Proceedings of the Edinburgh Mathematical Society (2021) **64**, 129–138
doi:10.1017/S0013091521000031

PEAKING AND INTERPOLATION BY COMPLEX POLYNOMIALS

THOMAS H. MACGREGOR[1] AND MICHAEL P. STERNER[2]

[1] *State University of New York at Albany, Professor Emeritus, 60 S. Washington St., Athens, NY 12015, USA*
[2] *Department of Biology, Chemistry, and Mathematics, University of Montevallo, Station 6493, Harman Hall, Montevallo, AL 35115, USA* (sternerm@montevallo.edu)

(Received 15 July 2019; first published online 5 May 2021)

Abstract Classical results about peaking from complex interpolation theory are extended to polynomials on a closed disk, and on the complement of its interior. New results are obtained concerning interpolation by univalent polynomials on a Jordan domain whose boundary satisfies certain smoothness conditions.

Keywords: peaking; univalent functions; interpolation; polynomials

2020 *Mathematics subject classification* Primary 30E05; 30C10;
Secondary 30C20

1. Introduction

This paper concerns finite peaking and boundary interpolation problems for complex-valued polynomials. The main results are about univalent polynomials.

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The following theorem is proven in [1, p. 101, Theorem 9].

Theorem 1.1. *Let Z be a finite set of n elements in $\partial\Delta$. Then there is a polynomial P of degree n such that $\Re P(z) > 0$ for $z \in \overline{\Delta} - Z$ and $\Re P(z) = 0$ for $z \in Z$.*

Using Theorem 1.1 and a suitable Möbius transformation implies that there is a rational function R of degree n such that $|R(z)| < 1$ for $z \in \overline{\Delta} - Z$ and $|R(z)| = 1$ for $z \in Z$. Theorem 2.1 shows that R may be replaced by a polynomial of degree n. Theorem 2.2 asserts that there is a polynomial P of degree n such that $|P(z)| > 1$ for $z \in \overline{\Delta} - Z$ and $|P(z)| = 1$ for $z \in Z$.

Our next result gives general information about finite boundary interpolation by univalent polynomials.

Theorem 1.2. *Suppose that Φ is a Jordan domain and $\partial\Phi$ is a curve belonging to C^2. Let $\{z_1, z_2, \ldots, z_n\}$ be distinct points on $\partial\Phi$ and let $\{w_1, w_2, \ldots, w_n\}$ be distinct points in*

$\mathbb{C}$. *Then there is a polynomial* P *which is univalent in a neighbourhood of* $\overline{\Phi}$ *and satisfies* $P(z_k) = w_k$ *for* $k = 1, 2, \ldots, n$.

Our proof of Theorem 1.2 shows that there are infinitely many polynomials P which provide the stated interpolation.

Interior interpolation by univalent functions is quite different. For functions f analytic and univalent in Δ, interpolation is possible when $n = 1$ if f satisfies the growth theorem [5, p. 33]. For general n, the numbers $\{z_1, z_2, \ldots, z_n\}$ in Δ and the numbers $\{f(z_1), f(z_2), \ldots, f(z_n)\}$ are restricted by the Goluzin inequalities [5, p. 128].

The following result was proved in [3, p. 568].

Theorem 1.3. *For* $k = 1, 2, \ldots, n$ *let* $z_k = e^{i\theta_k}$ *and* $w_k = e^{i\varphi_k}$ *where* $\theta_1 < \theta_2 < \cdots < \theta_n < \theta_1 + 2\pi$ *and* $\varphi_1 < \varphi_2 < \cdots < \varphi_n < \varphi_1 + 2\pi$. *Then there is a polynomial* P *which is univalent in* $\overline{\Delta}$ *and satisfies* $|P(z)| < 1$ *for* $|z| \leq 1$ *and* $z \neq z_k$ $(k = 1, 2, \ldots, n)$, *and* $P(z_k) = w_k$ *for* $k = 1, 2, \ldots, n$.

Theorem 3.1 generalizes Theorem 1.3 for univalent polynomials mapping a Jordan domain into another Jordan domain. Theorem 3.2 treats the more general situation in which the points $\{w_k\}$ for the interpolation $P(z_k) = w_k$ $(k = 1, 2, \ldots, n)$ may repeat and have any order on the boundary of the Jordan domain.

The references [2, 7, 8, 11, 13] also concern finite boundary interpolation and peaking by analytic functions whose domain or range is an open disk or an open half-plane.

The results in §2 rely in part on facts about self-inversive polynomials, that is polynomials p of degree n such that $p(z) = z^n\overline{p(1/\overline{z})}$. This relates to the material contained in §4.4 and Chapter 7 of [12].

2. Peaking on $\overline{\Delta}$ and on $\mathbb{C} - \Delta$

Theorem 2.1. *Let* Z *be any finite subset of* $\partial\Delta$ *of* n *elements. There is a polynomial* P *of degree* n *such that* $|P(z)| = 1$ *for* $z \in Z$ *and* $|P(z)| < 1$ *for* $z \in \overline{\Delta} - Z$.

Proof. Let $Z = \{z_1, z_2, \ldots, z_n\}$ and for $k = 1, 2, \ldots, n$ let $z_k = e^{i\theta_k}$ where $0 \leq \theta_k < 2\pi$. For θ real let $T(\theta) = \Pi_{k=1}^{n}[1 - \cos(\theta - \theta_k)]$. If U and V are trigonometric polynomials then so is the product UV and $\deg(UV) = \deg U + \deg V$. Since $1 - \cos(\theta - \theta_k)$ is a trigonometric polynomial of degree 1, this implies that T is a trigonometric polynomial of degree n. Also, $T(\theta_k) = 0$ for $k = 1, 2, \ldots, n$ and $T(\theta_k) > 0$ for $0 \leq \theta_k < 2\pi$ and $\theta \neq \theta_k$ $(k = 1, 2, \ldots, n)$.

