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SOME EXACT ALGEBRAIC EXPRESSIONS FOR THE TAILS OF TASOEV CONTINUED FRACTIONS

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Abstract

Denote the nth convergent of the continued fraction $\alpha = [a_0; a_1, a_2, \ldots]$ by $p_n/q_n = [a_0; a_1, \ldots, a_n]$. In this paper we give exact formulae for the quantities $D_n := q_n\alpha - p_n$ in several typical types of Tasoev continued fractions. A simple example of the type of Tasoev continued fraction considered is $\alpha = [0; ua, ua^2, ua^3, \ldots]$.

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1. Introduction

Denote by $\alpha = [a_0; a_1, a_2, \ldots]$ the regular (or simple) continued fraction expansion of a real number α, where

$$\alpha = a_0 + 1/\alpha_1, \quad a_0 = \lfloor \alpha \rfloor,$$
$$\alpha_n = a_n + 1/\alpha_{n+1}, \quad a_n = \lfloor \alpha_n \rfloor$$

with $n \geq 1$. Let $p_n/q_n = [a_0; a_1, \ldots, a_n]$ denote the nth convergent associated with α. It is well known that p_n/q_n (where $n = 0, 1, 2, \ldots$) are good approximations of α in the sense of the inequality

$$|q_n\alpha - p_n| < \frac{1}{a_{n+1}q_n}.$$

More precisely,

$$D_n := q_n\alpha - p_n = \frac{(-1)^n}{\alpha_{n+1}q_n + q_{n-1}} \quad \forall n \geq 0$$

(see, for instance, [1, Lemma 5.4]).

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Quasi-periodic continued fractions have the form

$$\begin{aligned}\alpha &= [a_0; a_1, \dots, a_n, \overline{Q_1(k), \dots, Q_p(k)}]_{k=1}^{\infty}\\ &= [a_0; a_1, \dots, a_n, Q_1(1), \dots, Q_p(1), Q_1(2), \dots, Q_p(2), Q_1(3), \dots],\end{aligned}$$

where a_0 is an integer, $a_1, \dots, a_n$ are positive integers, $Q_1, \dots, Q_p$ are functions that take positive integral values for all $k = 1, 2, \dots$, at least one of which is not constant. If each Q_j has polynomial form (or is constant), the expansion of α is called a *Hurwitz continued fraction* (see, for instance, [16]). Well-known examples are

$$\begin{aligned}e &= [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, \dots] = [2; \overline{1, 2k, 1}]_{k=1}^{\infty},\\ \tanh 1 &= \frac{e^2-1}{e^2+1} = [0; 1, 3, 5, 7, \dots] = [0; \overline{2k-1}]_{k=1}^{\infty},\\ \tan 1 &= [1; 1, 1, 1, 3, 1, 1, 5, 1, \dots] = [1; \overline{2k-1, 1}]_{k=1}^{\infty}.\end{aligned}$$

If each Q_j has exponential form (or is constant), the expansion of α is called a *Tasoev continued fraction* [3–7, 17]. In [4], the author found some more general Tasoev continued fractions:

$$[0; \overline{ua^k}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-2n-1} a^{-(n+1)^2} \prod_{i=1}^{n} (a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty} u^{-2n} a^{-n^2} \prod_{i=1}^{n} (a^{2i}-1)^{-1}}, \tag{1}$$

$$[0; ua-1, \overline{1, ua^{k+1}-2}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} (-1)^n u^{-2n-1} a^{-(n+1)^2} \prod_{i=1}^{n} (a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty} (-1)^n u^{-2n} a^{-n^2} \prod_{i=1}^{n} (a^{2i}-1)^{-1}}, \tag{2}$$

$$[0; \overline{ua^k, va^k}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-n-1} v^{-n} a^{-(n+1)(n+2)/2} \prod_{i=1}^{n} (a^{i}-1)^{-1}}{\sum_{n=0}^{\infty} u^{-n} v^{-n} a^{-n(n+1)/2} \prod_{i=1}^{n} (a^{i}-1)^{-1}} \tag{3}$$

and

$$\begin{aligned}&[0; ua-1, 1, va-2, \overline{1, ua^{k+1}-2, 1, va^{k+1}-2}]_{k=1}^{\infty}\\ &\quad= \frac{\sum_{n=0}^{\infty} (-1)^n u^{-n-1} v^{-n} a^{-(n+1)(n+2)/2} \prod_{i=1}^{n} (a^{i}-1)^{-1}}{\sum_{n=0}^{\infty} (-1)^n u^{-n} v^{-n} a^{-n(n+1)/2} \prod_{i=1}^{n} (a^{i}-1)^{-1}}.\end{aligned} \tag{4}$$

