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SOME COMPUTATIONS OF 1-COHOMOLOGY GROUPS AND CONSTRUCTION OF NON-ORBIT-EQUIVALENT ACTIONS

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Abstract For each group G having an infinite normal subgroup with the relative property (T) (e.g. $G = H \times K$, with H infinite with property (T) and K arbitrary) and each countable abelian group Λ we construct free ergodic measure-preserving actions σ_Λ of G on the probability space such that the first cohomology group of σ_Λ, $H^1(\sigma_\Lambda, G)$, is equal to $\mathrm{Char}(G) \times \Lambda$. We deduce that G has uncountably many non-stably orbit-equivalent actions. We also calculate 1-cohomology groups and show existence of 'many' non-stably orbit-equivalent actions for free products of groups as above.

Keywords: Property (T) groups; actions; cocycles

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0. Introduction

Let G be a countable discrete group and $\sigma : G \to \mathrm{Aut}(X, \mu)$ a free measure preserving (m.p.) action of G on the probability space (X, μ), which we also view as an integral preserving action of G on the abelian von Neumann algebra $A = L^\infty(X, \mu)$. A 1-*cocycle* for (σ, G) is a map $w : G \to \mathcal{U}(A)$, satisfying $w_g \sigma_g(w_h) = w_{gh}$, $\forall g, h \in G$, where $\mathcal{U}(A) = \{u \in A \mid uu^* = 1\}$ denotes the group of unitary elements in A. The set of 1-cocycles for σ is denoted $Z^1(\sigma, G)$ and is endowed with the Polish group structure given by point multiplication and pointwise convergence in the norm $\|\cdot\|_2$. The 1-*cohomology group* of σ, $H^1(\sigma, G)$, is the quotient of $Z^1(\sigma, G)$ by the subgroup of coboundaries $B^1(\sigma, G) = \{\sigma_g(u)u^* \mid u \in \mathcal{U}(A)\}$.

The group $H^1(\sigma, G)$ was first mentioned by Singer [**Si55**], related to his study of automorphisms of group measure space von Neumann algebras [**MvN43**]. Feldman and Moore extended the definition to countable, measurable equivalence relations and pointed out that $H^1(\sigma, G)$ depends only on the orbit-equivalence (OE) class of (σ, G) [**FM77-1**, **FM77-2**], thus being an OE invariant for actions. Schmidt showed in [**S80**, **S81**] that $H^1(\sigma, G)$ is Polish (i.e. $B^1(\sigma, G)$ closed in $Z^1(\sigma, G)$) if and only if σ is strongly ergodic, i.e. has no non-trivial asymptotically invariant sequences, and noticed that Bernoulli actions of non-amenable groups are always strongly ergodic (so their H^1 group is Polish). On the other hand, by [**D63**, **OW80**, **CFW81**] all free ergodic m.p. actions of infinite amenable groups are OE and non-strongly ergodic, thus having all the same ('wild') H^1-group. Results of Connes and Weiss in [**CW80**] and [**S81**] show that, for a fixed G, $H^1(\sigma, G)$ is countable discrete $\forall\sigma$ if and only if G has the property (T) of Kazhdan.

Moore produced the first examples of free ergodic m.p. actions of infinite groups with trivial 1-cohomology (see [**M82**, p. 220]; note that the groups in these examples have the property (T) of Kazhdan). Then in [**Ge87**] Gefter showed that if a Kazhdan group G can be densely embedded into a compact simply connected semi-simple Lie group $\mathcal{G}$ and $K \subset \mathcal{G}$ is a closed subgroup, then the action of G by left translation on $\mathcal{G}/K$ has H^1-group equal to $\mathrm{Char}(G \times K)$ (the character group of $G \times K$). But these initial calculations were not followed up upon and, in fact, after an intense activity during 1977–1987 [**FM77-1**, **FM77-2**, **OW80**, **S80**, **S81**, **CFW81**, **CW80**, **Z84**, **Ge87**, **GeGo88**, **P86**, **JS87**], the whole area of orbit-equivalence ergodic theory went through more than a decade of relative neglect.

In fact, even after the spectacular revival of this subject in recent years [**Fu99**, **G00**, **G02**, **Hj02**, **P01**, **P02**, **MoSh06**, **GP05**, **P04**], 1-cohomology was not really exploited as a tool to distinguish between orbit-inequivalent actions of groups. And this despite a new calculation of H^1-groups was obtained in [**P01**, **PSa03**], this time for Bernoulli actions σ of arbitrary property (T) groups, more generally for groups G having infinite normal subgroups with the relative property (T) of Kazhdan and Margulis (called *weakly rigid* in [**P01**, **P03**, **PSa03**]). Thus, it was shown in [**PSa03**] that for such (σ, G) one has $H^1(\sigma, G) = \mathrm{Char}(G)$.

In this paper we consider an even larger class of groups, denoted $w\mathcal{T}$, generalizing the weakly rigid groups of [**P01**, **PSa03**], and for each $G \in w\mathcal{T}$ calculate $H^1(\sigma', G)$ for a large family of quotients $\sigma' : G \to \mathrm{Aut}(X', \mu')$ of the Bernoulli actions σ of G on $(X, \mu) = \prod_{g \in G}(\mathbb{T}, \lambda)_g$. Thus, our main result shows that given any countable discrete abelian group Λ there exists a free ergodic action σ_Λ of G implemented by the restriction of σ to an appropriate σ-invariant subalgebra of $L^\infty(X, \mu)$, such that $H^1(\sigma_\Lambda, G) = \mathrm{Char}(G) \times \Lambda$ as topological groups. We also calculate the 1-cohomology for similar quotients of Bernoulli actions of free products of groups in $w\mathcal{T}$. We deduce that each $G \in w\mathcal{T}$, or G a free product of infinite Kazhdan groups, has a continuous family of free ergodic m.p. actions with mutually non-isomorphic H^1-groups, and which are thus OE inequivalent.

These results, together with prior ones in [**Ge87**, **PSa03**], establish 1-cohomology as an effective OE invariant, adding to the existing pool of methods used to differentiate orbit-inequivalent actions [**CoZ89**, **Fu99**, **G00**, **G02**, **MoSh06**]. Rather than depending on the group only, like the cost or ℓ^2-Betti numbers in [**G00**, **G02**], the H^1-invariant depends also on the action, proving particularly useful in distinguishing large classes of orbit-inequivalent actions of the same group.

Before stating the results in more details, let us define the class $w\mathcal{T}$ more precisely. It consists of all countable discrete groups G that have an infinite subgroup H with the following properties.

(a) $H \subset G$ has the *relative property* (T) of Kazhdan and Margulis (see [**Ma82**, **dHV89**]), i.e. all unitary representations of G that weakly contain the trivial representation of G must contain the trivial representation of H (when restricted to H).

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(b) H is *wq-normal* in G, i.e. given any intermediate subgroup $H \subset K \subsetneq G$ there exists $g \in G \setminus K$ with $gKg^{-1} \cap K$ infinite (see Definition 2.3 for other equivalent characterizations of this property).

For instance, if there exist finitely many subgroups $H = H_0 \subset H_1 \subset \cdots \subset H_n = G$ with all consecutive inclusions $H_j \subset H_{j+1}$ normal, then $H \subset G$ is wq-normal. Thus, weakly rigid groups are in the class $w\mathcal{T}$.

Theorem 0.1. *Let $G \in w\mathcal{T}$. Let σ be a Bernoulli action of G on the probability space $(X,\mu) = \prod_g (X_0,\mu_0)_g$ and β a free action of a group Γ on (X,μ) that commutes with σ and such that the restriction σ^Γ of σ to the fixed point algebra $\{a \in L^\infty(X,\mu) \mid \beta_h(a) = a,\ \forall h \in \Gamma\}$ is still a free action of G. Denote by $\mathrm{Char}_\beta(\Gamma)$ the group of characters γ on Γ for which there exist unitary elements $u \in L^\infty(X,\mu)$ with $\beta_h(u) = \gamma(h)u$, $\forall h \in \Gamma$. Then $\mathsf{H}^1(\sigma^\Gamma, G) = \mathrm{Char}(G) \times \mathrm{Char}_\beta(\Gamma)$ as topological groups.*

Since any countable abelian group Λ can be realized as $\mathrm{Char}_\beta(\Gamma)$, for some appropriate action β of a group Γ commuting with the Bernoulli action σ, and noticing that $\mathsf{H}^1(\sigma, G)$ are even invariant to stable orbit equivalence, we deduce the following corollary.

Corollary 0.2. *Let $G \in w\mathcal{T}$. Given any countable discrete abelian group Λ there exists a free ergodic m.p. action σ_Λ of G on the standard non-atomic probability space such that $\mathsf{H}^1(\sigma_\Lambda, G) = \mathrm{Char}(G) \times \Lambda$. Moreover, σ_Λ can be taken to be 'quotients' of Bernoulli G-actions. Thus, any $G \in w\mathcal{T}$ has a continuous family of mutually non-stably orbit-equivalent free ergodic m.p. actions on the probability space.*

Examples of groups in the class $w\mathcal{T}$ covered by the above results are the infinite property (T) groups, the groups $\mathbb{Z}^2 \rtimes \Gamma$, for $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ non-amenable (cf. [**K67**, **Ma82**, **B91**]) and the groups $\mathbb{Z}^N \rtimes \Gamma$ for suitable actions of arithmetic lattices Γ in $\mathrm{SU}(n,1)$ or $\mathrm{SO}(n,1), n \geqslant 2$ (cf. [**V05**]). Note that if $G \in w\mathcal{T}$ and K is a group acting on G by automorphisms then $G \rtimes K \in w\mathcal{T}$. Also, if $G \in w\mathcal{T}$ and K is an arbitrary group, then $G \times K \in w\mathcal{T}$. Moreover, if G has an infinite subgroup $H \subset G$ with the relative property (T) (not necessarily normal), then $G \times K$ is wq-rigid for all K infinite. In particular, any product between an infinite property (T) group and an arbitrary group is in the class $w\mathcal{T}$. Thus, Corollary 0.2 covers a recent result of Hjorth [**Hj02**], showing that infinite property (T) groups have uncountably many orbit-inequivalent actions. Moreover, rather than an existence result, Corollary 0.2 provides a concrete list of uncountably many inequivalent actions (indexed by the virtual isomorphism classes of all countable, discrete, abelian groups (see Corollary 2.13)).

Note also that if $G_0 \in w\mathcal{T}$ and K_0, K are arbitrary groups with K infinite, then $(G_0 * K_0) \times K \in w\mathcal{T}$. However, if $G = G_0 * K_0$ with $G_0 \in w\mathcal{T}$ and K_0 non-trivial, then G is not in the class $w\mathcal{T}$ (see Corollary 3.7). Yet we can still calculate in this case the 1-cohomology for the quotients of Bernoulli G-actions σ_Λ considered in Corollary 0.2. While $\mathsf{H}^1(\sigma_\Lambda, G)$ are 'huge' (non-locally compact) in this case, if we denote by $\tilde{\mathsf{H}}^1(\sigma_\Lambda, G)$ the quotient of $\mathsf{H}^1(\sigma_\Lambda, G)$ by the connected component of 1, then we get the following theorem.

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Theorem 0.3. *Let $\{G_n\}_{n\geqslant 0}$ be a sequence of countable groups such that each G_n is either amenable or belongs to the class $w\mathcal{T}$ and denote $G = *_{n\geqslant 0} G_n$. Assume the set J of indices $j \geqslant 0$ for which $G_j \in w\mathcal{T}$ is non-empty and that G_j has totally disconnected character group, $\forall j \in J$. If Λ is a countable abelian group, then*

$$\tilde{H}^1(\sigma_\Lambda, G) \simeq \prod_{j\in J} \mathrm{Char}(G_j) \times \Lambda^{|J|}$$

as Polish groups.

Since property (T) groups have finite (thus totally disconnected) character group, from the above theorem we get the following corollary.

Corollary 0.4. *Let $H_1, H_2, \dots, H_k$ be infinite property* (T) *groups and $0 \leqslant n \leqslant \infty$. The free product group $H_1 * H_2 * \cdots * H_k * \mathbb{F}_n$ has continuously many non-stably orbit-equivalent free ergodic m.p. actions.*

The use of von Neumann algebras framework and non-commutative analysis tools is crucial for the approach in this paper. Thus, the construction used in Theorem 0.1, as well as its proof, become quite natural in von Neumann algebra context, where similar ideas have been used in [**P01**] to compute the 1-cohomology and fundamental group for non-commutative (Connes–Størmer) Bernoulli actions of weakly rigid groups on the hyperfinite II_1 factor R, and in [**C75b**] to compute the approximately inner, centrally free part $\mathcal{X}(M)$ of the outer automorphism group of a II_1 factor M.

The paper is organized as follows. In § 1 we present some basic facts on 1-cohomology for actions, including a detailed discussion of the similar concept for full (pseudo)groups and equivalence relations. Also, we revisit the results on 1-cohomology in [**FM77-1**, **FM77-2**, **S80**, **S81**]. In § 2 we prove Theorem 0.1 and its consequences. In § 3 we consider actions of free product groups and prove Theorem 0.3.

