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TWO-PERSON RED-AND-BLACK GAME WITH LOWER LIMIT

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In this article we consider a two-person red-and-black game with lower limit. More precisely, assume each player holds an integral amount of chips. At each stage, each player can bet an integral amount between a fixed positive integer ℓ and his possession x if $x \geq \ell$; otherwise, he bets all of his own fortune. He might win his opponent's stakes with a probability that is a function of the ratio of his bet to the sum of both players' bets and is called a win probability function. The goal of each player is to maximize the probability of winning the entire fortune of his opponent by gambling repeatedly with suitably chosen stakes. We will give some suitable conditions on the win probability function such that it is a Nash equilibrium for the subfair player to play boldly and for the superfair player to play timidly.

1. INTRODUCTION

One of most interesting examples in Dubins and Savage [4] is the famous red-and-black gambling problem. In a discrete version of red-and-black game, a player beginning with a positive integral fortune of x units can stake any positive integer amount $a \leq x$. His fortune becomes $x + a$ if he wins with a fixed probability w $(0 < w < 1)$ and $x - a$ if he loses with probability $1 - w$. The player seeks to maximize the probability of reaching a prespecified goal M by gambling repeatedly with suitably chosen stakes. Dubins and Savage [4] showed that in the subfair case (i.e., $w \leq 1/2$), an optimal strategy is a bold play, which corresponds to always betting the entire fortune or just enough to reach the goal, whichever is smaller. This seems intuitively reasonable in that a shorter game seems to give a better chance to the subfair player since he will surely lose in the long run. In the superfair case (i.e., $w \geq 1/2$), Ross [8] proved that it is optimal for the player to bet timidly—that is, always to stake

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1 unit of his current fortune at each stage. Intuitively, if the player is superfair, to prolong the game is better for him.

The discrete version of red-and-black game has been extended in several ways. One of the extensions is a two-person red-and-black game that was introduced by Secchi [9]. Then Pontiggia [6] proposed two different formulations of two-person red-and-black models, in which two players hold a positive integer fortune and they both aim to win the entire fortune of his opponent. At each stage, each player's win probability is not fixed but depends on both players' bets. She showed that, in each model, a bold strategy is optimal for subfair player while the superfair player plays timidly, and a timid strategy is optimal for superfair player while the subfair player plays boldly; we usually say that it is a Nash equilibrium for the subfair player to play boldly and for the superfair player to play timidly. Chen and Hsiau [2] extended Pontiggia's results to a two-person red-and-black game with bet-dependent win probability functions. They showed that if the subfair player's win probability function f is convex and satisfies that $f(0) = 0$, $f(s) \leq s$, and $f(s)f(t) \leq f(st)$ for all $0 \leq t \leq s \leq 1$, then it is a Nash equilibrium for the subfair player to play boldly and for the superfair player to play timidly. Recently, Chen and Hsiau [3] also gave two new models of a two-person red-and-black game. One is called a bet-exchangeable game, in which at each stage there is a positive probability that two players exchange their bets. The other one is called a stage-dependent game, in which the win probability functions are stage-dependent. In each model, they showed that under some suitable conditions, it is a Nash equilibrium for the subfair player to play boldly and for the superfair player to play timidly.

Pontiggia [6] also introduced an N-person model, called a proportional N-person red-and-black game, and proposed a conjecture about it. For this, Chen and Hsiau [2] gave a counterexample of the conjecture and Chen [1] showed that the conjecture is true in a proportional three-person red-and-black game with suitable weights of players. In [7], Pontiggia proposed an N-person nonconstant sum game, for which she gave some suitable conditions on the winning probability function to ensure that it is a Nash equilibrium for each player to play boldly.

In this article we consider a two-person red-and-black game with a lower limit. More precisely, assume each player holds an integral amount of chips. At each stage, each player is allowed to bet an amount between a given number ℓ and his own current fortune x if $x \geq \ell$; otherwise, he bets all of his own current fortune.

Denote the two players by I and II. Let $M \geq 2$ be the total amount of chips in the system and let $S = \{0, 1, \ldots, M\}$ be the state space of fortune of each player in the game. If player I has x chips, let

$$A_{\mathrm{I}}(x) = \begin{cases} \{x\} & \text{if } x \in \{1, 2, \ldots, \ell - 1\} \\ \{\ell, \ell + 1, \ldots, x\} & \text{if } x \in \{\ell, \ldots, M - 1\} \\ \{0\} & \text{if } x \in \{0, M\} \end{cases}$$

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be the action set for player I and let

$$A_{\mathrm{II}}(x) = \begin{cases} \{M-x\} & \text{if } x \in \{M-\ell+1, M-\ell+2, \ldots, M-1\} \\ \{\ell, \ell+1, \ldots, M-x\} & \text{if } x \in \{1, \ldots, M-\ell\} \\ \{0\} & \text{if } x \in \{0, M\} \end{cases}$$

be that for player II. Assume that each player chooses his action without any knowledge of the action chosen by the other. The goal of each player is to maximize his probability of taking all the chips (i.e., reaching M).

