorem 3.1 (Gitler and Handel [12, Theorem 1.6]). *Let* $1 \leqslant k < n$ *and* $N = N(p,n,k)$. *Then*

$$H^*(X_k(\mathbb{R}^n);\mathbb{Z}/2) \cong \frac{\mathbb{Z}/2[u]}{\langle u^N \rangle} \otimes V(v_{n-k},\dots,v_{N-2},\widehat{v_{N-1}},v_N,\dots,v_{n-1})$$

as an algebra, where $\deg(u) = 1$ *and* $\deg(v_j) = j$.

Theorem 3.2 (Astey *et al.* [1, Theorems 1.1 and 1.2]). *Let* p *be a prime, let* $1 \leqslant k < n$ *and let* $N = N(p,n,k)$. *Then*

$$H^*(X_k(\mathbb{C}^n);\mathbb{Z}/p) \cong \frac{\mathbb{Z}/p[\boldsymbol{u}]}{\langle \boldsymbol{u}^N \rangle} \otimes V(\boldsymbol{v}_{n-k+1},\dots,\boldsymbol{v}_{N-1},\widehat{\boldsymbol{v}_N},\boldsymbol{v}_{N+1},\dots,\boldsymbol{v}_n)$$

as an algebra, where $\deg(\boldsymbol{u}) = 2$ *and* $\deg(\boldsymbol{v}_j) = 2j-1$.

4. Characteristic polynomials for bundles

Let $X \hookrightarrow E \xrightarrow{\pi} B$ be a fibre bundle with a fibre-preserving free action of a compact Lie group G such that the quotient bundle $\bar{X} \hookrightarrow \bar{E} \to B$ admits a cohomology extension of the fibre with respect to $\mathbb{Z}/p$. With this hypothesis, and following Dold [**7**], we define characteristic polynomials for the bundles under consideration.

4.1. Characteristic polynomials for $V_k(\mathbb{R}^n) \hookrightarrow E \to B$ with $\mathbb{Z}/2$-action

Let $\mathbb{Z}/2$ act freely on $V_k(\mathbb{R}^n)$ by the antipodal map

$$(x_1,\dots,x_k) \mapsto (-x_1,\dots,-x_k).$$

Then, by Theorem 3.1, $H^*(X_k(\mathbb{R}^n);\mathbb{Z}/2)$ is a free graded algebra over $\mathbb{Z}/2$ generated by the set

$$\{u^i v_{n-k}^{\varepsilon_{n-k}} \cdots v_{N-2}^{\varepsilon_{N-2}} v_N^{\varepsilon_N} \cdots v_{n-1}^{\varepsilon_{n-1}} \mid 0 \leqslant i \leqslant N-1 \text{ and } \varepsilon_j \in \{0,1\}\}$$

subject to the relations

$$
\begin{aligned}
&u^N = 0,\\
&v_j^2 = v_{2j} \quad \text{for } n-k \leqslant j \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor \text{ and } j \neq N-1,\\
&v_j^2 = 0 \qquad \text{for } \left\lceil \frac{n-1}{2} \right\rceil \leqslant j \leqslant n-1 \text{ and } j \neq N-1.
\end{aligned}
$$

Recall that $u \in H^1(X_k(\mathbb{R}^n);\mathbb{Z}/2)$ and $v_j \in H^j(X_k(\mathbb{R}^n);\mathbb{Z}/2)$. We assume that the quotient bundle $X_k(\mathbb{R}^n) \hookrightarrow \bar{E} \to B$ admits a cohomology extension of the fibre with respect to $\mathbb{Z}/2$. Therefore, by the Leray–Hirsch theorem there exist elements $a \in H^1(\bar{E})$ and $b_j \in H^j(\bar{E})$ such that the restriction to a typical fibre

$$H^*(\bar{E}) \to H^*(X_k(\mathbb{R}^n))$$

maps $a \mapsto u$ and $b_j \mapsto v_j$. Observe that the homomorphism induced by $\bar{\pi}\colon \bar{E} \to B$ makes $H^*(\bar{E})$ an $H^*(B)$-module with a basis

$$\{a^i b_{n-k}^{\varepsilon_{n-k}} \cdots b_{N-2}^{\varepsilon_{N-2}} b_N^{\varepsilon_N} \cdots b_{n-1}^{\varepsilon_{n-1}} \mid 0 \leqslant i \leqslant N-1 \text{ and } \varepsilon_j \in \{0,1\}\}.$$

For simplicity, we write

$$\varepsilon = (\varepsilon_{n-k}, \dots, \varepsilon_{N-2}, \varepsilon_N, \dots, \varepsilon_{n-1}) \quad \text{and} \quad b^\varepsilon = b_{n-k}^{\varepsilon_{n-k}} \cdots b_{N-2}^{\varepsilon_{N-2}} b_N^{\varepsilon_N} \cdots b_{n-1}^{\varepsilon_{n-1}}.$$

Then a basis for the $H^*(B)$-module $H^*(\bar{E})$ is

$$\{a^i b^\varepsilon \mid 0 \leqslant i \leqslant N-1 \text{ and } \varepsilon \in \{0,1\}^{k-1}\}. \tag{4.1}$$

Consider the elements

$$
\begin{aligned}
&a^N \in H^N(\bar{E}),\\
&b_j^2 + b_{2j} \in H^{2j}(\bar{E}) \quad \text{for } n-k \leqslant j \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor \text{ and } j \neq N-1,\\
&b_j^2 \in H^{2j}(\bar{E}) \quad \text{for } \left\lceil \frac{n-1}{2} \right\rceil \leqslant j \leqslant n-1 \text{ and } j \neq N-1.
\end{aligned}
$$