1. 1-cohomology for actions and equivalence relations

We recall here the definition and basic properties of the 1-cohomology groups for actions and equivalence relations, using the framework of von Neumann algebras. We revisit this way the results in [**S80**, **S81**, **FM77-1**, **FM77-2**] and prove the invariance of 1-cohomology groups to stable orbit equivalence. The von Neumann algebra setting leads us to adopt Dye's initial point of view [**D63**] of regarding equivalence relations as full (pseudo)groups and to use Singer's observation [**Si55**] that the group of 1-cocycles of an action is naturally isomorphic to the group of automorphisms of the associated group measure space von Neumann algebra that leave the Cartan subalgebra pointwise fixed.

1.1. 1-cohomology for actions

Let $\sigma : G \to \mathrm{Aut}(X, \mu)$ be a free measure preserving (m.p.) action of the (at most) countable discrete group G on the standard probability space (X, μ) and still denote by σ the action it implements on $A = L^\infty(X, \mu)$. Denote $\mathcal{U}(A) = \{u \in A \mid uu^* = 1\}$ the group of unitary elements of A. A function $w : G \to \mathcal{U}(A)$ satisfying $w_g \sigma_g(w_h) = w_{gh}$,

$\forall g, h \in G$, is called a 1-*cocycle* for σ. Note that a scalar valued function $w : G \to \mathcal{U}(A)$ is a 1-cocycle if and only if $w \in \mathrm{Char}(G)$.

Two 1-cocycles w, w' are *cohomologous*, $w \sim_c w'$, if there exists $u \in \mathcal{U}(A)$ such that $w'_g = u^* w_g \sigma_g(u)$, $\forall g \in G$. A 1-cocycle w is *coboundary* if $w \sim_c \mathbf{1}$, where $\mathbf{1}_g = 1$, $\forall g$.

Denote by $\mathsf{Z}^1(\sigma, G)$ (or simply $\mathsf{Z}^1(\sigma)$, when there is no risk of confusion) the set of 1-cocycles for σ, endowed with the structure of a topological (commutative) group given by point multiplication and pointwise convergence in norm $\|\cdot\|_2$. Denote by $\mathsf{B}^1(\sigma, G) = \mathsf{B}^1(\sigma) \subset \mathsf{Z}^1(\sigma)$ the subgroup of coboundaries and by $\mathsf{H}^1(\sigma, G) = \mathsf{H}^1(\sigma)$ the quotient group $\mathsf{Z}^1(\sigma)/\mathsf{B}^1(\sigma) = \mathsf{Z}^1(\sigma)/{\sim_c}$, called the *first cohomology group* of σ. Note that $\mathrm{Char}(G)$ with its usual topology can be viewed as a compact subgroup of $\mathsf{Z}^1(\sigma)$ and its image in $\mathsf{H}^1(\sigma)$ is a compact subgroup. If in addition σ is weakly mixing, then the image of $\mathrm{Char}(G)$ in $\mathsf{H}^1(\sigma)$ is faithful (see Lemma 2.4 (1)).

The groups $\mathsf{B}^1(\sigma)$, $\mathsf{Z}^1(\sigma)$, $\mathsf{H}^1(\sigma)$ were first considered in [**Si55**]. As noticed in [**Si55**], they can be identified with certain groups of automorphisms of the finite von Neumann algebra $M = A \rtimes_\sigma G$, as explained below. Note that there exists a unique normal faithful trace τ on M that extends the integral $\int \cdot \, \mathrm{d}\mu$ on A and that M is a factor if and only if σ is ergodic. For $x \in M$ we denote $\|x\|_2 = \tau(x^*x)^{1/2}$.

1.2. Automorphisms associated with 1-cocycles

Let $\mathrm{Aut}_0(M; A)$ denote the group of automorphisms of M that leave all elements of A fixed, endowed with the topology of pointwise convergence in norm $\|\cdot\|_2$ (the topology it inherits from $\mathrm{Aut}(M, \tau)$). If $\theta \in \mathrm{Aut}_0(M; A)$, then $w^\theta_g = \theta(u_g)u^*_g$, $g \in G$, is a 1-cocycle, where $\{u_g\}_g \subset M$ denote the canonical unitaries implementing the action σ. Conversely, if $w \in \mathsf{Z}^1(\sigma)$ then $\theta^w(au_g) = aw_g u_g$, $a \in A$, $g \in G$, defines an automorphism of M that fixes A. Clearly, $\theta \mapsto w^\theta$, $w \mapsto \theta^w$ are group morphisms and are inverse one another, thus identifying $\mathsf{Z}^1(\sigma)$ with $\mathrm{Aut}_0(M; A)$ as topological groups, with $\mathsf{B}^1(\sigma)$ corresponding to the inner automorphism group $\mathrm{Int}_0(M; A) = \{\mathrm{Ad}(u) \mid u \in \mathcal{U}(A)\}$. Thus, $\mathsf{H}^1(\sigma)$ is naturally isomorphic to

$$\mathrm{Out}_0(M; A) \stackrel{\mathrm{def}}{=} \mathrm{Aut}_0(M; A) / \mathrm{Int}_0(M; A).$$

The groups $\mathrm{Aut}_0(M; A)$, $\mathrm{Int}_0(M; A)$, $\mathrm{Out}_0(M; A)$ actually make sense for any inclusion $A \subset M$ consisting of a II_1 factor M with a *Cartan subalgebra* A, i.e. a maximal abelian $*$-subalgebra of M with *normalizer* $\mathcal{N}_M(A) \stackrel{\mathrm{def}}{=} \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ generating M. In order to interpret $\mathrm{Out}_0(M; A)$ as 1-cohomology group in this more general case, we will recall from [**D63**, **FM77-1**, **FM77-2**] two alternative, equivalent ways of viewing Cartan subalgebra inclusions $A \subset M$.

1.3. Full pseudogroups and equivalence relations

With $A \subset M$ as above, let

$$\mathcal{GN}_M(A) = \{v \in M \mid vv^*, v^*v \in \mathcal{P}(A),\ vAv^* = Avv^*\},$$

where $\mathcal{P}(A)$ denotes the idempotents (or projections) in A. Identify A with $L^\infty(X, \mu)$, for some probability space (X, μ), with μ corresponding to $\tau_{|A}$, where τ is the trace on

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M. We denote by $\mathcal{G} = \mathcal{G}_{A\subset M}$ the set of all local isomorphisms $\phi_v = \mathrm{Ad}(v)$, $v \in \mathcal{GN}_M(A)$, defined modulo sets of measure zero. For $\phi \in \mathcal{G}$ denote by $R(\phi) \subset X$ the domain of definition of ϕ and by $L(\phi) \subset X$ the range of ϕ. Thus, if $\phi = \phi_v$ then $v^*v = \chi_{R(\phi)}$, $vv^* = \chi_{L(\phi)}$.

We endow $\mathcal{G}$ with the natural *pseudogroup structure* given by composition: if $\phi, \psi \in \mathcal{G}$, then $\psi \cdot \phi$ is the local isomorphism with domain $R = \{x \in R(\phi) \mid \phi(x) \in R(\psi)\}$, defined on R by $\psi \cdot \phi(x) = \psi(\phi(x))$. Thus, if $\phi = \phi_v$ and $\psi = \psi_w$, then $\psi \cdot \phi = \phi_{wv}$. We call $(\mathcal{G}_{A\subset M}, \cdot)$ the *full pseudogroup* associated with $A \subset M$. Since $\{v_n\}_n \subset \mathcal{GN}_M(A)$ with $\{v_n v_n^*\}_n$, respectively $\{v_n^* v_n\}_n$, mutually orthogonal implies $\sum_n v_n \in \mathcal{GN}_M(A)$, it follows that $\mathcal{G} = \mathcal{G}_{A\subset M}$ satisfies the following axioms.

(1.3.0) $\mathrm{id}_X \in \mathcal{G}$ and if $\phi \in \mathcal{G}$, $Y \subset R(\phi)$ measurable, then $\phi_{|Y} \in \mathcal{G}$. Thus, the set of *units* (or *idempotents*) $\mathcal{G}_0$ of $\mathcal{G}$ coincides with the set $\{\mathrm{id}_Y \in \mathcal{G} \mid Y \subset X \text{ measurable}\}$.

(1.3.1) Let $R, L \subset X$ be measurable subsets with $\mu(R) = \mu(L)$ and $\phi : R \simeq L$ a measurable, measure preserving isomorphism. Then $\phi \in \mathcal{G}$ if and only if there exists a countable partition of R with measurable subsets $\{R_n\}_n$ such that $\phi_{|R_n} \in \mathcal{G}$, $\forall n$.

Note that the factoriality of M amounts to the ergodicity of the action of $\mathcal{G}$ on $L^\infty(X, \mu)$ and that M is separable in the norm $\|\cdot\|_2$ if and only if $\mathcal{G}$ is countably generated as a pseudogroup satisfying (1.3.0), (1.3.1). If $M = A \rtimes_\sigma G$ for some free m.p. action σ of a group G, then we denote $\mathcal{G}_{A\subset M}$ by $\mathcal{G}_\sigma$. Note that if $\phi : R \simeq L$ is an m.p. isomorphism, for some measurable subsets $R, L \subset X$ with $\mu(R) = \mu(L)$, then $\phi \in \mathcal{G}_\sigma$ if and only if there exist $g_n \in G$ and a partition of R with measurable subsets $\{R_n\}_n$ such that $\phi_{|R_n} = \sigma(g_n)_{|R_n}$, $\forall n$.

A pseudogroup $\mathcal{G}$ of m.p. local isomorphisms of the probability space (X, μ) satisfying (1.3.0), (1.3.1) is called an *abstract full pseudogroup*.

If an abstract full pseudogroup $\mathcal{G}$ acting on (X, μ) is given, then let $\mathbb{C}\mathcal{G}$ denote the algebra of formal finite linear combinations $\sum_\phi c_\phi \phi$. Let $\tau(\phi)$ denote the measure of the largest set on which ϕ acts as the identity and extend it by linearity to $\mathbb{C}\mathcal{G}$. Then define a sesquilinear form on $\mathbb{C}\mathcal{G}$ by $\langle x, y\rangle = \tau(y^*x)$ and denote by $L^2(\mathcal{G})$ the Hilbert space obtained by completing $\mathbb{C}\mathcal{G}/I_\tau$ in the norm $\|x\|_2 = \tau(x^*x)^{1/2}$, where $I_\tau = \{x \mid \langle x, x\rangle = 0\}$. Each $\phi \in \mathcal{G}$ acts on $L^2(\mathcal{G})$ as the operator u_ϕ of left multiplication by ϕ. Denote by $L(\mathcal{G})$ the von Neumann algebra generated by the operators $\{u_\phi, \phi \in \mathcal{G}\}$ and by $L(\mathcal{G}_0) \simeq L^\infty(X, \mu)$ the von Neumann subalgebra generated by the units $\mathcal{G}_0$.

It is easy to check that $L(\mathcal{G})$ is a finite von Neumann algebra with Cartan subalgebra $L(\mathcal{G}_0) = L^\infty(X, \mu)$ and faithful normal trace τ extending the integral on $L^\infty(X, \mu)$ and satisfying $\tau(u_\phi) = \tau(\phi)$, with $L^2(L(\mathcal{G})) = L^2(\mathcal{G})$ the standard representation of $(L(\mathcal{G}), \tau)$. Moreover, if $A = L(\mathcal{G}_0)$, $M = L(\mathcal{G})$, then

$$\mathcal{GN}_M(A) = \{au_\phi \mid \phi \in \mathcal{G},\ a \in \mathcal{I}(A)\},$$

where $\mathcal{I}(A)$ denotes the set of partial isometries in A. Thus, the full pseudogroup $\mathcal{G}_{A\subset M}$ associated with the Cartan subalgebra inclusion $L(\mathcal{G}_0) \subset L(\mathcal{G})$ can be naturally identified with the initial abstract pseudogroup $\mathcal{G}$. Also, note that $L(\mathcal{G})$ is a factor if and only if $\mathcal{G}$ is ergodic, in which case either $L(\mathcal{G}) \simeq M_{n\times n}(\mathbb{C})$ (when (X, μ) is the n-points probability space) or $L(\mathcal{G})$ is a II_1 factor (when (X, μ) has no atoms, equivalently when $\mathcal{G}$ has infinitely many elements).

If the abstract full pseudogroup $\mathcal{G}$ is generated by a countable set of local isomorphisms $\{\phi_n\}_n \subset \mathcal{G}$ and one considers a standard Borel structure on X with σ-field $\mathcal{X}$, then ϕ_n can be taken Borel. If one denotes $\mathcal{R} = \mathcal{R}_{\mathcal{G}}$ the equivalence relation implemented by the orbits of $\phi \in \mathcal{G}$, then each class of equivalence in $\mathcal{R}$ is countable and $\mathcal{R}$ lies in the σ-field $\mathcal{X} \times \mathcal{X}$. Moreover, all $\phi \in \mathcal{G}$ can be recuperated from $\mathcal{R}$ as graphs of local isomorphisms that lie in $\mathcal{R} \cap \mathcal{X} \times \mathcal{X}$. Such $\mathcal{R}$ is called a *countable measure preserving* (m.p.) *standard equivalence relation*. The m.p. standard equivalence relation $\mathcal{R}_{A\subset M}$ associated with a Cartan subalgebra inclusion $A \subset M$ is the equivalence relation implemented by the orbits of $\mathcal{G}_{A\subset M}$. In the case where $\mathcal{G}$ is given by an action σ of a countable group G, the orbits of $\mathcal{G}_\sigma$ coincide with the orbits of σ and one denotes the corresponding equivalence relation by $\mathcal{R}_\sigma$.

