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# Time-dependent linear water-wave scattering in two dimensions by a generalized eigenfunction expansion

MICHAEL H. MEYLAN†

Department of Mathematics, The University of Auckland, New Zealand

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We consider the solution in the time domain of the two-dimensional water-wave scattering by fixed bodies, which may or may not intersect with the free surface. We show how the problem with arbitrary initial conditions can be found from the single-frequency solutions using a generalized eigenfunction expansion, required because the operator has a continuous spectrum. From this expansion we derive simple formulas for the evolution in time of the initial surface conditions, and we present some examples of numerical calculations.

## 1. Introduction

This paper presents a *generalized eigenfunction* expansion for the time-dependent two-dimensional problem of linear water-wave scattering by fixed bodies. We derive a simple expression for the expansion of the time-dependent motion, in terms of the solutions for a single frequency. The generalized eigenfunction method goes back to the works of Povzner (1953) and Ikebe (1960). The generalized eigenfunction method has been applied to water-wave problems by Friedman & Shinbrot (1967), Hazard & Lenoir (2002), Meylan (2002), Hazard & Loret (2007) and Hazard & Meylan (2007). These papers were focused on either theoretical expressions, without numerical calculations, or to the problem of a floating elastic plate. The generalized eigenfunction method was also applied recently to the scattering from bottom-mounted cylinders by Meylan & Eatock Taylor (2009).

The generalized eigenfunction expansion method provides an alternative to other time-domain methods, such as the memory effect method (Mei 1989, Ch. 7), the time-dependent Green function method (Stoker 1957, Ch. 6) or the Laplace transform (which is closely related to the memory effect method Ogilvie 1964; Meylan & Sturova 2009). The generalized eigenfunction method has the advantage of calculating the solution from the single-frequency solutions for an incident wave, without any time stepping. The Laplace transform requires the solution of an equation with non-zero free surface for each frequency. The memory effect method and the time-dependent Green function method require explicit time stepping of the solution.

The generalized eigenfunction method is based on an inner product in which the evolution operator is self-adjoint. It follows from the self-adjointness that we can expand the solution in the eigenfunctions of the operator. These eigenfunctions are nothing more than the single-frequency solutions. The main difficulty is that the

† Email address for correspondence: meylan@math.auckland.ac.nz

eigenfunctions are associated with a continuous spectrum, and this requires that they be carefully normalized. Once this is done, we can derive simple expressions which allow the time-domain problem to be solved in terms of the single-frequency solutions. The mathematical ideas here have appeared previously in Hazard & Loret (2007). However, the emphasis here is not on proving the expansion, nor do we emphasize the role of the diagonalizing transformation derived from the eigenfunction, nor do we formally prove the self-adjointness of the operator or discuss the exact nature of the solution space. Instead, we use a normalizing condition involving the Dirac delta function to derive our expressions for the surface elevation, in keeping with our aim of developing numerical methods. The presentation here is in terms of the second-order equations. A derivation in terms of system of two first-order equations (without numerical calculations) was given in Meylan (2008*a*); however, the second-order equation in a single variable which is presented here is much simpler.

## 2. Equations for fixed bodies in the time domain

We consider a two-dimensional fluid domain of constant depth, which contains a finite number of fixed bodies of arbitrary geometry. We denote the fluid domain by $\Omega$, the boundary of the fluid domain which touches the fixed bodies by $\partial\Omega$, and the free surface by $F$. The $x$ and $z$ coordinates are such that $x$ is pointing in the horizontal direction and $z$ is pointing in the vertical upward direction (we denote $\boldsymbol{x}=(x,z)$). The free surface is at $z=0$ and the sea floor is at $z=-h$ (the theory would be almost identical if the sea floor depth varied within some finite region and was at $z=-h$ outside this region). The equations of motion in the time domain are

$$\Delta\Phi(\boldsymbol{x},t)=0, \quad \boldsymbol{x}\in\Omega, \tag{2.1a}$$

$$\partial_n\Phi=0, \quad \boldsymbol{x}\in\partial\Omega, \tag{2.1b}$$

$$\partial_n\Phi=0, \quad z=-h, \tag{2.1c}$$

where $\Phi$ is the velocity potential for the fluid. At the free surface, we have the kinematic condition