Let $M \geq \max T$ and for θ real let $S(\theta) = M - T(\theta)$. Then S is a nonnegative trigonometric polynomial of degree n. By the Fejér lemma [12, p. 150, Thm 4.3.5], there is a polynomial Q of degree n such that $|Q(e^{i\theta})|^2 = S(\theta)$ for θ real. See also [9, p. 77, problem 40]. Let $P = Q/\sqrt{M}$. Then P is a polynomial of degree n, $|P(z)| = 1$ for $z \in Z$ and $|P(z)| < 1$ for $z \in \partial\Delta - Z$. The maximum modulus theorem implies that $|P(z)| < 1$ for $|z| < 1$. □

Theorem 2.2. *Let* Z *be any finite subset of* $\partial\Delta$ *of* n *elements. There is a polynomial* P *of degree* n *such that* $|P(z)| = 1$ *for* $z \in Z$ *and* $|P(z)| > 1$ *for* $z \in \overline{\Delta} - Z$.

Proof. Let $Z = \{z_1, z_2, \ldots, z_n\}$ and let T denote the trigonometric polynomial defined in the proof of Theorem 2.1. For θ real let $S(\theta) = T(\theta) + 1$. Then S is a nonnegative trigonometric polynomial of degree n. Hence there is a polynomial P of degree n such that $|P(e^{i\theta})|^2 = S(\theta)$ for θ real. We have $|P(e^{i\theta_k})| = 1$ for $k = 1, 2, \ldots, n$ and $|P(e^{i\theta})| > 1$ for $0 \le \theta < 2\pi$ and $\theta \ne \theta_k$ $(k = 1, 2, \ldots, n)$. Suppose that P has no zeros in Δ. Then the minimum modulus theorem implies that $|P(z)| > 1$ for $z \in \Delta$. Hence, in this case, the proof is complete.

Now suppose that P has at least one zero in Δ. We claim that $P(0) \ne 0$. Otherwise, $Q(z) = P(z)/z$ defines a polynomial of degree $n-1$ and then $|Q(e^{i\theta})|^2$ defines a trigonometric polynomial of degree at most $n-1$. This trigonometric polynomial has a minimum at n values of θ in $[0, 2\pi)$. This is impossible. This verifies our claim and hence P has the form $P(z) = b(z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_\ell)(z-\sigma_1)(z-\sigma_2)\cdots(z-\sigma_m)$ where $b \ne 0$, $\ell \ge 1$, $m \ge 0$, $0 < |\zeta_k| < 1$ for $k = 1, 2, \ldots, \ell$, $|\sigma_k| > 1$ for $k = 1, 2, \ldots, m$ and $\ell + m = n$. The case $m = 0$ corresponds to the factors $(z - \sigma_k)$ not being present in the product. If $|z| = 1$ and $\zeta \in \mathbb{C}$ then $|z - \zeta| = |1 - \overline{\zeta} z|$. Hence the polynomial $R(z) = b(1-\overline{\zeta}_1 z)(1-\overline{\zeta}_2 z)\cdots(1-\overline{\zeta}_\ell z)(z-\sigma_1)(z-\sigma_2)\cdots(z-\sigma_m)$ satisfies $|R(e^{i\theta})| = |P(e^{i\theta})|$ for θ real. Since R has no zeros in Δ, the minimum modulus theorem implies that R satisfies the conclusions about P stated in the theorem. □

Theorems 2.1 and 2.2 do not hold for a polynomial of degree less than n. To see this, suppose that $n > 1$ and P is a polynomial of degree m where $0 < m < n$. Then $U(\theta) = |P(e^{i\theta})|^2$ for θ real defines a trigonometric polynomial of degree at most m. Hence there are at most m values of θ in $[0, 2\pi)$ where U has a relative maximum and at most m values of θ where U has a relative minimum.

We ask the question of describing all domains Ω such that for every positive integer n and for every set $\{z_1, z_2, \ldots, z_n\}$ of distinct numbers on $\partial\Delta$ there exists a polynomial P of degree n such that $P(z_k) \in \partial\Omega$ for $k = 1, 2, \ldots, n$ and $P(z) \in \Omega$ for $|z| \le 1$ and $z \ne z_k (k = 1, 2, \ldots, n)$. An open half-plane, an open disk, and the exterior of an open disk are such domains.

3. Boundary interpolation by univalent polynomials

Let n be a positive integer. An arc $w = \lambda(t)$, $a \le t \le b$ belongs to C^n provided that λ has n derivatives, $\lambda^{(n)}$ is continuous, and $\lambda'(t) \ne 0$. When Φ is a Jordan domain, the assumptions that $\partial\Phi \in C^n$ and $\lambda^{(n)}$ satisfies a Lipschitz condition together imply that each conformal mapping of Δ onto Φ has an extension to $\overline{\Delta}$ which has a continuous n^{th} derivative [10, p. 49].

Let Λ be the Jordan curve $w = \lambda(t)$, $a \le t \le b$, and let Φ denote the interior of Λ. Let F denote a conformal mapping of Δ onto Φ. Then F extends to a homeomorphism of $\overline{\Delta}$ onto $\overline{\Phi}$, and F gives a homeomorphism of $\partial\Delta$ onto Λ. We say that Λ is in conformal order provided that the mapping from $[\theta_0, \theta_0 + 2\pi)$ to $[a, b)$ given by $\theta \to w \to t$ where $w = F(e^{i\theta})$ and $t = \lambda^{-1}(w)$ and where $F(e^{i\theta_0}) = \lambda(a) = \lambda(b)$, is strictly increasing. When Λ is rectifiable this is equivalent to each point in Φ has index 1 with respect to Λ.