An isomorphism between two full pseudogroups (respectively m.p. equivalence relations) is an isomorphism of the corresponding probability spaces that takes one full pseudogroup (respectively m.p. equivalence relation) onto the other. Such an isomorphism clearly agrees with the correspondence between pseudogroups and equivalence relations described above. Two Cartan subalgebra inclusions $(A_1 \subset M_1, \tau_1)$, $(A_2 \subset M_2, \tau_2)$ are isomorphic if there exists $\theta : (M_1, \tau_1) \simeq (M_2, \tau_2)$ such that $\theta(A_1) = A_2$. Note that if this is the case then

$$\mathcal{G}_{A_1\subset M_1} \simeq \mathcal{G}_{A_2\subset M_2}, \qquad \mathcal{R}_{A_1\subset M_1} \simeq \mathcal{R}_{A_2\subset M_2}.$$

Conversely, if $\mathcal{G}_1 \simeq \mathcal{G}_2$ then $(L(G_{1,0}) \subset L(G_1)) \simeq (L(G_{2,0}) \subset L(G_2))$, by the way we have constructed a Cartan subalgebra inclusion from a full pseudogroup. In particular, two free ergodic m.p. actions $\sigma_i : G_i \to \mathrm{Aut}(X_i, \mu_i)$ are orbit equivalent if and only if

$$(A_1 \subset A_1 \rtimes_{\sigma_1} G_1) \simeq (A_2 \subset A_2 \rtimes_{\sigma_2} G_2).$$

1.4. Amplifications and stable orbit equivalence

If M is a II_1 factor and $t > 0$, then for any $n \geqslant m \geqslant t$ and any projections $p \in M_{n\times n}(M)$, $q \in M_{m\times m}(M)$ of (normalized) trace $\tau(p) = t/n$, $\tau(q) = t/m$, one has $pM_{n\times n}(M)p \simeq qM_{m\times m}(M)q$. Indeed, because if we regard $M_{m\times m}(M)$ as a 'corner' of $M_{n\times n}(M)$ then p, q have the same trace in $M_{n\times n}(M)$, so they are conjugate by a unitary U in $M_{n\times n}(M)$, which implements an isomorphism between $pM_{n\times n}(M)p$ and $qM_{m\times m}(M)q$. One denotes by M^t this common (up to isomorphism) II_1 factor and calls it the *amplification* of M by t.

Similarly, if $A \subset M$ is a Cartan subalgebra of the II_1 factor M, then $(A \subset M)^t = (A^t \subset M^t)$ denotes the (isomorphism class of the) Cartan subalgebra inclusion $p(A \otimes D_n \subset M \otimes M_{n\times n}(\mathbb{C}))p$ where $n \geqslant t$, D_n is the diagonal subalgebra of $M_{n\times n}(\mathbb{C})$ and $p \in A \otimes D_n$ is a projection of trace $\tau(p) = t/n$. In this case, the fact that the isomorphism class of $(A \subset M)^t$ does not depend on the choice of n, p follows from a lemma of Dye [**D63**], showing that if M_0 is a II_1 factor and $A_0 \subset M_0$ is a Cartan subalgebra, then two

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projections $p, q \in A_0$ having the same trace are conjugate by a unitary element in the normalizer of A_0 in M_0.

If $\mathcal{G}$ is an ergodic full pseudogroup on the non-atomic probability space, then $\mathcal{G}^t$ is the full pseudogroup obtained by restricting the full pseudogroup generated by $\mathcal{G} \times \mathcal{D}_n$ to a subset of measure t/n, where $\mathcal{D}_n$ is the pseudogroup of permutations of the n-points probability space with the counting measure. If $\mathcal{R}$ is an ergodic m.p. standard equivalence relation, then $\mathcal{R}^t$ is defined in a similar way. Again, $\mathcal{G}^t$, $\mathcal{R}^t$ are defined only up to isomorphism.

$(A \subset M)^t$ (respectively $\mathcal{G}^t$, $\mathcal{R}^t$) is called the *t-amplification* of $A \subset M$ (respectively of $\mathcal{G}$, $\mathcal{R}$). We clearly have

$$\mathcal{G}_{(A\subset M)^t} = \mathcal{G}^t_{(A\subset M)}, \qquad \mathcal{R}_{(A\subset M)^t} = \mathcal{R}^t_{(A\subset M)}$$

and if $\mathcal{G}$, $\mathcal{R}$ correspond with one another then so do $\mathcal{R}^t$, $\mathcal{G}^t$, $\forall t$. Note that

$$((A \subset M)^t)^s = (A \subset M)^{st}, \quad (\mathcal{G}^t)^s = \mathcal{G}^{ts}, \quad (\mathcal{R}^t)^s = \mathcal{R}^{ts}, \quad \forall t, s > 0.$$

Two ergodic full pseudogroups $\mathcal{G}_i$, $i = 1, 2$ (respectively ergodic equivalence relations $\mathcal{R}_i$, $i = 1, 2$), are *stably orbit equivalent* if $\mathcal{G}_1 \simeq \mathcal{G}_2^t$ (respectively $\mathcal{R}_1 \simeq \mathcal{R}_2^t$), for some $t > 0$. Two free ergodic m.p. actions (σ_i, G_i), $i = 1, 2$, are stably orbit equivalent if $\mathcal{G}_{\sigma_1} \simeq \mathcal{G}^t_{\sigma_2}$ for some t. Note that this is equivalent to the existence of subsets of positive measure $Y_i \subset X_i$ and of an isomorphism $\Psi : (Y_1, \mu_1/\mu_1(Y_1)) \simeq (Y_2, \mu_2/\mu_2(Y_2))$ such that $\Psi(\sigma_1(G_1)x \cap Y_1) = \sigma_2(G_2)\Psi(x) \cap Y_2$, a.e. in $x \in Y_1$.

1.5. 1-cohomology for full pseudogroups

Let $\mathcal{G}$ be a full pseudogroup acting on the probability space (X, μ) and denote $A = L^\infty(X, \mu)$. A 1-*cocycle* for $\mathcal{G}$ is a map $w : \mathcal{G} \to \mathcal{I}(A)$ satisfying the relation $w_\phi \phi(w_\psi) = w_{\phi\psi}$, $\forall \phi, \psi \in \mathcal{G}$. In particular, this implies that the support of w_ϕ, $w_\phi w_\phi^*$, is equal to the range $r(\phi)$ of ϕ. Thus, $w_{\mathrm{id}_Y} = \chi_Y$, $\forall Y \subset X$ measurable.

We denote by $Z^1(\mathcal{G})$ the set of all 1-cocycles and endow it with the (commutative) semigroup structure given by point multiplication. We denote by $\mathbf{1}$ the 1-cocycle given by $\mathbf{1}_\phi = r(\phi)$, $\forall \phi \in \mathcal{G}$. If we let $(w^{-1})_\phi = w_\phi{}^*$, then we clearly have $ww^{-1} = \mathbf{1}$ and $\mathbf{1}w = w$, $\forall w \in Z^1(\mathcal{G})$. Thus, together also with the topology given by pointwise norm $\|\cdot\|_2$-convergence, $Z^1(\mathcal{G})$ is a commutative Polish group.

Two 1-cocycles w_1, w_2 are *cohomologous*, $w_1 \sim_c w_2$, if there exists $u \in \mathcal{U}(A)$ such that $w_2(\phi) = u^* w_2(\phi)\phi(u)$, $\forall \phi \in \mathcal{G}$. A 1-cocycle w cohomologous to $\mathbf{1}$ is called a *coboundary* for $\mathcal{G}$ and the set of coboundaries is denoted $B^1(\mathcal{G})$. It is clearly a subgroup of $Z^1(\mathcal{G})$. We denote the quotient group

$$H^1(\mathcal{G}) \stackrel{\mathrm{def}}{=} Z^1(\mathcal{G})/B^1(\mathcal{G}) = Z^1(\mathcal{G})/{\sim_c}$$

and call it the *first cohomology group* of $\mathcal{G}$.

By the correspondence between countably generated full pseudogroups and countable m.p. standard equivalence relations described in § 1.3, one can alternatively view the

1-cohomology groups $Z^1(\mathcal{G})$, $B^1(\mathcal{G})$, $H^1(\mathcal{G})$ as associated to the equivalence relation $\mathcal{R} = \mathcal{R}_\mathcal{G}$, in which case one recovers the definition of $H^1(\mathcal{R})$ from p. 308 of [**FM77-1**,**FM77-2**].

Let now $A \subset M$ be a II_1 factor with a Cartan subalgebra. If $\theta \in \text{Aut}_0(M;A)$ and $\phi_v = \text{Ad}(v) \in \mathcal{G}_{A\subset M}$ for some $v \in \mathcal{GN}_M(A)$, then $w^\theta(\phi_v) = \theta(v)v^*$ is a well defined 1-cocycle for $\mathcal{G}$. Conversely, if $w \in H^1(\mathcal{G})$, then there exists a unique automorphism $\theta^w \in \text{Aut}_0(M;A)$ satisfying $\theta^w(av) = aw_{\phi_v}v$, $\forall a \in A, v \in \mathcal{GN}_M(A)$.

Proposition 1.5.1. *$\theta \mapsto w^\theta$ is an isomorphism of topological groups, from $\text{Aut}_0(M;A)$ onto $Z^1(\mathcal{G}_{A\subset M})$, that takes $\text{Int}_0(M;A) = \{\text{Ad}(u) \mid u \in \mathcal{U}(A)\}$ onto $B^1(\mathcal{G}_{A\subset M})$ and whose inverse is $w \mapsto \theta^w$. Thus, $\theta \mapsto w^\theta$ implements an isomorphism between the topological groups $\text{Out}_0(M;A) = \text{Aut}_0(M;A)/\,\text{Int}_0(M;A)$ and $H^1(\mathcal{G}_{A\subset M})$.*

Proof. This is trivial by the definitions. □

By a well-known lemma of Connes (see, for example, [**C75**]), if $\theta \in \text{Aut}_0(M;A)$ satisfies $\theta_{|pMp} = \text{Ad}(u)_{|pMp}$ for some $p \in \mathcal{P}(A)$, $u \in \mathcal{U}(A)$ then $\theta \in \text{Int}_0(M;A)$. Thus, $\theta \mapsto \theta_{|pMp}$ defines an isomorphism from $\text{Out}_0(M;A)$ onto $\text{Out}_0(pMp;Ap)$. Applying this to the Cartan subalgebra inclusion $L(\mathcal{G}_0) \subset L(\mathcal{G})$ for $\mathcal{G}$ an ergodic full pseudogroup acting on the non-atomic probability space, from Proposition 1.5.1 we obtain that $H^1(\mathcal{G})$ is naturally isomorphic to $H^1(\mathcal{G}^t)$, $\forall t > 0$. In particular, since Proposition 1.5.1 also implies $H^1(\sigma) = H^1(\mathcal{G}_\sigma)$, it follows that $H^1(\sigma)$ is invariant to stable orbit equivalence. We have thus shown the following corollary.

Corollary 1.5.2.

(1) *$H^1(\mathcal{G}^t)$ is naturally isomorphic to $H^1(\mathcal{G})$, $\forall t > 0$.*

(2) *If σ is a free ergodic measure preserving action, then $H^1(\sigma) = H^1(\mathcal{G}_\sigma)$ and $H^1(\sigma)$ is invariant to stable orbit equivalence. Also, $Z^1(\sigma) = Z^1(\mathcal{G}_\sigma)$ and $Z^1(\sigma)$ is invariant to orbit equivalence.*

Note that the equality $H^1(\sigma) = H^1(\mathcal{G}_\sigma)$ (and thus the invariance of $H^1(\sigma)$ to orbit equivalence) was already shown in [**FM77-1**, **FM77-2**].

1.6. The closure of $B^1(\mathcal{G})$ in $Z^1(\mathcal{G})$

Given any ergodic full pseudogroup $\mathcal{G}$, the groups $B^1(\mathcal{G}) \simeq \text{Int}_0(M;A)$ are naturally isomorphic to $\mathcal{U}(A)/\mathbb{T}$, where $A = L(\mathcal{G}_0)$, $M = L(\mathcal{G})$. But this isomorphism does not always carry the topology that $B^1(\mathcal{G})$ (respectively $\text{Int}_0(M;A)$) inherits from $Z^1(\mathcal{G})$ (respectively $\text{Aut}_0(M;A)$) onto the quotient of the $\|\cdot\|_2$-topology on $\mathcal{U}(A)/\mathbb{T}$. It was shown by Schmidt in [**S80**, **S81**] that the two topologies on $B^1(\sigma)$ coincide if and only if the action σ is strongly ergodic. We recall his result in the statement below, relating it to a result of Connes, showing that the group of inner automorphisms of a II_1 factor is closed if and only if the factor has no non-trivial central sequences [**C75**].

Proposition 1.6.1. *Let $A \subset M$ be a II_1 factor with a Cartan subalgebra. The following conditions are equivalent.*

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(a) $H^1(\mathcal{G}_{A\subset M})$ *is a Polish group (equivalently* $H^1(\mathcal{G}_{A\subset M})$ *is separate), i.e.* $B^1(\mathcal{G}_{A\subset M})$ *is closed in* $Z^1(\mathcal{G}_{A\subset M})$.