Let f be a function from $[0, 1]$ to $[0, 1]$ with $f(0) = 0$ and $f(s) \le s$. Suppose at stage m, player I has x_m chips and bets $a_m \in A_{\mathrm{I}}(x_m)$ chips, whereas player II bets $b_m \in A_{\mathrm{II}}(x_m)$ chips. The law of motion for player I is defined by

$$x_{m+1} = \begin{cases} x_m + b_m & \text{with probability } \; f\left(\dfrac{a_m}{a_m+b_m}\right) \\ x_m - a_m & \text{with probability } \; 1 - f\left(\dfrac{a_m}{a_m+b_m}\right), \end{cases}$$

for $1 \le x_m \le M-1$, and by $x_{m+1} = x_m$ with probability 1 for $x_m = 0$ or $x_m = M$. This means that once one of the players reaches M, the state of neither player can change. For convenience, we call this game an ℓ-lower-limit game.

Since $f(s) \le s$, we see that

$$\begin{aligned} E[x_{m+1}|x_m] &= (x_m + b_m) f\left(\frac{a_m}{a_m+b_m}\right) + (x_m - a_m)\left(1 - f\left(\frac{a_m}{a_m+b_m}\right)\right) \\ &= x_m + (a_m + b_m) f\left(\frac{a_m}{a_m+b_m}\right) - a_m \\ &\le x_m \end{aligned}$$

for $1 \le x_m \le M-1$ and that $E[x_{m+1}|x_m] = x_m$ if $x_m = 0$ or $x_m = M$. Therefore, the process, $\{x_m\}_{m\ge1}$, of the fortune of player I is a supermartingale. This means that the game is subfair (or unfavorable) to player I and superfair (or favorable) to player II. Notice that if $f(s) = s$, then $E[x_{m+1} \mid x_m] = x_m$, that is, if $f(s) = s$, then the game is fair to both players.

For an ℓ-lower-limit game, we need to modify a timid strategy as the strategy in which the gambler, whose current fortune is x, always stakes $\min\{\ell, x\}$ at each stage. A bold strategy is, as usual, that a player always stakes his entire fortune at each stage of the game. For convenience, the profile (bold, timid) will denote that player I always plays boldly and player II always plays timidly. If a bold strategy is optimal for player I while player II plays timidly and a timid strategy is optimal for player II while player I plays boldly, then we say that the profile (bold, timid) is a Nash equilibrium.

Note that if $\ell = 1$, this model is just the model proposed by Chen and Hsiau [2]. In this case, if the win probability function f is convex and satisfies that

$f(s)f(t) \leq f(st)$, then the profile (bold, timid) is a Nash equilibrium. However, for $\ell \geq 2$, a win probability function f satisfying the above two conditions cannot ensure that the profile (bold, timid) is a Nash equilibrium. For example, if $f(s) = sw$ with $0 < w < 1$, then f is convex and satisfies that $f(s)f(t) \leq f(st)$, but for $\ell \geq 2$, the profile (bold, timid) is not a Nash equilibrium; more precisely, if $M = \ell + 2$, $w = 3/4$, and player I has $\ell + 1$ units, then while player II plays timidly, for player I to bet ℓ units at first stage and then to play boldly is better than always playing boldly.

The structure of this article is as follows. In Section 2 we give the main result, in which some suitable conditions on the win probability function are proposed such that the profile (bold, timid) is a Nash equilibrium for an ℓ-lower-limit game. Moreover, some related results and several examples are given. Finally, the proof of the main result is provided in Section 3.

2. MAIN RESULTS

In the following, some suitable conditions on the win probability function are provided such that the profile (bold, timid) is a Nash equilibrium for an ℓ-lower-limit game.

THEOREM 2.1: *In an ℓ-lower-limit game, assume that the win probability function f satisfies that*

$$f(s) = \frac{s}{s + (1-s)h(s)}, \quad \text{where } h(s) \text{ is decreasing and } h(s) \geq 1, \tag{2.1}$$

$$f(u)f(v) \leq f(s)f(t) \quad \text{for all } 0 \leq s \leq u \leq v \leq t \leq 1 \text{ with } st = uv. \tag{2.2}$$

Then the profile (bold, timid) is a Nash equilibrium. If, in addition, h is strictly decreasing and $f(u)f(v) < f(s)f(t)$ for all $0 \leq s < u \leq v < t \leq 1$ with $st = uv$, then the profile (bold, timid) is the unique Nash equilibrium.

The proof of Theorem 2.1 will be given in Section 3. Note that if f_1 and f_2 are two win probability functions, then $f_1 \circ f_2(0) = f_1(0) = 0$ and $f_1 \circ f_2(s) \leq f_2(s) \leq s$, which implies that $f_1 \circ f_2$ is also a win probability function. This simple observation induces the following interesting result about how to produce more win probability functions such that the profile (bold, timid) is a Nash equilibrium for an ℓ-lower-limit game.

THEOREM 2.2: *Let f_1 and f_2 be two win probability functions. Suppose that for each $i = 1, 2$, f_i satisfies (2.1) and (2.2). Then the profile (bold, timid) is a Nash equilibrium for an ℓ-lower-limit game with the win probability function $f_1 \circ f_2$.*

PROOF: If we can prove that $f_1 \circ f_2$ satisfies that (2.1) and (2.2), then by Theorem 2.1, the profile (bold, timid) is a Nash equilibrium for an ℓ-lower-limit game with the win probability function $f_1 \circ f_2$.