These elements can be expressed uniquely in terms of the basis (4.1). Therefore, there exist unique elements

$$w_{i,\varepsilon}^0 \in H^{d^0(i,\varepsilon)}(B) \quad \text{for } 0 \leqslant i \leqslant N-1 \text{ and } \varepsilon \in \{0,1\}^{k-1},$$

and

$$w_{i,\varepsilon}^j \in H^{d^j(i,\varepsilon)}(B) \quad \text{for } 0 \leqslant i \leqslant N-1,\ \varepsilon \in \{0,1\}^{k-1} \text{ and } n-k \leqslant j \neq N-1 \leqslant n-1,$$

https://doi.org/10.1017/S0013091515000541 Published online by Cambridge University Press

such that

$$a^N = \sum_{i,\varepsilon} w^0_{i,\varepsilon} a^i b^\varepsilon,$$

$$b_j^2 + b_{2j} = \sum_{i,\varepsilon} w^j_{i,\varepsilon} a^i b^\varepsilon \quad \text{for } n-k \leqslant j \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor \text{ and } j \neq N-1,$$

$$b_j^2 = \sum_{i,\varepsilon} w^j_{i,\varepsilon} a^i b^\varepsilon \quad \text{for } \left\lceil \frac{n-1}{2} \right\rceil \leqslant j \leqslant n-1 \text{ and } j \neq N-1.$$

It is understood that

$$\deg(w^0_{i,\varepsilon}) = d^0(i,\varepsilon) = N - \deg(a^i b^\varepsilon)$$

and

$$\deg(w^j_{i,\varepsilon}) = d^j(i,\varepsilon) = 2j - \deg(a^i b^\varepsilon) \quad \text{for} \quad n-k \leqslant j \neq N-1 \leqslant n-1.$$

Let x be an indeterminate of degree 1. For each $n-k \leqslant j \neq N-1 \leqslant n-1$ let x_j be an indeterminate of degree j. For each $\varepsilon = (\varepsilon_{n-k}, \dots, \varepsilon_{N-2}, \varepsilon_N, \dots, \varepsilon_{n-1}) \in \{0,1\}^{k-1}$, set

$$x^\varepsilon = x_{n-k}^{\varepsilon_{n-k}} \cdots x_{N-2}^{\varepsilon_{N-2}} x_N^{\varepsilon_N} \cdots x_{n-1}^{\varepsilon_{n-1}}.$$

Then the characteristic polynomials in the indeterminates

$$\{x, x_{n-k}, \dots, x_{N-2}, x_N, \dots, x_{n-1}\},$$

associated with the fibre bundle $V_k(\mathbb{R}^n) \hookrightarrow E \to B$, are defined by

$$p_0(x, x_{n-k}, \dots, x_{N-2}, x_N, \dots, x_{n-1}) = x^N + \sum_{i,\varepsilon} w^0_{i,\varepsilon} x^i x^\varepsilon,$$

$$p_j(x, x_{n-k}, \dots, x_{N-2}, x_N, \dots, x_{n-1}) = x_j^2 + x_{2j} + \sum_{i,\varepsilon} w^j_{i,\varepsilon} x^i x^\varepsilon$$

$$\text{for } n-k \leqslant j \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor \text{ and } j \neq N-1,$$

$$p_j(x, x_{n-k}, \dots, x_{N-2}, x_N, \dots, x_{n-1}) = x_j^2 + \sum_{i,\varepsilon} w^j_{i,\varepsilon} x^i x^\varepsilon$$

$$\text{for } \left\lceil \frac{n-1}{2} \right\rceil \leqslant j \leqslant n-1 \text{ and } j \neq N-1.$$

On substituting the values for the indeterminates, we obtain the homomorphism of $H^*(B)$-algebras

$$H^*(B)[x, x_{n-k}, \dots, x_{N-2}, x_N, \dots, x_{n-1}] \to H^*(\bar{E})$$

given by

$$(x, x_{n-k}, \dots, x_{N-2}, x_N, \dots, x_{n-1}) \mapsto (a, b_{n-k}, \dots, b_{N-2}, b_N, \dots, b_{n-1}).$$

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Note that the kernel is the ideal $\langle p_0, p_{n-k}, \dots, p_{N-2}, p_N, \dots, p_{n-1}\rangle$. Hence, we have the following isomorphism of $H^*(B)$-algebras:

$$\frac{H^*(B)[x, x_{n-k}, \dots, x_{N-2}, x_N, \dots, x_{n-1}]}{\langle p_0, p_{n-k}, \dots, p_{N-2}, p_N, \dots, p_{n-1}\rangle} \cong H^*(\bar{E}). \tag{4.2}$$

4.2. Characteristic polynomials for $\mathbb{R}^m \hookrightarrow E' \to B$ with $\mathbb{Z}/2$-action

We now define the characteristic polynomial associated with the vector bundle $\mathbb{R}^m \hookrightarrow E' \to B$ equipped with a fibre preserving $\mathbb{Z}/2$-action on E' that is free outside the zero section. The construction is originally due to Dold [**7**] and Nakaoka [**25**]. Let

$$\mathbb{S}^{m-1} \hookrightarrow SE' \to B$$

be the associated sphere bundle. The free $\mathbb{Z}/2$-action on SE' gives the projective space bundle

$$\mathbb{R}P^{m-1} \hookrightarrow \overline{SE'} \to B$$

and the principal $\mathbb{Z}/2$-bundle $SE' \to \overline{SE'}$. It is well known that

$$H^*(\mathbb{R}P^{m-1}; \mathbb{Z}/2) \cong \frac{\mathbb{Z}/2[s']}{\langle s'^m\rangle}.$$