Let $\{w_1, w_2, \ldots, w_n\}$ be distinct points on a Jordan curve Λ. Let F be described as above and for each k let $F(z_k) = w_k$ where $|z_k| = 1$. We say that $\{w_1, w_2, \ldots, w_n\}$ is in conformal order on Λ provided that $z_k = e^{i\theta_k}$ and $\theta_1 < \theta_2 < \cdots < \theta_n < \theta_1 + 2\pi$.

https://doi.org/10.1017/S0013091521000031 Published online by Cambridge University Press

Proof of Theorem 1.2. Suppose the curve $\Lambda = \partial\Phi$ is given by $z = \lambda(t)$, $a \le t \le b$, where λ' and λ'' exist and are continuous and $\lambda'(t) \ne 0$, and that this parametrization gives Λ in conformal order. We may assume that $z_k = \lambda(t_k)$ for $k = 1, 2, \ldots, n$ where $a \le t_1 < t_2 < t_3 < \cdots < t_n < b$. We now construct small ε-perturbations $\Lambda' = \Lambda'(\varepsilon)$ of the curve Λ in the direction of the outward pointing normal and show that they are also Jordan curves, and, for sufficiently small ε, are disjoint from Λ.

For each positive real number ε, let the curve $\Lambda' = \Lambda'(\varepsilon)$ be defined by $w = \mu(t) = \lambda(t) - \varepsilon i\lambda'(t)$, $a \le t \le b$. Let $t_0 \in [a, b]$ and let $\lambda'(t_0) = e^{i\theta}|\lambda'(t_0)|$ where θ is real. Since $\lambda'(t_0) \ne 0$ and λ' is continuous there is a neighbourhood of t_0, say N, such that $\Re[e^{-i\theta}\lambda'(t)] \ge \frac{1}{2}|\lambda'(t_0)|$ for $t \in N$. Let $M = \max|\lambda''(t)|$. For $t \in N$ and ε small, we have $\Re[e^{-i\theta}\mu'(t)] = \Re[e^{-i\theta}\lambda'(t)] - \varepsilon\Re[ie^{-i\theta}\lambda''(t)] \ge \frac{1}{2}|\lambda'(t_0)| - \varepsilon M \ge \frac{1}{3}|\lambda'(t_0)|$. Hence, if $t_1, t_2 \in N$, $t_1 < t_2$ and ε is small, then $\Re[e^{-i\theta}(\mu(t_2) - \mu(t_1))] = \int_{t_1}^{t_2} \Re[e^{-i\theta}\mu'(t)]\,\mathrm{d}t \ge \frac{1}{3}|\lambda'(t_0)|(t_2 - t_1)$. This implies that $\mu(t_2) \ne \mu(t_1)$. Hence, for each t there is a neighbourhood of t such that μ is injective in that neighbourhood for all small ε.

We claim that μ is injective on $[a, b)$ for all small ε. Suppose this is false. Then there are sequences $\{\varepsilon_n\}$, $\{t_n\}$, and $\{t'_n\}$ such that $\varepsilon_n > 0$, $\varepsilon_n \to 0$, $\mu(t_n) = \mu(t'_n)$ and $t_n \ne t'_n$ for every n. By considering subsequences of $\{t_n\}$ and $\{t'_n\}$, we may assume that $t_0 = \lim t_n$ and $t'_0 = \lim t'_n$ exist. Since $\mu(t'_n) = \mu(t_n)$, this is the same as $\lambda(t'_n) - \varepsilon_n i\lambda'(t'_n) = \lambda(t_n) - \varepsilon_n i\lambda'(t_n)$, which yields $\lambda(t'_0) = \lambda(t_0)$. By re-parametrizing the curve as follows:

$$\lambda_1(t) = \begin{cases} \lambda(t) & \text{if } a + \delta \le t \le b \\ \lambda(t + a - b) & \text{if } b \le t \le b + \delta, \end{cases}$$

for some small $\delta > 0$, we may assume that $\{t_0, t'_0\}$ is distinct from the endpoints of the interval on which λ is defined. Note that this does not change any of the properties required of the curve. Because Λ is simple this implies $t'_0 = t_0$. The previous argument gives a neighbourhood of t_0, say N, such that μ is injective in N for all small ε. This contradicts $\mu(t'_n) = \mu(t_n)$ where $t'_n \ne t_n$ when n is large.

Because of the smoothness conditions on λ, we have that $\lim_{t\to a^+} \lambda'(t) = \lim_{t\to b^-} \lambda'(t)$, and so Λ' is closed. We have shown that Λ' is a Jordan curve for all small ε. Let Φ' denote the interior of Λ'. Also, $\mu'(t) = \lambda'(t) - \varepsilon i\lambda''(t)$ shows that μ' is continuous. Let $m = \min|\lambda'(t)|$ and $M = \max|\lambda''(t)|$. Then $|\mu'(t)| \ge m - \varepsilon M > 0$ for all small ε. Therefore, the curve Λ' belongs to C^1 for all small ε.

We claim that $\Lambda' \cap \Lambda = \emptyset$ for all small ε. Suppose this is false. Then there are sequences $\{\varepsilon_n\}$, $\{t_n\}$, and $\{t'_n\}$ such that $\varepsilon_n > 0$, $\varepsilon_n \to 0$, $\mu(t'_n) = \lambda(t_n)$ for every n, and $t_0 = \lim t_n$ and $t'_0 = \lim t'_n$ exist. Since $\mu(t'_n) = \lambda(t_n)$ is the same as $\lambda(t'_n) - \varepsilon_n i\lambda'(t'_n) = \lambda(t_n)$ this yields $\lambda(t'_0) = \lambda(t_0)$. As before, by a re-parametrization, we may assume that $\{t_0, t'_0\} \ne \{a, b\}$. Because Λ is simple we obtain $t'_0 = t_0$. We continue our argument assuming that $t_0 = 0$. By a translation and a rotation and a re-parametrization, we also may assume that $\lambda(t) = t + if(t)$ for t in a neighbourhood of 0, where the real-valued function f satisfies $f(0) = 0$, $f'(0) = 0$, and f'' is continuous at 0. Then $\mu(t) = t + \varepsilon f'(t) + i[f(t) - \varepsilon]$. In order to obtain a contradiction, we again consider the equality $\mu(s) = \lambda(t)$. We know it holds at least for the values $t = t_n$ and $s = t'_n$ mentioned above. We wish to analyse the local behaviour of f near $t = 0$ starting from this equality, which is equivalent to $s + \varepsilon f'(s) = t$ and $f(s) - \varepsilon = f(t)$, which implies $f(s) - \varepsilon = f[s + \varepsilon f'(s)]$. We have

https://doi.org/10.1017/S0013091521000031 Published online by Cambridge University Press