(b) $\mathrm{Int}_0(M;A)$ *is closed in* $\mathrm{Aut}_0(M;A)$.

(c) *The action of* $\mathcal{G}_{A\subset M}$ *on* A *is strongly ergodic, i.e. it has no non-trivial asymptotically invariant sequences.*

(d) $M'\cap A^\omega=\mathbb{C}$, *where* ω *is a free ultrafilter on* $\mathbb{N}$.

Moreover, if $M=A\rtimes_\sigma G$ *for some free action* σ *of a group* G *on* (A,τ)*, then the above conditions are equivalent to* σ *being strongly ergodic.*

Proof. (a) $\Leftrightarrow$ (b) follows from Proposition 1.5.1 and (c) $\Leftrightarrow$ (d) is well known (and trivial). Then notice that (b) $\Leftrightarrow$ (d) is a relative version of the result of Connes in [**C75**], showing that 'Int(N) is closed in Aut(N) if and only if N has no non-trivial central sequences' for II_1 factors N. Thus, a proof of (b) $\Leftrightarrow$ (d) is obtained by following the argument in [**C75**], but replacing everywhere Int(N) by $\mathrm{Int}_0(M;A)$, Aut(N) by $\mathrm{Aut}_0(M;A)$ and 'non-trivial central sequences of N' by 'non-trivial central sequences of M that are contained in A'.

To prove the last part, note that σ is strongly ergodic if and only if $\{u_g\}_g'\cap A^\omega=\mathbb{C}$, where $\{u_g\}_g\subset M$ denote the canonical unitaries implementing the action σ of G on A. But $\{u_g\}_g'\cap A^\omega=(A\cup\{u_g\}_g)'\cap A^\omega=M'\cap A^\omega$, hence strong ergodicity of σ is equivalent to (d). ☐

It was shown in [**S80, S81**] that arbitrary ergodic m.p. actions σ of infinite property (T) groups G are always strongly ergodic, and that in fact $B^1(\sigma)$ is always open (and thus also closed) in $Z^1(\sigma)$. The interpretation of the inclusion $B^1(\sigma)\subset Z^1(\sigma)$ as $\mathrm{Int}_0(M;A)\subset\mathrm{Aut}_0(M;A)$ makes this result into a relative version of the rigidity result in [**C80**], showing that for property (T) factors Int(M) is open in Aut(M). We notice here the following generalization.

Proposition 1.6.2. *Assume that* G *has an infinite subgroup* $H\subset G$ *such that the pair* (G,H) *has the relative property* (T). *If* σ *is a free m.p. action of* G *on the probability space such that* $\sigma_{|H}$ *is ergodic, then* σ *is strongly ergodic, equivalently* $B^1(\sigma)$ *is closed in* $Z^1(\sigma)$. *Moreover, the subgroup* $Z^1_H(\sigma)\overset{\mathrm{def}}{=}\{w\in Z^1(\sigma)\mid w_{|H}\sim_c \mathbf{1}_H\}$ *is open and closed in* $Z^1(\sigma)$.

Proof. Since (G,H) has the relative property (T), by [**Jol05**] there exist a finite subset $F\subset G$ and $\delta>0$ such that if $\pi:G\to\mathcal{U}(\mathcal{H})$, $\xi\in\mathcal{H}$, $\|\xi\|_2=1$ satisfy $\|\pi_g(\xi)-\xi\|_2\leqslant\delta$, $\forall g\in F$, then $\|\pi_h(\xi)-\xi\|_2\leqslant\frac{1}{2}$, $\forall h\in H$, and $\pi_{|H}$ has a non-trivial fixed vector.

If σ is not strongly ergodic, then there exists $p\in\mathcal{P}(A)$ such that $\tau(p)=\frac{1}{2}$ and $\|\sigma_g(p)-p\|_2\leqslant\frac{1}{2}\delta$, $\forall g\in F$. But then $u=1-2p$ satisfies $\tau(u)=0$ and $\|\sigma_g(u)-u\|_2\leqslant\delta$, $\forall g\in F$. Taking π to be the G-representation induced by σ on $L^2(A,\tau)\ominus\mathbb{C}1$, it follows that $L^2(A,\tau)\ominus\mathbb{C}1$ contains a non-trivial vector fixed by $\sigma_{|H}$. But this contradicts the ergodicity of $\sigma_{|H}$.

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Let now $M = A \rtimes_\sigma G$ and $\theta = \theta^w \in \mathrm{Aut}_0(M; A)$ be the automorphism associated to some $w \in Z^1(\sigma)$ satisfying

$$\|\theta(u_g) - u_g\|_2 = \|w_g - 1\|_2 \leqslant \delta, \quad \forall g \in F.$$

Then the unitary representation $\pi : G \to \mathcal{U}(L^2(M,\tau))$ defined by $\pi_g(\xi) = u_g \xi \theta(u_g^*)$ satisfies

$$\|\pi_g(\hat{1}) - \hat{1}\|_2 = \|w_g - 1\|_2 \leqslant \delta, \quad \forall g \in F.$$

Thus, $\|w_h - 1\|_2 = \|\pi_h(\hat{1}) - \hat{1}\|_2 \leqslant \frac{1}{2}$ implying

$$\|\theta(vu_h) - vu_h\|_2 = \|\theta(u_h) - u_h\|_2 \leqslant \tfrac{1}{2}, \quad \forall h \in H,\ v \in \mathcal{U}(A).$$

It follows that if b denotes the element of minimal norm $\|\cdot\|_2$ in $\overline{\mathrm{co}}^w\{u_h^* v^* \theta(vu_h) \mid h \in H,\ v \in \mathcal{U}(A)\}$, then $\|b - 1\|_2 \leqslant \frac{1}{2}$ and $vu_h b = b\theta(vu_h) = bw_h vu_h$, $\forall h \in H$, $v \in \mathcal{U}(A)$. But this implies $b \neq 0$ and $xb = b\theta(x)$, $\forall x \in N = A \rtimes_{\sigma_{|H}} H$. In particular $[b, A] = 0$ so $b \in A \subset N$. Since N is a factor (because $\sigma_{|H}$ is ergodic), this implies b is a scalar multiple of a unitary element u in A satisfying $w_h = u^* \sigma_h(u)$, $\forall h \in H$. Thus $w \in Z^1_H(\sigma)$, showing that $Z^1_H(\sigma)$ is open (thus closed too). □

Corollary 1.6.3. *If G has an infinite subgroup $H \subset G$ such that the pair (G, H) has the relative property* (T), *then* $\mathrm{Char}(G)/\{\gamma \in \mathrm{Char}(G) \mid \gamma_{|H} = \mathbf{1}_H\}$ *is a finite group.*

Proof. Take σ to be the (classic) Bernoulli G-action and view $\mathrm{Char}(G)$ as a subgroup of $Z^1(\sigma)$, as in § 1.1. By Proposition 1.6.2 it follows that $\gamma \in \mathrm{Char}(G)$ satisfies $\gamma_{|H} \sim_c \mathbf{1}_H$ if and only if $\gamma_{|H} = \mathbf{1}_H$ (because $\sigma_{|H}$ is mixing; see also Lemma 2.4 (1)). Thus, $Z^1_H(\sigma) \cap \mathrm{Char}(G) = \{\gamma \in \mathrm{Char}(G) \mid \gamma_{|H} = \mathbf{1}_H\}$ and the statement follows from Proposition 1.6.2. □

Other actions shown to be strongly ergodic in [**S80**, **S81**] are the Bernoulli actions of non-amenable groups and the action of $\mathrm{SL}(2, \mathbb{Z})$ on $(\mathbb{T}^2, \lambda)$. There is in fact a common explanation for these examples: in both cases the representation implemented by (σ, G) on $L^2(X, \mu) \ominus \mathbb{C}1$ can be realized as $\oplus_i \ell^2(G/\Gamma_i)$ with $\Gamma_i \subset G$ amenable subgroups. (If σ is a Bernoulli G-action, then Γ_i can even be taken finite (see also [**J83b**]). If in turn $G = \mathrm{SL}(2, \mathbb{Z})$, then the action σ of G on $L^2(\mathbb{T}^2, \lambda) \ominus \mathbb{C}1 \simeq \ell^2(\mathbb{Z}^2 \setminus \{(0,0)\})$ corresponds to the action of G on $\mathbb{Z}^2 \setminus \{(0,0)\}$ and Γ_i are stabilizers of elements $h_i \in \mathbb{Z}^2 \setminus \{(0,0)\}$, thus amenable.) Hence, if (σ, G) is not strongly ergodic, then the trivial representation of G is weakly contained in $\oplus_i \ell^2(G/H_i)$ and the following general observation applies.

Lemma 1.6.4. *Let G be a non-amenable group and $\{H_i\}_i$ the family of amenable subgroups of G. Then the trivial representation of G is not weakly contained in $\oplus_i \ell^2(G/H_i)$ (thus not weakly contained in $\oplus_i \ell^2(G/H_i) \bar{\otimes} \ell^2(\mathbb{N})$ either).*

Proof. This follows immediately from the continuity of induction of representations. Indeed, every $\ell^2(G/H_i)$ is equivalent to the induced from H_i to G of the trivial representation 1_{H_i} of H_i, $\mathrm{Ind}_{H_i}^G 1_{H_i}$. Since H_i is amenable, 1_{H_i} follows weakly contained in the left regular representation λ_{H_i} of H_i. Thus, $\mathrm{Ind}_{H_i}^G 1_{H_i}$ is weakly contained in $\mathrm{Ind}_{H_i}^G(\lambda_{H_i})$,

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which in turn is just the left regular representation λ_G of G. Altogether, this shows that if 1_G is weakly contained in $\oplus_i \ell^2(G/H_i)$, then it is weakly contained in a multiple of λ_G. Since the latter is weakly equivalent to λ_G, 1_G follows weakly contained in λ_G, implying that G is amenable, a contradiction. □

If one defines the property (τ) for a group G with respect to a family $\mathcal{L}$ of subgroups by requiring that the trivial representation of G is not an accumulation point of $\oplus_{H\in\mathcal{L}} \ell^2(G/H)$, like in [**L94**], then the above lemma can be restated as follows.

If G is non-amenable, then it has the property (τ) with respect to the family $\mathcal{L}$ of its amenable subgroups.

Let now G act by automorphisms on a discrete group H and denote by σ the action it implements on the finite group von Neumann algebra $(L(H), \tau)$, then note that the ensuing representation of G on $L^2(L(H) \ominus \mathbb{C}1) = \ell^2(H \setminus \{e\})$ is equal to $\oplus_h \ell^2(G/\Gamma_h)$, where $\Gamma_h \subset G$ denotes the stabilizer of $h \in H \setminus \{e\}$, $\Gamma_h = \{\gamma \in G \mid \gamma(h) = h\}$. Lemma 1.6.4 thus gives us the following corollary.

Corollary 1.6.5. *Assume G is non-amenable and the stabilizer of each $h \in H \setminus \{e\}$ is amenable. For any non-amenable $\Gamma \subset G$ the action $\sigma_{|\Gamma}$ of Γ on $(L(H), \tau)$ is strongly ergodic. In particular, if $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$ is non-amenable then the restriction to Γ of the canonical action of $\mathrm{SL}(2, \mathbb{Z})$ on $(\mathbb{T}^2, \mu)$ is strongly ergodic.*

Finally, note that if one takes $H = G$ and lets G act on itself by conjugation, then Lemma 1.6.4 implies that if G is non-amenable and the commutant in G of any $h \in G \setminus \{e\}$ is amenable, then G is not inner amenable either.

2. 1-cohomology for quotients of Bernoulli actions

In this section we consider groups G satisfying some 'mild' rigidity property and construct many examples of actions (σ', G) for which we can explicitly calculate the 1-cohomology. The actions σ' are quotients of the Bernoulli G-actions σ (or of other 'malleable' actions of G), obtained by restricting σ to subalgebras that are fixed points of groups of automorphisms in the commutant of σ. The construction is inspired from [**P01**], where a similar idea is used to produce actions on the hyperfinite II_1 factor that have prescribed fundamental group and prescribed 1-cohomology.

This calculation of $H^1(\sigma', G)$ works whenever the 1-cohomology group of the 'initial' action (σ, G) is equal to the character group of G. For G weakly rigid and σ Bernoulli action, $H^1(\sigma, G)$ was shown equal to $\mathrm{Char}(G)$ in [**PSa03**], by adapting to the commutative case the proof of the general result for non-commutative Bernoulli actions in [**P01**]. We begin by extending the result in [**PSa03**] to a more general context in which the argument in [**P01**,**PSa03**] still works. Recall in this respect that an action $\sigma : G \to \mathrm{Aut}(X, \mu)$ is weakly mixing if and only if $L^2(X, \mu)$ has no σ-invariant finite-dimensional subspaces.