One might say that Tasoev continued fractions are geometric and Hurwitz continued fractions are arithmetic [5]. Roughly speaking, ua^k in a Tasoev continued fraction corresponds to $u(a+bk)$ in a Hurwitz continued fraction, where $k = 1, 2, \dots$. The Tasoev continued fractions corresponding to e-type Hurwitz continued fractions were also derived in [6]:

$$\begin{aligned}&[0; \overline{ua^k-1, 1, v-1}]_{k=1}^{\infty}\\ &\quad= \frac{\sum_{n=0}^{\infty} u^{-2n-1} v^{-2n} a^{-(n+1)^2} \prod_{i=1}^{n} (a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty} ((uv)^{-2n} a^{-n^2} - (uv)^{-2n-1} a^{-(n+1)^2}) \prod_{i=1}^{n} (a^{2i}-1)^{-1}}\end{aligned} \tag{5}$$

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and

$$[0; \overline{v-1, 1, ua^k - 1}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty}(u^{-2n}v^{-2n-1}a^{-n^2} + u^{-2n-1}v^{-2n-2}a^{-(n+1)^2}) \prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(uv)^{-2n}a^{-n^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}. \tag{6}$$

The different types of Tasoev continued fractions with period 3 shown in [5] are

$$[0; \overline{ua^{2k-1} - 1, 1, va^{2k} - 1}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-n-1}v^{-n}a^{-(n+1)^2} \prod_{i=1}^{n}(a^{2i} - (-1)^i)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-n}v^{-n}a^{-n^2} \prod_{i=1}^{n}(a^{2i} - (-1)^i)^{-1}}, \tag{7}$$

$$[0; ua, \overline{va^{2k} - 1, 1, ua^{2k+1} - 1}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty}(-1)^n u^{-n-1}v^{-n}a^{-(n+1)^2} \prod_{i=1}^{n}(a^{2i} - (-1)^i)^{-1}}{\sum_{n=0}^{\infty} u^{-n}v^{-n}a^{-n^2} \prod_{i=1}^{n}(a^{2i} - (-1)^i)^{-1}} \tag{8}$$

and so on.

More Tasoev-type continued fractions have been found by Mc Laughlin [14] and by the author [9, 10].

Some Diophantine properties of Tasoev continued fractions are characterized as properties of Ramanujan-type q-continued fractions [13]. The purpose of this paper is to examine Diophantine properties of Tasoev continued fractions. In particular, the error values D_n for such continued fractions are explicitly given in Sections 2 and 3. A representative proof for one of these results is given in Section 4. Sections 5 and 6 consider Diophantine approximations of Hurwitz continued fractions. The error value D_n can be expressed in terms of integrals in the cases of some typical Hurwitz continued fractions [2, 11, 12, 15], and these expressions are given in Section 7.

2. Main results

Let p_n/q_n be the nth convergent of the Tasoev continued fraction

$$[0; \overline{ua^{2k-1}, va^{2k}}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-n-1}v^{-n}a^{-(n+1)^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(uv)^{-n}a^{-n^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}.$$

THEOREM 1. *For all $k \ge 1$,*

$$D_{2k-1} = -\frac{\sum_{n=0}^{\infty}(uv)^{-n-k}a^{-n^2-4kn-k(2k+1)} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(uv)^{-n}a^{-n^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}},$$

$$D_{2k} = \frac{\sum_{n=0}^{\infty} u^{-n-k-1}v^{-n-k}a^{-n^2-(4k+2)n-(k+1)(2k+1)} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(uv)^{-n}a^{-n^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}.$$

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REMARK 2. If $u = v$, then these identities become those of (1). If u is replaced by ua and a is replaced by $a^{1/2}$, then they become those of (3).