Here $s' = g^*(s)$, where $s \in H^1(B_G)$ and $g\colon \mathbb{R}P^{m-1} \to B_G$ is a classifying map for the principal $\mathbb{Z}/2$-bundle $\mathbb{S}^{m-1} \to \mathbb{R}P^{m-1}$. Let $h\colon \overline{SE'} \to B_G$ be a classifying map for the principal $\mathbb{Z}/2$-bundle $SE' \to \overline{SE'}$ and let $c = h^*(s) \in H^1(\overline{SE'})$. Then the $\mathbb{Z}/2$-module homomorphism

$$\theta\colon H^*(\mathbb{R}P^{m-1}) \to H^*(\overline{SE'})$$

given by $s' \mapsto c$ is a cohomology extension of the fibre. Therefore, by the Leray–Hirsch theorem $H^*(\overline{SE'})$ is an $H^*(B)$-module with a basis

$$\{1, c, c^2, \dots, c^{m-1}\}.$$

Hence, we can write $c^m \in H^m(\overline{SE'})$ as

$$c^m = w_m + w_{m-1}c + \cdots + w_1 c^{m-1},$$

where the $w_i \in H^i(B)$ are unique elements. Now the characteristic polynomial in the indeterminate x of degree 1 associated with $\mathbb{R}^m \hookrightarrow E' \to B$ is defined as

$$p(x) = w_m + w_{m-1}x + \cdots + w_1 x^{m-1} + x^m.$$

As earlier, the evaluation map $x \mapsto c$ gives the following isomorphism of $H^*(B)$-algebras:

$$\frac{H^*(B)[x]}{\langle p(x)\rangle} \cong H^*(\overline{SE'}).$$

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4.3. Characteristic polynomials for $V_k(\mathbb{C}^n) \hookrightarrow E \to B$ with $\mathbb{S}^1$-action

Let $\mathbb{S}^1$ act freely on $V_k(\mathbb{C}^n)$ in the standard way

$$(\zeta, (z_1, \dots, z_k)) \mapsto (\zeta z_1, \dots, \zeta z_k) \quad \text{for } \zeta \in \mathbb{S}^1 \text{ and } (z_1, \dots, z_k) \in V_k(\mathbb{C}^n).$$

Then, by Theorem 3.2, $H^*(X_k(\mathbb{C}^n); \mathbb{Z}/p)$ is a free graded algebra over $\mathbb{Z}/p$ generated by the set

$$\{\boldsymbol{u}^i \boldsymbol{v}_{n-k+1}^{\varepsilon_{n-k+1}} \cdots \boldsymbol{v}_{N-1}^{\varepsilon_{N-1}} \boldsymbol{v}_{N+1}^{\varepsilon_{N+1}} \cdots \boldsymbol{v}_n^{\varepsilon_n} \mid 0 \leqslant i \leqslant N-1 \text{ and } \varepsilon_j \in \{0,1\}\}$$

subject to the following relations depending on the parity of p. For p even, the relations

$$\begin{aligned} \boldsymbol{u}^N &= 0, \\ \boldsymbol{v}_j^2 &= \boldsymbol{v}_{2j} \quad \text{for } n-k+1 \leqslant j \leqslant \lfloor \tfrac{1}{2} n \rfloor \text{ and } j \neq N, \\ \boldsymbol{v}_j^2 &= 0 \qquad \text{for } \lceil \tfrac{1}{2} n \rceil \leqslant j \leqslant n \text{ and } j \neq N \end{aligned}$$

hold, and for p odd the relations

$$\begin{aligned} \boldsymbol{u}^N &= 0, \\ \boldsymbol{v}_j^2 &= 0 \quad \text{for } n-k+1 \leqslant j \neq N \leqslant n \end{aligned}$$

hold.

We assume that the quotient bundle $X_k(\mathbb{C}^n) \hookrightarrow \bar{E} \to B$ admits a cohomology extension of the fibre with respect to $\mathbb{Z}/p$. Since $\boldsymbol{u} \in H^2(X_k(\mathbb{C}^n); \mathbb{Z}/p)$ and $\boldsymbol{v}_j \in H^{2j-1}(X_k(\mathbb{C}^n); \mathbb{Z}/p)$, by the Leray–Hirsch theorem there exist elements $\boldsymbol{a} \in H^2(\bar{E})$ and $\boldsymbol{b}_j \in H^{2j-1}(\bar{E})$ such that the restriction to a typical fibre

$$H^*(\bar{E}) \to H^*(X_k(\mathbb{C}^n))$$

maps $\boldsymbol{a} \mapsto \boldsymbol{u}$ and $\boldsymbol{b}_j \mapsto \boldsymbol{v}_j$. The homomorphism induced by $\bar{\pi}\colon \bar{E} \to B$ makes $H^*(\bar{E})$ an $H^*(B)$-module with a basis

$$\{\boldsymbol{a}^i \boldsymbol{b}_{n-k+1}^{\varepsilon_{n-k+1}} \cdots \boldsymbol{b}_{N-1}^{\varepsilon_{N-1}} \boldsymbol{b}_{N+1}^{\varepsilon_{N+1}} \cdots \boldsymbol{b}_n^{\varepsilon_n} \mid 0 \leqslant i \leqslant N-1 \text{ and } \varepsilon_j \in \{0,1\}\}.$$

We simplify the notation by setting

$$\varepsilon = (\varepsilon_{n-k+1}, \dots, \varepsilon_{N-1}, \varepsilon_{N+1}, \dots, \varepsilon_n) \quad \text{and} \quad \boldsymbol{b}^\varepsilon = \boldsymbol{b}_{n-k+1}^{\varepsilon_{n-k+1}} \cdots \boldsymbol{b}_{N-1}^{\varepsilon_{N-1}} \boldsymbol{b}_{N+1}^{\varepsilon_{N+1}} \cdots \boldsymbol{b}_n^{\varepsilon_n}.$$

Then a basis for the $H^*(B)$-module $H^*(\bar{E})$ is

$$\{\boldsymbol{a}^i \boldsymbol{b}^\varepsilon \mid 0 \leqslant i \leqslant N-1 \text{ and } \varepsilon \in \{0,1\}^{k-1}\}. \tag{4.3}$$