Definition 2.1 (see [P01, P03]). An integral preserving action $\sigma : G \to \mathrm{Aut}(A, \tau)$ of G on $A \simeq L^\infty(X, \mu)$ is *w-malleable* if there exist a decreasing sequence of abelian von Neumann algebras $\{(A_n, \tau)\}_n$ containing A and actions $\sigma_n : G \to \mathrm{Aut}(A_n, \tau)$, such

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that $\cap A_n = A$, $\sigma_{n|A_{n+1}} = \sigma_{n+1}$, $\sigma_{n|A} = \sigma$, $\forall n$, and such that for each n the flip automorphism α_1 on $A_n \bar{\otimes} A_n$, defined by $\alpha_1(x \otimes y) = y \otimes x$, $x, y \in A_n$, is in the connected component of the identity in the Polish group $\tilde{\sigma}_n(G)' \cap \mathrm{Aut}(A_n \bar{\otimes} A_n, \tau \times \tau)$, where $\tilde{\sigma}_n$ is the automorphism on $A_n \bar{\otimes} A_n$ given by $\tilde{\sigma}_n(g) = \sigma_n(g) \otimes \sigma_n(g)$, $g \in G$. If $H \subset G$ is a subgroup, then the action σ is *w-malleable w-mixing/*H (respectively *w-malleable mixing*) if the extensions σ_n can be chosen so that $\sigma_{n|H}$ are weakly mixing (respectively so that σ_n are mixing), $\forall n$.

Example 2.1′. Let $(g, s) \mapsto gs$ be an action of the group G on a set S, (Y_0, ν_0) be a non-trivial standard probability space. Let $(X, \mu) = \prod_s (Y_0, \nu_0)_s$ and denote by σ the *Bernoulli action* of G on $L^\infty(X, \mu)$ implemented by $\sigma_g((x_s)_s) = (x'_s)_s$, where $x'_s = x_{g^{-1}s}$. If either of the following conditions is satisfied, then σ is free.

(2.1.1) (Y_0, ν_0) has no atoms and $\forall g \neq e$, $\exists s \in S$ with $gs \neq s$.

(2.1.1′) (Y_0, ν_0) is arbitrary but $\forall g \neq e$, $\exists$ infinitely many $s \in S$ with $gs \neq s$.

Moreover, if $H \subset G$ is a subgroup such that

(2.1.2) $\forall S_0 \subset S$ finite, $\exists F_\infty \subset H$, an infinite set with $hS_0 \cap S_0 = \emptyset$, $\forall h \in F_\infty$,

then $\sigma_{|H}$ is weakly mixing. Also, if

(2.1.2′) $\forall S_0 \subset S$ finite, $\exists F_0 \subset G$, a finite set with $hS_0 \cap S_0 = \emptyset$, $\forall h \in G \setminus F_0$,

then σ is (strongly) mixing. If $S = G$ and we let G act on itself by left multiplication, then σ is called a *classic Bernoulli G-action*.

Lemma 2.2. *Let G be a group with an infinite subgroup $H \subset G$. A Bernoulli action (σ, G) satisfying* (2.1.1), (2.1.2) *is free, w-malleable w-mixing/H. Also, a classic Bernoulli action is free, w-malleable mixing* (*on G*).

Proof. The proof of [**P03**, 1.6.1] or of [**PSa03**, Lemma 3.2] shows that if $(Y_0, \nu_0) \simeq (\mathbb{T}, \lambda)$ then σ is malleable. For general (Y_0, ν_0), the proof of 3.6 in [**PSa03**] or 5.15 in [**P01**] shows that the action σ can be 'approximated from above' by Bernoulli actions with base space $\simeq (\mathbb{T}, \lambda)$, all satisfying (2.1.2), thus being w-malleable w-mixing/H. If σ is a classic Bernoulli action, then (2.1.2′) is satisfied, so σ is mixing. □

Definition 2.3. Let G be a group. An infinite subgroup $H \subset G$ is *wq-normal* in G if there exists a countable ordinal $\imath$ and a well ordered family of intermediate subgroups $H = H_0 \subset H_1 \subset \cdots \subset H_\jmath \subset \cdots \subset H_\imath = G$ such that for each $\jmath < \imath$, $H_{\jmath+1}$ is the group generated by the elements $g \in G$ with $gH_\jmath g^{-1} \cap H_\jmath$ infinite and such that if $\jmath \leqslant \imath$ has no 'predecessor' then $H_\jmath = \bigcup_{n<\jmath} H_n$. Note that this condition is equivalent to the following.

(2.3′) There exists no intermediate subgroup $H \subset K \subsetneq G$ such that $gKg^{-1} \cap K$ is finite $\forall g \in G \setminus K$. Equivalently, for all $H \subset K \subsetneq G$ there exists $g \in G \setminus K$, $gKg^{-1} \cap K$ is infinite.

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Indeed, if $H \subset G$ satisfies (2.3′), then it clearly satisfies Definition 2.3, by (countable transfinite) induction. Conversely, assume $H \subset G$ satisfies Definition 2.3 and let $K \subsetneq G$ be a subgroup containing H such that $gKg^{-1} \cap K$ finite $\forall g \in G \setminus K$. We show that this implies $K = G$, giving a contradiction. It is sufficient to show that $H_\jmath \subset K$ implies $H_{\jmath+1} \subset K$. If there exists $g \in H_{\jmath+1} \setminus K = H_{\jmath+1} \setminus K \cap H_{\jmath+1}$, then we would have $gKg^{-1} \cap K$ finite so in particular $gH_\jmath g^{-1} \cap H_\jmath$ finite. But this implies that all $g \in H_{\jmath+1}$ for which $gH_\jmath g^{-1} \cap H_\jmath$ is infinite lie in $K \cap H_{\jmath+1}$, thus $K \supset H_{\jmath+1}$ by the way $H_{\jmath+1}$ was defined.

By Definition 2.3 we see that if $H \subset G$ is wq-normal and G is embedded as a normal subgroup in some larger group $\bar{G}$ (or even merely as a *quasi-normal* subgroup $G \subset \bar{G}$, i.e. so that $gGg^{-1} \cap G$ has finite index in G, $\forall g \in \bar{G}$) then $H \subset \bar{G}$ is wq-normal. Note that if a group G has an infinite subgroup $H \subset G$ with the relative property (T), but not necessarily normal, then $G \times K$ is wq-rigid for any infinite group K. Indeed, because the inclusions $H \subset H \times K \subset G \times K$ clearly check the condition in Definition 2.3. On the other hand, (2.3′) shows that an inclusion of groups of the form $H \subset G = H * H'$, with H infinite and H' non-trivial, is not wq-normal.

Lemma 2.4. *Let G be an infinite group and σ a free ergodic measure preserving action of G on the probability space.*

(1) *Assume σ is weakly mixing and for each $\gamma \in \mathrm{Char}(G)$ denote w^γ the 1-cocycle $w^\gamma_g = \gamma(g)1$, $g \in G$. Then the group morphism $\gamma \mapsto w^\gamma$ is 1 to 1 and continuous from $\mathrm{Char}(G)$ into $\mathsf{H}^1(\sigma, G)$.*

(2) *Assume $H \subset G$ is an infinite subgroup of G such that either H is normal in G and $\sigma_{|H}$ is weakly mixing or H is wq-normal in G and $\sigma_{|H}$ is mixing. If $w \in \mathsf{Z}^1(\sigma, G)$ is such that $w_{|H} \in \mathrm{Char}(H)$, then $w \in \mathrm{Char}(G)$.*

Proof. (1) If $w_1(g) = u^* w_2(g) \sigma_g(u)$, $\forall g \in G$, then $\sigma_g(u) \in \mathbb{C}u$, $\forall g \in G$ and since σ is weakly mixing, this implies $u \in \mathbb{C}1$ so $w_1 = w_2$.

(2) In both cases, it is clearly sufficient to prove that if $g_0 \in G$ is such that $H' = g_0^{-1} H g_0 \cap H$ is infinite and σ is weakly mixing on H' with $w_{|H} = \gamma \in \mathrm{Char}(H)$, then $w_{g_0} \in \mathbb{C}1$. To see this, take $k \in H'$ and put $h = g_0 k g_0^{-1} \in H$. Then $hg_0 = g_0 k$. The 1-cocycle relation yields $w_h \sigma_h(w_{g_0}) = w_{g_0} \sigma_{g_0}(w_k)$. Since $w_h, w_k \in \mathbb{C}1$, this implies $\sigma_h(w_{g_0}) \in \mathbb{C}w_{g_0}$. Thus, $\sigma_h(w_{g_0}) \in \mathbb{C}w_{g_0}$, $\forall h \in g_0 H' g_0^{-1}$. Since $\sigma_{|g_0 H' g_0^{-1}}$ is weakly mixing (because $\sigma_{|H'}$ is weakly mixing), this implies $w_{g_0} \in \mathbb{C}1$. □

Corollary 2.5. *Let $H \subset G$, σ be as in Lemma 2.4 (2). If the restriction to H of any $w \in \mathsf{Z}^1(\sigma, G)$ is cohomologous to a character of H, then $\mathsf{H}^1(\sigma, G) = \mathrm{Char}(G)$.*

Theorem 2.6. *Let G be a countable discrete group with an infinite subgroup $H \subset G$ such that (G, H) has the relative property (T). Let σ be a free ergodic m.p. action of G on the probability space. Assume that σ is w-malleable w-mixing/H. Then the restriction to H of any 1-cocycle w for (σ, G) is cohomologous to a character on H. If in addition we assume that either H is normal in G, or $\sigma_{|H}$ is mixing with H wq-normal in G, then $\mathsf{H}^1(\sigma, G) = \mathrm{Char}(G)$.*

Proof. Let $w \in Z^1(\sigma, G)$. The proof that if σ is a classic (left) Bernoulli action (thus w-malleable mixing by Lemma 2.2) then $H^1(\sigma, G) = \mathrm{Char}(G)$ in [**P01**,**PSa03**] only uses the condition that σ is w-malleable w-mixing/H to derive that $w_{|H}$ is cohomologous to a character of H. But then Lemma 2.4 shows that w is cohomologous to a character of G, so $H^1(\sigma, G) = \mathrm{Char}(G)$ by Corollary 2.5. □

Lemma 2.7. *Let G, Γ be discrete groups with G infinite. Let σ be a free, weakly mixing m.p. action of G on the probability space and β a free measure preserving action of Γ on the same probability space which commutes with σ. If*

$$A^\Gamma \stackrel{\mathrm{def}}{=} \{a \in A \mid \beta_h(a) = a,\ \forall h \in \Gamma\},$$

then $\sigma_g(A^\Gamma) = A^\Gamma$, $\forall g \in G$, so $\sigma_g^\Gamma \stackrel{\mathrm{def}}{=} \sigma_{g|A^\Gamma}$ defines an integral preserving action of G on A^Γ.

Proof. Since $\beta_h(\sigma_g(a)) = \sigma_g(\beta_h(a)) = \sigma_g(a)$, $\forall h \in \Gamma$, $a \in A^\Gamma$, it follows that σ_g leaves A^Γ invariant $\forall g \in G$. □

Lemma 2.8. *With G, Γ, σ, β, A^Γ, σ^Γ as in Lemma 2.7, assume the action σ^Γ of G on A^Γ is free. For each $\gamma \in \mathrm{Char}(\Gamma)$ denote*

$$\mathcal{U}_\gamma \stackrel{\mathrm{def}}{=} \{v \in \mathcal{U}(A) \mid \beta_h(v) = \gamma(h)v,\ \forall h \in \Gamma\} \quad \text{and} \quad \mathrm{Char}_\beta(\Gamma) \stackrel{\mathrm{def}}{=} \{\gamma \in \mathrm{Char}(\Gamma) \mid \mathcal{U}_\gamma \neq \emptyset\}.$$

Then the following holds.

(1) *$\mathcal{U}_\gamma \mathcal{U}_{\gamma'} = \mathcal{U}_{\gamma\gamma'}$, $\forall \gamma, \gamma' \in \mathrm{Char}(G)$, and $\mathrm{Char}_\beta(\Gamma)$ is a countable group.*

(2) *If $\gamma_0 \in \mathrm{Char}(G)$, $\gamma \in \mathrm{Char}_\beta(\Gamma)$ and $v \in \mathcal{U}_\gamma$, then $w^{\gamma_0,\gamma}(g) \stackrel{\mathrm{def}}{=} \sigma_g(v)v^*\gamma_0(g)$ lies in A^Γ, for all $g \in G$, and $w^{\gamma_0,\gamma}$ defines a 1-cocycle for (σ^Γ, G) whose class in $H^1(\sigma^\Gamma, G)$ does not depend on the choice of $v \in \mathcal{U}_\gamma$.*

Proof. (1) If $v \in \mathcal{U}_\gamma$, $v' \in \mathcal{U}_{\gamma'}$, then $\beta_h(vv') = \beta_h(v)\beta_h(v') = \gamma(h)\gamma'(h)vv'$, so $vv' \in \mathcal{U}_{\gamma\gamma'}$. This also implies that $\mathrm{Char}_\beta(\Gamma)$ is a group. Noticing that $\{\mathcal{U}_\gamma\}_\gamma$ are mutually orthogonal in $L^2(A, \tau) = L^2(X, \mu)$, by the separability of $L^2(X, \mu)$, $\mathrm{Char}_\beta(\Gamma)$ follows countable.