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Note that

$$f_1 \circ f_2(s) = \frac{f_2(s)}{f_2(s) + (1 - f_2(s))h_1(f_2(s))}$$
$$= \frac{s}{s + (1-s)[s(1 - f_2(s))/(1-s)f_2(s)]h_1(f_2(s))}$$
$$= \frac{s}{s + (1-s)h_2(s)h_1(f_2(s))}$$
$$= \frac{s}{s + (1-s)h(s)},$$

where $h(s) = h_2(s)(h_1 \circ f_2)(s)$. Since $h_i(s) \geq 1$ for all $i = 1, 2$, it follows that $h_1 \circ f_2(s) \geq 1$ and so $h(s) \geq 1$. Moreover, because h_2 is decreasing, it follows that (1/t-1) $h_2(t) \leq (1/s - 1)h_2(s)$ if $t \geq s > 0$, which implies that

$$f_2(t) = \frac{t}{t + (1-t)h_2(t)} \geq \frac{s}{s + (1-s)h_2(s)} = f_2(s)$$

if $t \geq s > 0$. For $t \geq s = 0$, it is clear that $0 = f_2(s) \leq f_2(t)$. Therefore, f_2 is increasing and so $(h_1 \circ f_2)(t) \leq (h_1 \circ f_2)(s)$ since h_1 is decreasing. This implies that $h(t) = h_2(t)(h_1 \circ f_2)(t) \leq h_2(s)(h_1 \circ f_2)(s) = h(s)$ if $t \geq s$; that is, h is decreasing. So, $f_1 \circ f_2$ satisfies (2.1).

To prove that $f_1 \circ f_2$ satisfies (2.2), suppose $0 \leq s \leq u \leq v \leq t \leq 1$ with $st = uv$. If $f_2(t) = 0$, then $f_2(u) = f_2(v) = f_2(s) = 0$ since f_2 is increasing. Therefore, $[f_1 \circ f_2(u)][f_1 \circ f_2(v)] = 0 = [f_1 \circ f_2(s)][f_1 \circ f_2(t)]$.

If $f_2(t) > 0$, let $r = f_2(u)f_2(v)/f_2(t)$. Then $rf_2(t) = f_2(u)f_2(v) \leq f_2(s)f_2(t)$ and so $0 \leq r \leq f_2(s)$. Since f_2 is increasing and $s \leq u \leq v \leq t$, we see that $f_2(s) \leq f_2(u) \leq f_2(v) \leq f_2(t)$ and so $r \leq f_2(u) \leq f_2(v) \leq f_2(t)$. Therefore, $f_1(f_2(u))f_1(f_2(v)) \leq f_1(r)f_1(f_2(t))$. Moreover, $f_1(r) \leq f_1(f_2(s))$ since f_1 is increasing. This implies that $f_1(r)f_1(f_2(t)) \leq f_1(f_2(s))f_1(f_2(t))$ and so $f_1 \circ f_2$ satisfies (2.2). Hence, the proof is complete. ■

Next, we apply Theorems 2.1 and 2.2 to two specified win probability functions, which are proposed by Pontiggia [6] and Chen and Hsiau [2].

Example 2.1: Let $h(s) = \bar{w}/w$ with $0 < w < 1/2$ and $\bar{w} = 1 - w$. Then the win probability function is

$$f(s) = \frac{s}{s + (1-s)(\bar{w}/w)} = \frac{sw}{sw + (1-s)\bar{w}}.$$

It is clear that $h(s)$ is decreasing and $h(s) \geq 1$, so f satisfies (2.1). Moreover, for all $0 \leq s \leq u \leq v \leq t \leq 1$ with $st = uv$,

$$f(s)f(t) - f(u)f(v) = \left(\frac{sw}{sw + \bar{s}\bar{w}}\right)\left(\frac{tw}{tw + \bar{t}\bar{w}}\right) - \left(\frac{uw}{uw + \bar{u}\bar{w}}\right)\left(\frac{vw}{vw + \bar{v}\bar{w}}\right)$$
$$= \frac{w^2\bar{w}(1-2w)st(s+t-u-v)}{[sw + \bar{s}\bar{w}][tw + \bar{t}\bar{w}][uw + \bar{u}\bar{w}][vw + \bar{v}\bar{w}]},$$

where $\bar{y} = 1 - y$. Since $st = uv$, we have $t(s + t - u - v) = (t - u)(t - v) \geq 0$, which implies that $f(s)f(t) - f(u)f(v) \geq 0$ and so f satisfies (2.2). According to Theorem 2.1, we see that the profile (bold, timid) is a Nash equilibrium for an ℓ-lower-limit game with win probability function $f(s) = sw/[sw + (1-s)\bar{w}]$, $0 < w < 1/2$ and $\bar{w} = 1 - w$.