$f(s) = bs^2 + o(s^2)$ and $f'(s) = 2bs + o(s)$ as $s \to 0$, where $b = \frac{1}{2}f''(0)$. Whenever we consider a fixed but sufficiently small ε, we will arrive at a contradiction. The equalities above give $-\varepsilon + bs^2 + o(s^2) = bs^2 + 4b^2\varepsilon s^2[1 + b\varepsilon] + o(s^2)$. This implies $\varphi(s) = 0$ where $\varphi(s) = \varepsilon + 4b^2\varepsilon s^2[1 + b\varepsilon] + o(s^2)$. If $b \geq 0$ then $\varphi(s) > 0$ for all small s. This gives a contradiction when $\varepsilon = \varepsilon_n$, $s = t'_n$ and n is large. Now consider the case $b < 0$. Then for small $\varepsilon > 0$ we have $1 + b\varepsilon > 0$. Again we get the contradiction that $\varphi(s) > 0$ where $\varepsilon = \varepsilon_n$, $s = t'_n$ and n is large. This completes the proof that $\Lambda' \cap \Lambda = \emptyset$ for all small ε.

By the last result, we see that either $\Lambda' \subset \Phi$ for all small ε or $\Lambda' \subset \mathbb{C} - \overline{\Phi}$ for all small ε. For any t_0 the vector $-\varepsilon i\lambda'(t_0)$ is in the direction of an outer normal to $\overline{\Phi}$ at $z_0 = \lambda(t_0)$. Because Λ belongs to C^2 and is simple, it follows that $w = \lambda(t_0) - \varepsilon i\lambda'(t_0)$ belongs to $\mathbb{C} - \overline{\Phi}$ for all small ε. Therefore, $\Lambda' \subset \mathbb{C} - \overline{\Phi}$ for all small ε.

Let $\zeta \in \Phi$. Then

$$\int_{\Lambda'} \frac{1}{w - \zeta}\,\mathrm{d}w = \int_a^b \frac{\mu'(t)}{\mu(t) - \zeta}\,\mathrm{d}t = \int_a^b \frac{\lambda'(t) - \varepsilon i\lambda''(t)}{\lambda(t) - \varepsilon i\lambda'(t) - \zeta}\,\mathrm{d}t.$$

For all small $\varepsilon > 0$, $w = \mu(t) \in \mathbb{C} - \overline{\Phi}$ and Φ is open. Hence, there is a constant $d > 0$ such that $|\mu(t) - \zeta| \geq d$ for all t and for all small ε. This implies that the limit of the last integral exists as $\varepsilon \to 0^+$. That limit equals $\int_a^b (\lambda'(t)/(\lambda(t) - \zeta))\,\mathrm{d}t$, which is the same as $\int_\Lambda (1/(z - \zeta))\,\mathrm{d}z$. Since $(1/2\pi i)\int_\Lambda (1/(z - \zeta))\,\mathrm{d}z = 1$ and $(1/2\pi i)\int_{\Lambda'} (1/(w - \zeta))\,\mathrm{d}w$ is an integer for all small ε, we conclude that $(1/2\pi i)\int_{\Lambda'} (1/(w - \zeta))\,\mathrm{d}w = 1$ for all small ε. Therefore, $\zeta \in \Phi'$. We have shown that $\Phi \subset \Phi'$. Because $\Lambda' \cap \Lambda = \emptyset$ this yields $\overline{\Phi} \subset \Phi'$.

Henceforth, we let ε be a fixed positive real number such that the various properties of Φ' described above are valid. In what follows, we observe that the doubly connected domain bounded by the curves Λ and $\Lambda'(\varepsilon)$ is the union of disjoint line segments and, using this fact, we construct a new smooth Jordan curve Λ'' inscribed in this domain. Part of its boundary will consist of circular arcs. Later on, it will be important to consider the conformal map onto the interior domain of this new curve. The part of the boundary consisting of circular arcs will allow us to use Schwarz reflection to extend the mapping analytically to a larger domain precisely around the points related to our interpolation.

Let $z \in \Lambda$. We will obtain closed disks D such that $D \cap \overline{\Phi} = z$. We do this by making a translation and rotation of Φ so that $z = 0$ and a subarc of Λ containing 0 is given by $y = g(x)$, $-c \leq x \leq c$, where $c > 0$ and the function g satisfies $g(0) = 0$, $g'(0) = 0$ and g'' is continuous at 0. Also, the orientation of this subarc corresponds to decreasing values of x. For $r > 0$ let $D(r)$ denote the closed disk with centre $(0, r)$ and radius r. Then $h(x) = r - \sqrt{r^2 - x^2}$, $-r \leq x \leq r$, gives the lower semicircle on $\partial D(r)$. Hence $h(x) = (1/2r)x^2 + O(x^4)$ as $x \to 0$. Also $g(x) = bx^2 + o(x^2)$ as $x \to 0$, where $b = \frac{1}{2}g''(0)$. Thus, if $1/2r > b$ then $h(x) > g(x)$ for all small $x \neq 0$. Therefore, there is a real number d such that $0 < d < c$ and the closed disk $D(r)$ does not meet the subarc of Λ given by $y = g(x)$, $-d \leq x \leq d$, except at 0, for all small $r > 0$. Because Λ is simple and the subarc has the stated orientation, this implies that $D(r) \cap \Lambda = \emptyset$ and then $D(r) \cap \overline{\Phi} = \emptyset$ for all small r. Hence, for each k there is a closed disk D_k such that $D_k \cap \overline{\Phi} = z_k$, $D_k \subset \Phi'$ and the collection $\{D_1, D_2, \ldots, D_n\}$ is pairwise disjoint.

