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ON MINIMAL RESTRICTED ASYMPTOTIC BASES

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Abstract

Let $h \geq 2$ be a positive integer. We introduce the concept of minimal restricted asymptotic bases and obtain some examples of minimal restricted asymptotic bases of order h.

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1. Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers. For $A \subseteq \mathbb{N}$, $h \geq 2$, the h-fold sum of A, denoted hA, is the set of sums of h not necessarily distinct elements of A and $h^\wedge A$ is the set of sums of h distinct elements of A. Let W be a nonempty subset of $\mathbb{N}$. Denote by $F^*(W)$ the set of all finite, nonempty subsets of W. Given positive integers $g, h \geq 2$, denote

$$A_g(W) = \Big\{ \sum_{f \in F} a_f g^f : 1 \leq a_f \leq g-1, F \in F^*(W) \Big\}.$$

For $i = 0, 1, \ldots, h-1$, let $W_i^{(h)} = \{n \in \mathbb{N} : n \equiv i \pmod h\}$ and let

$$\mathcal{A}_{g,h} = A_g(W_0^{(h)}) \cup A_g(W_1^{(h)}) \cup \cdots \cup A_g(W_{h-1}^{(h)}).$$

The set A is an asymptotic basis of order h if hA contains all sufficiently large integers. An asymptotic basis A of order h is minimal if $A \setminus \{a\}$ is not an asymptotic basis of order h for every nonnegative integer $a \in A$. In 1974, Nathanson [4] first gave an explicit construction of a minimal asymptotic basis of order 2 by using properties of binary representations. In 2010, Jańczak and Schoen [3] constructed a dense minimal asymptotic basis of order two. Nathanson's method has been widely used in the construction of minimal asymptotic bases. For related problems concerning minimal asymptotic bases, see [2, 6, 7]. The study of asymptotic bases and minimal

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asymptotic bases is closely related to the famous Erdős–Turán conjecture in additive number theory (see [1, 5]).

It is natural to introduce a parallel concept of minimal restricted asymptotic bases. We call A a restricted asymptotic basis of order h if $h^\wedge A$ contains all sufficiently large integers. A restricted asymptotic basis A of order h is minimal if $A \setminus \{a\}$ is not a restricted asymptotic basis of order h for every nonnegative integer $a \in A$. Does there exist a minimal restricted asymptotic basis of order h?

Our study begins with a result of Sun and Tao on minimal asymptotic bases.

THEOREM 1.1 [8]. *Let $h \geq 2$. Then for any $g \geq h$, the set $\mathcal{A}_{g,h}$ is a minimal asymptotic basis of order h.*

We obtain the following results.

PROPOSITION 1.2. *For $k \geq 0$, $g \geq 2$:*

(1) $2g^{2k+1} \notin 2^\wedge \mathcal{A}_{g,2}$;
(2) $2g^{3k+2} + 1 \notin 3^\wedge \mathcal{A}_{g,3}$;
(3) $2 \cdot 4^{4k+3} + 5 \notin 4^\wedge \mathcal{A}_{4,4}$.

THEOREM 1.3. *Let $h \geq 4$. Then for any $g \geq \max\{h, 5\}$, the set $\mathcal{A}_{g,h}$ is a minimal restricted asymptotic basis of order h.*

2. A preliminary lemma

LEMMA 2.1. *Given positive integers $h \geq 2$, $g \geq 5$ and $u \geq 2$, let the g-adic representation of n be*

$$n = e_1 g^{i_1} + \cdots + e_{u-1} g^{i_{u-1}} + g^{i_u},$$

where $0 \leq i_1 < \cdots < i_u$ and $1 \leq e_j \leq g-1$ for $j = 1, \ldots, u-1$. Then $n \in (u+1)^\wedge \mathcal{A}_{g,h}$.

PROOF. If $i_{u-1} < i_u - 1$, or if $i_{u-1} = i_u - 1$, $e_{u-1} \notin \{1, g-1\}$, then because $g \geq 5$ and

$$n = e_1 g^{i_1} + \cdots + e_{u-1} g^{i_{u-1}} + g^{i_u - 1} + (g-1) g^{i_u - 1},$$

we see that $n \in (u+1)^\wedge \mathcal{A}_{g,h}$.

If $i_{u-1} = i_u - 1$ and $e_{u-1} = 1$, then because $g \geq 5$ and

$$n = e_1 g^{i_1} + \cdots + e_{u-2} g^{i_{u-2}} + g^{i_u - 1} + 2g^{i_u - 1} + (g-2) g^{i_u - 1},$$

we see that $n \in (u+1)^\wedge \mathcal{A}_{g,h}$.