Example 2.2: Let $h(s) = (1 - s^\delta)/(s^{\delta-1} - s^\delta)$ for some $\delta > 1$. Then the win probability function is

$$f(s) = \frac{s}{s + (1-s)h(s)} = \frac{s}{s + (1-s^\delta)/s^{\delta-1}} = s^\delta.$$

It is easy to check that $h(s) \geq 1$. Note that $h'(s) = (\delta s - s^\delta + 1 - \delta)/[s^\delta(1-s)^2]$ and $\delta s - s^\delta + 1 - \delta < 0$ for all $0 < s < 1$ and $\delta > 1$. Thus, $h'(s) < 0$ on $(0, 1)$ and so $h(s)$ is strictly decreasing. Moreover, for all $0 \leq s \leq u \leq v \leq t \leq 1$ with $st = uv$, it is clear that f satisfies (2.2). By Theorem 2.1, we see that the profile (bold, timid) is a Nash equilibrium for an ℓ-lower-limit game with the win probability function $f(s) = s^\delta$, $\delta > 1$.

Example 2.3: Let $f_1(s) = sw/[sw + (1-s)\bar{w}]$ and $f_2(s) = s^\delta$, where $0 < w < 1/2$, $\bar{w} = 1 - w$, and $\delta > 1$. From Examples 2.1 and 2.2 and Theorem 2.2, the profile (bold, timid) is a Nash equilibrium for an ℓ-lower-limit game with win probability function $f_1 \circ f_2$.

Furthermore, note that

$$f_1 \circ f_2(s) = \frac{s^\delta w}{s^\delta w + (1-s^\delta)\bar{w}} = \frac{s}{s + (1-s)h(s)},$$

where

$$h(s) = \left(\frac{1 - s^\delta}{s^{\delta-1} - s^\delta}\right)\left(\frac{\bar{w}}{w}\right).$$

It is not difficult to show that $h(s) \geq 1$ and

$$h'(s) = \left[\frac{\delta s - s^\delta + 1 - \delta}{s^\delta(1-s)^2}\right]\left(\frac{\bar{w}}{w}\right).$$

Since $\delta s - s^\delta + 1 - \delta < 0$ for all $0 < s < 1$ and $\delta > 1$, this implies that $h'(s) < 0$ and so h is strictly decreasing. Moreover, for all $0 \leq s < u \leq v < t \leq 1$ with $st = uv$,

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$$\begin{aligned}
&(f_1 \circ f_2(s))(f_1 \circ f_2(t)) - (f_1 \circ f_2(u))(f_1 \circ f_2(v)) \\
&\quad = \frac{s^\delta t^\delta w^2}{[s^\delta w + (1 - s^\delta)\bar{w}][t^\delta w + (1 - t^\delta)\bar{w}]} - \frac{u^\delta v^\delta w^2}{[u^\delta w + (1 - u^\delta)\bar{w}][v^\delta w + (1 - v^\delta)\bar{w}]} \\
&\quad = \frac{w^2 \bar{w}(1 - 2w)s^\delta t^\delta (s^\delta + t^\delta - u^\delta - v^\delta)}{[s^\delta w + (1 - s^\delta)\bar{w}][t^\delta w + (1 - t^\delta)\bar{w}][u^\delta w + (1 - u^\delta)\bar{w}][v^\delta w + (1 - v^\delta)\bar{w}]}.
\end{aligned}$$

Since $1 - 2w > 0$ and $t^\delta(s^\delta + t^\delta - u^\delta - v^\delta) = (t^\delta - u^\delta)(t^\delta - v^\delta) > 0$ for all $0 < w < 1/2$ and $0 \le s < u \le v < t \le 1$ with $st = uv$, it follows that $(f_1 \circ f_2(s))(f_1 \circ f_2(t)) > (f_1 \circ f_2(u))(f_1 \circ f_2(v))$. By Theorem 2.1, it follows that the profile (bold, timid) is the unique Nash equilibrium for an ℓ-lower-limit game with win probability function $f_1 \circ f_2$.

3. PROOF OF THEOREM 2.1

To prove that the profile (bold, timid) is a Nash equilibrium is to prove that a bold strategy is optimal for player I while player II plays timidly and a timid strategy is optimal for player II while player I plays boldly. Therefore, we first prove that if f satisfies (2.1), then a bold strategy is optimal for player I while player II plays timidly in an ℓ-lower-limit game.

THEOREM 3.1: *In an ℓ-lower-limit game, if f satisfies that $f(s) = s/[s + (1 - s)h(s)]$, where $h(s)$ is decreasing and $h(s) \ge 1$, then a bold strategy is optimal for player I while player II uses a timid strategy.*

PROOF: Assume player II plays a timid strategy. If player I uses a bold strategy, set

$$Q(x) = P(\text{player I reaches } M \text{ with an initial fortune } x).$$

The corresponding law of motion at stage m for player I having x_m units and playing boldly is given by

$$x_{m+1} = \begin{cases} x_m + \ell & \text{with probability} \quad f\left(\dfrac{x_m}{x_m + \ell}\right) \\ 0 & \text{with probability} \quad 1 - f\left(\dfrac{x_m}{x_m + \ell}\right) \end{cases}$$

if $1 \le x_m < M - \ell$, by

$$x_{m+1} = \begin{cases} M & \text{with probability} \quad f\left(\dfrac{x_m}{M}\right) \\ 0 & \text{with probability} \quad 1 - f\left(\dfrac{x_m}{M}\right) \end{cases}$$

if $M - \ell \le x_m < M - 1$, and by $x_{m+1} = x_m$ with probability 1 if $x_m = 0$ or M.