$$\partial_t\zeta=\partial_n\Phi, \quad z=0, \quad x\in F, \tag{2.1d}$$

and the dynamic condition (the linearized Bernoulli equation)

$$\zeta=-\partial_t\Phi, \quad z=0, \quad x\in F, \tag{2.1e}$$

where $\zeta$ is the free-surface elevation. Equations (2.1*a*)–(2.1*e*) are in non-dimensional form (so that the fluid density and gravity are both unity). They are also subject to initial conditions

$$\zeta|_{t=0}=\zeta_0(x) \quad\text{and}\quad \partial_t\zeta|_{t=0}=v_0(x). \tag{2.2}$$

Figure 1 is a schematic diagram of the problem.

### 2.1. *Single-frequency equations*

The single-frequency solution is based on the assumption that all time-dependence is given by $\mathrm{e}^{-\mathrm{i}\omega t}$ and that the system is excited by an incident wave. Then we can write

$$\Phi(\boldsymbol{x},t)=\Phi_\kappa(\boldsymbol{x},\omega)\mathrm{e}^{-\mathrm{i}\omega t} \quad\text{and}\quad \zeta(x,t)=\zeta_\kappa(x,\omega)\mathrm{e}^{-\mathrm{i}\omega t}, \tag{2.3}$$

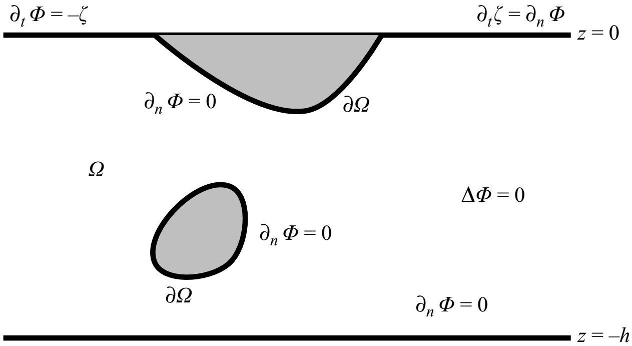


FIGURE 1. Schematic diagram showing the time-dependent equations.

where $\kappa = 1$ for waves excited by an incident wave from the left and $-1$ for waves incident from the right. Under these assumptions, (2.1*a*)–(2.1*e*) become

$$\Delta\Phi_\kappa(\boldsymbol{x}, \omega) = 0, \quad \boldsymbol{x} \in \Omega, \tag{2.4a}$$

$$\partial_n \Phi_\kappa = 0, \quad \boldsymbol{x} \in \partial\Omega, \tag{2.4b}$$

$$\partial_n \Phi_\kappa = 0, \quad z = -h, \tag{2.4c}$$

$$-\mathrm{i}\omega\zeta_\kappa = \partial_n \Phi_\kappa, \quad z = 0,\ x \in F, \tag{2.4d}$$

$$\zeta_\kappa = \mathrm{i}\omega\Phi_\kappa, \quad z = 0, \quad x \in F. \tag{2.4e}$$

We must also specify radiations conditions, which are given by

$$\Phi_1 = \frac{1}{\mathrm{i}\omega}(\mathrm{e}^{\mathrm{i}kx} + R_1\mathrm{e}^{-\mathrm{i}kx})\frac{\cosh k(z+h)}{\cosh kh}, \quad \text{as } x \to -\infty, \tag{2.5a}$$

$$\Phi_1 = \frac{1}{\mathrm{i}\omega}T_1\mathrm{e}^{\mathrm{i}kx}\frac{\cosh k(z+h)}{\cosh kh}, \quad \text{as } x \to \infty, \tag{2.5b}$$

$$\Phi_{-1} = \frac{1}{\mathrm{i}\omega}T_{-1}\mathrm{e}^{-\mathrm{i}kx}\frac{\cosh k(z+h)}{\cosh kh}, \quad \text{as } x \to \infty, \tag{2.5c}$$

$$\Phi_{-1} = \frac{1}{\mathrm{i}\omega}(\mathrm{e}^{-\mathrm{i}kx} + R_{-1}\mathrm{e}^{\mathrm{i}kx})\frac{\cosh k(z+h)}{\cosh kh}, \quad \text{as } x \to -\infty. \tag{2.5d}$$

