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RIEMANN MINIMAL SURFACES IN HIGHER DIMENSIONS

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Abstract We prove the existence of a one parameter family of minimal embedded hypersurfaces in $\mathbb{R}^{n+1}$, for $n \geqslant 3$, which generalize the well known two-dimensional 'Riemann minimal surfaces'. The hypersurfaces we obtain are complete, embedded, simply periodic hypersurfaces which have infinitely many parallel hyperplanar ends. By opposition with the two-dimensional case, they are not foliated by spheres.

Résumé Nous prouvons l'existence d'une famille à un paramètre d'hypersurfaces de $\mathbb{R}^{n+1}$, pour $n \geqslant 3$, qui sont minimales et qui généralisent les surfaces minimales de Riemann. Les hypersurfaces que nous obtenons sont des hypersurfaces complètes, simplement périodiques et qui ont une infinité de bouts hyperplans parallèles. Contrairement au cas des surfaces, i.e. $n = 2$, ces hypersurfaces ne sont pas feuilletées par des sphères.

Keywords: Riemann minimal surface; minimal hypersurface; connected sum method

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1. Introduction and statement of results

In three-dimensional Euclidean space, the minimal surfaces known as 'Riemann minimal surfaces' belong to a one parameter family of minimal surfaces which are embedded, have planar ends and are simply periodic (i.e. are invariant under a discrete one parameter group of translations). Moreover, in the quotient space, they have the topology of a 2-torus and have finite total curvature.

These minimal surfaces have been discovered by Riemann in the nineteenth century and each element of this family is foliated by circles or straight lines. In fact, up to some rigid motion and dilation, a fundamental piece of any of these surfaces can be parametrized by

$$X(t, \theta) := (a(t) + \rho(t)\cos\theta, \rho(t)\sin\theta, t)$$

for $(t, \theta) \in \mathbb{R} \times S^1$ in which case the functions a and ρ are non-constant solutions of the following system of first order nonlinear ordinary differential equations

$$(\partial_t \rho)^2 + 1 = \mu\rho^2 + \rho^4$$

and

$$\partial_t a = \rho^2,$$

where $\mu \in \mathbb{R}$ is a parameter. The corresponding Riemann minimal surface is invariant under some translation $\boldsymbol{d}_\mu \in \mathbb{R}^3$, and, in the quotient space $\mathbb{R}^3/(\mathbb{Z}\boldsymbol{d}_\mu)$, is topologically equivalent to a torus $S^1 \times S^1$. In addition, in the quotient space, this surface has finite total curvature.

In this paper, we prove that this one parameter family of minimal surfaces can be generalized to any dimension $n \geqslant 3$. More precisely, we show that there exists a one parameter family of minimal hypersurfaces in $\mathbb{R}^{n+1}$, which are embedded, have infinitely many hyperplanar ends and are invariant under some one parameter discrete group of translations.

The canonical basis of $\mathbb{R}^{n+1}$ will be denoted by $\boldsymbol{e}_j$, for $j = 1, \dots, n+1$, and coordinates of $x \in \mathbb{R}^n$ will be denoted by $(x^1, \dots, x^n)$. In order to state our result precisely, we introduce the subgroup $\mathfrak{G} \subset O(n+1)$ which is generated by elements of the form

$$R := \begin{pmatrix} -1 & 0 & 0 \\ 0 & \bar{R} & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where $\bar{R} \in O(n-1)$.

Our main result reads as follows.

Theorem 1.1. *There exists a one parameter family of embedded minimal hypersurfaces $(\Sigma_\varepsilon)_{\varepsilon \in (0,\varepsilon_0)}$ which have horizontal hyperplanar ends and are simply periodic. These hypersurfaces are invariant under the discrete group of translations $\mathbb{Z}\boldsymbol{d}_\varepsilon$, where $\boldsymbol{d}_\varepsilon = \boldsymbol{e}_1 + h_\varepsilon \boldsymbol{e}_{n+1}$ with $h_\varepsilon > 0$, and they are also invariant under the action of $\mathfrak{G}$. In the quotient space $\mathbb{R}^{n+1}/(\mathbb{Z}\boldsymbol{d}_\varepsilon)$ the hypersurface Σ_ε is topologically equivalent to $S^{n-1} \times S^1$ and has finite total scalar curvature.*

When $n \geqslant 3$ and by opposition to the case of surfaces, it follows from the result of Jagy [**3**] that our hypersurfaces are not foliated anymore by $(n-1)$-dimensional spheres and hence it seems unlikely that these hypersurfaces could be recovered by solving some system of nonlinear ordinary differential equations. In fact, near one of its ends, the hypersurface Σ_ε we construct is close to the vertical graph of the function

$$x \in \mathbb{R}^n \to \frac{\varepsilon}{n-2}(|x - x^*|^{2-n} - |x + x^*|^{2-n}),$$

where $x^* := (1, 0, \dots, 0) \in \mathbb{R}^n$ and one can check that the level sets of this function are not spheres. As already mentioned, near their ends, the minimal hypersurfaces we construct are not exactly vertical graphs of these functions but vertical graphs of some small perturbation of these functions and it turns out that, in dimension $n \geqslant 3$, the perturbations are small enough so that the corresponding level sets are not spheres. This is in striking contrast with the two-dimensional case where the corresponding two-dimensional construction (leading to the construction of Riemann minimal surfaces) yields a minimal surface which, near one of its ends, is close to the vertical graph of the function

$$x \in \mathbb{R}^2 \to \varepsilon(\log|x + x^*| - \log|x - x^*|)$$

and this time one can check that the level sets of this function are either circles or straight lines.

Let us emphasize that the hypersurfaces we construct do not describe the full family of such hypersurfaces. Indeed, we only describe the elements of this family when the translation period $\boldsymbol{d}_\varepsilon$ is close to $\boldsymbol{e}_1$.

In order to construct these hypersurfaces, the main observation is that the moduli space of Riemann's minimal surfaces is one dimensional (once the action of rigid motions and dilations has been taken into account) and non-compact. In particular, one can investigate the behaviour of these surfaces close to one of the two ends of the moduli space. It turns out that surfaces belonging to one end of the moduli space, when they are properly rescaled, can be understood as infinitely many parallel planes which are connected together by small catenoidal necks. Hence, even though this would not be worth the effort, these surfaces could be recovered using the connected sum result by Traizet [**12**]. This is this connected sum result which allows us to describe part of the moduli space of the n-dimensional analogues of Riemann's minimal surfaces.

This work complements previous works which have been done to generalize, in higher dimensions, some classical families of minimal surfaces. For example, in [**1**], the minimal k-noids, which are complete minimal surfaces with catenoidal ends have been generalized to any dimension. This is also the case for Scherk's second surfaces which have been generalized to any dimension in [**9**].

In § 2 we give the definition of the n-catenoid, which generalizes the usual catenoid to any dimension. We then proceed with a perturbation of the truncated n-catenoid to produce an infinite-dimensional family of minimal hypersurfaces which are parametrized by their boundary data. Section 3 is devoted to the perturbation of the hyperplane with two balls removed. Again we find an infinite-dimensional family of minimal hypersurfaces which are parametrized by their boundary data on the boundaries of the two excised balls. In § 4, we explain how these infinite-dimensional families can be connected together to produce the n-dimensional analogues of Riemann's minimal surfaces.

2. The n-catenoid and minimal hypersurfaces close to it

From now on, we assume that $n \geqslant 3$ is fixed. We recall some well known facts concerning the definition and properties of the n-catenoid C, a minimal hypersurface of revolution which generalizes in $\mathbb{R}^{n+1}$, the standard catenoid in three-dimensional Euclidean space. We also give a rather explicit expansion formula for the mean curvature of any hypersurface which is close enough to C.

The n-catenoid C is a hypersurface of revolution about the x^{n+1}-axis. It will be convenient to consider a parametrization $X : \mathbb{R} \times S^{n-1} \to \mathbb{R}^{n+1}$ of C for which the induced metric is conformal to the product metric on $\mathbb{R} \times S^{n-1}$. This parametrization is given by

$$X(t, z) := (\varphi(t)z, \psi(t)), \tag{2.1}$$

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where $t \in \mathbb{R}$, $z \in S^{n-1}$ and where the functions φ and ψ are explicitly given by

$$\varphi(t) := (\cosh((n-1)t))^{1/(n-1)} \quad \text{and} \quad \psi(t) := \int_0^t \varphi^{2-n}\,\mathrm{d}s.$$

It is easy to check that the induced metric on C is given by

$$g := \varphi^2(\mathrm{d}t^2 + g_{S^{n-1}})$$

and, if the orientation of C is chosen so that the unit normal vector field is given by

$$\boldsymbol{n} := (-\varphi^{1-n} z, \partial_t \ln \varphi), \tag{2.2}$$

then, the second fundamental form of C is given by

$$b := \varphi^{2-n}((1-n)\,\mathrm{d}t^2 + g_{S^{n-1}}).$$

From these expressions, it is easy to check that the hypersurface parametrized by X is indeed minimal.

2.1. The Jacobi operator about the n-catenoid

We now consider the hypersurfaces which can be parametrized as normal graphs over C, namely they can be parametrized by

$$X_w := X + w\boldsymbol{n} \tag{2.3}$$

for some small (sufficiently smooth) function w. Let us denote by $H(w)$ the mean curvature of the hypersurface parametrized by X_w. The Jacobi operator, which is nothing but the linearized mean curvature operator, appears in the second variation of the n-volume functional. It is given by the general formula

$$J := \Delta_g + |A|^2,$$

where Δ_g denotes the Laplace–Beltrami operator and A is the shape operator of the hypersurface. In the case of the n-catenoid and in the above defined parametrization, the Jacobi operator about C is given by

$$J = \varphi^{-n}\partial_t(\varphi^{n-2}\partial_t\cdot) + \varphi^{-2}\Delta_{S^{n-1}} + n(n-1)\varphi^{-2n}.$$

It turns out that it is easier (and equivalent) to study the mapping properties of the conjugate operator L which is defined by

$$L := \varphi^{(2+n)/2} J \varphi^{(2-n)/2}.$$

We have, explicitly,

$$L := \partial_t^2 + \Delta_{S^{n-1}} - (\tfrac{1}{2}(n-2))^2 + \tfrac{1}{4}n(3n-2)\varphi^{2-2n}.$$

The next lemma is borrowed from [**1**]. It explains the structure of the expansion of the mean curvature operator $w \longmapsto H(w)$ in terms of the function w and its derivatives.