The elements $\boldsymbol{a}^N \in H^{2N}(\bar{E})$, $\boldsymbol{b}_j^2 + \boldsymbol{b}_{2j} \in H^{4j-2}(\bar{E})$ and $\boldsymbol{b}_j^2 \in H^{4j-2}(\bar{E})$ can be expressed uniquely in terms of the basis (4.3). Therefore, there exist unique elements

$$\boldsymbol{w}_{i,\varepsilon}^0 \in H^{\boldsymbol{d}^0(i,\varepsilon)}(B) \quad \text{for } 0 \leqslant i \leqslant N-1 \text{ and } \varepsilon \in \{0,1\}^{k-1},$$

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and

$$\boldsymbol{w}_{i,\varepsilon}^{j} \in H^{\boldsymbol{d}^{j}(i,\varepsilon)}(B) \quad \text{for } 0 \leqslant i \leqslant N-1,\ \varepsilon \in \{0,1\}^{k-1} \text{ and } n-k+1 \leqslant j \neq N \leqslant n,$$

such that for p even

$$\begin{aligned} \boldsymbol{a}^{N} &= \sum_{i,\varepsilon} \boldsymbol{w}_{i,\varepsilon}^{0} \boldsymbol{a}^{i} \boldsymbol{b}^{\varepsilon}, \\ \boldsymbol{b}_{j}^{2} + \boldsymbol{b}_{2j} &= \sum_{i,\varepsilon} \boldsymbol{w}_{i,\varepsilon}^{j} \boldsymbol{a}^{i} \boldsymbol{b}^{\varepsilon} \quad \text{for } n-k+1 \leqslant j \leqslant \lfloor \tfrac{1}{2}n \rfloor \text{ and } j \neq N, \\ \boldsymbol{b}_{j}^{2} &= \sum_{i,\varepsilon} \boldsymbol{w}_{i,\varepsilon}^{j} \boldsymbol{a}^{i} \boldsymbol{b}^{\varepsilon} \quad \text{for } \lceil \tfrac{1}{2}n \rceil \leqslant j \leqslant n \text{ and } j \neq N, \end{aligned}$$

and for p odd

$$\begin{aligned} \boldsymbol{a}^{N} &= \sum_{i,\varepsilon} \boldsymbol{w}_{i,\varepsilon}^{0} \boldsymbol{a}^{i} \boldsymbol{b}^{\varepsilon}, \\ \boldsymbol{b}_{j}^{2} &= \sum_{i,\varepsilon} \boldsymbol{w}_{i,\varepsilon}^{j} \boldsymbol{a}^{i} \boldsymbol{b}^{\varepsilon} \quad \text{for } n-k+1 \leqslant j \neq N \leqslant n. \end{aligned}$$

Let y be an indeterminate of degree 2. For each $n-k+1 \leqslant j \neq N \leqslant n$, let y_j be an indeterminate of degree $2j-1$. And for each $\varepsilon = (\varepsilon_{n-k+1}, \dots, \varepsilon_{N-1}, \varepsilon_{N+1}, \dots, \varepsilon_n) \in \{0,1\}^{k-1}$ set

$$y^{\varepsilon} = y_{n-k+1}^{\varepsilon_{n-k+1}} \cdots y_{N-1}^{\varepsilon_{N-1}} y_{N+1}^{\varepsilon_{N+1}} \cdots y_{n}^{\varepsilon_{n}}.$$

Then the characteristic polynomials in the indeterminates

$$\{y, y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n\},$$

associated with the fibre bundle $V_k(\mathbb{C}^n) \hookrightarrow E \to B$, for p even are defined by

$$\begin{aligned} \boldsymbol{p}_0(y, y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n) &= y^{N} - \sum_{i,\varepsilon} \boldsymbol{w}_{i,\varepsilon}^{0} y^{i} y^{\varepsilon}, \\ \boldsymbol{p}_j(y, y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n) &= y_j^{2} + y_{2j} - \sum_{i,\varepsilon} \boldsymbol{w}_{i,\varepsilon}^{j} y^{i} y^{\varepsilon} \\ &\qquad \text{for } n-k+1 \leqslant j \leqslant \lfloor \tfrac{1}{2}n \rfloor \text{ and } j \neq N, \\ \boldsymbol{p}_j(y, y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n) &= y_j^{2} - \sum_{i,\varepsilon} \boldsymbol{w}_{i,\varepsilon}^{j} y^{i} y^{\varepsilon} \\ &\qquad \text{for } \lceil \tfrac{1}{2}n \rceil \leqslant j \leqslant n \text{ and } j \neq N, \end{aligned}$$

and for p odd are defined by

$$\begin{aligned} \boldsymbol{p}_0(y, y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n) &= y^{N} - \sum_{i,\varepsilon} \boldsymbol{w}_{i,\varepsilon}^{0} y^{i} y^{\varepsilon}, \\ \boldsymbol{p}_j(y, y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n) &= y_j^{2} - \sum_{i,\varepsilon} \boldsymbol{w}_{i,\varepsilon}^{j} y^{i} y^{\varepsilon} \\ &\qquad \text{for } n-k+1 \leqslant j \neq N \leqslant n. \end{aligned}$$

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The evaluation map

$$(y, y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n) \mapsto (\boldsymbol{a}, \boldsymbol{b}_{n-k+1}, \dots, \boldsymbol{b}_{N-1}, \boldsymbol{b}_{N+1}, \dots, \boldsymbol{b}_n)$$

defines a homomorphism of $H^*(B)$-algebras

$$H^*(B)[y, y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n] \to H^*(\bar{E}),$$

whose kernel is $\langle \boldsymbol{p}_0, \boldsymbol{p}_{n-k+1}, \dots, \boldsymbol{p}_{N-1}, \boldsymbol{p}_{N+1}, \dots, \boldsymbol{p}_n \rangle$. Hence, we have the following isomorphism of $H^*(B)$-algebras:

$$\frac{H^*(B)[y, y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n]}{\langle \boldsymbol{p}_0, \boldsymbol{p}_{n-k+1}, \dots, \boldsymbol{p}_{N-1}, \boldsymbol{p}_{N+1}, \dots, \boldsymbol{p}_n \rangle} \cong H^*(\bar{E}). \tag{4.4}$$