If $i_{u-1} = i_u - 1$ and $e_{u-1} = g-1$, then because $g \geq 5$ and

$$n = e_1 g^{i_1} + \cdots + e_{u-2} g^{i_{u-2}} + g^{i_u - 1} + (g-2) g^{i_u - 1} + g^{i_u},$$

again $n \in (u+1)^\wedge \mathcal{A}_{g,h}$.

This completes the proof of Lemma 2.1. □

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3. Proof of Proposition 1.2

(1) Assume that $2g^{2k+1} = a_0 + a_1 \in 2^\wedge\mathcal{A}_{g,2}$ with $0 < a_0 < a_1$. Then $a_0 < g^{2k+1}$ and $a_1 < 2g^{2k+1}$. Since $a_0, a_1 \in \mathcal{A}_{g,2}$,

$$\begin{aligned} a_0 + a_1 &\le (g-1)(g^0 + \cdots + g^{2k-2} + g^{2k}) + (g-1)(g^1 + \cdots + g^{2k-3} + g^{2k-1}) + g^{2k+1} \\ &< 2g^{2k+1}, \end{aligned}$$

which is a contradiction. Hence, $2g^{2k+1} \notin 2^\wedge\mathcal{A}_{g,2}$ for all $g \ge 2$.
(2) Assume that $2g^{3k+2} + 1 \in 3^\wedge\mathcal{A}_{g,3}$. Write

$$2g^{3k+2} + 1 = a_0 + a_1 + a_2, \quad a_0 < a_1 < a_2. \tag{3.1}$$

Then $a_1 < g^{3k+2}$ and $a_2 < 2g^{3k+2}$. By (3.1), there exists at least one $a_i \in A_g(W_0^{(3)})$.

If $a_2 \notin A_g(W_2^{(3)})$, then $a_2 \le (g-1)(g^1 + g^4 + \cdots + g^{3k+1})$. Thus,

$$\begin{aligned} a_0 + a_1 + a_2 &< (g-1)(g^0 + g^3 + \cdots + g^{3k}) + 2(g-1)(g^1 + g^4 + \cdots + g^{3k+1}) \\ &< 2g^{3k+2} + 1, \end{aligned}$$

which is a contradiction.

If $a_2 \in A_g(W_2^{(3)})$, then $a_2 \le (g-1)(g^2 + g^5 + \cdots + g^{3k-1}) + g^{3k+2}$. Thus,

$$\begin{aligned} a_0 + a_1 + a_2 &\le (g-1)(g^0 + g^3 + \cdots + g^{3k}) + (g-1)(g^1 + g^4 + \cdots + g^{3k+1}) \\ &\quad + (g-1)(g^2 + g^5 + \cdots + g^{3k-1}) + g^{3k+2} \\ &< 2g^{3k+2} + 1, \end{aligned}$$

which is a contradiction.

Hence, $2g^{3k+2} + 1 \notin 3^\wedge\mathcal{A}_{g,3}$ for all $g \ge 2$.
(3) Assume that $2 \cdot 4^{4k+3} + 5 \in 4^\wedge\mathcal{A}_{4,4}$. Write

$$2 \cdot 4^{4k+3} + 5 = a_0 + a_1 + a_2 + a_3, \quad a_0 < a_1 < a_2 < a_3. \tag{3.2}$$

Then $a_3 < 2 \cdot 4^{4k+3}$.

If $a_3 \notin A_4(W_3^{(4)})$, then $a_3 \le 3(4^2 + 4^6 + \cdots + 4^{4k+2})$. Since $2 \cdot 4^{4k+3} + 5 \equiv 5 \pmod{16}$, it follows from (3.2) that

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 &< 4^0 + 3 \times (4^4 + \cdots + 4^{4k}) + 4^1 + 3 \times (4^5 + \cdots + 4^{4k+1}) \\ &\quad + 2 \times 3 \times (4^2 + 4^6 + \cdots + 4^{4k+2}) \\ &< 2 \cdot 4^{4k+3} + 5, \end{aligned}$$

which is a contradiction.

If $a_3 \in A_4(W_3^{(4)})$, then $a_3 \le 3(4^3 + 4^7 + \cdots + 4^{4k-1}) + 4^{4k+3}$. If $a_2 \ge 4^{4k+3}$, then we have $a_2 + a_3 > 2 \cdot 4^{4k+3} + 5$, which is a contradiction. It follows that $a_2 < 4^{4k+3}$. Again by (3.2) and $2 \cdot 4^{4k+3} + 5 \equiv 5 \pmod{16}$,

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 \leq 4^0 &+ 3 \times (4^4 + \cdots + 4^{4k}) + 4^1 + 3 \times (4^5 + \cdots + 4^{4k+1}) \\ &+ 3(4^2 + 4^6 + \cdots + 4^{4k+2}) + 3(4^3 + 4^7 + \cdots + 4^{4k-1}) + 4^{4k+3} \\ &< 2 \cdot 4^{4k+3} + 5, \end{aligned}$$

which is a contradiction.