Lemma 2.1 (Fakhi and Pacard [1]). *The equation $H(w)=0$ is equivalent to*

$$Lw = \varphi^{(2-n)/2} Q_2(\varphi^{-n/2} w) + \varphi^{n/2} Q_3(\varphi^{-n/2} w), \tag{2.4}$$

where the operators Q_2 and Q_3 enjoy the following property: there exists a constant $c>0$ such that for all $t\in\mathbb{R}$ and for all $w_1, w_2 \in \mathcal{C}^{2,\alpha}([t-1,t+1]\times S^{n-1})$, we have

$$\|Q_2(w_2)-Q_2(w_1)\|_{\mathcal{C}^{0,\alpha}} \leqslant c(\|w_2\|_{\mathcal{C}^{2,\alpha}} + \|w_1\|_{\mathcal{C}^{2,\alpha}})\|w_2-w_1\|_{\mathcal{C}^{2,\alpha}}, \tag{2.5}$$

and, provided $\|w_1\|_{\mathcal{C}^{2,\alpha}} + \|w_2\|_{\mathcal{C}^{2,\alpha}} \leqslant 1$, we also have

$$\|Q_3(w_2)-Q_3(w_1)\|_{\mathcal{C}^{0,\alpha}} \leqslant c(\|w_2\|_{\mathcal{C}^{2,\alpha}} + \|w_1\|_{\mathcal{C}^{2,\alpha}})^2\|w_2-w_1\|_{\mathcal{C}^{2,\alpha}}. \tag{2.6}$$

Here all norms are understood on the domain of definition of the functions.

Proof. We recall the main lines of the proof of this critical lemma for the sake of completeness. We set

$$\tilde{\boldsymbol{n}} := \varphi \boldsymbol{n} \quad \text{and} \quad \tilde{w} := \frac{w}{\varphi},$$

so that the hypersurface parametrized by X_w is also parametrized by $\tilde{X}_w := X + \tilde{w}\tilde{\boldsymbol{n}}$. Now, the first fundamental form g_w of the hypersurface parametrized by $\tilde{X}_w$ is explicitly given by

$$\begin{aligned} g_w = \varphi^2(\mathrm{d}t^2 + g_{S^{n-1}}) &+ 2\varphi^{3-n}\tilde{w}((n-1)\,\mathrm{d}t^2 - g_{S^{n-1}}) \\ &+ 2\varphi\partial_t\varphi\tilde{w}(\partial_t\tilde{w}\,\mathrm{d}t^2 + \partial_{z^i}\tilde{w}\,\mathrm{d}t\,\mathrm{d}z^i) + \varphi^{4-2n}\tilde{w}^2(n(n-2)\,\mathrm{d}t^2 + g_{S^{n-1}}) \\ &+ \varphi^2((\tilde{w}^2 + (\partial_t\tilde{w})^2)\,\mathrm{d}t^2 + 2\partial_t\tilde{w}\partial_{z^i}\tilde{w}\,\mathrm{d}t\,\mathrm{d}z^i + \partial_{z^i}\tilde{w}\partial_{z^j}\tilde{w}\,\mathrm{d}z^i\,\mathrm{d}z^j). \end{aligned}$$

Making use of the expansion

$$\det(I+B) = 1 + \operatorname{Tr} B + \tfrac{1}{2}((\operatorname{Tr} B)^2 - \operatorname{Tr}(B^2)) + \mathcal{O}(|B|^3),$$

and changing back $\tilde{w}$ into w/φ, we obtain the expansion

$$\sqrt{\det g_w} = \varphi^n + \tfrac{1}{2}\varphi^{n-2}|\nabla w|^2 - \tfrac{1}{2}n(n-1)\varphi^{-n}w^2 + \varphi\tilde{Q}_3(\varphi^{-1}w) + \varphi^n\tilde{Q}_4(\varphi^{-1}w),$$

where $\tilde{Q}_3$ is homogeneous of degree 3 and where $\tilde{Q}_4$ collects all the higher order terms. The key point is that the Taylor's coefficients of $\tilde{Q}_i$ are bounded functions of t and z and so are the derivatives of any order of these functions, with respect to the vector fields ∂_t and ∂_{z^j} to any order.

The result then follows from the variational characterization of minimal hypersurfaces as critical points of the functional

$$\mathcal{E}(w) = \int \sqrt{\det g_w}\,\mathrm{d}s\,\mathrm{d}z.$$

It is easy to check that critical points of $\mathcal{E}$ are solutions of the nonlinear elliptic equation

$$\partial_t(\varphi^{n-2}\partial_t w) + \varphi^{n-2}\Delta_{S^{n-1}} w + n(n-1)\varphi^{-n} w = Q_2(\varphi^{-1}w) + \phi^{n-1}Q_3(\varphi^{-1}w),$$

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where Q_2 is homogeneous of degree 2 and where Q_3 collects all the higher order terms. Again, the Taylor's coefficients of Q_i are bounded functions of t and z and so are the derivatives of any order of these functions, with respect to the vector fields ∂_t and ∂_{z^j}. To complete the proof, it is enough to perform the conjugacy which was used to define L starting from J. □

Let us briefly comment on this result. The first estimate reflects the fact that the operator Q_2 is a nonlinear second order differential operator which is homogenous of degree 2 in w and its derivatives, and has coefficients which are bounded functions of t. The second estimate reflects the fact that the nonlinear operator Q_3 is a nonlinear second order differential operator whose Taylor expansion at $w = 0$ does not involve any constant, linear nor quadratic term and has coefficients which are bounded functions of t.

Observe that the n-catenoid is invariant under the action of the group $\mathfrak{G}$ and if one looks for hypersurfaces which are invariant under the action of $\mathfrak{G}$ then this amounts to consider normal variations of the n-catenoid for some functions w which enjoy the following invariance property

$$w(-t,-z) = w(t,z) \quad \text{and} \quad w(t,z) = w(t,Rz) \tag{2.7}$$

for all $R \in O(n)$ of the form

$$R := \begin{pmatrix} 1 & 0 \\ 0 & \bar{R} \end{pmatrix},$$

where $\bar{R} \in O(n-1)$. Clearly, the Jacobi operator and its conjugate preserve this invariance, i.e. if a function w satisfies (2.7) then so does the function Lw. Since the mean curvature is invariant under the action of isometries, the nonlinear operator which appears on the right-hand side of (2.4) also enjoys a similar invariance property.

2.2. Linear analysis about the n-catenoid

We study the mapping properties of the conjugate Jacobi operator L.

Given $n \geqslant 2$, we denote by $\lambda_j = j(n-2+j)$, $j \in \mathbb{N}$, the eigenvalues of the Laplace–Beltrami operator on S^{n-1} and we denote by E_j the corresponding eigenspace. That is

$$\Delta_{S^{n-1}} \phi = -\lambda_j \phi,$$

for all $\phi \in E_j$.

The *indicial roots* of L describe the asymptotic behaviour, at infinity, of the solutions of the homogeneous problem

$$Lw = 0 \quad \text{in } \mathbb{R} \times S^{n-1}.$$

If $w \in \mathcal{C}^{2,\alpha}(\mathbb{R} \times S^{n-1})$ is a solution of the homogeneous problem $Lw = 0$ in $\mathbb{R} \times S^{n-1}$, we may consider the eigenfunction decomposition of w as

$$w(t,z) = \sum_{j \in \mathbb{N}} w_j(t,z),$$

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where, for each $t \in \mathbb{R}$, the function $w_j(t, \cdot) \in E_j$. Then the E_j-valued function w_j is a solution of $L_j v = 0$ on $\mathbb{R}$ where the operator L_j is defined by

$$L_j := \partial_t^2 - (\tfrac{1}{2}(n-2) + j)^2 + \tfrac{1}{4}n(3n-2)\varphi^{2-2n}.$$

Since φ^{2-2n} tends exponentially to 0 at $\pm\infty$, it is easy to check that there are exactly two independent solutions of the homogenous problem $L_j v = 0$, which we denote by v_j^+ and v_j^- and which satisfy

$$\lim_{t \to +\infty} \mathrm{e}^{-\delta_j t} v_j^+(t) = 1 \quad \text{and} \quad \lim_{t \to +\infty} \mathrm{e}^{\delta_j t} v_j^-(t) = 1,$$

where

$$\delta_j := \tfrac{1}{2}(n-2) + j,$$

for all $j \in \mathbb{N}$. The real numbers $\pm\delta_j$ are usually referred to as the *indicial roots* of L at both $+\infty$ and $-\infty$.

We define the operator

$$\Delta_0 := \partial_t^2 + \Delta_{S^{n-1}} - (\tfrac{1}{2}(n-2))^2.$$

Observe that the indicial roots of Δ_0 are equal to the indicial roots of the operator L.

The solutions of the homogeneous problem $Jw = 0$ are usually called *Jacobi fields*. Some Jacobi fields which correspond to explicit one-parameter families of minimal hypersurfaces to which C belongs, are explicitly known. These Jacobi fields are obtained by projecting over the normal vector field the Killing vector fields associated to rigid motions and the vector field associated to dilations. With slight abuse of terminology we shall also refer to solutions of $Lw = 0$ as Jacobi fields. One has to multiply the Jacobi fields associated to J by $\varphi^{(n-2)/2}$ to obtain the expression of the corresponding Jacobi fields for L, in doing so one takes into account the fact that L is conjugate to J. Using this recipe we obtain the following Jacobi fields (for the operator L).

(i) The function

$$\Phi^{0,-} := \varphi^{(n-4)/2} \partial_t \varphi,$$

which is associated to the translation of C along its axis.

(ii) The function

$$\Phi^{0,+} := \varphi^{(n-4)/2} (\varphi \partial_t \psi - \psi \partial_t \varphi),$$

which is associated to the dilation of C.

(iii) The function

$$\Phi_{\boldsymbol{e}}^{1,-} := \varphi^{-n/2} (z \cdot \boldsymbol{e}) \quad \text{for } \boldsymbol{e} \in \mathbb{R}^n \times \{0\},$$

which is associated to the translation of C along the direction $\boldsymbol{e}$ orthogonal to its axis.

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(iv) The function

$$\Phi_{\boldsymbol{e}}^{1,+} := \varphi^{(n-4)/2}(\psi\partial_t\psi + \varphi\partial_t\varphi)(z\cdot\boldsymbol{e}) \quad \text{for } \boldsymbol{e} \in \mathbb{R}^n \times \{0\},$$

which is associated to the rotation of the axis of C in a direction $\boldsymbol{e}$ orthogonal to its axis.

These constitute $2(n+1)$ linearly independent Jacobi fields.