Note that $R_\kappa$ and $T_\kappa$ are the reflection and transmission coefficients, respectively, and that we have normalized so that the amplitude (in displacement) is unity. The wavenumber $k$ is the positive real solution of the dispersion equation $k \tanh kh = \omega^2$, and we will consider both $k(\omega)$ and $\omega(k)$ in what follows as required. The solution of the single-frequency equation may be computationally challenging and for the generalized eigenfunction expansion the major numerical work is to determine the single-frequency solutions. We do not discuss here methods to find the single-frequency solution. Details of various methods can be found in Newman (1977), Mei (1989) and Linton & McIver (2001).

## 3. Time-domain calculations

The solution in the frequency domain can be used to construct the solution in the time domain. This is well known for the case of a plane incident wave which is initially far from the body, and in this case the solution can be calculated straightforwardly using the standard Fourier transform. However, when we consider an initial displacement which is non-zero around the bodies, we cannot express the time-dependent solution in terms of the single-frequency solutions by a standard Fourier transform.

### 3.1. *Calculation in the time domain for arbitrary initial conditions*

We begin with the equations in the time domain (2.1*a*)–(2.1*e*), subject to the initial conditions given by (2.2). Denoting the potential at the surface by

$$\phi(x,t) = \Phi(\boldsymbol{x},t)|_{z=0}, \tag{3.1}$$

we introduce the operator $\boldsymbol{G}$ which maps the surface potential to the potential throughout the fluid domain. The operator $\boldsymbol{G}\psi$ is found by solving

$$\Delta\Psi(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Omega, \tag{3.2a}$$

$$\partial_n\Psi = 0, \quad \boldsymbol{x} \in \partial\Omega, \tag{3.2b}$$

$$\partial_n\Psi = 0, \quad z = -h, \tag{3.2c}$$

$$\Psi = \psi, \quad z = 0, \quad x \in F, \tag{3.2d}$$

and is defined by $\boldsymbol{G}\psi = \Psi$. The operator $\partial_n\boldsymbol{G}$, which maps the surface potential to the normal derivative of potential at the surface (called the Dirichlet-to-Neumann map), is given by

$$\partial_n\boldsymbol{G}\psi = \partial_n\Psi|_{z=0}. \tag{3.3}$$

Therefore, (2.1*a*)–(2.1*e*) can be written as

$$\partial_t^2\zeta + \partial_n\boldsymbol{G}\zeta = 0. \tag{3.4}$$

where we can recover the potential using the operator $\boldsymbol{G}$. The evolution operator $\partial_n\boldsymbol{G}$ is symmetric in the Hilbert space given by the following inner product

$$\langle\zeta,\eta\rangle_{\mathscr{H}} = \int_F \zeta\eta^*\,\mathrm{d}x, \tag{3.5}$$

where $^*$ denotes complex conjugate, and we assume that this symmetry implies that the operator is self-adjoint. We can prove the symmetry by using Green's second identity

$$\begin{aligned}\int_F (\partial_n\boldsymbol{G}\zeta)(\eta)^*\,\mathrm{d}x &= \int_F (\partial_n\boldsymbol{G}\partial_t\phi)(\partial_t\psi)^*\,\mathrm{d}x \\ &= \int_F (\partial_t\phi)(\partial_n\boldsymbol{G}\partial_t\psi)^*\,\mathrm{d}x \\ &= \int_F (\zeta)(\partial_n\boldsymbol{G}\eta)^*\,\mathrm{d}x,\end{aligned} \tag{3.6}$$

where $\phi$ and $\psi$ are the surface potentials associated with $\zeta$ and $\eta$, respectively.