For $\varepsilon > 0$ and $t \in [a, b)$ let $L(t) = L(t, \varepsilon)$ denote the closed line segment $\{\zeta : \zeta = \lambda(t) - xi\lambda'(t), 0 \leq x \leq \varepsilon\}$. Then for all small ε, $L(s) \cap L(t) = \emptyset$ for $s \neq t$. This can be shown

https://doi.org/10.1017/S0013091521000031 Published online by Cambridge University Press

using an argument similar to that used to prove $\Lambda' \cap \Lambda = \emptyset$ for all small ε. We also see that $\overline{\Phi'} - \Phi = \bigcup_{a \le t < b} L(t)$.

For each k let β_k denote the closed semicircle on ∂D_k having the midpoint z_k and oriented clockwise. Let q_k and r_k denote the endpoints of β_k with q_k, z_k, r_k in clockwise order. Then $q_k \in L(t_k)$ for some t_k. Let p_k denote the endpoint of $L(t_k)$ which is on Λ'. Let α_k denote the closed line segment from p_k to q_k. We have $r_k = L(t'_k)$ for some t'_k. Let s_k denote the endpoint of $L(t'_k)$ on Λ'. Let γ_k denote the closed line segment from r_k to s_k. For $k = 1, 2, \ldots, n-1$ let δ_k denote the subarc of Λ' from s_k to p_{k+1} with the same orientation as Λ'. Let δ_n denote the subarc of Λ' from s_n to p_1 with the same orientation as Λ'. Let Λ'' denote the curve obtained by successively joining the arcs $\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \ldots, \alpha_n, \beta_n, \gamma_n$, and δ_n. Then Λ'' is a Jordan curve in conformal order. Let Φ'' denote the interior of Λ''. Then $\Phi \subset \Phi''$ and $\overline{\Phi} \cap \Lambda'' = \{z_1, z_2, \ldots, z_n\}$.

For each k let η_k denote a closed semicircle of radius r having the midpoint w_k and a counterclockwise order. We can choose the radius r small enough that $\eta_1, \eta_2, \ldots, \eta_n$ are pairwise disjoint. Let w'_k and w''_k denote the endpoints of η_k with w'_k, w_k, w''_k in counterclockwise order. There is a simple arc ν_1 from w''_1 to w'_2 which does not meet $\bigcup_{k=1}^n \eta_k$ except for its endpoints. There is a simple arc ν_2 from w''_2 to w'_3 which does not meet $\bigcup_{k=1}^n \eta_k \bigcup \nu_1$ except for its endpoints. There is a simple arc ν_3 from w''_3 to w'_4 which does not meet $(\bigcup_{k=1}^n \eta_k) \bigcup (\bigcup_{j=1}^2 \nu_j)$ except for its endpoints. We continue in this way ending with a simple arc ν_{n-1} from w''_{n-1} to w'_n which does not meet $(\bigcup_{k=1}^n \eta_k) \bigcup (\bigcup_{j=1}^{n-2} \nu_j)$ except for its endpoints. Let Θ denote the curve obtained by successively joining the arcs $\eta_1, \nu_1, \eta_2, \nu_2, \ldots, \eta_{n-1}, \nu_{n-1}, \eta_n$. This is a simple arc. There are such arcs $\nu_1, \nu_2, \ldots, \nu_{n-1}$ which are as smooth as we like and where ν_k joins smoothly at w''_k and at w'_{k+1}. In particular, there are such arcs for which $\Theta \in C^2$. The arc Θ starts at w'_1 and ends at w''_n. Let Θ be given by $w = \xi(t)$, $a \le t \le b$, where ξ' and ξ'' exist and are continuous and $\xi'(t) \ne 0$. For $\varepsilon > 0$ let the curve Θ' be defined by $w = \psi(t) = \xi(t) + i\varepsilon\xi'(t)$, $a \le t \le b$. By arguments given earlier about Λ', we see that for all small ε the curve Θ' is simple and $\Theta' \cap \Theta = \emptyset$. Henceforth, we let ε be a fixed positive real number for which these properties of Θ' are valid. Let $\psi_1 = \psi(a)$ and $\psi_2 = \psi(b)$. Let L_1 denote the closed line segment from ψ_1 to w'_1, let L_2 denote the closed line segment from w''_n to ψ_2, and let Θ'' denote the opposite arc of Θ'. Let Γ denote the curve obtained by successively joining Θ, L_2, Θ'', and L_1. Then Γ is a Jordan curve. Let Ω denote the interior of Γ. Because the vector $i\varepsilon\xi'(t)$ is in the direction of an inner normal to $\overline{\Omega}$ at $\xi(t)$ for $a < t < b$, we see that the points $w_1, w_2, \ldots, w_n$ are in conformal order on Γ.

Let F denote a conformal mapping of Δ onto Φ''. Then F extends to a homeomorphism of $\overline{\Delta}$ onto $\overline{\Phi''}$. For each k let $F(\sigma_k) = z_k$ where $|\sigma_k| = 1$. Since $\{z_1, z_2, \ldots, z_n\}$ is in conformal order on Φ'' we have $\sigma_k = e^{i\theta_k}$ and $\theta_1 < \theta_2 < \cdots < \theta_n < \theta_1 + 2\pi$.

Let G denote a conformal mapping of Δ onto Ω. Then G extends to a homeomorphism of $\overline{\Delta}$ onto $\overline{\Omega}$. For each k let $G(\tau_k) = w_k$ where $|\tau_k| = 1$. Since $\{w_1, w_2, \ldots, w_n\}$ is in conformal order on Γ, we have $\tau_k = e^{i\varphi_k}$ and $\varphi_1 < \varphi_2 < \cdots < \varphi_n < \varphi_1 + 2\pi$.