The operator L does not satisfy the maximum principle. Indeed, one checks that the Jacobi fields $\Phi_{\boldsymbol{e}}^{1,-}$ decay exponentially at $\pm\infty$. Nevertheless, if the operator L is restricted to a suitable subspace of functions, some version of the maximum principle is still available. This is the content of the following result whose proof can be found in [**1**] (see also [**4**]).

Proposition 2.2 (Fakhi and Pacard [1]). *Assume that $\delta < -\frac{1}{2}n$. Let w be a solution of*

$$Lw = 0 \quad \textit{in } \mathbb{R}\times S^{n-1},$$

which is bounded by a constant times $(\cosh t)^\delta$. Then $w \equiv 0$.

Proof. A simple proof of this result can be obtained as follows. Proceed with the eigenfunction decomposition of w, the solution of $Lw = 0$, so that $w = \sum_j w_j$. Then w_j is a solution of $L_j w_j = 0$ which is bounded by a constant times $(\cosh t)^\delta$. When $j = 0$ or $j = 1$, all solutions are linear combinations of the solutions that are explicitly known and are described above. It is easy to check that no such solution is bounded by a constant times $(\cosh t)^\delta$ unless it is identically equal to 0 since we have chosen $\delta < -\frac{1}{2}n$. Now, when $j \geqslant 2$ we write $w_j(t,z) = v_j(t)\phi_j(z)$ where $\phi_j \in E_j$. As already observed the function v_j being bounded by a constant times $(\cosh t)^\delta$ for $\delta < -\frac{1}{2}n$ has to decay at infinity like $(\cosh t)^{-\delta_j}$. We have

$$\partial_t^2 v_j - (\tfrac{1}{2}(n-2)+j)^2 v_j + \tfrac{1}{4}n(3n-2)\varphi^{2-2n}v_j = 0$$

and also (since $\Phi_{\boldsymbol{e}}^{1,-}$ are Jacobi fields)

$$\partial_t^2 \varphi^{-n/2} - (\tfrac{1}{2}(n-2)+1)^2\varphi^{-n/2} + \tfrac{1}{4}n(3n-2)\varphi^{2-2n}\varphi^{-n/2} = 0.$$

We claim that $v_j \equiv 0$. Assume the contrary, for all $s \in \mathbb{R}$, we set $v_s := \varphi^{-n/2} - s v_j$. Using the above equations, we have

$$\partial_t^2 v_s - (\tfrac{1}{2}(n-2)+j)^2 v_s + \tfrac{1}{4}n(3n-2)\varphi^{2-2n}v_s = -(j-1)(n-1+j)\varphi^{-n/2}. \tag{2.8}$$

For all $s \in \mathbb{R}$, v_s is positive near $\pm\infty$ (because the function v_j tends to 0 at $\pm\infty$ much faster than the function $\varphi^{-n/2}$). We choose s to be the sup of the reals for which $v_s \geqslant 0$. Then v_s vanishes in $\mathbb{R}$ and at this point, which is a minimum point for v_s, (2.8) yields $\partial_t^2 v_s < 0$. A contradiction. This completes the proof of the result. □

Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$, the space $\mathcal{C}^{k,\alpha}_{\delta}(\mathbb{R} \times S^{n-1})$ is defined to be the space of functions $w \in \mathcal{C}^{k,\alpha}(\mathbb{R} \times S^{n-1})$ for which the following norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_{\delta}(\mathbb{R}\times S^{n-1})} := \|(\cosh t)^{-\delta} w\|_{\mathcal{C}^{k,\alpha}(\mathbb{R}\times S^{n-1})}$$

is finite.

We show in the next result that, provided the weight parameter δ is suitably chosen and $\alpha \in (0,1)$ is fixed, one can define a right inverse for L.

Proposition 2.3. *Assume that $\delta \in (\frac{1}{2}n, \frac{1}{2}(n+2))$ and $\alpha \in (0,1)$ are fixed. Then, there exists a continuous operator*

$$G : \mathcal{C}^{0,\alpha}_{\delta}(\mathbb{R} \times S^{n-1}) \to \mathcal{C}^{2,\alpha}_{\delta}(\mathbb{R} \times S^{n-1})$$

which is a right inverse for the operator L. Moreover, if the function f satisfies (2.7) then so does $G(f)$.

Proof. The existence of G follows from standard results and we refer to [**8**] and [**5**] for a proof (see also [**10**]). According to Proposition 2.2 and the description of the geometric Jacobi fields, we see that the operator

$$L : \mathcal{C}^{2,\alpha}_{\delta}(\mathbb{R} \times S^{n-1}) \to \mathcal{C}^{0,\alpha}_{\delta}(\mathbb{R} \times S^{n-1})$$

is injective for all $\delta < -\frac{1}{2}n$. Hence according to [**8**] and [**5**], the operator L is surjective for any $\delta > \frac{1}{2}n$ which is not an indicial root. This proves the existence of a right inverse. Observe that, for $\delta > \frac{1}{2}n$ there is no uniqueness of the right inverse and in order to define a right inverse which preserves (2.7) it is enough to average over the orbit of the group and define

$$G(f)(t,z) = \frac{1}{2}\int_{O(n-1)} (\tilde{G}(f)(t,(z^1,\bar{R}\bar{z})) + \tilde{G}(f)(-t,(-z^1,\bar{R}\bar{z})))\,\mathrm{d}\sigma_{\bar{R}},$$

where $z = (z^1, \bar{z}) \in S^{n-1}$ and where $\tilde{G}$ is *any* right inverse for L. Here $\mathrm{d}\sigma_{\bar{R}}$ is the standard Haar measure on $O(n-1)$ (normalized so that the volume of $O(n-1)$ is equal to 1). □

The last result we will need is concerned with the Poisson operator associated to the operator

$$\Delta_0 = \partial_t^2 + \Delta_{S^{n-1}} - (\tfrac{1}{2}(n-2))^2,$$

which acts on functions defined on the cylinder $\mathbb{R} \times S^{n-1}$. A similar result has already been proven in [**1**] but we give here a new very short self-contained proof.

Lemma 2.4. *There exists a constant $c = c(n) > 0$ such that for all $h \in \mathcal{C}^{2,\alpha}(S^{n-1})$, which is $L^2(S^{n-1})$-orthogonal to E_0 and E_1, there exists a unique $w_h \in \mathcal{C}^{2,\alpha}([0,+\infty) \times S^{n-1})$ solution of*

$$\begin{cases} \Delta_0 w_h = 0 & \text{in } [0,+\infty) \times S^{n-1}, \\ w_h = h & \text{on } \{0\} \times S^{n-1}, \end{cases}$$

which tends to 0 as t tends to $+\infty$. Furthermore,

$$\|\mathrm{e}^{((n+2)/2)t} w_h\|_{\mathcal{C}^{2,\alpha}([0,+\infty)\times S^{n-1})} \leqslant c\|h\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}.$$

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Proof. We perform the eigenfunction decomposition of h

$$h = \sum_{j \geqslant 2} h_j,$$

where $h_j \in E_j$. Then we have the explicit expression of w_h given by

$$w_h(t, z) = \sum_{j \geqslant 2} \mathrm{e}^{-\delta_j t} h_j(z).$$

Using elliptic estimates together the fact that h_j is an eigenfunction of $\Delta_{S^{n-1}}$, we get the rough estimate

$$\|h_j\|_{L^\infty(S^{n-1})} \leqslant c(1+j)^{p_n} \|h_j\|_{L^2(S^{n-1})}$$

for some exponent $p_n \geqslant 0$ which only depends on the dimension n. Furthermore, using the fact that the dimension of E_j grows polynomially with j [**11**], we get

$$\|h_j\|_{L^2(S^{n-1})} \leqslant c(1+j)^{q_n} \|h\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}$$

for some exponent $q_n \geqslant 0$ which only depends on the dimension n. Collecting these, we conclude that

$$\mathrm{e}^{((n+2)/2)t} |w_h(t, z)| \leqslant c \left(\sum_{j \geqslant 2} \mathrm{e}^{(2-j)t} (1+j)^{p_n + q_n} \right) \|h\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}.$$

It is easy to check that the series converges uniformly when $t \geqslant 1$. This provides the bound

$$\sup_{t \geqslant 1,\, z \in S^{n-1}} \mathrm{e}^{((n+2)/2)t} |w_h(t, z)| \leqslant c \|h\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}.$$

Finally, applying the maximum principle we get

$$\sup_{t \geqslant 0, z \in S^{n-1}} \mathrm{e}^{((n+2)/2)t} |w_h(t, z)| \leqslant c \|h\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}.$$

The estimate for the derivatives follows from Schauder's elliptic estimates. □

Observe that if the function h enjoys the following invariance property

$$h(z) = h(Rz) \tag{2.9}$$

for all $R \in O(n)$ of the form

$$R := \begin{pmatrix} 1 & 0 \\ 0 & \bar{R} \end{pmatrix},$$

where $\bar{R} \in O(n-1)$, then so does the function w_h, namely

$$w_h(t, z) = w_h(t, Rz) \tag{2.10}$$

for all $R \in O(n)$ as above.

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2.3. Nonlinear analysis

For all $\varepsilon \in (0,1)$, we define $t_\varepsilon > 0$ by

$$\varphi^{n-1}(t_\varepsilon) = \varepsilon^{-n/(3n-2)}$$

and $r_\varepsilon > 0$ is defined by

$$r_\varepsilon := \varepsilon^{1/(n-1)}\varphi(t_\varepsilon) = \varepsilon^{2/(3n-2)}.$$

It will be convenient to slightly modify the normal vector field $\boldsymbol{n}$ on C close into a transverse vector field $\boldsymbol{n}_\varepsilon$ which is defined by

$$\boldsymbol{n}_\varepsilon(t,z) := \chi_\varepsilon(t)\boldsymbol{n}(t,z) + \operatorname{sgn}(t)(1-\chi_\varepsilon(t))\boldsymbol{e}_{n+1}, \tag{2.11}$$

where χ_ε is a cutoff function equal to 0 for $|t| > t_\varepsilon - 1$ and equal to 1 when $|t| < t_\varepsilon - 2$. Observe that, as ε tends to 0, $\boldsymbol{n}_\varepsilon$ is a small perturbation of $\boldsymbol{n}$. This is made quantitatively precise in the estimate

$$|\nabla^k(\boldsymbol{n}_\varepsilon \cdot \boldsymbol{n} - 1)| \leqslant c_k\varphi^{2-2n}, \tag{2.12}$$

which holds for $|t| \geqslant t_\varepsilon - 2$ and $z \in S^{n-1}$, for some constant $c_k > 0$ only depending on $k \in \mathbb{N}$.