(2) Since σ, β commute, $\sigma_g(\mathcal{U}_\gamma) = \mathcal{U}_\gamma$, $\forall g \in G$, $\gamma \in \mathrm{Char}_\beta(\Gamma)$. In particular, $\sigma_g(v)v^* \in \mathcal{U}_1 = \mathcal{U}(A^\Gamma)$, $\forall g \in G$, showing that the function $w^{\gamma_0,\gamma}$ takes values in $\mathcal{U}(A^\Gamma)$. Since $w^{\gamma_0,\gamma}$ is clearly a 1-cocycle for σ (in fact $w^{\gamma_0,\gamma} \sim_c \gamma_0 1$ as elements in $Z^1(\sigma, G)$), it follows that $w^{\gamma_0,\gamma} \in Z^1(\sigma^\Gamma, G)$.

If v' is another element in $\mathcal{U}_\gamma$, then $u = v'v^* \in \mathcal{U}(A^\Gamma)$ and the associated 1-cocycles $w^{\gamma_0,\gamma}$ constructed out of v, v' follow cohomologous via u, in $Z^1(\sigma^\Gamma, G)$. □

Theorem 2.9. *Let (σ, G), (β, Γ) be commuting, free m.p. actions on the same probability space, with G infinite and σ weakly mixing (as in Lemmas 2.7 and 2.8). Let A^Γ, (σ^Γ, G) be defined as in Lemma 2.7 and $\mathrm{Char}_\beta(\Gamma)$ as in Lemma 2.8. Also, for $\gamma_0 \in \mathrm{Char}(\Gamma)$, $\gamma \in \mathrm{Char}_\beta(\Gamma)$ let $w^{\gamma_0,\gamma}$ be defined as in Lemma 2.8 (2). If $\mathrm{Char}_\beta(\Gamma)$ is given the discrete topology, then $\Delta : \mathrm{Char}(G) \times \mathrm{Char}_\beta(\Gamma) \to H^1(\sigma^\Gamma, G)$, defined by letting $\Delta(\gamma_0, \gamma)$ be the class of $w^{\gamma_0,\gamma}$ in $H^1(\sigma^\Gamma, G)$, is a 1 to 1 continuous group morphism. If in addition $H^1(\sigma, G) = \mathrm{Char}(G)$, then Δ is an isomorphism of topological groups.*

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Proof. The map Δ is clearly a group morphism and continuous. To see that it is 1 to 1 let $\gamma_0 \in \mathrm{Char}(G)$, $\gamma \in \mathrm{Char}_\beta(\Gamma)$ and $v \in \mathcal{U}_\gamma$ and represent the element $\Delta(\gamma_0, \gamma) \in H^1(\sigma^\Gamma, G)$ by the 1-cocycle $w_g^{\gamma_0,\gamma} = \sigma_g(v)v^*\gamma_0(g)$, $g \in G$. If $w^{\gamma_0,\gamma} \sim_c \mathbf{1}$, then there exists $u \in \mathcal{U}(A^\Gamma)$ such that $\sigma_g(u)u^* = \sigma_g(v)v^*\gamma_0(g)$, $\forall g \in G$. Thus, if we denote $u_0 = uv^* \in \mathcal{U}(A)$ then $\sigma_g(u_0)u_0^* = \gamma_0(g)1$, $\forall g$. It follows that $\sigma_g(\mathbb{C}u_0) = \mathbb{C}u_0$, $\forall g \in G$, and since σ is weakly mixing this implies $u_0 \in \mathbb{C}1$ and $\gamma_0 = 1$. Thus, $v \in \mathbb{C}u \subset \mathcal{U}(A^\Gamma) = \mathcal{U}_1$, showing that $\gamma = 1$ as well.

If we assume $H^1(\sigma, G) = \mathrm{Char}(G)$ and take $w \in Z^1(\sigma^\Gamma, G)$, then we can view w as a 1-cocycle for σ. But then $w \sim_c \gamma_0 1$, for some $\gamma_0 \in \mathrm{Char}(G)$. Since σ is ergodic, there exists a unique $v \in \mathcal{U}(A)$ (up to multiplication by a scalar) such that $w_g = \sigma_g(v)v^*\gamma_0(g)$, $\forall g \in G$. Since w is A^Γ-valued, $\sigma_g(v)v^* \in \mathcal{U}(A^\Gamma)$, $\forall g$. Thus $\sigma_g(v)v^* = \beta_h(\sigma_g(v)v^*) = \sigma_g(\beta_h(v))\beta_h(v)^*$, $\forall g$. By the uniqueness of v this implies that $\beta_h(v) = \gamma(h)v$, for some scalar $\gamma(h)$. The map $\Gamma \ni h \mapsto \gamma(h)$ is easily seen to be a character, so $w = w^{\gamma_0,\gamma}$ showing that $(\gamma_0, \gamma) \mapsto w^{\gamma_0,\gamma}$ is onto.

Since $H^1(\sigma, G) = \mathrm{Char}(G)$ is compact, by § 1.1 and Proposition 1.6.1 σ is strongly ergodic so σ^Γ is also strongly ergodic. Thus $H^1(\sigma^\Gamma, G)$ is Polish, with $\Delta(\mathrm{Char}(G))$ a closed subgroup, implying that $\Delta(\mathrm{Char}_\beta(\Gamma)) \simeq H^1(\sigma^\Gamma)/\Delta(\mathrm{Char}(G))$ is Polish. Since it is also countable, it is discrete. Thus, Δ is an isomorphism of topological groups. □

Note that in the above proof, from the hypothesis $H^1(\sigma, G) = \mathrm{Char}(G)$ we only used the following fact.

(2.9′) There exists a continuous group morphism $H^1(\sigma, G) \ni \hat{w}_0 \mapsto w_0 \in Z^1(\sigma, G)$ retract of the quotient map $Z^1(\sigma, G) \to H^1(\sigma, G)$ such that each w_0 is A^Γ-valued (so that it can be viewed as an element $w_0 \in Z^1(\sigma^\Gamma, G)$).

Thus, the above proof of Theorem 2.9 shows that $H^1(\sigma^\Gamma, G) \simeq H^1(\sigma, G) \times \mathrm{Char}_\beta(\Gamma)$ whenever condition (2.9′) is satisfied.

Lemma 2.10. *Let G be an infinite group and σ be the Bernoulli action of G on $(X, \mu) = \prod_g(\mathbb{T}, \lambda)_g$. With the notation of Lemma 2.8 and Theorem 2.9, for any countable abelian group Λ there exists a countable abelian group Γ and a free action β of Γ on (X, μ) such that $\mathrm{Char}_\beta(\Gamma) = \Lambda$, $[\sigma, \beta] = 0$ and $\sigma_{|A^\Gamma}$ is a free action of G. Moreover, if Λ is finite then one can take $\Gamma = \Lambda$ and β to be any action of $\Gamma = \Lambda$ on (X, μ) that commutes with σ and such that $\sigma \times \beta$ is a free action of $G \times \Gamma$.*

Proof. Let Γ be a countable dense subgroup in the (second countable) compact group $\hat{\Lambda}$ and let μ_0 be the Haar measure on $\hat{\Lambda}$. Let β_0 denote the action of Γ on $L^\infty(\hat{\Lambda}, \mu_0) = L(\Lambda)$ given by $\beta_0(h)(u_\gamma) = \gamma(h)u_\gamma$, $\forall h \in \Gamma$, where $\{u_\gamma\}_{\gamma \in \Lambda} \subset L(\Lambda)$ denotes the canonical basis of unitaries in the group von Neumann algebra $L(\Lambda)$ and $\gamma \in \Lambda$ is viewed as a character on $\Gamma \subset \hat{\Lambda}$. Denote $A_0 = L^\infty(\hat{\Lambda}, \mu_0) \bar{\otimes} L^\infty(\mathbb{T}, \lambda)$ and let τ_0 be the state on A_0 given by the product measure $\mu_0 \times \lambda$. Let β denote the product action of Γ on $\bar{\otimes}_{g \in G}(A_0, \tau_0)_g$ given by $\beta(h) = \otimes_g(\beta_0(h) \otimes \mathrm{id})_g$.

Since

$$(A_0, \tau_0) \simeq \left(L^\infty(\mathbb{T}, \lambda), \int \cdot \, \mathrm{d}\lambda\right),$$

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we can view σ as the Bernoulli action of G on $A = \bar{\otimes}_g (A_0, \tau_0)_g$. By the construction of β we have $[\sigma, \beta] = 0$. Also, the fixed point algebra A^Γ contains a σ-invariant subalgebra on which σ acts as the (classic) Bernoulli action. Thus, the restriction $\sigma^\Gamma = \sigma_{|A^\Gamma}$ is a free, mixing action of G. Finally, we see by construction that $\mathrm{Char}_\beta(\Gamma) = \Lambda$.

The last part is trivial, once we notice that if the action $\sigma \times \beta$ of $G \times \Gamma$ on A is free, then the action σ^Γ of G on A^Γ is free. □

From now on, it will be convenient to use the following notation.

2.11. Notation

We denote by $w\mathcal{T}$ the class of discrete countable groups G which have infinite, wq-normal subgroups $H \subset G$ such that the pair (G, H) has the relative property (T).

Note that all infinite property (T) groups are in the class $w\mathcal{T}$. Also, by Definition 2.3 it follows that $w\mathcal{T}$ is closed to inductive limits and normal extensions (i.e. if $G \in w\mathcal{T}$ and $G \subset \bar{G}$ is a normal inclusion of groups then $\bar{G} \in w\mathcal{T}$). In particular, if $G \in w\mathcal{T}$ and K is a group acting on G by automorphisms then $G \rtimes K \in w\mathcal{T}$. For instance, if G is infinite with property (T) and K is an arbitrary group, then $G \times K \in w\mathcal{T}$. Other examples of groups in the class $w\mathcal{T}$ are $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ [**K67**, **Ma82**], and more generally $\mathbb{Z}^2 \rtimes \Gamma$ for $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$ non-amenable (cf. [**B91**]).

Corollary 2.12. *Let $G \in w\mathcal{T}$. Given any countable discrete abelian group Λ there exists a free ergodic m.p. action σ_Λ of G on the standard non-atomic probability space such that $\mathsf{H}^1(\sigma_\Lambda, G) = \mathrm{Char}(G) \times \Lambda$. Moreover, if σ denotes the Bernoulli action of G on $(X, \mu) = \prod_g (\mathbb{T}, \mu)_g$ then all σ_Λ can be taken to be quotients of $(\sigma, (X, \mu))$ and such that the exact sequences of 1-cohomology groups $\mathsf{B}^1(\sigma_\Lambda) \hookrightarrow \mathsf{Z}^1(\sigma_\Lambda) \to \mathsf{H}^1(\sigma_\Lambda) \to 1$ are split. Thus, $\mathsf{Z}^1(\sigma_\Lambda) \simeq \mathsf{H}^1(\sigma_\Lambda) \times \mathsf{B}^1(\sigma_\Lambda)$.*

Proof. Since Bernoulli actions with base space $(\mathbb{T}, \mu)$ are malleable mixing, by [**PSa03**] or Theorem 2.6 above we have $\mathsf{H}^1(\sigma, G) = \mathrm{Char}(G)$ and the statement follows by Lemma 2.10 and Theorem 2.9. The fact that σ_Λ can be constructed so that its exact sequence of 1-cohomology groups is split is clear from the construction in the proof of Lemma 2.10, which shows that one can select $u_\gamma \in \mathcal{U}_\gamma$ such that $u_\gamma u_{\gamma'} = u_{\gamma\gamma'}$, $\forall \gamma, \gamma' \in \Lambda = \mathrm{Char}_\beta(\Gamma)$. □

Corollary 2.13. *If $G \in w\mathcal{T}$, then G has a continuous family of mutually non-stably orbit-equivalent free ergodic m.p. actions on the probability space, indexed by the classes of virtual isomorphism of all countable, discrete, abelian groups.*

Proof. If we denote $K = \mathrm{Char}(G)$, then K is compact and open in $K \times \Lambda$. Thus, for any isomorphism $\theta : K \times \Lambda_1 \simeq K \times \Lambda_2$, $\theta(K) \cap K$ has finite index both in K and in $\theta(K)$. Thus, Λ_1, Λ_2 must be virtually isomorphic. It is trivial to see that there are continuously many virtually non-isomorphic countable, discrete, abelian groups, for instance by considering all groups $\sum_{n \in I} \mathbb{Z}/p_n\mathbb{Z}_n$ with $I \subset \mathbb{N}$ and p_n the prime numbers and noticing that there are only countably many groups in each virtual isomorphism class. □

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Proof of Theorem 0.1 and Corollary 0.2. Theorem 0.1 now follows trivially from Theorems 2.6 and 2.9, while Corollary 0.2 is an immediate consequence of Theorem 0.1 and Lemma 2.10. □

Corollary 2.14. *Let G be an infinite property* (T) *group and σ the Bernoulli action of G on $\prod_g (\mathbb{T}, \lambda)_g$. Denote $\sigma_n = \sigma^{\mathbb{Z}/n\mathbb{Z}}$, where $\mathbb{Z}/n\mathbb{Z}$ acts as a (diagonal) product action on $\prod_g (\mathbb{T}, \lambda)_g$ and $\sigma^{\mathbb{Z}/n\mathbb{Z}}$ is defined out of σ as in Lemma 2.7. Then σ_n is not w-malleable, $\forall n \geqslant 2$. If in addition G is an infinite conjugacy class (ICC) group, then the inclusion of factors $N = A^{\mathbb{Z}/n\mathbb{Z}} \rtimes_{\sigma_n} G \subset A \rtimes_\sigma G = M$ has Jones index $[M : N] = n$ [**J83**], with M constructed from a Bernoulli action of an ICC Kazhdan group, while N cannot be constructed from such data, i.e. N cannot be realized as $N = A_0 \rtimes_{\sigma_0} G_0$ with G_0 ICC Kazhdan group and (σ_0, G_0) a Bernoulli action.*

Proof. If σ_n were w-malleable, then by Theorem 2.6 we would have $H^1(\sigma_n, G) = \mathrm{Char}(G)$. But by Theorem 2.9 we have $H^1(\sigma_n, G) = \mathrm{Char}(G) \times \mathbb{Z}/n\mathbb{Z}$ and since G has (T), $\mathrm{Char}(G)$ is finite so $\mathrm{Char}(G) \ncong \mathrm{Char}(G) \times \mathbb{Z}/n\mathbb{Z}$ for $n \geqslant 2$, a contradiction.