4.4. Characteristic polynomials for $\mathbb{R}^m \hookrightarrow E' \to B$ with $\mathbb{S}^1$-action

Just as in the real case, we can define the characteristic polynomial associated with the vector bundle $\mathbb{R}^m \hookrightarrow E' \to B$ equipped with a fibre-preserving $\mathbb{S}^1$-action that is free outside the zero section. Recall that m is even here. Let

$$\mathbb{S}^{m-1} \hookrightarrow SE' \to B$$

be the associated sphere bundle and let

$$\mathbb{C}P^{(m-2)/2} \hookrightarrow \overline{SE'} \to B$$

be the projective space bundle obtained by free $\mathbb{S}^1$-action. We also obtain the principal bundle

$$\mathbb{S}^1 \hookrightarrow SE' \to \overline{SE'}.$$

It is known that

$$H^*(\mathbb{C}P^{(m-2)/2}; \mathbb{Z}/p) \cong \frac{\mathbb{Z}/p[t']}{\langle t'^{m/2} \rangle}.$$

Here $t' = g^*(t)$, where $t \in H^2(B_G)$ and $g\colon \mathbb{C}P^{(m-2)/2} \to B_G$ is a classifying map for the principal bundle $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{m-1} \to \mathbb{C}P^{(m-2)/2}$. Let $h\colon \overline{SE'} \to B_G$ be a classifying map for the principal bundle $\mathbb{S}^1 \hookrightarrow SE' \to \overline{SE'}$, and let $\boldsymbol{c} = h^*(t) \in H^2(\overline{SE'})$. Then the $\mathbb{Z}/p$-module homomorphism

$$\theta\colon H^*(\mathbb{C}P^{(m-2)/2}) \to H^*(\overline{SE'})$$

given by $t' \mapsto \boldsymbol{c}$ is a cohomology extension of the fibre. Therefore, $H^*(\overline{SE'})$ is an $H^*(B)$-module with a basis

$$\{1, \boldsymbol{c}, \boldsymbol{c}^2, \dots, \boldsymbol{c}^{(m-2)/2}\},$$

and we can write $\boldsymbol{c}^{m/2} \in H^m(\overline{SE'})$ as

$$\boldsymbol{c}^{m/2} = \boldsymbol{w}_m + \boldsymbol{w}_{m-2}\boldsymbol{c} + \cdots + \boldsymbol{w}_2 \boldsymbol{c}^{(m-2)/2},$$

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where the $\boldsymbol{w}_i \in H^i(B)$ are unique elements. Now the characteristic polynomial in the indeterminate y of degree 2 associated with $\mathbb{R}^m \hookrightarrow E' \to B$ is defined as

$$\boldsymbol{p}(y) = \boldsymbol{w}_m + \boldsymbol{w}_{m-2}y + \cdots + \boldsymbol{w}_2 y^{(m-2)/2} + y^{m/2}.$$

The evaluation map $y \mapsto \boldsymbol{c}$ gives the isomorphism of $H^*(B)$-algebras

$$\frac{H^*(B)[y]}{\langle \boldsymbol{p}(y)\rangle} \cong H^*(\overline{SE'}).$$

5. Proofs of theorems

Let $X \hookrightarrow E \to B$ be a fibre bundle with a fibre-preserving free action by a compact Lie group G such that the quotient bundle $\bar{X} \hookrightarrow \bar{E} \to B$ admits a cohomology extension of the fibre. Let $\mathbb{R}^m \hookrightarrow E' \to B$ be a vector bundle with a fibre-preserving G-action on E' that is free outside the zero section. For a fibre-preserving G-equivariant map

$$f\colon E \to E'$$

define the zero set of f as

$$Z_f = \{x \in E \mid f(x) = 0\}.$$

Since the set Z_f is G-invariant, we denote by

$$\bar{Z}_f = Z_f/G$$

the quotient of Z_f by G. For brevity, let $\mathcal{X}$ denote a collection of indeterminates, and let $\mathcal{P}(\mathcal{X})$ denote a collection of polynomials on $\mathcal{X}$. In § 4 we defined characteristic polynomials $\mathcal{P}(\mathcal{X})$ in $H^*(B)[\mathcal{X}]$ associated with certain fibre bundles, and showed that

$$\frac{H^*(B)[\mathcal{X}]}{\langle \mathcal{P}(\mathcal{X})\rangle} \cong H^*(\bar{E})$$

as $H^*(B)$-algebras. Therefore, each polynomial $q(\mathcal{X})$ in $H^*(B)[\mathcal{X}]$ defines an element of $H^*(\bar{E})$, which we denote by $q(\mathcal{X})|_{\bar{E}}$. Let $q(\mathcal{X})|_{\bar{Z}_f}$ denote the image of $q(\mathcal{X})|_{\bar{E}}$ under the homomorphism

$$H^*(\bar{E}) \to H^*(\bar{Z}_f)$$

induced by the inclusion $\bar{Z}_f \hookrightarrow \bar{E}$. Next we prove our theorems for real and complex Stiefel bundles.

5.1. The real case

We set

$$\mathcal{X} = \{x, x_{n-k}, \dots, x_{N-2}, x_N, \dots, x_{n-1}\} \quad \text{and} \quad J = \{0, n-k, \dots, N-2, N, \dots, n-1\}.$$

With this notation, following Dold [**7**], we prove the following results for real Stiefel bundles.