We now look for minimal hypersurfaces which are close to C and which can be parametrized by

$$X_{\varepsilon,w} = X + \varphi^{(2-n)/2} w \boldsymbol{n}_\varepsilon,$$

for some (small) function w. We also ask that these minimal hypersurfaces are invariant under the action of $\mathfrak{G}$.

Using the result of Lemma 2.1, one can check that the hypersurface parametrized by $X_{\varepsilon,w}$ is minimal if and only if the function w satisfies

$$Lw = L_\varepsilon w + \varphi^{(2-n)/2} Q_{2,\varepsilon}(\varphi^{-n/2} w) + \varphi^{n/2} Q_{3,\varepsilon}(\varphi^{-n/2} w). \tag{2.13}$$

This formula is not exactly identical to (2.4) since the normal vector field $\boldsymbol{n}$ has been modified into the transverse vector field $\boldsymbol{n}_\varepsilon$. Observe that $L_\varepsilon \equiv 0$, $Q_{2,\varepsilon} \equiv Q_2$ and $Q_{3,\varepsilon} \equiv Q_3$ when $|t| \leqslant t_\varepsilon - 2$ since $\boldsymbol{n} = \boldsymbol{n}_\varepsilon$ in this range. Moreover, it follows from (2.12) that the coefficients of the linear second order operator L_ε are bounded by a constant times φ^{2-2n} (in any $\mathcal{C}^{k,\alpha}$ topology). Indeed, $\tilde{L} := L - L_\varepsilon$ is the corresponding linearized mean curvature operator when hypersurfaces close to C are parametrized as graphs over C using the vector field $\boldsymbol{n}_\varepsilon$. It is easy to check [**1**] that, since C has constant mean curvature (equal to 0) then

$$\tilde{L}w = L(\boldsymbol{n}_\varepsilon \cdot \boldsymbol{n} w).$$

The estimates of the coefficients of L_ε follow at once from this identity using (2.12). Finally, the operators $Q_{2,\varepsilon}$ and $Q_{3,\varepsilon}$ enjoy properties which are similar to those enjoyed by Q_2 and Q_3, uniformly for $\varepsilon \in (0, \frac{1}{2})$. Details can be found for example in [**1**].

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Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$, the space $\mathcal{C}^{k,\alpha}_\delta([-t_\varepsilon, t_\varepsilon] \times S^{n-1})$ is defined to be the space of functions $w \in \mathcal{C}^{k,\alpha}([-t_\varepsilon, t_\varepsilon] \times S^{n-1})$ which is endowed with the norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\delta([-t_\varepsilon,t_\varepsilon]\times S^{n-1})} := \|(\cosh t)^{-\delta} w\|_{\mathcal{C}^{k,\alpha}([-t_\varepsilon,t_\varepsilon]\times S^{n-1})}.$$

Given $\tilde{h} \in \mathcal{C}^{2,\alpha}(S^{n-1})$ which is $L^2(S^{n-1})$-orthogonal to E_0 and E_1, we define $w_{\tilde{h}}$ to be the harmonic extension of $\tilde{h}$ in a half cylinder, for the operator Δ_0, which is given by Lemma 2.4. Then we set

$$\tilde{w}_{\tilde{h}}(t,z) := w_{\tilde{h}}(t_\varepsilon - t, z) + w_{\tilde{h}}(t_\varepsilon + t, -z)$$

for all $(t,z) \in [-t_\varepsilon, t_\varepsilon] \times S^{n-1}$. Granted the estimate provided in Lemma 2.4, it is easy to check that there exists a constant $c = c(n) > 0$ such that

$$\|\tilde{w}_{\tilde{h}}\|_{\mathcal{C}^{2,\alpha}_{(n+2)/2}([-t_\varepsilon,t_\varepsilon]\times S^{n-1})} \leqslant c\varphi^{-(n+2)/2}(t_\varepsilon)\|\tilde{h}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}. \tag{2.14}$$

It will be convenient to define an extension operator

$$\mathcal{E}_\varepsilon : \mathcal{C}^{0,\alpha}_\delta([-t_\varepsilon, t_\varepsilon] \times S^{n-1}) \to \mathcal{C}^{0,\alpha}_\delta(\mathbb{R} \times S^{n-1})$$

as follows.

(i) For all $(t,z) \in (-t_\varepsilon, t_\varepsilon) \times S^{n-1}$, we set

$$\mathcal{E}_\varepsilon(f)(t,z) = f(t,z).$$

(ii) For all $(t,z) \in (-t_\varepsilon - 1, -t_\varepsilon) \times S^{n-1}$ we set

$$\mathcal{E}_\varepsilon(f)(t,z) = \chi(-(t_\varepsilon + t)) f(-t_\varepsilon, z).$$

(iii) For all $(t,z) \in (t_\varepsilon, t_\varepsilon + 1) \times S^{n-1}$ we set

$$\mathcal{E}_\varepsilon(f)(t,z) = \chi(t - t_\varepsilon) f(t_\varepsilon, z).$$

(iv) And finally, for all $(t,z) \notin (-t_\varepsilon - 1, t_\varepsilon + 1) \times S^{n-1}$ we set

$$\mathcal{E}_\varepsilon(f)(t,z) = 0.$$

In all of these instances, $\chi : \mathbb{R} \to [0,1]$ is a smooth cutoff function identically equal to 0 for $t \geqslant 1$ and identically equal to 1 for $t \leqslant 0$.

Obviously, there exists a constant $c = c(n,\delta) > 0$ such that

$$|||\mathcal{E}_\varepsilon||| \leqslant c.$$

We denote by I_ε the canonical imbedding

$$I_\varepsilon : \mathcal{C}^{2,\alpha}_\delta(\mathbb{R} \times S^{n-1}) \to \mathcal{C}^{2,\alpha}_\delta([-t_\varepsilon, t_\varepsilon] \times S^{n-1}).$$

We fix $\delta \in (\frac{1}{2}n, \frac{1}{2}(n+2))$ and we look for a solution w of (2.13) which is defined in $[-t_\varepsilon, t_\varepsilon] \times S^{n-1}$ and which can be decomposed as $w = \tilde{w}_{\tilde{h}} + v$ where $v \in \mathcal{C}^{2,\alpha}_\delta([-t_\varepsilon, t_\varepsilon] \times S^{n-1})$ is small. Thanks to the result of Proposition 2.3, we see that it is enough to find $v \in \mathcal{C}^{2,\alpha}_\delta([-t_\varepsilon, t_\varepsilon] \times S^{n-1})$ solution of

$$v = A_{\varepsilon,\tilde{h}}(v), \tag{2.15}$$

where we have defined

$$\begin{aligned} A_{\varepsilon,\tilde{h}}(v) := I_\varepsilon \circ G \circ \mathcal{E}_\varepsilon(L_\varepsilon(\tilde{w}_{\tilde{h}} + v) + \varphi^{(2-n)/2} Q_{2,\varepsilon}(\varphi^{-n/2}(\tilde{w}_{\tilde{h}} + v)) \\ + \varphi^{n/2} Q_{3,\varepsilon}(\varphi^{-n/2}(\tilde{w}_{\tilde{h}} + v)) - L\tilde{w}_{\tilde{h}}). \end{aligned}$$

The existence of a solution v to this fixed point problem will be a consequence of the following lemma and the application of a fixed point theorem for contraction mapping.

Lemma 2.5. *Given $\kappa > 0$, there exists $\varepsilon_\kappa > 0$ and $c_\kappa = c(n, \delta, \kappa) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\kappa)$, and all $\tilde{h} \in C^{2,\alpha}(S^{n-1})$ satisfying $\|\tilde{h}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leqslant \kappa\varepsilon r_\varepsilon \varphi^{n/2}(t_\varepsilon)$, we have*

$$\|A_{\varepsilon,\tilde{h}}(0)\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\varepsilon,t_\varepsilon]\times S^{n-1})} \leqslant c_\kappa \varepsilon r_\varepsilon \varphi^{-1}(t_\varepsilon)$$

and

$$\|A_{\varepsilon,\tilde{h}}(v_2) - A_{\varepsilon,\tilde{h}}(v_1)\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\varepsilon,t_\varepsilon]\times S^{n-1})} \leqslant \tfrac{1}{2}\|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\varepsilon,t_\varepsilon]\times S^{n-1})}$$

for all $v_1, v_2 \in \mathcal{C}^{2,\alpha}_\delta([-t_\varepsilon, t_\varepsilon] \times S^{n-1})$ satisfying

$$\|v_i\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\varepsilon,t_\varepsilon]\times S^{n-1})} \leqslant 2c_\kappa \varepsilon r_\varepsilon \varphi^{-1}(t_\varepsilon).$$

Proof. We use (2.14) together with the properties of L_ε to get

$$\|L_\varepsilon \tilde{w}_{\tilde{h}}\|_{\mathcal{C}^{0,\alpha}_\delta([-t_\varepsilon,t_\varepsilon]\times S^{n-1})} \leqslant c\varphi^{2-2n-\delta}(t_\varepsilon)\|\tilde{h}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leqslant c_\kappa \varepsilon r_\varepsilon \varphi^{((4-3n)/2)-\delta}(t_\varepsilon)$$

and also

$$\|L_\varepsilon v\|_{\mathcal{C}^{0,\alpha}_\delta([-t_\varepsilon,t_\varepsilon]\times S^{n-1})} \leqslant c\varphi^{2-2n}(t_\varepsilon)\|v\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\varepsilon,t_\varepsilon]\times S^{n-1})}.$$

Next, observe that $L - \Delta_0 = \frac{1}{4}n(3n-2)\varphi^{2-2n}$, hence

$$\begin{aligned} \|L\tilde{w}_{\tilde{h}}\|_{\mathcal{C}^{0,\alpha}_\delta([-t_\varepsilon,t_\varepsilon]\times S^{n-1})} &= \|(L - \Delta_0)\tilde{w}_{\tilde{h}}\|_{\mathcal{C}^{0,\alpha}_\delta([-t_\varepsilon,t_\varepsilon]\times S^{n-1})} \\ &\leqslant c\varphi^{-(n+2)/2}(t_\varepsilon)\|\tilde{h}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \\ &\leqslant c_\kappa \varepsilon r_\varepsilon \varphi^{-1}(t_\varepsilon). \end{aligned}$$

Using the properties of $Q_{2,\varepsilon}$, we can estimate, for all ε small enough (say $\varepsilon \in (0, \varepsilon_\kappa)$),

$$\|\varphi^{(2-n)/2} Q_{2,\varepsilon}(\varphi^{-n/2}\tilde{w}_{\tilde{h}})\|_{\mathcal{C}^{0,\alpha}_\delta([-t_\varepsilon,t_\varepsilon]\times S^{n-1})} \leqslant c\varphi^{-2-n}(t_\varepsilon)\|\tilde{h}\|^2_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leqslant c_\kappa \varepsilon r_\varepsilon \varphi^{1-3n}(t_\varepsilon)$$

and

$$\begin{aligned} \|\varphi^{(2-n)/2}(Q_{2,\varepsilon}(\varphi^{-n/2}(\tilde{w}_{\tilde{h}} + v_2)) - Q_{2,\varepsilon}(\varphi^{-n/2}(\tilde{w}_{\tilde{h}} + v_1)))\|_{\mathcal{C}^{0,\alpha}_\delta([-t_\varepsilon,t_\varepsilon]\times S^{n-1})} \\ \leqslant c\varepsilon r_\varepsilon \varphi^{-1}(t_\varepsilon)\|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\varepsilon,t_\varepsilon]\times S^{n-1})} \end{aligned}$$

provided $v_1, v_2 \in \mathcal{C}^{2,\alpha}_\delta([-t_\varepsilon, t_\varepsilon] \times S^{n-1})$ satisfy the assumption of the statement.