There is a function H which is analytic and univalent in $\overline{\Delta}$ such that $|H(\sigma)| \le 1$ for $|\sigma| \le 1$ and $H(\sigma_k) = \tau_k$ for $k = 1, 2, \ldots, n$ [3, p. 559]. Let $I = G \circ H \circ F^{-1}$. Then I is analytic and univalent in Φ'' and maps Φ'' onto Ω, with $I(z_k) = w_k$ for $k = 1, 2, \ldots, n$. For each k, I maps a subarc of β_k containing z_k in its interior continuously and injectively onto a subarc of η_k containing w_k in its interior. By the reflection principle, I extends

https://doi.org/10.1017/S0013091521000031 Published online by Cambridge University Press

analytically to a neighbourhood of z_k. Because of how the reflection takes place, there is a neighbourhood of z_k, say N_k, such that I is analytic and univalent in $\Psi = \Phi'' \bigcup (\bigcup_{k=1}^{n} N_k)$. Moreover, for small N_k, Ψ is simply connected.

Below and in the proof of Theorem 3.1, we make use of the following fact: If the functions f_n are analytic on an open set Ψ, $f_n \to f$ uniformly on compact subsets of Ψ, and f is univalent in Ψ, then for each fixed compact subset Σ of Ψ there is an integer N such that f_n is univalent in Σ for all $n \geq N$. To see this, for each function g analytic in Ψ let D_g denote the difference quotient of g, that is, for $z \in \Psi$ and $w \in \Psi$, we have $D_g(z, w) = (g(z) - g(w))/(z - w)$ if $w \neq z$ and $D_g(z, z) = g'(z)$. Then D_g is analytic in $\Psi \times \Psi$. If g is univalent in Ψ, then D_g does not vanish. Let Σ be any compact subset of Ψ. We claim that $D_{f_n} \to D_f$ uniformly on $\Sigma \times \Sigma$. It suffices to show that for each $(z_0, w_0) \in \Sigma \times \Sigma$, there is uniform convergence on $C \times D$, where C and D are open disks in Ψ with C centred at z_0 and D centred at w_0. This follows from the Heine–Borel Theorem, where the collection $\{C \times D\}$ is the open covering of the compact set $\Sigma \times \Sigma$. We first consider the case $w_0 = z_0$. Let C be an open disk in Ψ centred at z_0 with $\overline{C} \subset \Psi$. Let E be a closed disk in Ψ centred at z_0 and having radius greater than the radius of C. Suppose that $(z, w) \in C \times C$. If $w \neq z$ then Cauchy's formula yields

$$D_{f_n}(z, w) = \frac{1}{2\pi i} \int_{\partial E} \frac{f_n(\zeta)}{(\zeta - z)(\zeta - w)} \, \mathrm{d}\zeta.$$

This equality also holds when $w = z$. This implies that

$$D_{f_n}(z, w) \to \frac{1}{2\pi i} \int_{\partial E} \frac{f(\zeta)}{(\zeta - z)(\zeta - w)} \, \mathrm{d}\zeta$$

uniformly on $C \times C$. The last integral equals $D_f(z, w)$. Next, we consider the case $w_0 \neq z_0$. Let C be an open disk in Ψ centred at z_0 and let D be an open disk in Ψ centred at w_0 with $\overline{C} \subset \Psi$, $\overline{D} \subset \Psi$, and $\overline{C} \cap \overline{D} = \phi$. Then there is a positive constant c such that $|z - w| > c$ for $(z, w) \in C \times D$. Therefore,

$$\frac{f_n(z) - f_n(w)}{z - w} \to \frac{f(z) - f(w)}{z - w}$$

uniformly on $C \times D$. The number $d = \min |D_f(z, w)|$ where (z, w) varies in $\Sigma \times \Sigma$ is positive. Thus the uniform convergence $D_{f_n} \to D_f$ on $\Sigma \times \Sigma$ implies that there is an integer N such that $|D_{f_n}(z, w)| \geq d/2$ for $z \in \Sigma$, $w \in \Sigma$ and $n \geq N$. Therefore, f_n is univalent in Σ for $n \geq N$.

By Runge's theorem, there is a sequence of polynomials $\{P_m\}$ such that $P_m \to I$ uniformly on each compact subset of Ψ. Hence $P_m \to I$ on Σ, the closure of some neighbourhood of $\overline{\Phi}$. This implies that there is a sequence of polynomials $\{Q_m\}$ such that $Q_m \to I$ uniformly on Σ and $Q_m(z_k) = I(z_k)$ for $m = 1, 2, 3, \ldots$ and $k = 1, 2, \ldots, n$ [4, p. 121]. Since $Q_m \to I$ uniformly on the compact set Σ and I is analytic and univalent on the open set $\Psi \supset \Sigma$, it follows that Q_m is univalent in Σ for all large m. Therefore, for all large m, Q_m satisfies all the requirements for P stated in the theorem. □

Theorem 3.1. *Suppose that Φ and Ω are Jordan domains, and the curves $\Lambda = \partial\Phi$ and $\Gamma = \partial\Omega$ belong to C^2 and are in conformal order. Let $z_1, z_2, \ldots, z_n$ be distinct points*

https://doi.org/10.1017/S0013091521000031 Published online by Cambridge University Press

on Λ *in conformal order and let* $w_1, w_2, \ldots, w_n$ *be distinct points on* Γ *in conformal order. Then there is a polynomial* P *which is univalent in a neighbourhood of* $\overline{\Phi}$ *and satisfies* $P(z_k) = w_k$ *for* $k = 1, 2, \ldots, n$ *and* $P(\overline{\Phi} - \cup_{k=1}^{n}\{z_k\}) \subset \Omega$.

Proof. Let Λ'' denote the Jordan curve defined in the proof of Theorem 1.2, and let Φ'' denote the interior of Λ''. Let D_k denote the same disks described there.