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From this, it is not difficult to derive the recursive relations:

$$Q(x) = f\left(\frac{x}{x+\ell}\right)Q(x+\ell) \quad \text{if} \quad 1 \le x < M - \ell \tag{3.1}$$

and

$$Q(x) = f\left(\frac{x}{M}\right)Q(M) \quad \text{if} \quad M - \ell \le x < M. \tag{3.2}$$

Note that $Q(0) = 0$ and $Q(M) = 1$. Therefore, $Q(x) = f(x/M)$ if $M - \ell \le x < M$. For $1 \le x < M - \ell$, there exists a positive integer n such that $n\ell < M - x \le n\ell + \ell$. Using (3.1) repeatedly yields that

$$Q(x) = f(r_x)Q(x+\ell) = f(r_x)f(r_{x+\ell})Q(x+2\ell) = \cdots = \left(\prod_{i=0}^{n-1} f(r_{x+i\ell})\right)Q(x+n\ell).$$

Since $M - \ell \le x + n\ell < M$, it follows that for $1 \le x < M - \ell$,

$$Q(x) = \left(\prod_{i=0}^{n-1} f(r_{x+i\ell})\right)Q(x+n\ell) = \left(\prod_{i=0}^{n-1} f(r_{x+i\ell})\right)f\left(\frac{x+n\ell}{M}\right). \tag{3.3}$$

In order to prove that a bold strategy is optimal for player I while player II plays timidly, it suffices to show that $Q(\cdot)$ is excessive (see [5, Thm.]) or, equivalently, that the following two inequalities hold:

$$f\left(\frac{a}{a+\ell}\right)Q(x+\ell) + \left(1 - f\left(\frac{a}{a+\ell}\right)\right)Q(x-a) \le Q(x) \tag{3.4}$$

for every $x \in \{1, \ldots, M - \ell - 1\}$ and every $a \in A_{\text{I}}$;

$$f\left(\frac{a}{a+M-x}\right)Q(M) + \left(1 - f\left(\frac{a}{a+M-x}\right)\right)Q(x-a) \le Q(x) \tag{3.5}$$

for every $x \in \{M - \ell, \ldots, M - 1\}$ and every $a \in A_{\text{I}}$.

To prove (3.4), let $r_x = x/(x+\ell)$. From (3.1), we have $Q(x+\ell) = Q(x)/f(r_x)$ and so (3.4) becomes

$$(1 - f(r_a))Q(x-a) \le \left(1 - \frac{f(r_a)}{f(r_x)}\right)Q(x). \tag{3.6}$$

If $1 \le x < \ell$, then $a \in A_{\text{I}} = \{x\}$ and hence (3.4) holds for $1 \le x < \ell$. If $x \in \{\ell, \ldots, M - \ell - 1\}$, then $a \ge \ell$. This implies that there exists a nonnegative integer m

such that $n\ell + m\ell < M - x + a \le n\ell + m\ell + \ell$. Let $g(s) = 1/[s + (1-s)h(s)]$. Then $f(s) = sg(s)$. By (3.3), it follows that

$$Q(x) = \left(\prod_{i=0}^{n-1} r_{x+i\ell}\right)\left(\prod_{i=0}^{n-1} g(r_{x+i\ell})\right)\left(\frac{x+n\ell}{M}\right) g\left(\frac{x+n\ell}{M}\right)$$
$$= \frac{x}{M}\left(\prod_{i=0}^{n-1} g(r_{x+i\ell})\right) g\left(\frac{x+n\ell}{M}\right)$$

and

$$Q(x-a) = \frac{x-a}{M}\left(\prod_{i=0}^{n+m-1} g(r_{x-a+i\ell})\right) g\left(\frac{x-a+n\ell+m\ell}{M}\right).$$

Therefore, (3.6) becomes

$$(1 - f(r_a))\left(\frac{x-a}{M}\right)\left(\prod_{i=0}^{n+m-1} g(r_{x-a+i\ell})\right) g\left(\frac{x-a+n\ell+m\ell}{M}\right)$$
$$\le \left(1 - \frac{f(r_a)}{f(r_x)}\right)\left(\frac{x}{M}\right)\left(\prod_{i=0}^{n-1} g(r_{x+i\ell})\right) g\left(\frac{x+n\ell}{M}\right),$$

which is equivalent to

$$(x-a)(1 - f(r_a))\left(\prod_{i=0}^{n+m-1} g(r_{x-a+i\ell})\right) g\left(\frac{x-a+n\ell+m\ell}{M}\right)$$
$$\le x\left(1 - \frac{f(r_a)}{f(r_x)}\right)\left(\prod_{i=0}^{n-1} g(r_{x+i\ell})\right) g\left(\frac{x+n\ell}{M}\right).$$