### 3.2. *Eigenfunctions of* $\partial_n\boldsymbol{G}$

The eigenfunctions of $\partial_n\boldsymbol{G}$ satisfy

$$\partial_n\boldsymbol{G}\zeta = \omega^2\zeta. \tag{3.7}$$

Equation (3.7) is nothing more than (2.4*a*)–(2.4*e*). This means that to solve the eigenfunctions of $\partial_n\boldsymbol{G}$ we need to solve the frequency-domain equations, and the radian frequency $\omega$ is exactly the eigenvalue. To actually calculate the eigenfunctions of $\partial_n \boldsymbol{G}$ we need to specify the incident wave potential, and for each frequency we have two eigenfunctions (waves incident from the left and from the right). It is possible for there to exist point spectra for this operator which correspond to the existence of a trapped mode (McIver 1996), but this case is not discussed here. Note that the presence of a trapped mode requires that the generalized eigenfunction expansion we derive must be modified.

### 3.3. *Normalization of the eigenfunctions*

The eigenfunctions of $\partial_n \boldsymbol{G}$ (with eigenvalue $\omega$) are denoted by $\zeta_\kappa(x, k(\omega))$. As mentioned previously, determining $\zeta_\kappa$ is the major computation of the generalized eigenfunction method, but we simply assume that they are known. We know that the eigenfunctions are orthogonal for different $\omega$ (from the self-adjointness of $\partial_n \boldsymbol{G}$), and that the waves incident from the left and right with the same $\omega$ are orthogonal from the identity (Mei 1989)

$$R_1 T_{-1}^* + R_{-1}^* T_1 = 0.$$

It therefore follows that

$$\langle (\zeta_\kappa(x, k(\omega_1))), \zeta_{\kappa'}(x, k(\omega_2)) \rangle_{\mathscr{H}} = \Lambda_n(\omega_1)\, \delta(\omega_1 - \omega_2)\, \delta_{\kappa\kappa'}, \tag{3.8}$$

but we need to determine the normalizing function $\Lambda_n(\omega_n)$. This is achieved by using the result that the eigenfunctions satisfy the same normalizing condition with and without the scatterers present. This result, the proof of which is quite technical, is well known and has been shown for many different situations. The original proof was for Schrödinger's equation and was due to Povzner (1953) and Ikebe (1960). A proof for the case of Helmholtz equation was given by Wilcox (1975). Recently, the proof was given for water waves by Hazard & Lenoir (2002) and Hazard & Loret (2007).

Since the eigenfunctions satisfy the same normalizing condition with and without the scatterers, we normalize with the scatterers absent. This means that the eigenfunctions are simply the incident waves, and the free surface $F$ is the entire axis. This allows us to derive

$$\langle (\zeta_\kappa(x, k(\omega_1))), \zeta_{\kappa'}(x, k(\omega_2)) \rangle_{\mathscr{H}} = \int_{\mathbb{R}} (\mathrm{e}^{\kappa \mathrm{i} k_1 x})(\mathrm{e}^{\kappa' \mathrm{i} k_2 x})^* \, \mathrm{d}x \tag{3.9}$$

$$= 2\pi \delta_{\kappa\kappa'} \delta(k_1 - k_2) \tag{3.10}$$

$$= 2\pi \delta_{\kappa\kappa'} \delta(\omega_1 - \omega_2) \left. \frac{\mathrm{d}\omega}{\mathrm{d}k} \right|_{\omega=\omega_1}. \tag{3.11}$$

This result allows us to calculate the time-dependent solution in the eigenfunctions (or single-frequency solutions).

### 3.4. *Expansion in eigenfunctions*

We expand the solution for the displacement in the time domain as

$$\zeta(x, t) = \int_{\mathbb{R}^+} \sum_{\kappa \in \{-1,1\}} \left\{ f_\kappa(\omega) \cos(\omega t) + g_\kappa(\omega) \frac{\sin(\omega t)}{\omega} \right\} \zeta_\kappa(x, k)\, \mathrm{d}\omega, \tag{3.12}$$

where $f_\kappa$ and $g_\kappa$ will be determined from the initial conditions. Note that here, and in subsequent equations, we are assuming that $k$ is a function of $\omega$ or that $\omega$ is a function of $k$ as required. If we take the inner product with respect to the eigenfunctions $\zeta_\kappa$ we obtain

$$\langle \zeta_0(x), \zeta_\kappa(x, k) \rangle_{\mathscr{H}} = 2\pi f_\kappa(\omega) \frac{\mathrm{d}\omega}{\mathrm{d}k}, \tag{3.13}$$