If $N = A_0 \rtimes_{\sigma_0} G_0$ for some Bernoulli action σ_0 of an ICC property (T) group G_0, then by the superrigidity result [**P04**, 7.6] it would follow that (σ_n, G) and (σ_0, G_0) are conjugate actions, with $G \simeq G_0$, showing in particular that $H^1(\sigma_0, G_0) = H^1(\sigma_n, G)$. Since σ_0 is a Bernoulli G_0-action and $G_0 \simeq G$ has property (T), by [**PSa03**] we have $H^1(\sigma_0, G_0) = \mathrm{Char}(G)$, while, by Corollary 2.12 therein, $H^1(\sigma_n, G) = \mathrm{Char}(G) \times \mathbb{Z}/n\mathbb{Z}$, a contradiction. □

3. 1-cohomology for actions of free products of groups

We now use Theorem 2.9 to calculate the 1-cohomology for quotients of Bernoulli G-action in the case G is a free product of groups, $G = *_{n \geqslant 0} G_n$, with all G_n either amenable or in the class $w\mathcal{T}$, at least one of them with this latter property. Rather than locally compact as in Theorem 2.9, the H^1-group is 'huge' in this case, having either $\mathcal{U}(A)$ or $\mathcal{U}(A)/\mathbb{T}$ as direct summand. Since, however, $\mathcal{U}(A)$, $\mathcal{U}(A)/\mathbb{T}$ are easily seen to be connected, the quotient of H^1 by the connected component of 1 provides a 'nicer' group which is calculable and still an invariant to stable orbit equivalence. This allows us to distinguish many actions for each such G.

Lemma 3.1. *Let $G_0, G_1, \ldots$ be a sequence of groups and $G = *_{n \geqslant 0} G_n$ their free product. Let $\sigma : G \to \mathrm{Aut}(X, \mu)$ be a free ergodic measure preserving action of G on the probability space.*

(1) *For each sequence $w = (w_i)_{i \geqslant 0}$ with $w_i \in Z^1(\sigma_{|G_i}, G_i)$, $i \geqslant 0$, there exists a unique $\Delta(w) \in Z^1(\sigma, G)$ such that $\Delta(w)_{|G_i} = w_i$, $\forall i \geqslant 0$. The map $w \mapsto \Delta(w)$ is an isomorphism between the Polish groups $\prod_{i \geqslant 0} Z^1(\sigma_{|G_i}, G_i)$ and $Z^1(\sigma, G)$.*

(2) *If $\sigma_{|G_0}$ is ergodic and $K \subset Z^1(\sigma_{|G_0})$ is a Polish subgroup which maps 1 to 1 onto its image $K/\sim_c$ in $H^1(\sigma_{|G_0}, G_0)$, then Δ defined in* (1) *implements a 1 to 1 continuous morphism Δ' from $K \times \prod_{j \geqslant 1} Z^1(\sigma_{|G_j}, G_j)$ into $H^1(\sigma, G)$. If in addition $K/\sim_c = H^1(\sigma_{|G_0}, G_0)$ is also surjective (so if the 1-cohomology exact sequence for*

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$H^1(\sigma_{|G_0}, G_0)$ is split), then Δ' is an onto isomorphism of topological groups. In particular, if $\sigma_{|G_0}$ is weakly mixing then Δ implements a 1 to 1 continuous morphism Δ' from $\mathrm{Char}(G_0) \times \prod_{j\geqslant 1} Z^1(\sigma_{|G_j}, G_j)$ into $H^1(\sigma, G)$ and if $H^1(\sigma_{|G_0}, G_0) = \mathrm{Char}(G_0)$ then Δ' is an onto isomorphism of Polish groups.

Proof. Part (1) is evident by the isomorphism between $Z^1(\sigma, G)$ and $\mathrm{Aut}_0(A \rtimes_\sigma G; A)$ or by noticing that a function $w : G \to \mathcal{U}(A)$ is a 1-cocycle for σ if and only if $\{w_g u_g\}_{g\in G} \subset A \rtimes_\sigma G$ is a representation of G.

(2) If there existed $u \in \mathcal{U}(A)$ such that $\Delta(w)_g = \sigma_g(u)u^*$, $\forall g$, where $w = (w_0, w_1, \dots)$ for some $w_0 \in K$, $w_i \in Z^1(\sigma_{|G_i})$, $i \geqslant 1$, then $\sigma_{g_0}(u)u^* = w_0(g_0)$, $\forall g_0 \in G_0$, implying that $w_0 = 1$. By the ergodicity of $\sigma_{|G_0}$ this implies $u \in \mathbb{C}1$. Thus $w_i = 1$, $\forall i$, so that $w = (1, 1, \dots, 1)$. If in addition $H^1(\sigma_{|G_0}) \simeq K$, then Δ' follows onto because Δ is onto and because of the way Δ is defined. $\square$

From Lemma 3.1 (2) above we see that in case $\sigma_{|G_0}$ is weakly mixing, then in order to calculate $H^1(\sigma, G)$ for $G = *_{n\geqslant 0} G_n$ we need to know $H^1(\sigma_{|G_0}, G_0)$ and $Z^1(\sigma_{|G_i}, G_i)$ for $i \geqslant 1$. By Corollary 2.12, all these groups can be calculated if σ is a Bernoulli G-action, or certain quotients of it, and $G_0 \in w\mathcal{T}$. The groups Z^1 can in fact be calculated for amenable equivalence relations as well, as shown below.

For convenience, we denote by $\mathbb{G}$ the Polish group $\mathcal{U}(A)$ (with the topology given by convergence in norm $\|\cdot\|_2$), where $A = L^\infty(\mathbb{T}, \mu)$ as usual, and by $\mathbb{G}_0$ the 'pointed' space $\mathbb{G}/\mathbb{T}$. It is easy to see that $\mathbb{G}$ is contractible (use for instance the proofs in [**PT93**]), so that both $\mathbb{G}$, $\mathbb{G}_0$ are connected. Also, $\mathbb{G}^\infty \simeq \mathbb{G}$ and $\mathbb{G} \times \mathbb{G}_0 \simeq \mathbb{G}_0$.

Lemma 3.2.

(1) *If $G_i \in w\mathcal{T}$, Λ is a countable discrete abelian group and σ' is an action of G_i of the form σ_Λ, as constructed in Lemma 2.10, then $Z^1(\sigma') \simeq \mathbb{G}_0 \times \mathrm{Char}(G_i) \times \Lambda$.*

(2) *If σ' is a free m.p. action of a finite group with $n \geqslant 2$ elements on the probability space (X, μ) and $Y \subset X$ is a measurable subset with $\mu(Y) = (n-1)/n$, then $H^1(\sigma') = \{\mathbf{1}\}$ and $B^1(\sigma') = Z^1(\sigma') \simeq \mathcal{U}(L^\infty(Y, \mu))$. In particular, if (X, μ) is non-atomic, then $Z^1(\sigma') \simeq \mathbb{G}$.*

(3) *If σ' is a free ergodic m.p. action of an infinite amenable group, then $Z^1(\sigma') \simeq \mathbb{G}$. Moreover, $B^1(\sigma')$ is proper and dense in $Z^1(\sigma')$.*

Proof. Part (1) is clear by Theorems 2.6 and 2.9, Lemma 2.10 and the last part of Corollary 2.12, while (2) is a folklore result.

(3) By 1.4 and the results of Dye and Ornstein–Weiss [**D63, OW80**], we may assume the infinite amenable group is equal to $\mathbb{Z}$ and that the action is mixing (say a Bernoulli action). Identify $Z^1(\sigma', \mathbb{Z})$ with $\mathrm{Aut}_0(A \rtimes \mathbb{Z}, A)$ and notice that if $u = u_1 \in M = A \rtimes_{\sigma'} \mathbb{Z}$ denotes the canonical unitary implementing the single automorphism $\sigma'(1)$ of A then any $v \in \mathcal{U}(A)$ implements a unique automorphism $\theta^v \in \mathrm{Aut}_0(M, A)$ satisfying $\theta^v(au) = avu$. Also, it is trivial to see that $\mathcal{U}(A) \ni v \mapsto \theta^v \in \mathrm{Aut}_0(M, A)$ is an isomorphism

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of topological groups. The fact that $B^1(\sigma')$ is dense in $Z^1(\sigma')$ is immediate to deduce from [**OW80**, **CFW81**] and part (2). Also, by Lemma 2.4 (1) we have $\mathbb{T} \subset H^1(\sigma')$, so the subgroup $B^1(\sigma')$ is proper in $Z^1(\sigma')$. $\square$

Theorem 3.3. *Let $\{G_n\}_{n\geqslant 0}$ be a sequence of groups, at least two of them non-trivial, and denote $G = *_{n\geqslant 0} G_n$ their free product. Let $J = \{j \geqslant 0 \mid G_j \in w\mathcal{T}\}$ and assume $0 \in J$ and G_j amenable for all j not in J. Let Λ be a countable discrete abelian group and denote by σ_Λ the action of the group G constructed in Corollary 2.12, as a quotient of the classic Bernoulli G-action. We have the following isomorphisms of Polish groups.*

(1) *If $J = \{0\}$, then $H^1(\sigma_\Lambda, G) \simeq \mathbb{G} \times \mathrm{Char}(G_0) \times \Lambda$.*

(2) *If $J \neq \{0\}$ (i.e. $|J| \geqslant 2$), then*

$$H^1(\sigma_\Lambda, G) \simeq \mathbb{G}_0^{|J|-1} \times \prod_{j\in J} \mathrm{Char}(G_j) \times \Lambda^{|J|}.$$

Proof. This is now trivial by Lemma 3.1 and 3.2 and by the properties of $\mathbb{G}$, $\mathbb{G}_0$. $\square$

Definition 3.4. Let σ be a free ergodic m.p. action of an infinite countable discrete group G on a standard probability space. We denote by $\tilde{H}^1(\sigma, G)$ the quotient of $Z^1(\sigma, G)$ by the connected component $Z_0^1(\sigma, G)$ of $\mathbf{1}$ in $Z^1(\sigma, G)$. Since $Z_0^1(\sigma, G)$ is a closed subgroup in $H^1(\sigma, G)$, $\tilde{H}^1(\sigma, G)$ with its quotient topology is a totally disconnected Polish group. Note that, since $B^1(\sigma, G)$ is connected (being the image of the connected topological group $\mathbb{G}$), one has $B^1(\sigma, G) \subset Z_0^1(\sigma, G)$ and $\tilde{H}^1(\sigma, G)$ coincides with the quotient of $H^1(\sigma, G)$ by the connected component of $\mathbf{1}$ in $H^1(\sigma, G)$. Also, since $H^1(\sigma, G)$ is invariant to stable orbit equivalence, so is $\tilde{H}^1(\sigma, G)$. If $\mathcal{G}$ is an ergodic full pseudogroup as in § 1.3, then $\tilde{H}^1(\mathcal{G})$ is defined similarly and has similar properties.

Corollary 3.5. *Under the same assumptions as in Theorem 3.3, if all G_j, $j \in J$, have finite character group (for instance if they have the property (T)), or more generally if $\mathrm{Char}(G_j)$ is totally disconnected $\forall j \in J$, then $\tilde{H}^1(\sigma_\Lambda, G) \simeq \prod_{j\in J} \mathrm{Char}(G_j) \times \Lambda^{|J|}$ as Polish groups.*

Proof. Trivial by Theorem 3.3 and the comments in Definition 3.4. $\square$

Corollary 3.6. *Let $H_1, H_2, \dots, H_k$ be infinite property (T) groups and $0 \leqslant n \leqslant \infty$. The free product group $H_1 * H_2 * \cdots * H_k * \mathbb{F}_n$ has uncountably many non-stably orbit-equivalent free ergodic m.p. actions.*

Proof. Clear by Corollary 3.5 and by the argument in the proof of Corollary 2.13. $\square$

Note that the groups $G = *_{n\geqslant 0} G_n$ for which we calculated the 1-cohomology for quotients of Bernoulli G-actions in this section do have infinite subgroups $H_0 \subset G$ such that (G, H_0) has the relative property (T): for instance, if $H_0 \subset G_0$ is the infinite wq-normal subgroup of $G_0 \in w\mathcal{T}$ such that (G_0, H_0) has the relative property (T), then (G, H_0) has the relative property (T). It is trivial to see though that $gG_0g^{-1} \cap G_0$ is

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finite $\forall g \in G \setminus G_0$, so that by (2.3′) H_0 is not wq-normal in G. Furthermore, from Theorem 2.6 and Lemma 3.2, we deduce the following result (which can in fact also be proven using Bass–Serre theory).