Theorem 5.1. *Let $V_k(\mathbb{R}^n) \hookrightarrow E \to B$ be a fibre bundle with a fibre-preserving free $\mathbb{Z}/2$-action such that the induced action on each fibre is the antipodal action. Suppose that the quotient bundle $X_k(\mathbb{R}^n) \hookrightarrow \bar{E} \to B$ admits a cohomology extension of the fibre with respect to $\mathbb{Z}/2$. Let $\mathbb{R}^m \hookrightarrow E' \to B$ be a vector bundle with a fibre-preserving $\mathbb{Z}/2$-action that is free outside the zero section and let $f\colon E \to E'$ be a fibre-preserving $\mathbb{Z}/2$-equivariant map. If $q(\mathcal{X})$ in $H^*(B)[\mathcal{X}]$ is a polynomial such that $q(\mathcal{X})|_{\bar{Z}_f} = 0$, then there exist polynomials $\{r_j(\mathcal{X})\}_{j\in J}$ in $H^*(B)[\mathcal{X}]$ such that*

$$q(\mathcal{X})p(x) = \sum_{j\in J} r_j(\mathcal{X})p_j(\mathcal{X}).$$

Proof. Let $q(\mathcal{X})$ in $H^*(B)[\mathcal{X}]$ be such that $q(\mathcal{X})|_{\bar{Z}_f} = 0$. By the continuity property of the Čech cohomology theory, there exists an open subset V of $\bar{E}$ such that $\bar{Z}_f \subset V$ and $q(\mathcal{X})|_V = 0$. Let

$$j_1\colon \bar{E} \hookrightarrow (\bar{E}, V)$$

be the natural inclusion. Then we have the following long exact cohomology sequence for the pair $(\bar{E}, V)$,

$$\cdots \to H^*(\bar{E}, V) \xrightarrow{j_1^*} H^*(\bar{E}) \to H^*(V) \to H^*(\bar{E}, V) \to \cdots.$$

Since $q(\mathcal{X})|_V = 0$, there exists an element $\mu \in H^*(\bar{E}, V)$ such that $j_1^*(\mu) = q(\mathcal{X})|_{\bar{E}}$. Let

$$\bar{f}\colon \bar{E} - \bar{Z}_f \to \overline{E'} - 0$$

be the map induced by f on passing to the quotient. Then the induced map in cohomology

$$\bar{f}^*\colon H^*(\overline{E'} - 0) \to H^*(\bar{E} - \bar{Z}_f)$$

is an $H^*(B)$-algebra homomorphism. Recall that $p(c) = 0$. Therefore,

$$p(x)|_{\bar{E}-\bar{Z}_f} = p(a) = p(\bar{f}^*(c)) = \bar{f}^*(p(c)) = 0.$$

Now consider the pair $(\bar{E}, \bar{E} - \bar{Z}_f)$. Let

$$j_2\colon \bar{E} \hookrightarrow (\bar{E}, \bar{E} - \bar{Z}_f)$$

be the natural inclusion. Then we have the following long exact cohomology sequence

$$\cdots \to H^*(\bar{E}, \bar{E} - \bar{Z}_f) \xrightarrow{j_2^*} H^*(\bar{E}) \to H^*(\bar{E} - \bar{Z}_f) \to H^*(\bar{E}, \bar{E} - \bar{Z}_f) \to \cdots.$$

Since $p(x)|_{\bar{E}-\bar{Z}_f} = 0$, there exists an element $\eta \in H^*(\bar{E}, \bar{E} - \overline{Z_f})$ such that $j_2^*(\eta) = p(x)|_{\bar{E}}$. Now, by naturality of the cup product, we get

$$q(\mathcal{X})p(x)|_{\bar{E}} = j_1^*(\mu)j_2^*(\eta) = j^*(\mu\eta).$$

Observe that

$$\mu\eta \in H^*(\bar{E}, V \cup (\bar{E} - \bar{Z}_f)) = H^*(\bar{E}, \bar{E}) = 0,$$

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and hence $q(\mathcal{X})p(x)|_{\bar{E}} = 0$. Therefore, by (4.2), there exist polynomials

$$r_0(\mathcal{X}), r_{n-k}(\mathcal{X}), \dots, r_{N-2}(\mathcal{X}), r_N(\mathcal{X}), \dots, r_{n-1}(\mathcal{X})$$

in $H^*(B)[\mathcal{X}]$ such that

$$q(\mathcal{X})p(x) = \sum_{j\in J} r_j(\mathcal{X})p_j(\mathcal{X}).$$

□

As a consequence of the above theorem, we obtain the announced parametrized Borsuk–Ulam theorem (Theorem 1.3) for real Stiefel bundles with free $\mathbb{Z}/2$-action.

Proof of Theorem 1.3. Let $n-k \geqslant m$. We first show that the $H^*(B)$-algebra homomorphism

$$\bigoplus_{i=0}^{n-k-m} H^*(B)x^i \to H^*(\bar{Z}_f)$$

given by $x^i \to x^i|_{\bar{Z}_f}$ is a monomorphism. Let $q(x)$ in $H^*(B)[\mathcal{X}]$ be a non-zero polynomial such that $\deg(q(x)) \leqslant n-k-m$. If $q(x)|_{\bar{Z}_f} = 0$, then by Theorem 5.1 we have

$$q(x)p(x) = \sum_{j\in J} r_j(\mathcal{X})p_j(\mathcal{X}).$$

Note that $\deg(p(x)) = m$, $\deg(p_0(\mathcal{X})) = N$ and $\deg(p_j(\mathcal{X})) = 2j$ for each $n-k \leqslant j \neq N-1 \leqslant n-1$. Since

$$\deg(q(x)) + m = \max_{j\in J}\{\deg(r_j(\mathcal{X})) + \deg(p_j(\mathcal{X}))\},$$

we get

$$\deg(q(x)) + m \geqslant \deg(r_0(\mathcal{X})) + \deg(p_0(\mathcal{X})) \geqslant \deg(p_0(\mathcal{X})) = N.$$

This implies that $\deg(q(x)) \geqslant N-m \geqslant n-k-m+1$, which is a contradiction. Therefore, $q(x)|_{\bar{Z}_f} \neq 0$ and

$$\bigoplus_{i=0}^{n-k-m} H^*(B)x^i \to H^*(\bar{Z}_f)$$

is a monomorphism. This together with Theorem 2.1 gives

$$\operatorname{cohom.dim}(Z_f) \geqslant \operatorname{cohom.dim}(B) + (n-k-m).$$

This proves the theorem. □

Note that Dold [**7**] considered sphere bundles associated with vector bundles. Taking $k = 1$ in Theorem 1.3 gives a result for arbitrary sphere bundles. In particular, we obtain Dold's theorem (Theorem 1.1).