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Similarly, using the properties of $\tilde{Q}_{3,\varepsilon}$, we can estimate, for all ε small enough (say $\varepsilon \in (0, \varepsilon_\kappa)$),

$$\begin{aligned}\|\varphi^{n/2} Q_{3,\varepsilon}(\varphi^{-n/2}\tilde{w}_{\tilde{h}})\|_{\mathcal{C}^{0,\alpha}_\delta([-t_\varepsilon,t_\varepsilon]\times S^{n-1})} &\leqslant c\varphi^{-n-\delta}(t_\varepsilon)\|\tilde{h}\|^3_{\mathcal{C}^{2,\alpha}(S^{n-1})}\\ &\leqslant c_\kappa \varepsilon r_\varepsilon \varphi^{(12-11n)/2-\delta}(t_\varepsilon)\end{aligned}$$

and

$$\begin{aligned}\|\varphi^{n/2}(Q_{3,\varepsilon}(\varphi^{-n/2}(\tilde{w}_{\tilde{h}}+v_2)) - Q_{3,\varepsilon}(\varphi^{-n/2}(\tilde{w}_{\tilde{h}}+v_1)))\|_{\mathcal{C}^{0,\alpha}_\delta([-t_\varepsilon,t_\varepsilon]\times S^{n-1})}&\\ \leqslant c\varepsilon^2 r_\varepsilon^2 \|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\varepsilon,t_\varepsilon]\times S^{n-1})}&\end{aligned}$$

provided $v_1, v_2 \in \mathcal{C}^{2,\alpha}_\delta([-t_\varepsilon, t_\varepsilon] \times S^{n-1})$ satisfy the assumption of the statement.

The result follows at once from these estimates together with the fact that the norms of G, I_ε and $\mathcal{E}_\varepsilon$ are bounded independently of ε. □

Collecting the previous results, we conclude that, given $\kappa > 0$, there exists $\varepsilon_\kappa > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\kappa)$, the mapping $A_{\varepsilon,\tilde{h}}$ is a contraction from

$$\{v \in \mathcal{C}^{2,\alpha}_\delta([-t_\varepsilon, t_\varepsilon] \times S^{n-1}) : \|v\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\varepsilon,t_\varepsilon]\times S^{n-1})} \leqslant 2c_\kappa \varepsilon r_\varepsilon \varphi^{-1}(t_\varepsilon)\}$$

into itself and hence has a unique fixed point $v_{\varepsilon,\tilde{h}}$ in this set.

The hypersurface parametrized by

$$X_{\varepsilon,\tilde{h}} = \varepsilon^{1/(n-1)}(X + \varphi^{(2-n)/2}(\tilde{w}_{\tilde{h}} + v_{\varepsilon,\tilde{h}})\boldsymbol{n}_\varepsilon),$$

for $(t, z) \in [-t_\varepsilon, t_\varepsilon] \times S^{n-1}$ is a minimal hypersurface which will be denoted by $\tilde{C}_{\varepsilon,h}$. To summarize, we have produced an infinite-dimensional family of minimal hypersurfaces which are close to the piece of the catenoid C which is the image of $[-t_\varepsilon, t_\varepsilon] \times S^{n-1}$ by X. This family is parametrized by the boundary data $\tilde{h}$.

Observe that, if one wants to produce a hypersurface which is invariant under the action of $\mathfrak{G}$ it is enough to restrict our attention to the subspace of functions $\tilde{h} \in \mathcal{C}^{2,\alpha}(S^{n-1})$ which enjoy the following invariance property

$$\tilde{h}(z) = \tilde{h}(Rz) \tag{2.16}$$

for all $R \in O(n)$ of the form

$$R := \begin{pmatrix} 1 & 0 \\ 0 & \bar{R} \end{pmatrix}$$

where $\bar{R} \in O(n-1)$.

2.4. Local description of the hypersurface $\tilde{C}_{\varepsilon,h}$ near its boundaries

We first recall the asymptotic expansion of the parametrization of the ends of the n-catenoid. Starting from the parametrization of the n-catenoid which was given in (2.1), we perform the change of variables

$$x = \varphi(t)z \in \mathbb{R}^n.$$

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The lower end of the n-catenoid can be parametrized as a vertical graph over $\mathbb{R}^n \times \{0\}$ for some function u which can be expanded as

$$u(x) = -d_0 + \frac{1}{n-2}|x|^{2-n} + \mathcal{O}(|x|^{4-3n}),$$

for $|x|$ large enough, where

$$d_0 := \lim_{t\to+\infty} \psi(t).$$

This expansion follows at once from the definition of φ and ψ. We refer to [**1**] for the details.

We now consider the n-catenoid which has been scaled by a factor $\varepsilon^{1/(n-1)}$. Its lower end can be parametrized by

$$x \to \varepsilon^{1/(n-1)}\left(x, -d_0 + \frac{1}{n-2}|x|^{2-n} + \mathcal{O}(|x|^{4-3n})\right),$$

for $|x|$ large enough. Changing $\varepsilon^{1/(n-1)}x$ into x, we see that the lower end of the scaled n-catenoid can also be parametrized by

$$x \to \left(x, -\varepsilon^{1/(n-1)}d_0 + \frac{\varepsilon}{n-2}|x|^{2-n} + \mathcal{O}(\varepsilon^3|x|^{4-3n})\right). \tag{2.17}$$

We apply the analysis of the previous section and collect the results. Close to its lower boundary, the minimal hypersurface $C_{\varepsilon,\tilde{h}}$ can be described as a vertical graph over an annulus in the horizontal hyperplane $x^{n+1} = 0$, this was the purpose of changing the normal vector field $\boldsymbol{n}$ into $\boldsymbol{n}_\varepsilon$. To make this precise, recall that we have defined

$$r_\varepsilon := \varepsilon^{1/(n-1)}\varphi(t_\varepsilon) = \varepsilon^{2/(3n-2)}.$$

Given $h \in \mathcal{C}^{2,\alpha}(S^{n-1})$, we define $\tilde{h}$ by

$$\tilde{h} = \varepsilon^{-1/(n-1)}\varphi^{(n-2)/2}(t_\varepsilon)h$$

and we write the hypersurface $C_{\varepsilon,h} := \tilde{C}_{\varepsilon,\tilde{h}}$, close to its lower boundary, as the graph over $\bar{B}(0, r_\varepsilon) - B(0, r_\varepsilon/2)$ for a function $u_{\varepsilon,h}$.

We set

$$u^0_{\varepsilon,h}(x) := -\varepsilon^{1/(n-1)}d_0 + \frac{\varepsilon}{n-2}|x|^{2-n} + w^i_h(x/r_\varepsilon),$$

where w^i_h is the harmonic extension of h in $B(0, 1)$ and we set

$$v_{\varepsilon,h} := u_{\varepsilon,h} - u^0_{\varepsilon,h}. \tag{2.18}$$

Following the construction of the previous section, we obtain the following lemma.

Lemma 2.6. *There exists $c = c(n, \delta) > 0$ and, for all $\kappa > 0$, there exists $\varepsilon_\kappa > 0$ such that, for all $h \in \mathcal{C}^{2,\alpha}(S^{n-1})$ which is $L^2(S^{n-1})$-orthogonal to E_0 and E_1 and which satisfies $\|h\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leqslant \kappa\varepsilon r^2_\varepsilon$, we have*

$$\|v_{\varepsilon,h}(r_\varepsilon\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}(0,1)-B(0,\frac{1}{2}))} \leqslant c\varepsilon r^2_\varepsilon.$$

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In addition,

$$\|(v_{\varepsilon,h_2} - v_{\varepsilon,h_1})(r_\varepsilon \cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}(0,1)-B(0,\frac{1}{2}))} \leqslant c\varphi^{\delta-(n+2)/2}(t_\varepsilon)\|h_2 - h_1\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}$$

if $h_2, h_1 \in \mathcal{C}^{2,\alpha}(S^{n-1})$ *are* $L^2(S^{n-1})$*-orthogonal to* E_0 *and* E_1 *and satisfy*

$$\|h_i\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leqslant \kappa\varepsilon r_\varepsilon^2.$$

The key and crucial point is that the constant $c > 0$ does not depend on κ. The proof of this estimate follows from a simple but tedious computation following the steps of the construction of $C_{\varepsilon,h}$. Observe that, when $h = 0$, the difference between $u_{\varepsilon,0}$ and $u^0_{\varepsilon,0}$ comes from the term $\mathcal{O}(\varepsilon^3|x|^{4-3n})$, with $|x| = r_\varepsilon$, which appears in (2.17) the expansion of the lower end of the n-catenoid.