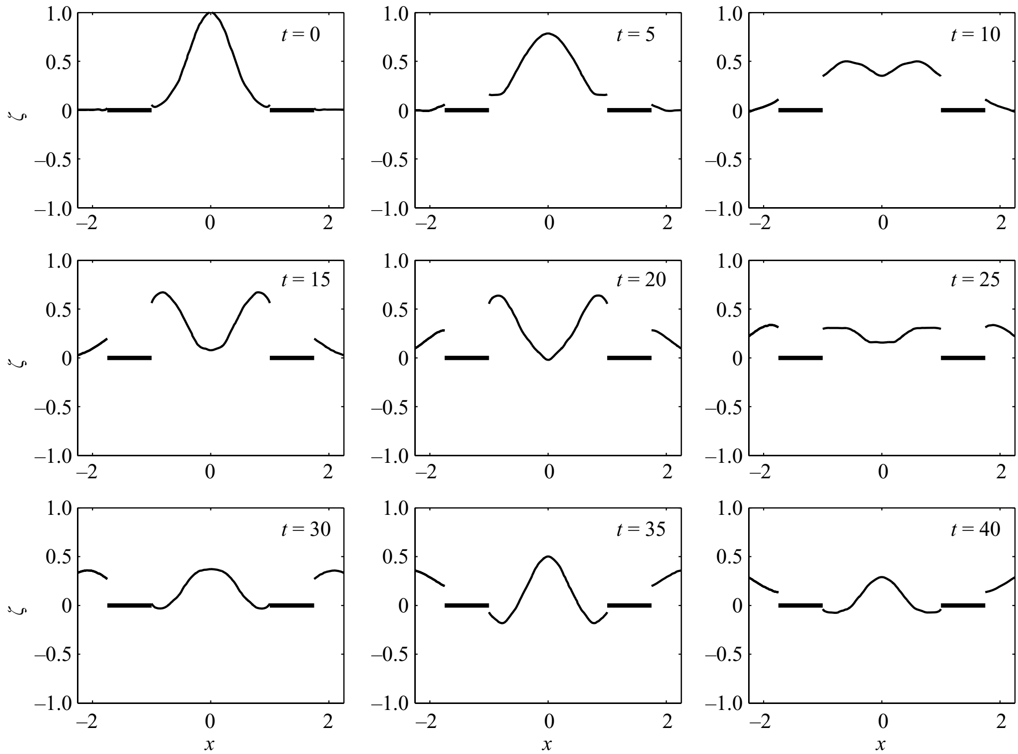


FIGURE 2. The surface displacement for the initial conditions given by (4.1) for the times shown. The water depth is $h = -2$, and the dock lengths are $L_1 = 1$ and $L_2 = 1.75$. The dock submergence is $d = 0$. The dock position is shown by the dark line for illustrative purposes only.

and

$$\langle v_0(x), \zeta_\kappa(x,k)\rangle_{\mathscr{H}} = 2\pi g_\kappa(\omega)\frac{\mathrm{d}\omega}{\mathrm{d}k}. \tag{3.14}$$

We can therefore write (3.12), changing the variable of integration to $k$, as

$$\zeta(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^+}\sum_{\kappa\in\{-1,1\}}\left\{\langle \zeta_0(x), \zeta_\kappa(x,k)\rangle_{\mathscr{H}}\cos(\omega t) + \langle v_0(x), \zeta_\kappa(x,k)\rangle_{\mathscr{H}}\frac{\sin(\omega t)}{\omega}\right\}\zeta_\kappa(x,k)\,\mathrm{d}k. \tag{3.15}$$