Since $h(s) \ge 1$ is decreasing, we have $0 \le h(s) - 1 \le h(t) - 1$ for all $0 \le t \le s \le 1$ and this implies that $(1-s)(h(s)-1) \le (1-t)(h(t)-1)$, which is equivalent to $s + (1-s)h(s) \le t + (1-t)h(t)$. Therefore, $g(s) \ge g(t)$ for all $0 \le t \le s \le 1$; that is, g is increasing. Then we deduce that

$$\prod_{i=0}^{n-1} g(r_{x-a+i\ell}) \le \prod_{i=0}^{n-1} g(r_{x+i\ell}) \tag{3.7}$$

since $r_{x-a+i\ell} \le r_{x+i\ell}$ for $i \in \{0, 1, \ldots, n-1\}$ and that

$$\max\left\{g\left(\frac{x-a+n\ell}{M}\right),\ g(r_{x-a+n\ell})\right\} \le g\left(\frac{x+n\ell}{M}\right)$$

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since $x - a + n\ell < x + n\ell$ and

$$r_{x-a+n\ell} = \frac{x-a+n\ell}{x-a+n\ell+\ell} \le \frac{x+n\ell}{x+n\ell+\ell} \le \frac{x+n\ell}{M}.$$

Moreover, since $h(s) \ge 1$, it follows that

$$g(s) = \frac{1}{s+(1-s)h(s)} \le \frac{1}{s+(1-s)} = 1$$

and so

$$\begin{aligned} &g\left(\frac{x-a+n\ell+m\ell}{M}\right)\prod_{i=n}^{n+m-1} g(r_{x-a+i\ell}) \\ &\le \max\left\{g\left(\frac{x-a+n\ell}{M}\right),\ g(r_{x-a+n\ell})\right\} \\ &\le g\left(\frac{x+n\ell}{M}\right). \end{aligned} \tag{3.8}$$

Here we have used the convention that $\prod_{i=j}^{j+k} a_i = 1$ if $k < 0$.

According to (3.7) and (3.8), it remains to show that

$$(x-a)(1-f(r_a)) \le x\left(1-\frac{f(r_a)}{f(r_x)}\right),$$

which is equivalent to

$$\frac{a(1-f(r_a))}{f(r_a)} \ge \frac{x(1-f(r_x))}{f(r_x)}. \tag{3.9}$$

Since $f(s) = s/[s+(1-s)h(s)]$, it follows that

$$h(s) = \frac{s(1-f(s))}{(1-s)f(s)}.$$

For $a \le x$, we see that $r_a \le r_x$ and so

$$\frac{a(1-f(r_a))}{\ell f(r_a)} = \frac{r_a(1-f(r_a))}{(1-r_a)f(r_a)} = h(r_a) \ge h(r_x) = \frac{r_x(1-f(r_x))}{(1-r_x)f(r_x)} = \frac{x(1-f(r_x))}{\ell f(r_x)}$$

since $h(s)$ is decreasing. This implies that (3.9) holds for $x \in \{\ell, \ldots, M-\ell-1\}$ and hence (3.4) holds.

For $x \in \{M-\ell, \ldots, M-1\}$, the proof of (3.5) is similar to that of (3.4) and is omitted here. ■

Remark 3.1: If h is strictly decreasing, then (3.4) and (3.5) are actually an equality if and only if $a = x$. This states that in this case, the bold strategy is the unique optimal strategy for player I while player II plays timidly.

Next, we prove that if f satisfies (2.2), then a timid strategy is optimal for player II while player I plays boldly in an ℓ-lower-limit game.

THEOREM 3.2: *In an ℓ-lower-limit game, if f satisfies $f(u)f(v) \le f(s)f(t)$ for all $0 \le s \le u \le v \le t \le 1$ with $st = uv$, then a timid strategy is optimal for player II while player I uses a bold strategy.*

PROOF: Assume that player I plays a bold strategy. If player II uses a timid strategy, then let

$$T(x) = P(\text{player II reaches M with an initial fortune } M - x).$$

The corresponding law of motion at stage m for player I with x_m units is given by

$$x_{m+1} = \begin{cases} x_m + \ell & \text{with probability} \quad f\left(\dfrac{x_m}{x_m+\ell}\right) \\ 0 & \text{with probability} \quad 1 - f\left(\dfrac{x_m}{x_m+\ell}\right) \end{cases}$$

for $1 \le x_m < M - \ell$, by

$$x_{m+1} = \begin{cases} M & \text{with probability} \quad f\left(\dfrac{x_m}{M}\right) \\ 0 & \text{with probability} \quad 1 - f\left(\dfrac{x_m}{M}\right) \end{cases}$$

for $M - \ell \le x_m \le M - 1$, and by $x_{m+1} = x_m$ with probability 1 for $x_m = 0$ or M. In this case of the games, one player reaches the goal and the other goes broke with probability 1; hence, $T(x) = 1 - Q(x)$, where $Q(x)$ is given in the proof of Theorem 3.1.