When $h \neq 0$, there are many discrepancies to take into account. The first comes from the fact that, by construction, the function $\varepsilon^{1/(n-1)}\varphi^{(2-n)/2}\tilde{w}_{\tilde{h}}$ is not exactly equal to the function h when $t = t_\varepsilon$ but the difference between these two functions is bounded by a constant (depending on κ) times $\varphi^{-2-n}(t_\varepsilon)\varepsilon r_\varepsilon^2$, and hence which is uniformly bounded by a constant (independent of κ) times $\varepsilon r_\varepsilon^2$ if ε is taken small enough. Next, one has to take into account the fact that, for t close to t_ε, the coordinates (t, z) are not the usual cylindrical coordinates (s, z) with $|x| = \varepsilon^{1/(n-1)}\mathrm{e}^s$ in $\mathbb{R}^n$ and hence the vertical graph of $\varepsilon^{1/(n-1)}\varphi^{(2-n)/2}\tilde{w}_{\tilde{h}}$ is not exactly equal to the vertical graph of w^i_h. This induces in (2.18) another discrepancy which is bounded by a constant (depending on κ) times $\varphi^{2-2n}(t_\varepsilon)\varepsilon r_\varepsilon^2$, and again is uniformly bounded by a constant (independent of κ) times $\varepsilon r_\varepsilon^2$ if ε is taken small enough. Finally, there is a term which comes from $v_{\varepsilon,\tilde{h}}$, the solution of the nonlinear problem, and this induces a discrepancy which is bounded by a constant (depending on κ) times $\varphi^{\delta-(n+2)/2}(t_\varepsilon)\varepsilon r_\varepsilon^2$, and since $\delta - (n+2)/2 < 0$, is uniformly bounded by a constant (independent of κ) times $\varepsilon r_\varepsilon^2$ if ε is taken small enough.

3. Minimal hypersurfaces which are graphs over a hyperplane

3.1. The mean curvature for graphs

Assume that a hypersurface is a vertical graph over the hyperplane $x^{n+1} = 0$ for some function w, i.e. this hypersurface is parametrized by

$$x \in \mathbb{R}^n \to (x, w(x)) \in \mathbb{R}^{n+1}.$$

We recall that this hypersurface is minimal if and only if w is a solution of

$$\mathrm{div}\left(\frac{\nabla w}{(1 + |\nabla w|^2)^{1/2}}\right) = 0.$$

It will be more convenient to write this equation as

$$\Delta w = \frac{\nabla^2 w(\nabla w, \nabla w)}{1 + |\nabla w|^2}. \tag{3.1}$$

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We will be interested in vertical graphs which are invariant under the action of the group $\mathfrak{G}$ which has been defined in the introduction. This amounts to restricting our attention to functions w which enjoy the following invariance property

$$w(-x) = -w(x) \quad \text{and} \quad w(x) = w(Rx) \tag{3.2}$$

for all $R \in O(n)$ of the form

$$R := \begin{pmatrix} 1 & 0 \\ 0 & \bar{R} \end{pmatrix},$$

where $\bar{R} \in O(n-1)$. Again the Laplacian preserves this invariance, i.e. if a function w satisfies (3.2) then so does the function Δw and since the mean curvature is invariant under the action of isometries, the nonlinear operator which appears on the right-hand side of (3.1) also enjoys a similar invariance property.

3.2. Linear analysis of the Laplacian in weighted spaces

We set $x^* := (1, 0, \dots, 0) \in \mathbb{R}^n$ and we define

$$\mathbb{R}^n_* := \mathbb{R}^n - \{x^*, -x^*\}.$$

Given $k \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\mu, \nu \in \mathbb{R}$, the space $\mathcal{C}^{k,\alpha}_{\mu,\nu}(\mathbb{R}^n_*)$ is defined to be the space of functions $w \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R}^n_*)$ for which the following norm

$$\begin{aligned}\|w\|_{\mathcal{C}^{k,\alpha}_{\mu,\nu}(\mathbb{R}^n_*)} := \sup_{s\in(0,1/2)} & s^{-\nu} \|w(s\cdot + x^*)\|_{\mathcal{C}^{k,\alpha}(\bar{B}(0,2)-B(0,1))} \\ & + \sup_{s\in(0,1/2)} s^{-\nu} \|w(s\cdot - x^*)\|_{\mathcal{C}^{k,\alpha}(\bar{B}(0,2)-B(0,1))} \\ & + \|w\|_{\mathcal{C}^{k,\alpha}(\bar{B}(0,4)-(B(x^*,\frac{1}{2})\cup B(-x^*,\frac{1}{2})))} \\ & + \sup_{s\in(2,+\infty)} s^{-\mu} \|w(s\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}(0,2)-B(0,1))}\end{aligned}$$

is finite. Therefore, the weight parameter ν controls the behaviour of the function w near the points $\pm x^*$ and the weight parameter μ controls its behaviour at infinity.

The following result follows from [**8**] and [**5**] but is also a simple consequence of the maximum principle.

Proposition 3.1. *Assume that $\mu, \nu \in (2-n, 0)$ are fixed. Then, there exists a constant $c = c(n, \mu, \nu) > 0$ and, for all $r \in (0, \frac{1}{2})$, there exists a continuous operator*

$$\varGamma : \mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}(\mathbb{R}^n_*) \to \mathcal{C}^{2,\alpha}_{\mu,\nu}(\mathbb{R}^n_*)$$

such that, for all $f \in \mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}(\mathbb{R}^n_*)$, the function $w := \varGamma(f)$ is a solution of

$$\Delta w = f$$

in $\mathbb{R}^n_*$. In addition, if f satisfies (3.2) then so does $\varGamma(f)$.

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Proof. As mentioned, the existence of Γ follows from the results in [**8**] and [**5**] (see also [**10**]). Clearly, the operator

$$\Delta : \mathcal{C}^{2,\alpha}_{\mu,\nu}(\mathbb{R}^n_*) \to \mathcal{C}^{2,\alpha}_{\mu-2,\nu-2}(\mathbb{R}^n_*)$$

is injective when $\nu > 2-n$ and $\mu < 0$. Hence, according to the results in [**8**] and [**5**], the operator is surjective when $\nu < 0$ and $\mu > 2-n$ are not indicial roots of the Laplacian (namely are not of the form j or $2-n-j$ for some $j \in \mathbb{N}$). This completes the proof of the result.

We also provide a simple proof of this result based on the maximum principle. Observe that, for all $\lambda \in \mathbb{R}$, we have

$$\Delta |x|^\lambda = \lambda(n-2+\lambda)|x|^{\lambda-2}$$

and that $\lambda(n-2+\lambda) < 0$ precisely when $\lambda \in (2-n, 0)$.

Now, let us first assume that f_1 is supported in $B(0,4) - \{x^*, -x^*\}$. The above observation implies that the function

$$x \to |x-x^*|^\nu + |x+x^*|^\nu$$

can be used as a barrier function to prove both the existence w_1 solution of $\Delta w_1 = f_1$ in $\mathbb{R}^n_*$ as well as the pointwise bound

$$|w_1(x)| \leqslant c\|f_1\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}(\mathbb{R}^n_*)}(|x-x^*|^\nu + |x+x^*|^\nu)$$

for some constant $c = c(n,\nu) > 0$. However, since f_1 is supported in $B(0,4) - \{x^*, -x^*\}$, w_1 is harmonic in $\mathbb{R}^n - B(0,4)$ and hence, it follows from the maximum principle that the function $x \to |x|^{2-n}$ can be used as a barrier function to prove that

$$|w_1(x)| \leqslant \sup_{\partial B(0,4)} |w_1| 4^{n-2}|x|^{2-n} \leqslant c\|f_1\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}(\mathbb{R}^n_*)},$$

for all $x \in \mathbb{R}^n - B(0,4)$, where $c = c(n,\nu) > 0$.

Finally, let us assume that f_2 is supported in $\mathbb{R}^n - B(0,2)$. Then, the above observation implies that the function

$$x \to |x-x^*|^\mu + |x+x^*|^\mu$$

can be used as a barrier function to prove both the existence w_2 solution of $\Delta w_2 = f_2$ in $\mathbb{R}^n_*$ as well as the pointwise bound

$$|w_2(x)| \leqslant c\|f_2\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}(\mathbb{R}^n_*)}(|x-x^*|^\mu + |x+x^*|^\mu)$$

for some constant $c = c(n,\mu) > 0$. However, since f_2 is supported in $\mathbb{R}^n - B(0,2)$, w_2 is harmonic in $B(0,2)$ and hence, it follows from the maximum principle that

$$|w_2(x)| \leqslant \sup_{\partial B(0,2)} |w_2| \leqslant c\|f_2\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}(\mathbb{R}^n_*)},$$

for all $x \in B(0,2)$, where $c = c(n,\mu) > 0$.

The existence of $\Gamma(f)$ follows from these considerations, first decomposing $f = f_1 + f_2$ where f_1 is a function supported in $B(0,4)$ and f_2 a function supported in $\mathbb{R}^n - B(0,4)$. We obtain a function $\Gamma(f) = w_1 + w_2$ where w_1 and w_2 are defined as above. Collecting the above estimates, we know that

$$|w(x)| \leqslant c\|f\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}(\mathbb{R}^n_*)}(|x - x^*|^\nu + |x + x^*|^\nu)$$

in $B(0,4) - \{x^*, -x^*\}$ and

$$|w(x)| \leqslant c\|f\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}(\mathbb{R}^n_*)}|x|^\mu$$

in $\mathbb{R}^n - B(0,4)$, for some constant $c = c(n,\mu,\nu) > 0$. Once the existence and the pointwise control of w have been obtained, the estimates for the derivatives of w follow from Schauder's estimates. This completes the proof of the result. □

The following lemma is the counterpart of Lemma 2.4. Observe that, this time we do not impose any constraint on the boundary data and the proof is again a simple application of the maximum principle.

Lemma 3.2. *Assume that $\alpha \in (0,1)$ is fixed. For all $h \in \mathcal{C}^{2,\alpha}(S^{n-1})$, there exists $\bar{w}_h \in \mathcal{C}^{2,\alpha}_{2-n}(\mathbb{R}^n - B(0,1))$ satisfying*

$$\begin{cases} \Delta \bar{w}_h = 0 & \text{in } \mathbb{R}^n - \bar{B}(0,1), \\ \bar{w}_h = h & \text{on } \partial B(0,1). \end{cases}$$

Furthermore,

$$\|\bar{w}_h\|_{\mathcal{C}^{2,\alpha}_{2-n}(\mathbb{R}^n - B(0,1))} \leqslant c\|h\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}$$

for some constant $c = c(n) > 0$.

Proof. The proof of this result follows at once from the observation that $x \to |x|^{2-n}$ can be used as a barrier function to prove both the existence of $\bar{w}_h$ and the pointwise bound

$$|\bar{w}_h(x)| \leqslant \sup_{\partial B(0,1)} |h||x|^{2-n}.$$

We then apply Schauder's estimates to get the relevant estimates for the derivatives of $\bar{w}_h$. □

Observe that if the function h enjoys the invariance property (2.9), namely

$$h(z) = h(Rz)$$

for all $R \in O(n)$ of the form

$$R := \begin{pmatrix} 1 & 0 \\ 0 & \bar{R} \end{pmatrix},$$

where $\bar{R} \in O(n-1)$, then so does the function $\bar{w}_h$, namely

$$\bar{w}_h(x) = \bar{w}_h(Rx) \tag{3.3}$$

for all $R \in O(n)$ as above.