If we take the case when $v_0(x) = 0$ and write the integral given by the inner product explicitly, we obtain

$$\zeta(x,t) = \int_{\mathbb{R}^+}\left\{\sum_{\kappa\in\{-1,1\}}\left(\frac{1}{2\pi}\int_F \zeta_0(x')\,\zeta_\kappa(x',k)^*\,\mathrm{d}x'\right)\zeta_\kappa(x,k)\right\}\cos(\omega t)\mathrm{d}k. \tag{3.16}$$

### 3.5. *An identity linking waves from the left and right*

A consequence of the requirement that the displacement be real, if the initial displacement and initial derivative of displacement is real, is that

$$\sum_{\kappa\in\{-1,1\}}\langle \zeta_0(x), \zeta_\kappa(x,k)\rangle_{\mathscr{H}}\,\zeta_\kappa(x,k),$$

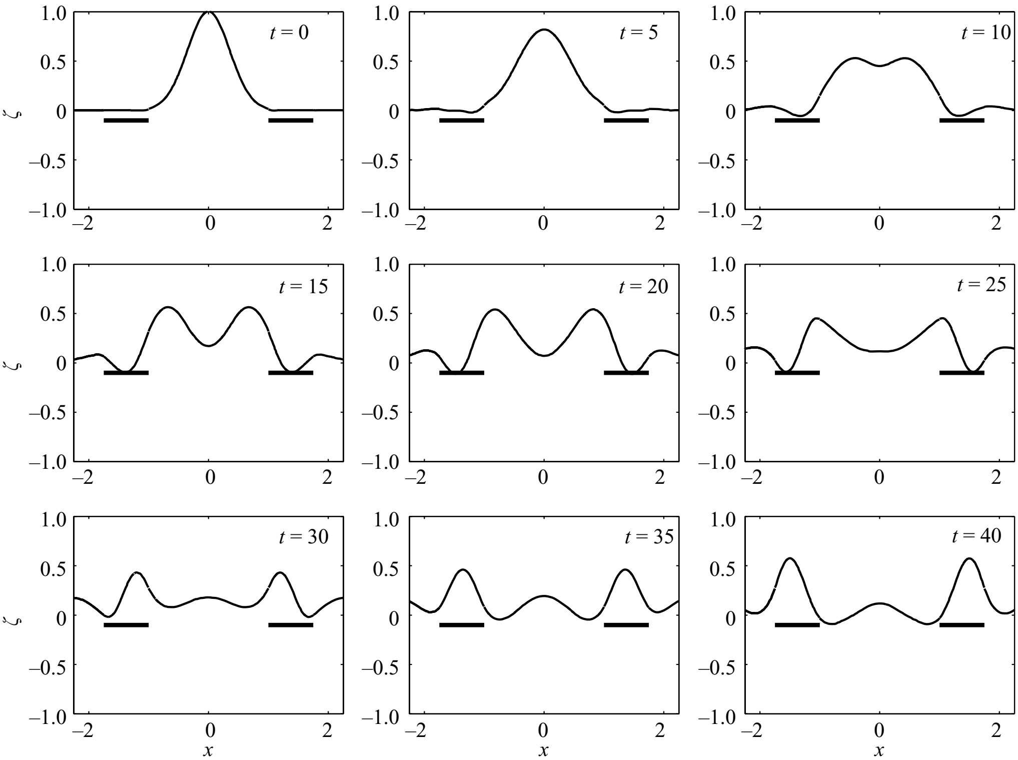


FIGURE 3. As in figure 2, except that the dock submergence is $d = 0.1$.

must be purely real. This can be true only if

$$\operatorname{Im}\left\{\zeta_1(x',k)^*\zeta_1(x,k)\right\} = -\operatorname{Im}\left\{\zeta_{-1}(x',k)^*\zeta_{-1}(x,k)\right\}, \quad x, x' \in F. \tag{3.17}$$

To the best of the author's knowledge, this result has not appeared previously.