Hence, $2 \cdot 4^{4k+3} + 5 \notin 4^{\wedge}\mathcal{A}_{4,4}$.

This completes the proof of Proposition 1.2.

4. Proof of Theorem 1.3

By Theorem 1.1, $\mathcal{A}_{g,h}$ is a minimal asymptotic basis of order h. Thus, we only need to prove that $\mathcal{A}_{g,h}$ is a restricted asymptotic basis of order h.

Let $n \geq g^{(h-2)^2+1}$ and let the g-adic representation of n be

$$n = e_1 g^{i_1} + \cdots + e_{t-1} g^{i_{t-1}} + e_t g^{i_t},$$

where $0 \leq i_1 < \cdots < i_t$ and $1 \leq e_j \leq g-1$ for $j = 1, \ldots, t$.

Case 1: $t = 1$. Then $i_1 \geq h$. Note that

$$n = (e_1 - 1)g^{i_1} + (g-1)g^{i_1 - 1} + \cdots + (g-1)g^{i_1-(h-1-\delta)} + g^{i_1-(h-1-\delta)},$$

where $\delta = 0$ if $e_1 = 1$ and otherwise $\delta = 1$. Hence, $n \in h^{\wedge}\mathcal{A}_{g,h}$.

Case 2: $2 \leq t \leq h-2$. If there exists a $k \in \{2, \ldots, t\}$ such that $i_k - i_{k-1} > h - t$, then

$$n = (e_k - 1)g^{i_k} + (g-1)g^{i_k - 1} + \cdots + (g-1)g^{i_k-(h-t-\delta)} + g^{i_k-(h-t-\delta)} + \sum_{j \in \{1,\ldots,t\} \setminus \{k\}} e_j g^{i_j},$$

where $\delta = 0$ if $e_k = 1$ and otherwise $\delta = 1$. Hence, $n \in h^{\wedge}\mathcal{A}_{g,h}$.

If $i_k - i_{k-1} \leq h - t$ for all $k \in \{2, \ldots, t\}$, then by $n \geq g^{(h-2)^2+1}$, we have $i_1 \geq h - t$. Otherwise, if $i_1 < h - t$, then $i_t < t(h-t)$, so that

$$n = e_1 g^{i_1} + \cdots + e_{t-1} g^{i_{t-1}} + e_t g^{i_t} < g^{i_t+1} < g^{(h-2)^2+1},$$

which is a contradiction. Thus,

$$n = (e_1 - 1)g^{i_1} + (g-1)g^{i_1 - 1} + \cdots + (g-1)g^{i_1-(h-t-\delta)} + g^{i_1-(h-t-\delta)} + \sum_{j \in \{2,\ldots,t\}} e_j g^{i_j},$$

where $\delta = 0$ if $e_1 = 1$ and otherwise $\delta = 1$. Hence, $n \in h^{\wedge}\mathcal{A}_{g,h}$.

Case 3: $t = h - 1$. Then

$$n = e_1 g^{i_1} + e_2 g^{i_2} + \cdots + e_{h-1} g^{i_{h-1}},$$

where $0 \leq i_1 < \cdots < i_{h-1}$, $1 \leq e_j \leq g-1$ for $j = 1, \ldots, h-1$.

If there exists a $k \in \{1, \ldots, h-1\}$ such that $3 \leq e_k \leq g-1$, then

$$n = \sum_{j \in I \setminus \{k\}} e_j g^{i_j} + g^{i_k} + (e_k - 1)g^{i_k},$$

where $I = \{1, \ldots, h-1\}$, and thus $n \in h^{\wedge}\mathcal{A}_{g,h}$.

Now we consider what happens if $1 \leq e_j \leq 2$ for $j = 1, \ldots, h-1$.

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(a) Suppose $e_{h-1} = 1$. Then by Lemma 2.1, $n \in h^\wedge \mathcal{A}_{g,h}$.
(b) Suppose $e_1 = e_2 = \cdots = e_{h-2} = e_{h-1} = 2$.

If there exist $i_u < i_v$ with $1 \le u, v \le h-1$ such that $i_u \equiv i_v \pmod h$, then because

$$2g^{i_u} + 2g^{i_v} = (2g^{i_u} + g^{i_v}) + g^{i_v-1} + (g-1)g^{i_v-1}$$

and $i_v \ge h$, we have $n \in h^\wedge \mathcal{A}_{g,h}$.