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3.3. Nonlinear analysis

Recall that we have defined

$$r_\varepsilon := \varepsilon^{2/(3n-2)}.$$

For all $r \in (0,1)$, we also define

$$D_r := \mathbb{R}^n - (B(x^*, r) \cup B(-x^*, r)).$$

Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu, \nu \in \mathbb{R}$, we define the space $\mathcal{C}^{k,\alpha}_{\mu,\nu}(D_r)$ to be the space of functions $w \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(D_r)$ for which the following norm

$$\begin{aligned} \|w\|_{\mathcal{C}^{k,\alpha}_{\mu,\nu}(D_r)} := \sup_{s\in(r,\frac{1}{2})} s^{-\nu} \|w(s\cdot + x^*)\|_{\mathcal{C}^{k,\alpha}(\bar{B}(0,2)-B(0,1))} \\ + \sup_{s\in(r,\frac{1}{2})} s^{-\nu} \|w(s\cdot - x^*)\|_{\mathcal{C}^{k,\alpha}(\bar{B}(0,2)-B(0,1))} \\ + \|w\|_{\mathcal{C}^{k,\alpha}(\bar{B}(0,4)-(B(x^*,\frac{1}{2})\cup B(-x^*,\frac{1}{2})))} \\ + \sup_{s\in(2,+\infty)} s^{-\mu} \|w(s\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}(0,2)-B(0,1))} \end{aligned}$$

is finite.

The remainder of the analysis parallels what we have already done to perturb the truncated rescaled n-catenoid. Given $\bar{h} \in \mathcal{C}^{2,\alpha}(S^{n-1})$, we use the result of Lemma 3.2 to define the function

$$\hat{w}_{\bar{h}} := \bar{w}_{\bar{h}}((\cdot - x^*)/r_\varepsilon) - \bar{w}_{\bar{h}}(-(\cdot + x^*)/r_\varepsilon).$$

Using the estimate provided by Lemma 3.2, we conclude that

$$\|\hat{w}_{\bar{h}}\|_{\mathcal{C}^{2,\alpha}_{2-n,2-n}(D_{r_\varepsilon})} \leqslant c r_\varepsilon^{n-2} \|\bar{h}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}$$

for some constant $c = c(n) > 0$.

Remark 3.3. Observe that the decay at infinity of $\hat{w}_{\bar{h}}$ can be slightly improved and we have

$$\|\hat{w}_{\bar{h}}\|_{\mathcal{C}^{2,\alpha}_{1-n,2-n}(D_{r_\varepsilon})} \leqslant c r_\varepsilon^{n-2} \|\bar{h}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}.$$

This follows at once from the fact that the function $\bar{w}_{\bar{h}}$, being harmonic and tending to 0 at infinity, can be decomposed as

$$\bar{w}_{\bar{h}}(x) = a_{\bar{h}} |x|^{2-n} + \bar{w}'_{\bar{h}}(x),$$

where

$$|a_{\bar{h}}| + \|\bar{w}'_{\bar{h}}\|_{\mathcal{C}^{2,\alpha}_{1-n}(\mathbb{R}^n - B(0,1))} \leqslant c \|\bar{h}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}.$$

We define, for all $\rho \in \mathbb{R}$ and $\varepsilon > 0$ the function

$$w_{\varepsilon,\rho}(x) := (\rho - 2^{1-n}\varepsilon)x^1 + \frac{\varepsilon}{n-2}(|x - x^*|^{2-n} - |x + x^*|^{2-n}).$$

We look for a solution of (3.1) of the form

$$w := w_{\varepsilon,\rho} + \hat{w}_{\bar{h}} + v,$$

where v is a small function.

It will be convenient to define an extension operator

$$\bar{\mathcal{E}}_\varepsilon : \mathcal{C}^{0,\alpha}_{\mu,\nu}(D_{r_\varepsilon}) \to \mathcal{C}^{0,\alpha}_{\mu,\nu}(\mathbb{R}^n_*)$$

as follows.

(i) For all $x \in D_{r_\varepsilon}$, we set

$$\bar{\mathcal{E}}_\varepsilon(f)(x) = f(x).$$

(ii) For all $x \in B(x^*, r_\varepsilon) - \bar{B}(x^*, r_\varepsilon/2)$ we set

$$\bar{\mathcal{E}}_\varepsilon(f)(x) = \bar{\chi}(|x - x^*|/r_\varepsilon) f(x^* + r_\varepsilon(x - x^*)/|x - x^*|).$$

(iii) For all $x \in B(-x^*, r_\varepsilon) - \bar{B}(-x^*, r_\varepsilon/2)$ we set

$$\bar{\mathcal{E}}_\varepsilon(f)(x) = \bar{\chi}(|x + x^*|/r_\varepsilon) f(-x^* + r_\varepsilon(x + x^*)/|x + x^*|).$$

(iv) And finally, for all $x \in B(x^*, r_\varepsilon/2) \cup B(-x^*, r_\varepsilon/2)$ we set

$$\bar{\mathcal{E}}_\varepsilon(f)(x) = 0.$$

In all of these instances, $\bar{\chi} : \mathbb{R} \to [0, 1]$ is a smooth cutoff function identically equal to 1 for $s \geqslant 1$ and identically equal to 0 for $s \leqslant 1/2$.

Obviously, there exists a constant $c = c(n, \nu) > 0$ such that

$$|||\bar{\mathcal{E}}_\varepsilon||| \leqslant c.$$

We denote by $\bar{I}_\varepsilon$ the canonical imbedding

$$\bar{I}_\varepsilon : \mathcal{C}^{2,\alpha}_{\mu,\nu}(\mathbb{R}^n_*) \to \mathcal{C}^{2,\alpha}_{\mu,\nu}(D_{r_\varepsilon}).$$

We fix $\mu, \nu \in (2 - n, 0)$. Using the result of Proposition 3.1, we rephrase the solvability of (3.1) as a fixed point problem. It is now enough to find $v \in \mathcal{C}^{2,\alpha}_{\mu,\nu}(D_{r_\varepsilon})$ solution of

$$v = B_{\varepsilon,\rho,\bar{h}}(v), \tag{3.4}$$

where

$$B_{\varepsilon,\rho,\bar{h}}(v) := \bar{I}_\varepsilon \circ \Gamma \circ \bar{\mathcal{E}}_\varepsilon(\Xi(w_{\varepsilon,\rho} + \hat{w}_{\bar{h}} + v))$$

and where we have set

$$\Xi(w) := \frac{\nabla^2 w(\nabla w, \nabla w)}{1 + |\nabla w|^2}.$$

The existence of a fixed point for $B_{\varepsilon,\rho,\bar{h}}$ relies on the following lemma.

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Lemma 3.4. *There exists $c = c(n, \mu, \nu) > 0$ and for all $\kappa > 0$ there exists $\varepsilon_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, for all $\rho \in \mathbb{R}$ and for all $\bar{h} \in \mathcal{C}^{2,\alpha}(S^{n-1})$ satisfying*

$$r_\varepsilon|\rho| + \|\bar{h}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leqslant \kappa\varepsilon r_\varepsilon^2,$$

we have

$$\|B_{\varepsilon,\rho,\bar{h}}(0)\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}(D_{r_\varepsilon})} \leqslant c\varepsilon r_\varepsilon^{2-\nu}.$$

Moreover,

$$\|B_{\varepsilon,\rho,\bar{h}}(v_1) - B_{\varepsilon,\rho,\bar{h}}(v_2)\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}(D_{r_\varepsilon})} \leqslant \tfrac{1}{2}\|v_1 - v_2\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}(D_{r_\varepsilon})}$$

for all $v_1, v_2 \in \mathcal{C}^{2,\alpha}_{\mu,\nu}(D_{r_\varepsilon})$ satisfying

$$\|v_i\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}(D_{r_\varepsilon})} \leqslant 2c\varepsilon r_\varepsilon^{2-\nu}.$$

Proof. The first estimate follows from the result of Proposition 3.1 together with the estimate

$$\|\Xi(w_{\varepsilon,\rho} + \hat{w}_{\bar{h}})\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}(D_{r_\varepsilon})} \leqslant c\varepsilon r_\varepsilon^{2-\nu},$$

which follows from the construction of $w_{\varepsilon,\rho} + \hat{w}_{\bar{h}}$. The second estimate follows from

$$\|\Xi(w_{\varepsilon,\rho} + \hat{w}_{\bar{h}} + v_2) - \Xi(w_{\varepsilon,\rho} + \hat{w}_{\bar{h}} + v_1)\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}(D_{r_\varepsilon})} \leqslant c r_\varepsilon^n \|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}(D_{r_\varepsilon})}.$$

Details are left to the reader. □

Collecting the previous results, we conclude that, given $\kappa > 0$, there exists $\varepsilon_\kappa > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\kappa)$ the mapping $B_{\varepsilon,\rho,\bar{h}}$ is a contraction from

$$\{v \in \mathcal{C}^{2,\alpha}_{\mu,\nu}(D_{r_\varepsilon}) : \|v\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}(D_{r_\varepsilon})} \leqslant 2c\varepsilon r_\varepsilon^{2-\nu}\}$$

into itself and hence has a unique fixed point $w_{\varepsilon,\rho,\bar{h}}$ in this set. We define

$$\bar{u}_{\varepsilon,\rho,\bar{h}} := w_{\varepsilon,\rho} + \hat{w}_{\bar{h}} + w_{\varepsilon,\rho,\bar{h}}.$$

The hypersurface parametrized by $x \to (x, \bar{u}_{\varepsilon,\rho,\bar{h}}(x))$ for $x \in D_{r_\varepsilon}$ is a minimal hypersurface and will be denoted by $\Sigma_{\varepsilon,\rho,\bar{h}}$.

The important fact is that the constant $c > 0$ which appears in Lemma 3.4 does not depend on κ provided ε is chosen small enough.

If one looks for minimal hypersurfaces which are invariant under the action of the group $\mathfrak{G}$, it is enough to restrict our attention to the set of functions $\bar{h} \in \mathcal{C}^{2,\alpha}(S^{n-1})$ which are invariant under

$$\bar{h}(z) = \bar{h}(Rz)$$

for all $R \in O(n)$ of the form

$$R := \begin{pmatrix} 1 & 0 \\ 0 & \bar{R} \end{pmatrix},$$

where $\bar{R} \in O(n-1)$.