To prove that a timid strategy is optimal for player II while player I plays boldly, as earlier it suffices to show that $T(\cdot)$ is excessive—that is, to prove that the following inequality holds for every $x \in \{1, \cdots, M-1\}$ and every $b \in A_{\text{II}}(x)$:

$$f\left(\frac{x}{x+b}\right)T(x+b) + \left(1 - f\left(\frac{x}{x+b}\right)\right)T(0) \le T(x).$$

Since $T(x) = 1 - Q(x)$, the above inequality is equivalent to

$$Q(x) \le f\left(\frac{x}{x+b}\right)Q(x+b) \tag{3.10}$$

for every $x \in \{1, \ldots, M-1\}$ and every $b \in A_{\text{II}}(x)$.

For the case $M - \ell \le x < M$, we have $0 < M - x \le \ell$ and so $b = M - x$. Hence, (3.10) follows by (3.2).

For the case $1 \le x < M - \ell$, we have $\ell < M - x \le M - 1$ and it follows that either $b = \ell$ or $\ell < b \le M - x$. If $b = \ell$, then (3.10) holds by (3.1). If $\ell < b \le M - x$,

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there exists a positive integer n such that $n\ell < b \le n\ell + \ell$. Next, we divide our discussion into two cases: (i) $b = M - x$ and (ii) $\ell < b < M - x$.

(i) If $b = M - x$, then from (3.3), we have $Q(x + b) = Q(M) = 1$ and

$$Q(x) = \left(\prod_{i=0}^{n-1} f(r_{x+i\ell})\right) f\left(\frac{x + n\ell}{M}\right).$$

Since f satisfies (2.2) and $f(1) \le 1$, it follows that $f(s)f(t) \le f(st)$ for all $s, t \in [0, 1]$. This implies that

$$Q(x) \le f\left(\frac{x}{M}\right) = f\left(\frac{x}{x+b}\right)$$

since $(\prod_{i=0}^{n-1} r_{x+i\ell})(x + n\ell)/M = x/M$.

(ii) If $\ell < b < M - x$, then $M - x - b > 0$ and so there exists a nonnegative integer m such that $m\ell < M - x - b \le m\ell + \ell$. Since $n\ell < b \le n\ell + \ell$ and $m\ell < M - x - b \le m\ell + \ell$, it follows that $n\ell + m\ell < M - x \le n\ell + m\ell + 2\ell$. Using (3.1) repeatedly yields that

$$\begin{aligned} Q(x) &= \left(\prod_{i=0}^{n+m-1} f(r_{x+i\ell})\right) Q(x + n\ell + m\ell) \\ &= \left(\prod_{i=0}^{n-1} f(r_{x+i\ell})\right)\left(\prod_{j=0}^{m-1} f(r_{x+n\ell+j\ell})\right) Q(x + n\ell + m\ell), \end{aligned}$$

and from (3.3), we have

$$Q(x + b) = \left(\prod_{j=0}^{m-1} f(r_{x+b+j\ell})\right) f\left(\frac{x + b + m\ell}{M}\right).$$

Therefore, (3.10) becomes

$$\begin{aligned} &\left(\prod_{i=0}^{n-1} f(r_{x+i\ell})\right)\left(\prod_{j=0}^{m-1} f(r_{x+n\ell+j\ell})\right) Q(x + n\ell + m\ell) \\ &\qquad \le f\left(\frac{x}{x+b}\right)\left(\prod_{j=0}^{m-1} f(r_{x+b+j\ell})\right) f\left(\frac{x + b + m\ell}{M}\right). \end{aligned} \tag{3.11}$$

Since $f(s)f(t) \le f(st)$ for all $s, t \in [0, 1]$, this implies that

$$\prod_{i=0}^{n-1} f(r_{x+i\ell}) \le f\left(\frac{x}{x + n\ell}\right) \tag{3.12}$$

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since $\prod_{i=0}^{n-1} r_{x+i\ell} = x/(x+n\ell)$. With (3.12), (3.11) holds if we can prove that

$$f\left(\frac{x}{x+n\ell}\right)\left(\prod_{j=0}^{m-1} f(r_{x+n\ell+j\ell})\right)Q(x+n\ell+m\ell)$$
$$\le f\left(\frac{x}{x+b}\right)\left(\prod_{j=0}^{m-1} f(r_{x+b+j\ell})\right)f\left(\frac{x+b+m\ell}{M}\right). \qquad \textbf{(3.13)}$$

Let $s_0 = x/(x+b)$, $t_0 = x/(x+n\ell)$, $s_j = r_{x+b+j\ell-\ell}$, and $t_j = r_{x+n\ell+j\ell-\ell}$ for $j \ge 1$. Then (3.13) becomes

$$\left(\prod_{j=0}^{m} f(t_j)\right)Q(x+n\ell+m\ell) \le \left(\prod_{j=0}^{m} f(s_j)\right)f\left(\frac{x+b+m\ell}{M}\right). \qquad \textbf{(3.14)}$$