## 4. Results

We present here results for a very simple geometry, so that frequency domain calculations are straightforward. We consider a pair of rigid plates of negligible thickness with the boundary of the structure given by

$$\partial\Omega = \left\{-L_2 \leqslant x \leqslant -L_1, z = -d\right\} \cup \left\{L_1 \leqslant x \leqslant L_2, z = -d\right\}.$$

The method used to solve the equations is based on the matched eigenfunction expansion method. Full details and computer code can be found on Meylan (2008*b*) and the problem is also discussed in Linton & Evans (1991). The initial displacement is given by

$$\zeta_0(x) = \exp(-4x^2) \text{ and } v_0(x) = 0. \tag{4.1}$$

We fix the water depth $h = -2$, and the dock lengths are $L_1 = 1$ and $L_2 = 1.75$. We consider the case when $d = 0$ (figure 2), $d = 0.1$ (figure 3) and $d = 0.2$ (figure 4). Note that the dock is also shown as a dark line in the figures, but this is only shown for illustrative purposes and has no relationship with the free surface. The solutions are also shown in Movies 1, 4 and 5. Further solutions with different dock lengths

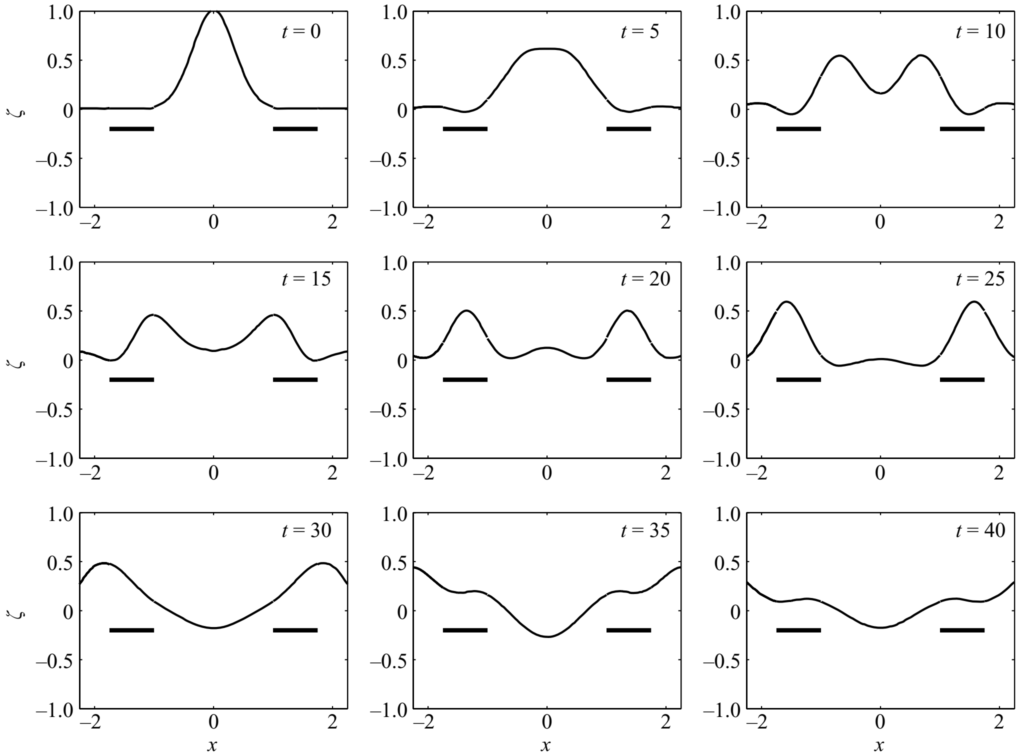


FIGURE 4. As in figure 2, except that the dock submergence is $d = 0.2$.

and submergences are illustrated in Movies 2, 3 and 6. These figures show interesting resonant effects, which are strongly a function of the depth of submergence.

## 5. Summary

We have considered the time-dependent problem of a two-dimensional domain with fixed bodies. We have shown that the solution in the time-domain can be calculated from the single-frequency solutions using a generalized eigenfunction expansion. The resulting formula for the time-dependent solution is relatively simple, and involves an integral over the free surface and an integral in frequency. The method outlined here would generalize easily to other situations, such as three dimensions or floating rather than fixed bodies.

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