Since the curve Γ belongs to C^2, it is given by $w = \xi(t)$, $a \le t \le b$, where ξ' and ξ'' exist and are continuous and $\xi'(t) \neq 0$. We may assume that this parametrization gives Γ in conformal order and we have $w_k = \xi(t_k)$ for $k = 1, 2, \ldots, n$ where $a \le t_1 < t_2 < \cdots < t_n < b$. For each positive real number ε let the curve $\Gamma' = \Gamma'(\varepsilon)$ be defined by $w = \psi(t) = \xi(t) + \varepsilon i \xi'(t)$, $a \le t \le b$. Using arguments similar to those given in the proof of Theorem 1.2, we find that for all small ε, Γ' is a Jordan curve and belongs to C^1. Letting Ω' denote the interior of Γ' we have $\overline{\Omega'} \subset \Omega$. If ε is small, then for each k there is a closed disk E_k such that $w_k \in \partial E_k$, $E_k - \{w_k\} \subset \Omega$ and the collection $\{E_1, E_2, \ldots, E_n\}$ is pairwise disjoint. Furthermore, there is a Jordan curve Γ'' containing each w_k and $\Gamma'' - \bigcup_{k=1}^{n} w_k \subset \Omega - \Omega'$. Note that Γ'' is made up of the semicircle on E_k with centre w_k, line segments which connect the end points of this semicircle to points on Γ', and arcs on Γ'. Let Ω'' denote the interior of Γ''. Then $\Omega'' \subset \Omega$ and $\Gamma'' \cap \Gamma = \{w_1, w_2, \ldots, w_n\}$. Also, $\{w_1, w_2, \ldots, w_n\}$ is in conformal order on Γ''.

Let F denote a conformal mapping of Δ onto Φ'' and let G denote a conformal mapping of Δ onto Ω''. Then F extends to a homeomorphism of $\overline{\Delta}$ onto $\overline{\Phi''}$ and G extends to a homeomorphism of $\overline{\Delta}$ onto $\overline{\Omega''}$. For each k let $F(\zeta_k) = z_k$ where $|\zeta_k| = 1$ and $G(\eta_k) = w_k$ where $|\eta_k| = 1$. Because $\{z_1, z_2, \ldots, z_n\}$ is in conformal order on Λ and $\{w_1, w_2, \ldots, w_n\}$ is in conformal order on Γ, we have $\zeta_k = e^{i\theta_k}$ where $\theta_1 < \theta_2 < \cdots < \theta_n < \theta_1 + 2\pi$ and $\eta_k = e^{i\varphi_k}$ where $\varphi_1 < \varphi_2 < \cdots < \varphi_n < \varphi_1 + 2\pi$.

There is a function H which is analytic and univalent in $\overline{\Delta}$ such that $H(\zeta_k) = \eta_k$ for $k = 1, 2, \ldots, n$ and $|H(\zeta)| < 1$ for $|\zeta| \le 1$ and $\zeta \neq \zeta_k$ $(k = 1, 2, \ldots, n)$ [3, p. 559]. Let $I = G \circ H \circ F^{-1}$. Then I is analytic and univalent in Φ'', $I(\Phi'') = \Omega''$, and $I(z_k) = w_k$ for $k = 1, 2, \ldots, n$. We have that I is continuous on a subarc of ∂D_k containing z_k in its interior and maps that subarc onto a subarc of ∂E_k containing w_k in its interior. The reflection principle implies that I extends analytically to some neighbourhood of z_k. For each k there is a neighbourhood of z_k, say N_k, so that this extension of I is univalent in $\Psi = \Phi'' \bigcup_{k=1}^{n} N_k$ and Ψ is simply connected. The univalence of I implies $I'(z) \neq 0$ for $z \in \Psi$.

By Runge's theorem, there is a sequence of polynomials $\{P_m\}$ such that $P_m \to I$ uniformly on each compact subset of Ψ. Let Σ denote the closure of a neighbourhood of $\overline{\Phi}$ with $\Sigma \subset \Psi$. Then $P_m \to I$ uniformly on Σ. Therefore, there is a sequence of polynomials $\{Q_m\}$ such that $Q_m \to I$ uniformly on Σ, and $Q_m(z_k) = I(z_k)$, $Q'_m(z_k) = I'(z_k)$, and $Q''_m(z_k) = I''(z_k)$ for $k = 1, 2, \ldots, n$ and $m = 1, 2, 3, \ldots$ [3, p. 566]. Since $Q_m \to I$ uniformly on the compact set Σ and I is analytic and univalent on the open set $\Psi \supset \Sigma$, it follows that Q_m is univalent in Σ for all large m. Hence Q_m is univalent in a neighbourhood of $\overline{\Phi}$ for all large m, and so $Q'_m(z) \neq 0$ for every z in that neighbourhood. For each positive integer m, let $\Gamma_m = Q_m(\Lambda'')$. We have that Γ_m is a Jordan curve for all large m.

Let μ_k denote the angle of the common ordered tangents to Γ, Γ'', and Γ_m at w_k. We introduce a Cartesian coordinate system with origin at w_k and with coordinates (u, v) such that the direction of the positive u axis equals μ_k; that is, we let $u + iv =$

$(w - w_k)e^{-i\mu_k}$. In some deleted neighbourhood of the origin in this system, the points on Γ'' are above the points on Γ.

For $w \in \Gamma$ let $\kappa(w)$ denote the curvature of Γ at w, for $w \in \Gamma''$ let $\sigma(w)$ denote the curvature of Γ'' at w, and for $w \in \Gamma_m$ let $\tau_m(w)$ denote the curvature of Γ_m at w. Since $\Gamma \in C^2$ it follows that κ is continuous on Γ. For $w \in \Gamma''$ in a small neighbourhood of w_k we have $\sigma(w) = 1/r_k$ where r_k denotes the radius of E_k. Let b_k denote the maximum of $\kappa(w)$ for points w in Γ and in the closure of that neighbourhood. Since the disks E_k may have arbitrarily small radii, we may let r_k be small enough that $1/r_k > b_k$ for $k = 1, 2, \ldots, n$. From [6, p. 527, Equation (4)], we obtain

$$\tau_m(w) = \frac{1}{|Q'_m(z)|}\left\{-\frac{1}{r_k} + \Im\left(e^{i\alpha(z)}\frac{Q''_m(z)}{Q'_m(z)}\right)\right\}$$

for $w \in \Gamma_m$ in some neighbourhood of w_k, where $w = Q_m(z)$ and $\alpha(z)$ denotes the angle of the directed tangent to Λ'' at z. Also

$$\sigma(w) = \frac{1}{|I'(z)|}\left\{-\frac{1}{r_k} + \Im\left(e^{i\alpha(z)}\frac{I''(z)}{I'(z)}\right)\right\}$$

for $w \in \Gamma''$ in some neighbourhood of w_k, where $w = I(z)$. Since $Q'_m(z_k) = I'(z_k)$ and $Q''_m(z_k) = I''(z_k)$, letting $w = w_k$ in these formulas, we obtain $\tau_m(w_k) = \sigma(w_k)$ for $k = 1, 2, \ldots, n$ and m large.