If $i_s \not\equiv i_t \pmod h$ for $1 \le s \ne t \le h-1$, then because $h \ge 4$, there exist i_v (≥ 1), i_u with $u, v \in \{1, 2, \ldots, h-1\}$ such that $i_v \equiv i_u + 1 \pmod h$. Since $g \ge 5$ and

$$2g^{i_u} + 2g^{i_v} = (2g^{i_u} + g^{i_v-1}) + (g-1)g^{i_v-1} + g^{i_v},$$

$n \in h^\wedge \mathcal{A}_{g,h}$.
(c) Suppose $e_{h-1} = 2$, $e_k = 1$ for some $k \in \{2, \ldots, h-2\}$. Then

$$n = \sum_{j\in K} e_j g^{i_j} + g^{i_k} + \sum_{j\in I} e_j g^{i_j},$$

where $K = \{1, \ldots, k-1\}$ and $I = \{k+1, \ldots, h-1\}$. By Lemma 2.1,

$$\sum_{j\in K} e_j g^{i_j} + g^{i_k} \in (k+1)^\wedge \mathcal{A}_{g,h}$$

and so $n \in h^\wedge \mathcal{A}_{g,h}$.
(d) Suppose $e_{h-1} = e_{h-2} = \cdots = e_2 = 2$, $e_1 = 1$. If $i_1 > 0$, then

$$n = g^{i_1-1} + (g-1)g^{i_1-1} + \sum_{j\in\{2,\ldots,h-1\}} 2g^{i_j}$$

and so $n \in h^\wedge \mathcal{A}_{g,h}$.

If $i_1 = 0$, then

$$n = 1 + \sum_{j\in\{2,\ldots,h-1\}} 2g^{i_j}.$$

(d1) There exists a $k \in \{2, \ldots, h-1\}$ such that $i_k \equiv 0 \pmod h$. Then

$$n = (1 + g^{i_k}) + g^{i_k-1} + (g-1)g^{i_k-1} + \sum_{j\in\{2,\ldots,h-1\}\setminus\{k\}} 2g^{i_j}.$$

Thus, $n \in h^\wedge \mathcal{A}_{g,h}$.
(d2) Suppose $i_k \not\equiv 0 \pmod h$ for all $k \in \{2, \ldots, h-1\}$.

If $h \ge 5$, then one of the following two cases must occur.

Case (i): There exist $i_u < i_v$ with $2 \le u, v \le h-1$ such that $i_u \equiv i_v \pmod h$. Since

$$2g^{i_u} + 2g^{i_v} = (2g^{i_u} + g^{i_v}) + g^{i_v-1} + (g-1)g^{i_v-1}$$

and $i_v > h$, we have

$$\sum_{j\in\{2,\ldots,h-1\}} 2g^{i_j} \in (h-1)^\wedge (\mathcal{A}_{g,h}\setminus\{1\}).$$

Thus, $n \in h^\wedge \mathcal{A}_{g,h}$.

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Case (ii): $i_s \not\equiv i_t \pmod{h}$ for $2 \le s \ne t \le h-1$. Then there exist i_v (≥ 1), i_u for some $u, v \in \{2, \ldots, h-1\}$ such that $i_v \equiv i_u + 1 \pmod{h}$. Because $g \ge 5$ and

$$2g^{i_u} + 2g^{i_v} = (2g^{i_u} + g^{i_v-1}) + (g-1)g^{i_v-1} + g^{i_v},$$

we have

$$\sum_{j \in \{2,\ldots,h-1\}} 2g^{i_j} \in (h-1)^{\wedge}(\mathcal{A}_{g,h} \backslash \{1\}).$$

Thus, $n \in h^{\wedge}\mathcal{A}_{g,h}$.

Now suppose $h = 4$. Since $i_2, i_3 \not\equiv 0 \pmod{4}$, there is one further case in addition to (i) and (ii), namely, $\{i_2 \pmod{4}, i_3 \pmod{4}\} = \{1 \pmod{4}, 3 \pmod{4}\}$. We may assume that $i_2 \equiv 1 \pmod{4}$. Because $g \ge 5$ and

$$n = (1 + g^{i_2-1}) + (g-1)g^{i_2-1} + g^{i_2} + 2g^{i_3},$$

we have $n \in 4^{\wedge}\mathcal{A}_{g,4}$.

Case 4: $t \ge h$.

Write $I = \{i_1, \ldots, i_t\}$. Since $|I| \ge h$, it is possible to write I as a union of h nonempty sets $I_1, \ldots, I_h$, where each I_j is a subset of some $W_k^{(h)}$. It follows that $n \in h^{\wedge}\mathcal{A}_{g,h}$.

This completes the proof of Theorem 1.3.

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