Taking B to be a point yields an extension of a result of Komiya [**21**, Corollary 5.7] and Hara [**13**, Corollary 4.4].

Corollary 5.2. *Let $\mathbb{Z}/2$ act antipodally on both $V_k(\mathbb{R}^n)$ and $\mathbb{R}^m$ and let $f\colon V_k(\mathbb{R}^n) \to \mathbb{R}^m$ be a $\mathbb{Z}/2$-equivariant map. If $(n-k) \geqslant m$, then*

$$\operatorname{cohom.dim}(Z_f) \geqslant (n-k-m).$$

Remark 5.3. It is worth mentioning that the bound in Theorem 1.3 can be improved for $k > 1$ by taking the polynomial $r_j(\mathcal{X})$ of maximal degree.

5.2. The complex case

Setting

$$\mathcal{X} = \{y, y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n\} \quad \text{and} \quad J = \{0, n-k+1, \dots, N-1, N+1, \dots, n\},$$

and using similar hypotheses and notation to the real case, we prove the following results for complex Stiefel bundles.

Theorem 5.4. *Let $V_k(\mathbb{C}^n) \hookrightarrow E \to B$ be a fibre bundle with a fibre-preserving free $\mathbb{S}^1$-action such that the induced action on each fibre is the standard action. Suppose that the quotient bundle $X_k(\mathbb{C}^n) \hookrightarrow \bar{E} \to B$ admits a cohomology extension of the fibre. Let $\mathbb{R}^m \hookrightarrow E' \to B$ be an even-dimensional vector bundle with a fibre-preserving $\mathbb{S}^1$-action that is free outside the zero section, and let $f\colon E \to E'$ be a fibre-preserving $\mathbb{S}^1$-equivariant map. If $q(y)$ in $H^*(B)[\mathcal{X}]$ is a polynomial such that $q(y)|_{\bar{Z}_f} = 0$, then there exist polynomials $\{\boldsymbol{r}_j(\mathcal{X})\}_{j\in J}$ in $H^*(B)[\mathcal{X}]$ such that*

$$q(y)\boldsymbol{p}(y) = \sum_{j\in J} \boldsymbol{r}_j(\mathcal{X})\boldsymbol{p}_j(\mathcal{X}).$$

Proof. The proof is similar to that of Theorem 5.1, and hence left to the reader. □

As a consequence, we obtain the parametrized Borsuk–Ulam theorem for complex Stiefel bundles (Theorem 1.4).

Proof of Theorem 1.4. Let $q(y) \in H^*(B)[\mathcal{X}]$ be a non-zero polynomial such that $\deg(q(y)) \leqslant (2n-2k-m+1)$. Suppose that $q(y)|_{\bar{Z}_f} = 0$. Then, by Theorem 5.4, there exist polynomials $\{\boldsymbol{r}_j(\mathcal{X})\}_{j\in J}$ in $H^*(B)[\mathcal{X}]$ such that

$$q(y)\boldsymbol{p}(y) = \sum_{j\in J} \boldsymbol{r}_j(\mathcal{X})\boldsymbol{p}_j(\mathcal{X}).$$

Note that $\deg(\boldsymbol{p}(y)) = m$, $\deg(\boldsymbol{p}_0(\mathcal{X})) = 2N$ and $\deg(\boldsymbol{p}_j(\mathcal{X})) = 4j-2$ for each $n-k+1 \leqslant j \neq N \leqslant n$. Now

$$\deg(q(y)) + m = \max_{j\in J}\{\deg(\boldsymbol{r}_j(\mathcal{X})) + \deg(\boldsymbol{p}_j(\mathcal{X}))\}$$

implies that

$$\deg(q(y)) + m \geqslant \deg(\boldsymbol{r}_0(\mathcal{X})) + \deg(\boldsymbol{p}_0(\mathcal{X})) \geqslant \deg(\boldsymbol{p}_0(\mathcal{X})) = 2N.$$

This furthermore implies that $\deg(q(y)) \geqslant 2N - m \geqslant 2n - 2k - m + 2$, which is a contradiction. Therefore, $q(y)|_{\bar{Z}_f} \neq 0$ and the homomorphism

$$\bigoplus_{i=0}^{(2n-2k-m+1)/2} H^*(B)y^i \to H^*(\bar{Z}_f)$$

is a monomorphism. This together with Theorem 2.1 gives the bound

$$\text{cohom.}\dim(Z_f) \geqslant \text{cohom.}\dim(B) + (2n - 2k - m + 1).$$

This proves the theorem. □

Taking B to be a point yields the following extension of a result of Hara [**13**, Corollary 4.10].

Corollary 5.5. *Let $\mathbb{S}^1$ act in the standard way on both $V_k(\mathbb{C}^n)$ and $\mathbb{C}^m$, and let $f\colon V_k(\mathbb{C}^n) \to \mathbb{C}^m$ be an $\mathbb{S}^1$-equivariant map. If $(n - k) \geqslant m$, then*

$$\text{cohom.}\dim(Z_f) \geqslant (2n - 2k - m + 1).$$

Remark 5.6. As in the real case, the bound in Theorem 1.4 can be improved for $k > 1$ by choosing the polynomial $\boldsymbol{r}_j(\mathcal{X})$ of maximal degree.