The functions $f_m = Q_m \circ I^{-1}$ are defined in a small neighbourhood of w_k. Since $Q_m \to I$ uniformly in a small neighbourhood of z_k, we see that $f_m(w) \to w$ uniformly on a neighbourhood of w_k. Also $Q'_m \to I'$ and $Q''_m \to I''$ uniformly in a neighbourhood of z_k. Hence the formulas above for τ_m and σ imply that $\tau_m(f_m(w)) \to \sigma(w)$ uniformly for $w \in \Gamma''$ and in some neighbourhood of w_k. This implies that there is a neighbourhood of w_k and a constant $c_k > b_k$ such that $\tau_m(w) \geq c_k$ for large m and for $w \in \Gamma_m$ in that neighbourhood.

We have $\kappa(w) \leq b_k$ for $w \in \Gamma$ near w_k, $\tau_m(w) \geq c_k$ for $w \in \Gamma_m$ near w_k, and $c_k > b_k$. This implies that the curve Γ_m is above the curve Γ in the u-v plane in some deleted neighbourhood of the origin. Because Γ is continuous and simple, we conclude that there is a neighbourhood of z_k, say N'_k, such that $Q_m(z) \in \Omega$ for $z \in \Lambda'' \cap N'_k$ and $z \neq z_k$ and for all large m. Hence $Q_m(z) \in \Omega$ for $z \in \Lambda'' \cap N'_k$ and $z \neq z_k$ for large m and for $k = 1, 2, \ldots, n$.

Let $\Lambda''' = \Lambda'' - \bigcup_{k=1}^n (\Lambda'' \cup N'_k)$. Since $Q_m \to I$ uniformly on Λ''' and $I(\Lambda''')$ is a compact subset of Ω, it follows that $Q_m(\Lambda''') \subset \Omega$ for all large m. Therefore, $Q_m(\Lambda'' - \bigcup_{k=1}^n \{z_k\}) \subset \Omega$ for all large m. This implies that $Q_m(\overline{\Phi''} - \bigcup_{k=1}^n \{z_k\}) \subset \Omega$ and hence $Q_m(\overline{\Phi} - \bigcup_{k=1}^n \{z_k\}) \subset \Omega$. We have shown that for all large m, Q_m satisfies all the requirements for P stated in the theorem. □

Our proof shows that there are infinitely many polynomials P which satisfy Theorem 3.1.

The next theorem treats interpolation and peaking as in Theorem 3.1 but the numbers $w_1, w_2, \ldots, w_n$ are not restricted to being distinct nor in conformal order. The resulting polynomial P need not be univalent.

https://doi.org/10.1017/S0013091521000031 Published online by Cambridge University Press

Theorem 3.2. *Suppose that Φ and Ω are Jordan domains, and the curves $\Lambda = \partial\Phi$ and $\Gamma = \partial\Omega$ belong to C^2. Let $z_1, z_2, \ldots, z_n$ be distinct points on Λ and let $w_1, w_2, \ldots, w_n$ be points on Γ. Then there is a polynomial P such that $P(z_k) = w_k$ for $k = 1, 2, \ldots, n$ and $P(\overline{\Phi} - \cup_{k=1}^{n} z_k) \subset \Omega$.*

Proof. Theorem 3.2 is proven using arguments similar to those given for Theorems 1.2 and 3.1. The arguments rely on the analogous result for analytic maps of Δ into Δ given in [3, p. 560, Prop. 1]. □

Acknowledgments. For the reviewer's many helpful comments and suggestions, the authors wish to express their thanks.

References

1. L. Brickman, T. H. MacGregor and D. R. Wilken, Convex hulls of some classical families of univalent functions, *Trans. Am. Math. Soc.* **156** (1971), 91–107.
2. D. J. Cantor and R. R. Phelphs, An elementary interpolation theorem, *Proc. Am. Math. Soc.* **16** (1965), 523–525.
3. J. G. Clunie, D. J. Hallenbeck and T. H. MacGregor, A peaking and interpolation problem for univalent functions, *J. Math Anal. Appl.* **111** (1985), 559–570.
4. P. J. Davis, *Interpolation and approximation* (Ginn (Blaisdell), Waltham, MA, 1963).
5. P. L. Duren, *Univalent functions* (Springer-Verlag, New York, 1983).
6. A. W. Goodman, Curvature under an analytic transformation, *J. London Math. Soc.* **43** (1968), 527–533.
7. W. B. Jones and S. Ruscheweyh, Blaschke product interpolation and its application to the design of digital filters, *Constr. Approx.* **3** (1987), 405–409.
8. T. H. MacGregor and D. E. Tepper, Finite boundary interpolation by univalent functions, *J. Math. Anal. Appl.* **52** (1988), 315–321.
9. G. Pólya and G. Szegö, *Problems and theorems in analysis II* (Springer-Verlag, Berlin, 1976).
10. Ch. Pommerenke, *Boundary behaviour of conformal maps* (Springer-Verlag, New York, 1992).
11. V. V. Savin, On the moduli of Riemann surfaces, *Soviet Math. Dokl.* **12** (1971), 267–270.
12. T. Sheil-Small, *Complex polynomials*, Cambridge Studies in Advanced Mathematics, **Volume 75** (Cambridge University Press, Cambridge, 2002).
13. R. Younis, Interpolation by a finite Blaschke product, *Proc. Am. Math. Soc.* **78** (1980), 451–452.