Corollary 3.7. *If $G = K_1 * K_2$ with K_1, K_2 non-trivial groups, then G is not in the class $w\mathcal{T}$.*

Proof. If K_1, K_2 are finite, then G has the Haagerup approximation property [**H79**], so it cannot contain an infinite subgroup with the relative property (T) (see, for example, [**P03**]). If say K_1 is infinite and we let σ be a Bernoulli G-action, then by Lemma 3.1 (2) and Lemma 3.2, $H^1(\sigma, G)$ contains either $\mathbb{G}$ or $\mathbb{G}_0$ as closed subgroups. Since the latter are not compact (not even locally compact), this contradicts Theorem 2.6. □

Remarks 3.8. (1) Let $\bar{H}^1(\sigma, G)$ denote the quotient of $Z^1(\sigma, G)$ by the closure of $B^1(\sigma, G)$ in $Z^1(\sigma, G)$, or equivalently the quotient of $H^1(\sigma, G)$ by the closure of $\hat{\mathbf{1}}$ in $H^1(\sigma, G)$. We see by the definition that $\bar{H}^1(\sigma, G)$ is invariant to stable orbit equivalence. One can use arguments similar to the ones in [**P01**, **P03**] to prove that if $G \in w\mathcal{T}$ has an infinite amenable quotient K with $\pi : G \to K$ the quotient map, and $\sigma_g = \sigma_0(g) \otimes \sigma_1(\pi(g))$, where σ_0 is a Bernoulli G-action and σ_1 a Bernoulli K-action, then $\bar{H}^1(\sigma, G) = \mathrm{Char}(G)$, while σ is not strongly ergodic in this case. By using the construction in the proof of Theorem 2.9, from the action σ one can then construct free ergodic m.p. actions σ_Λ of G such that $\bar{H}^1(\sigma_\Lambda, G) = \mathrm{Char}(G) \times \Lambda$, for any countable abelian groups Λ.

(2) Corollary 2.12 and Theorem 3.3 provide computations of the 1-cohomology group $H^1(\sigma_\Lambda, G)$ for the family of actions σ_Λ constructed in Corollary 2.12, for most groups G having infinite subgroups with the relative property (T). However, groups having the Haagerup compact approximation property [**H79**], such as the free groups $\mathbb{F}_n$, $2 \leqslant n \leqslant \infty$, do not contain infinite subgroups with the relative property (T) (see, for example, [**P02**]). The problem of calculating the H^1-groups for Bernoulli G-actions and their quotients σ_Λ when G are free groups, or other non-amenable groups with the Haagerup property, remains open. Note, however, that by Lemma 3.1 (1) and Lemma 3.2 (3) if σ is an arbitrary free ergodic m.p. action of $\mathbb{F}_n$ on the probability space then $Z^1(\sigma, \mathbb{F}_n) \simeq \mathcal{U}(A)^n = \mathbb{G}^n \simeq \mathbb{G}$, so $\tilde{H}^1(\sigma, G) = \{1\}$. (In fact Z^1 is even contractible.) Also, by Lemma 3.1 (2) one has an embedding of $\mathbb{T} \times \mathbb{G}^{n-1}$ into $H^1(\sigma, \mathbb{F}_n)$ whenever one of the generators of $\mathbb{F}_n$ acts weak mixing (e.g. when σ is a classic Bernoulli action). All this indicates that the H^1-invariant may be less effective in recognizing orbit-inequivalent actions of the free groups.

Related to this, our last result below emphasizes the limitations of the ‘deformation/rigidity’ techniques of [**P01**, **P03**] when trying to prove that $H^1(\sigma, G) = \mathrm{Char}(G)$ for arbitrary (commutative and non-commutative) Bernoulli G-actions, beyond the class of w-rigid groups G dealt with in [**P01**, **PSa03**] and the class $w\mathcal{T}$ in this paper.

Thus, we let this time G be an arbitrary non-amenable group and σ be the action of G on the finite von Neumann algebra $(N, \tau) = \bar{\otimes}_g (N_0, \tau_0)_g$ by (left) Bernoulli shifts, with the ‘base’ (N_0, τ_0) either the diffuse abelian von Neumann algebra $L^\infty(\mathbb{T}, \lambda)$, or a

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finite-dimensional factor $M_{n\times n}(\mathbb{C})$, or the hyperfinite II_1 factor R. By [**P01**, **P03**] σ is malleable. More precisely, there exists a continuous action α of $\mathbb{R}$ on $(N \mathbin{\bar\otimes} N, \tau\otimes\tau)$ such that $[\alpha, \tilde\sigma] = 0$ and $\alpha_1(N\otimes 1) = 1\otimes N$, where $\tilde\sigma_g = \sigma_g\otimes\sigma_g$, $g\in G$.

Proposition 3.9. *Let $w \in Z^1(\sigma, G)$. The following conditions are equivalent.*

(i) *w is cohomologous to a character of G.*

(ii) *For sufficiently small $|t|$, the representation π_t of G on $L^2(N,\tau)\mathbin{\bar\otimes} L^2(N,\tau)$ given by $f\mapsto (w_g\otimes 1)\tilde\sigma_g(f)\alpha_t(w_g^*\otimes 1)$ is a direct sum between a multiple of the trivial representation of G and a subrepresentation of a multiple of the left regular representation of G.*

Proof. If $w_g = \gamma(g)u\sigma_g(u^*)$, $g\in G$, for some $\gamma\in \mathrm{Char}(G)$ and $u\in\mathcal{U}(N)$, then $U_t(f) = (u\otimes 1)f\alpha_t(u^*\otimes 1)$, for $f\in L^2(N,\tau)\mathbin{\bar\otimes} L^2(N,\tau)$ defines a unitary operator that intertwines the representations π_0 and π_t, which thus follow equivalent. But $\pi_0 = \tilde\sigma$ is a direct sum between one copy of the trivial representation of G and a subrepresentation of a multiple of the left regular representation of G (see, for example, [**S80**, **J83b**]). This shows that (i) $\Longrightarrow$ (ii).

Conversely, since $\lim_{t\to 0}\|\pi_t(g)(1) - 1\|_2 = 0$, $\forall g$, where $1 = 1_N\otimes 1_N$, if π_t satisfy (ii), then for t small enough $\pi_t(g)(1)$ follows close to 1 uniformly in $g\in G$, i.e. $(w_g\otimes 1)\alpha_t(w_g^*\otimes 1)$, $g\in G$, is uniformly close to 1 (in the norm $\|\cdot\|_2$). But then the argument in [**P01**] or [**PSa03**] shows that w is cohomologous to a character, thus (ii) $\Longrightarrow$ (i). □

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References

[B91] M. Burger, Kazhdan constants for $SL(3,\mathbb{Z})$, *J. Reine Angew. Math.* **413** (1991), 36–67.

[C75] A. Connes, Outer conjugacy classes of automorphisms of factors, *Annls Scient. Éc. Norm. Sup.* **8** (1975), 383–419.

[C75b] A. Connes, Sur la classification des facteurs de type II, *C. R. Acad. Sci. Paris Sér. I* **281** (1975), 13–15.

[C80] A. Connes, A type II_1 factor with countable fundamental group, *J. Operat. Theory* **4** (1980), 151–153.

[CW80] A. Connes and B. Weiss, Property (T) and asymptotically invariant sequences, *Israel J. Math.* **37** (1980), 209–210.

[CFW81] A. Connes, J. Feldman and B. Weiss, An amenable equivalence relation is generated by a single transformation, *Ergod. Theory Dynam. Syst.* **1** (1981), 431–450.

[CoZ89] M. Cowling and R. Zimmer, Actions of lattices in $Sp(n,1)$, *Ergod. Theory Dynam. Syst.* **9** (1989), 221–237.

[dHV89] P. de la Harpe and A. Valette, *La propriété T de Kazhdan pour les groupes localement compacts*, Astérisque, Volume 175 (Société Mathématique de France, Paris, 1989).

[D63] H. Dye, On groups of measure preserving transformations, II, *Am. J. Math.* **85** (1963), 551–576.

[FM77-1] J. Feldman and C. C. Moore, Ergodic equivalence relations, cohomology, and von Neumann algebras, I, *Trans. Am. Math. Soc.* **234** (1977), 289–324.

[FM77-2] J. Feldman and C. C. Moore, Ergodic equivalence relations, cohomology, and von Neumann algebras, II, *Trans. Am. Math. Soc.* **234** (1977), 325–359.

[Fu99] A. Furman, Orbit equivalence rigidity, *Ann. Math.* **150** (1999), 1083–1108.

[G00] D. Gaboriau, Cout des rélations d'équivalence et des groupes, *Invent. Math.* **139** (2000), 41–98.

[G02] D. Gaboriau, Invariants ℓ^2 de rélations d'équivalence et de groupes, *Publ. Math. IHES* **95** (2002), 93–150.

[GP05] D. Gaboriau and S. Popa, An uncountable family of non orbit equivalent actions of $\mathbb{F}_n$, *J. Am. Math. Soc.* **18** (2005), 547–559.

[Ge87] S. L. Gefter, On cohomologies of ergodic actions of a T-group on homogeneous spaces of a compact Lie group (in Russian), in *Operators in functional spaces and questions of function theory*, Collected Scientific Works, pp. 77–83 (Naukova Dumka, Kiev, 1987).

[GeGo88] S. L. Gefter and V. Y. Golodets, Fundamental groups for ergodic actions and actions with unit fundamental groups, *Publ. RIMS Kyoto* **6** (1988), 821–847.

[H79] U. Haagerup, An example of a non-nuclear C^*-algebra which has the metric approximation property, *Invent. Math.* **50** (1979), 279–293.

[Hj02] G. Hjorth, A converse to Dye's theorem, *Trans. Am. Math. Soc.* **357** (2005), 3083–3103.

[Jol05] P. Jolissaint, On property (T) for pairs of topological groups, *Enseign. Math.* **51** (2005), 31–45.

[J83] V. F. R. Jones, Index for subfactors, *Invent. Math.* **72** (1983), 1–25.

[J83b] V. F. R. Jones, A converse to Ocneanu's theorem, *J. Operat. Theory* **10** (1983), 121–123.

[JS87] V. F. R. Jones and K. Schmidt, Asymptotically invariant sequences and approximate finiteness, *Am. J. Math.* **109** (1987), 91–114.

[K67] D. Kazhdan, Connection of the dual space of a group with the structure of its closed subgroups, *Funct. Analysis Applic.* **1** (1967), 63–65.

[L94] A. Lubotzky, *Discrete groups, expanding graphs and invariant measures* (Birkhäuser, Basel, 1994).

[Ma82] G. Margulis, Finitely-additive invariant measures on Euclidian spaces, *Ergod. Theory Dynam. Syst.* **2** (1982), 383–396.

[MoSh06] N. Monod and Y. Shalom, Orbit equivalence rigidity and bounded cohomology, *Ann. Math.*, in press.

[M82] C. C. Moore, *Ergodic theory and von Neumann algebras*, Proceedings of Symposia of Pure Mathematics, Volume 38, pp. 179–226 (American Mathematical Society, Providence, RI, 1982).

[MvN43] F. Murray and J. von Neumann, Rings of operators, IV, *Ann. Math.* **44** (1943), 716–808.

[OW80] D. Ornstein and B. Weiss, Ergodic theory of amenable group actions, I, The Rohlin lemma, *Bull. Am. Math. Soc.* **2** (1980), 161–164.

[P86] S. Popa, Correspondences, INCREST preprint (1986, unpublished).

[P01] S. Popa, Some rigidity results for non-commutative Bernoulli shifts, *J. Funct. Analysis* **230** (2006), 273–328.

[P02] S. Popa, On a class of type II_1 factors with Betti numbers invariants, *Ann. Math.*, in press.

[P03] S. Popa, Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups, I, *Invent. Math.*, in press.

[P04] S. Popa, Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups, II, *Invent. Math.*, in press.

[PSa03] S. Popa and R. Sasyk, On the cohomology of Bernoulli actions, preprint math.OA/0310211.

[PT93] S. Popa and M. Takesaki, The topological structure of the unitary and automorphism groups of a factor, *Commun. Math. Phys.* **155** (1993), 93–101.

[S80] K. Schmidt, Asymptotically invariant sequences and an action of $SL(2,\mathbb{Z})$ on the 2-sphere, *Israel J. Math.* **37** (1980), 193–208.

[S81] K. Schmidt, Amenabilty, Kazhdan's property T, strong ergodicity and invariant means for ergodic group-actions, *Ergod. Theory Dynam. Syst.* **1** (1981), 223–236.

[Si55] I. M. Singer, Automorphisms of finite factors, *Am. J. Math.* **177** (1955), 117–133.

[V05] A. Valette, Group pairs with the relative property (T) from arithmetic lattices, *Geom. Dedicata* **112** (2005), 183–196.

[Z84] R. Zimmer, *Ergodic theory and semisimple groups* (Birkhäuser, Boston, MA, 1984).