6. Cohomological dimension of coincidence point set

Let $V_k(\mathbb{R}^n) \hookrightarrow E \to B$ be a fibre bundle with a fibre-preserving free $\mathbb{Z}/2$-action such that the induced action on each fibre is the antipodal action. Suppose that the quotient bundle $X_k(\mathbb{R}^n) \hookrightarrow \bar{E} \to B$ admits a cohomology extension of the fibre. Let $E' \to B$ be an m-dimensional vector bundle and let $f\colon E \to E'$ be a fibre-preserving map. Here we do not assume that E' has an involution. If $T\colon E \to E$ is a generator of the $\mathbb{Z}/2$-action, then the $\mathbb{Z}/2$-coincidence set of f is defined as

$$A_f = \{x \in E \mid f(x) = f(T(x))\}.$$

Notice that $\mathbb{Z}/2$ act on $V = E' \oplus E'$ by permuting the coordinates, and the m-dimensional diagonal sub-bundle D of V is the fixed-point set of this action. Furthermore, the orthogonal complement $D^\perp$ of D is also an m-dimensional subbundle of V, and the induced action on $D^\perp$ is free outside the zero section. Consider the $\mathbb{Z}/2$-equivariant map $F\colon E \to V$ given by

$$F(x) = (f(x), f(T(x))).$$

The linear projection along the diagonal defines a $\mathbb{Z}/2$-equivariant fibre-preserving map

$$g\colon V \to D^\perp$$

such that $g(V - D) \subset D^\perp - 0$. Let h be the composition

$$(E, E - A_f) \xrightarrow{F} (V, V - D) \xrightarrow{g} (D^\perp, D^\perp - 0).$$

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Then $h\colon E \to D^{\perp}$ is a fibre-preserving $\mathbb{Z}/2$-equivariant map and

$$Z_h = h^{-1}(0) = F^{-1}(D) = A_f.$$

Now applying Theorem 1.3 to h gives the following theorem.

Theorem 6.1. *Let $V_k(\mathbb{R}^n) \hookrightarrow E \to B$ be a fibre bundle with a fibre-preserving free $\mathbb{Z}/2$-action such that the induced action on each fibre is the antipodal action. Suppose that the quotient bundle $X_k(\mathbb{R}^n) \hookrightarrow \bar{E} \to B$ admits a cohomology extension of the fibre. Let $E' \to B$ be an m-dimensional vector bundle and let $f\colon E \to E'$ be a fibre-preserving map. If $(n-k) \geqslant m$, then*

$$\operatorname{cohom.dim}(A_f) \geqslant \operatorname{cohom.dim}(B) + (n-k-m).$$

We conclude by giving a bound for the covering dimension of the coincidence set. Let X be a space with a $G = \mathbb{Z}/2$ action. Consider the Leray–Serre spectral sequence $\{E_r^{*,*}, d_r\}$ associated with the Borel fibration $X \hookrightarrow X_G \to B_G$. In [**35**], Volovikov defined a numerical index of the involution, denoted by $i(X)$, to be the integer s such that

$$E_2^{*,0} = \cdots = E_s^{*,0} \neq E_{s+1}^{*,0}.$$

The index $i(X)$ is defined to be ∞ if

$$E_2^{*,0} = \cdots = E_\infty^{*,0}.$$

In [**33**] Yang also defined a numerical index for a space X with a free involution, which we denote by $\operatorname{Yang.index}(X)$. Among other things, Yang proved the following important result.

Theorem 6.2 (Yang [33, Theorem 4.1]). *Let $T\colon X \to X$ be a free involution and let $f\colon X \to \mathbb{R}^m$ be a continuous map. If* $\operatorname{Yang.index}(X) \geqslant m$, *then*

$$\operatorname{cov.dim}(A_f) \geqslant \operatorname{Yang.index}(X) - m.$$

Volovikov [**35**] observed that for a space X with a free $\mathbb{Z}/2$-action the two invariants are related as follows:

$$i(X) = \operatorname{Yang.index}(X) + 1.$$

Using this, we prove the following result.

Theorem 6.3. *Let $\mathbb{Z}/2$ act antipodally on $V_k(\mathbb{R}^n)$ and let $f\colon V_k(\mathbb{R}^n) \to \mathbb{R}^m$ be a continuous map. If $(N-1) \geqslant m$, then*

$$\operatorname{cov.dim}(A_f) \geqslant (n-k-m).$$

Proof. It was proved by Borel [**2**] that

$$H^*(V_k(\mathbb{R}^n); \mathbb{Z}/2) \cong V(v_{n-k}, \ldots, v_{n-1}),$$

https://doi.org/10.1017/S0013091515000541 Published online by Cambridge University Press

where $\deg(v_j) = j$. Now consider the Leray–Serre spectral sequence associated with the Borel fibration

$$V_k(\mathbb{R}^n) \hookrightarrow V_k(\mathbb{R}^n)_{\mathbb{Z}/2} \to \mathbb{R}P^\infty.$$

It was proved by Gitler and Handel [**12**, Theorem 1.6] that the first element of $H^*(V_k(\mathbb{R}^n); \mathbb{Z}/2)$ that does not survive to E_∞ is v_{N-1}. As a consequence,

$$E_2^{*,0} = \cdots = E_N^{*,0} \neq E_{N+1}^{*,0}.$$

Hence, $i(V_k(\mathbb{R}^n)) = N$ and $\text{Yang.index}(V_k(\mathbb{R}^n)) = N - 1$. If $(N - 1) \geqslant m$, then Yang's theorem gives the desired bound $\text{cov.dim}(A_f) \geqslant (n - k - m)$. □

Acknowledgements. The author thanks the anonymous referee for many comments that improved the presentation of the paper. This work started when the author visited the International Centre for Theoretical Physics (ICTP) during March to May of 2013. The author is grateful to the ICTP for providing financial support and excellent facilities. The author also acknowledges support from the DST INSPIRE Faculty Scheme (Grant IFA-11MA-01/2011) and the DST Fast Track Scheme (Grants SR/FTP/MS-027/2010 and DST/INT/RFBR/P-137).

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