Let $v_0 = t_0$ and

$$v_j = \frac{x+b+j\ell-\ell}{x+n\ell+j\ell} \qquad \text{for} \quad j \ge 1.$$

It follows that $s_j v_{j+1} = t_{j+1} v_j$ and $s_j \le \min\{v_j, t_{j+1}\} \le \max\{v_j, t_{j+1}\} \le v_{j+1}$ for all $j \ge 0$. Therefore, we have for all $j \ge 0$,

$$f(t_{j+1})f(v_j) \le f(s_j)f(v_{j+1}) \qquad \textbf{(3.15)}$$

and this imply that $\prod_{j=0}^{m-1}(f(t_{j+1})f(v_j)) \le \prod_{j=0}^{m-1}(f(s_j)f(v_{j+1}))$, which is equivalent to

$$f(v_0)\left(\prod_{j=1}^{m} f(t_j)\right) = \left(\prod_{j=0}^{m} f(t_j)\right) \le f(v_m)\left(\prod_{j=0}^{m-1} f(s_j)\right) \qquad \textbf{(3.16)}$$

since $v_0 = t_0$. If we can prove that

$$f(v_m)Q(x+n\ell+m\ell) \le f(s_m)f\left(\frac{x+b+m\ell}{M}\right), \qquad \textbf{(3.17)}$$

then (3.14) holds by (3.16).

Notice that $n\ell + m\ell < M - x \le n\ell + m\ell + 2\ell$. If $0 < M - x - n\ell - m\ell \le \ell$, then from (3.2), we have $Q(x+n\ell+m\ell) = f((x+n\ell+m\ell)/M)$. Note that $v_m(x+n\ell+m\ell)/M = s_m(x+b+m\ell)/M$ and

$$s_m \le \min\left\{v_m, \frac{x+n\ell+m\ell}{M}\right\} \le \max\left\{v_m, \frac{x+n\ell+m\ell}{M}\right\} \le \frac{x+b+m\ell}{M}.$$

It follows that

$$f(v_m)Q(x+n\ell+m\ell) = f(v_m)f\left(\frac{x+n\ell+m\ell}{M}\right) \le f(s_m)f\left(\frac{x+b+m\ell}{M}\right);$$

that is, (3.17) holds.

If $\ell < M - x - n\ell - m\ell \leq 2\ell$, then, from (3.3), we have

$$Q(x + n\ell + m\ell) = f(r_{x+n\ell+m\ell})f\left(\frac{x + n\ell + m\ell + \ell}{M}\right)$$
$$= f(t_{m+1})f\left(\frac{x + n\ell + m\ell + \ell}{M}\right).$$

According to (3.15) with $j = m$, we have $f(t_{m+1})f(v_m) \leq f(s_m)f(v_{m+1})$ and so

$$f(v_m)Q(x + n\ell + m\ell) \leq f(s_m)f(v_{m+1})f\left(\frac{x + n\ell + m\ell + \ell}{M}\right).$$

Since $f(s)f(t) \leq f(st)$ for all $s, t \in [0, 1]$, it follows that

$$f(v_{m+1})f\left(\frac{x + n\ell + m\ell + \ell}{M}\right)$$
$$= f\left(\frac{x + b + m\ell}{x + n\ell + m\ell + \ell}\right)f\left(\frac{x + n\ell + m\ell + \ell}{M}\right)$$
$$\leq f\left(\frac{x + b + m\ell}{M}\right)$$

and so (3.17) holds. Hence, the proof is complete. ■

Remark 3.2: Suppose that the win probability function f satisfies $f(u)f(v) < f(s)f(t)$ for all $0 \leq s < u \leq v < t \leq 1$ with $st = uv$. Then we can prove that (3.10) is actually an equality if and only if $b = \ell$ as $1 \leq x < M - \ell$. This means that, in this case, the timid strategy is the unique optimal strategy for player II while player I plays boldly.

Remark 3.3: For the case $\ell = 1$, Chen and Hsiau [2] proved a similar result as Theorem 3.2 under a weaker condition that $f(st) \geq f(s)f(t)$ for all $0 \leq s \leq t \leq 1$. However, for $\ell \geq 2$, this condition is not sufficient. For example, for $\ell \geq 2$ and $M \geq 2\ell + 1$, suppose $f(s) = s(2 - s)/2$ and player II has 2ℓ units. It is easy to verify that $f(st) \geq f(s)f(t)$ holds for all $0 \leq s \leq t \leq 1$. However, we can prove that while player I plays boldly, it is better for player II to bet $\ell + 1$ units at first stage and then to play timidly than always to play timidly.

By Theorems 3.1 and 3.2 and Remarks 3.1 and 3.2, we now can prove Theorem 2.1.

Proof of Theorem 2.1: If f satisfies (2.1) and (2.2), then, by Theorem 3.1, a bold strategy is optimal for player I while player II plays timidly; moreover, by Theorem 3.2, a timid strategy is optimal for player II while player I plays boldly. Therefore, the profile (bold, timid) is a Nash equilibrium.

For the second part of the theorem, if h is strictly decreasing and $f(u)f(v) < f(s)f(t)$ for all $0 \leq s < u \leq v < t \leq 1$ with $st = uv$, then, by Remarks 3.1 and 3.2,

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we can see that the bold strategy is the unique optimal strategy for player I if player II plays timidly and that the timid strategy is the unique optimal strategy for player II if player I plays boldly. Now, from [6, Lemma A.1], it follows that the profile (bold, timid) is the unique Nash equilibrium. ■

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