3.4. Local description of the hypersurface $\Sigma_{\varepsilon,\rho,\bar h}$ near its boundaries

We would like to analyse $\bar u_{\varepsilon,\rho,\bar h}$ close to x^*. To this aim, we define $y := x - x^*$ and the function

$$\bar u^0_{\varepsilon,\rho,\bar h}(y) := \rho - \frac{n}{n-2} 2^{1-n}\varepsilon + \frac{\varepsilon}{n-2}|y|^{2-n} + \rho y^1 + \bar w^e_{\bar h}(y/r_\varepsilon),$$

where

$$w^e_{\bar h} := \bar w_{\bar h}$$

is the harmonic extension defined in Lemma 3.2. We also define

$$\bar v_{\varepsilon,\rho,\bar h} := \bar u_{\varepsilon,\rho,\bar h} - \bar u^0_{\varepsilon,\rho,\bar h}.$$

Following the construction of $\bar u_{\varepsilon,\rho,\bar h}$, we obtain the following lemma.

Lemma 3.5. *There exists $c = c(n,\mu,\nu) > 0$ and for all $\kappa > 0$ there exists $\varepsilon_\kappa > 0$ such that, for all $\varepsilon \in (0,\varepsilon_\kappa)$, for all $\rho \in \mathbb{R}$ and for all $\bar h \in \mathcal{C}^{2,\alpha}(S^{n-1})$ satisfying*

$$r_\varepsilon|\rho| + \|\bar h\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leqslant \kappa\varepsilon r_\varepsilon^2$$

we have

$$\|\bar v_{\varepsilon,\rho,\bar h}(x^* + r_\varepsilon\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar B(0,2)-B(0,1))} \leqslant c\varepsilon r_\varepsilon^2.$$

Moreover,

$$\|(\bar v_{\varepsilon,\rho_2,\bar h_2} - \bar v_{\varepsilon,\rho_1,\bar h_1})(x^* + r_\varepsilon\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar B(0,2)-B(0,1))} \leqslant c r_\varepsilon^n (r_\varepsilon|\rho_2 - \rho_1| + \|\bar h_2 - \bar h_1\|_{\mathcal{C}^{2,\alpha}(S^{n-1})})$$

for all

$$r_\varepsilon|\rho_i| + \|\bar h_i\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leqslant \kappa\varepsilon r_\varepsilon^2.$$

Again, the important fact is that the constant $c > 0$ in the first estimate does not depend on κ.

4. The connected sum construction

We fix κ large enough and apply the results of the previous sections.

Assume that we are given $h, \bar h \in \mathcal{C}^{2,\alpha}(S^{n-1})$ satisfying

$$\|h\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leqslant \kappa\varepsilon r_\varepsilon^2 \quad \text{and} \quad \|\bar h\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leqslant \kappa\varepsilon r_\varepsilon^2.$$

We decompose

$$h = h_0 + h_1 + h^\perp,$$

where $h_0 \in E_0$, $h_1 \in E_1$ and $h^\perp$ is $L^2(S^{n-1})$-orthogonal to E_0 and E_1. We further assume that both h and $\bar h$ satisfy (2.16). In particular, this implies that $h_1 \in \operatorname{Span}\{x \cdot \boldsymbol{e}_1\}$ and if we decompose

$$\bar h = \bar h_0 + \bar h_1 + \bar h^\perp$$

this also implies that $\bar h_1 \in \operatorname{Span}\{x \cdot \boldsymbol{e}_1\}$.

Granted the above decomposition, we choose from now on

$$\rho := -r_\varepsilon^{-1} h_1$$

and consider the hypersurface $\Sigma_{\varepsilon,\rho,\bar{h}}$ which has been constructed in the previous section. Next, we define

$$t := \left(\varepsilon^{1/(n-1)} d_0 + \rho - \frac{n}{n-2} 2^{1-n} \varepsilon + h_0 \right)$$

and consider $\boldsymbol{e}_1 + t\boldsymbol{e}_{n+1} + C_{\varepsilon,h^\perp}$ the hypersurface defined in § 2.4 which has been translated by $\boldsymbol{e}_1 + t\boldsymbol{e}_{n+1}$. The lower boundary of $\boldsymbol{e}_1 + t\boldsymbol{e}_{n+1} + C_{\varepsilon,h^\perp}$ and the 'upper boundary' of $\Sigma_{\varepsilon,\rho,\bar{h}}$ are close on to the other and we will now show that, for all ε small enough, it is possible to find $\bar{h}$ and h in such a way that the union of $\Sigma_{\varepsilon,\rho,\bar{h}}$ and $\boldsymbol{e}_1 + t\boldsymbol{e}_{n+1} + C_{\varepsilon,h^\perp}$ is a $\mathcal{C}^1$ hypersurface. Since this hypersurface has piecewise mean curvature equal to 0 and is $\mathcal{C}^1$, regularity theory then implies that it is a smooth minimal hypersurface with two boundaries and one end asymptotic to $x \to (x, \rho x^1)$.

To complete the construction of the generalized Riemann's minimal hypersurface, it will remain to first apply a suitable rotation so that the end of the hypersurface $\Sigma_{\varepsilon,\rho,\bar{h}} \cup (\boldsymbol{e}_1 + t\boldsymbol{e}_{n+1} + C_{\varepsilon,h^\perp})$ which is asymptotic to $x \to (x, \rho x^1)$ becomes horizontal and next to extend this hypersurface so that it becomes a simply periodic hypersurface (which is invariant under the action of $\mathfrak{G}$).

In order to produce a $\mathcal{C}^1$ hypersurface, we consider the two summands as vertical graphs over annular regions in the hyperplane $x^{n+1} = 0$ and ask that the Cauchy data of these two graphs coincide. This condition can be translated into the following set of equations

$$\begin{cases} u_{\varepsilon,h^\perp}(r_\varepsilon \cdot) + t = \bar{u}_{\varepsilon,\rho,\bar{h}}(x^* + r_\varepsilon \cdot), \\ \partial_r u_{\varepsilon,h^\perp}(r_\varepsilon \cdot) = \partial_r \bar{u}_{\varepsilon,\rho,\bar{h}}(x^* + r_\varepsilon \cdot) \end{cases} \tag{4.1}$$

on S^{n-1}. However, given the expansions of $u_{\varepsilon,h^\perp}$ and $\bar{u}_{\varepsilon,\rho,\bar{h}}$ this is equivalent to solving

$$\begin{cases} w_h^i - w_{\bar{h}}^e = \bar{v}_{\varepsilon,\rho,\bar{h}}(x^* + r_\varepsilon \cdot) - v_{\varepsilon,h^\perp}(r_\varepsilon \cdot), \\ \partial_r(w_h^i - w_{\bar{h}}^e) = \partial_r(\bar{v}_{\varepsilon,\rho,\bar{h}}(x^* + r_\varepsilon \cdot) - v_{\varepsilon,h^\perp}(r_\varepsilon \cdot)) \end{cases} \tag{4.2}$$

on S^{n-1}. Here w_h^i is the harmonic extension of h in the unit ball of $\mathbb{R}^n$ and $w_{\bar{h}}^e$ is the harmonic extension, decaying at infinity, of $\bar{h}$ in $\mathbb{R}^n - B(0,1)$.

We recall the following result [**6**].

Lemma 4.1. *The mapping*

$$\begin{aligned} \mathcal{P} : \mathcal{C}^{2,\alpha}(S^{n-1}) &\to \mathcal{C}^{1,\alpha}(S^{n-1}), \\ h &\mapsto \partial_r(w_h^e - w_h^i) \end{aligned}$$

is an isomorphism.

Using this result, the solvability of (4.2) reduces to a fixed point problem which can be written as

$$(h, \bar{h}) = S_\varepsilon(h, \bar{h}).$$

It follows from the estimates of Lemma 2.6 and Lemma 3.5 that

$$\|S_\varepsilon(h,\bar{h})\|_{(\mathcal{C}^{2,\alpha}(S^{n-1}))^2} \leqslant c_0 \varepsilon r_\varepsilon^2$$

for some constant $c_0 > 0$ which does not depend on κ provided ε is small enough. In addition

$$\|S_\varepsilon(h_2,\bar{h}_2) - S_\varepsilon(h_1,\bar{h}_1)\|_{(\mathcal{C}^{2,\alpha}(S^{n-1}))^2} \leqslant \tfrac{1}{2}\|(h_2-h_1,\bar{h}_2-\bar{h}_1)\|_{(\mathcal{C}^{2,\alpha}(S^{n-1}))^2}$$

provided ε is chosen small enough.

To conclude, we choose $\kappa = 2c_0$ and use a fixed point theorem for contraction mappings which will ensure the existence of one fixed point for the mapping S_ε in

$$\{(h,\bar{h}) \in (\mathcal{C}^{2,\alpha}(S^{n-1}))^2 : \|(h,\bar{h})\|_{(\mathcal{C}^{2,\alpha}(S^{n-1}))^2} \leqslant \kappa\varepsilon r_\varepsilon^2\}$$

provided ε is chosen small enough, say $\varepsilon \in (0,\varepsilon_0]$. This completes our proof of the existence of a fixed point for S_ε and hence the existence of Riemann minimal hypersurfaces in any dimension.

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References

1. S. Fakhi and F. Pacard, Existence of complete minimal hypersurfaces with finite total curvature, *Manuscripta Math.* **103** (2000), 465–512.
2. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order* (Springer, 2001).
3. W. C. Jagy, Minimal hypersurfaces foliated by spheres, *Michigan Math. J.* **38** (1991), 255–270.
4. M. Jleli, Constant mean curvature hypersurfaces, PhD thesis, University of Paris 12 (2004).
5. R. Mazzeo, Elliptic theory of edge operators, I, *Commun. PDE* **16**(10) (1991), 1616–1664.
6. R. Mazzeo and F. Pacard, Constant scalar curvature metrics with isolated singulaties, *Duke Math. J.* **99**, (1999), 353–418.
7. W. Meeks, J. Perez and A. Ros, Uniqueness of the Riemann minimal examples, *Invent. Math.* **131** (1998), 107–132.
8. R. Melrose, *The Atiyah–Patodi–Singer index theorem*, Research Notes in Mathematics, Volume 14 (Springer, 1993).
9. F. Pacard, Higher dimensional Scherk's hypersurfaces, *J. Math. Pures Appl.* **9**(3) (2002), 241–258.
10. F. Pacard and T. Rivière, *Linear and nonlinear aspects of vortices: the Ginzburg–Landau model*, Progress in Nonlinear Differential Equations, Volume 39 (Birkäuser, Basel, 2000).
11. M. A. Shubin, *Pseudodifferential operators and spectral theory* (Springer, 1987).
12. M. Traizet, Adding handles to Riemann minimal examples, *J. Inst. Math. Jussieu* **1**(1) (2002), 145–174.