Let p_n/q_n be the nth convergent of the Tasoev continued fraction

$$[0; ua-1, \overline{1, va^{2k}-2, 1, ua^{2k+1}-2}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty}(-1)^n u^{-n-1} v^{-n} a^{-(n+1)^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(-1)^n (uv)^{-n} a^{-n^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}.$$

THEOREM 3. *For all* $k \geq 1$,

$$D_{4k-2} = \frac{\sum_{n=0}^{\infty}(-1)^n (uv)^{-n-k} a^{-n^2-4kn-k(2k+1)} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(-1)^n (uv)^{-n} a^{-n^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}},$$

$$D_{4k} = \frac{\sum_{n=0}^{\infty}(-1)^n u^{-n-k-1} v^{-n-k} a^{-n^2-(4k+2)n-(k+1)(2k+1)} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(-1)^n (uv)^{-n} a^{-n^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}.$$

REMARK 4. For odd k, the recurrence relation $D_k = D_{k+1} - D_{k-1}$ gives a formula for D_k.

If $u = v$, then these identities become those of (2). If u is replaced by ua and a is replaced by $a^{1/2}$, then these become those of (4).

3. *e*-type Tasoev continued fractions

Let p_n/q_n be the nth convergent of the Tasoev continued fraction (5):

$$[0; \overline{ua^k-1, 1, v-1}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-2n-1} v^{-2n} a^{-(n+1)^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}((uv)^{-2n} a^{-n^2} - (uv)^{-2n-1} a^{-(n+1)^2}) \prod_{i=1}^{n}(a^{2i}-1)^{-1}}.$$

THEOREM 5. *For all* $k \geq 1$,

$$\begin{aligned} D_{3k-1} = (-1)^k \sum_{n=0}^{\infty} &((uv)^{-2n-k-1} a^{-n^2-2(k+1)n-(k+1)(k+2)/2} \\ &- (uv)^{-2n-k} a^{-n^2-2kn-k(k+1)/2}) \prod_{i=1}^{n}(a^{2i}-1)^{-1} \\ &\times \Big(\sum_{n=0}^{\infty}((uv)^{-2n} a^{-n^2} - (uv)^{-2n-1} a^{-(n+1)^2}) \prod_{i=1}^{n}(a^{2i}-1)^{-1}\Big)^{-1}, \end{aligned}$$

$$D_{3k} = \frac{(-1)^k \sum_{n=0}^{\infty} u^{-2n-k-1} v^{-2n-k} a^{-n^2-2(k+1)n-(k+1)(k+2)/2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}((uv)^{-2n} a^{-n^2} - (uv)^{-2n-1} a^{-(n+1)^2}) \prod_{i=1}^{n}(a^{2i}-1)^{-1}}.$$

REMARK 6. The recurrence relation $D_{3k-2} = D_{3k-1} - D_{3k-3}$ gives a formula for D_{3k-2}.

Let p_n/q_n be the nth convergent of the Tasoev continued fraction (6):

$$[0; \overline{v-1, 1, ua^k - 1}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty}(u^{-2n}v^{-2n-1}a^{-n^2} + u^{-2n-1}v^{-2n-2}a^{-(n+1)^2}) \prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(uv)^{-2n}a^{-n^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}.$$

THEOREM 7. *For all $k \geq 1$,*

$$D_{3k-1} = \frac{(-1)^{k-1} \sum_{n=0}^{\infty}(uv)^{-2n-k}a^{-n^2-2kn-k(k+1)/2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(uv)^{-2n}a^{-n^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}},$$

$$\begin{aligned} D_{3k} = (-1)^k \sum_{n=0}^{\infty}(&u^{-2n-k}v^{-2n-k-1}a^{-n^2-2kn-k(k+1)/2} \\ &+ u^{-2n-k-1}v^{-2n-k-2}a^{-n^2-2(k+1)n-(k+1)(k+2)/2}) \prod_{i=1}^{n}(a^{2i}-1)^{-1} \\ &\times \Big(\sum_{n=0}^{\infty}(uv)^{-2n}a^{-n^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}\Big)^{-1}. \end{aligned}$$

REMARK 8. The recurrence relation $D_{3k-2} = D_{3k-1} - D_{3k-3}$ gives a formula for D_{3k-2}.

Let p_n/q_n be the nth convergent of the Tasoev continued fraction (7):

$$[0; \overline{ua^{2k-1} - 1, 1, va^{2k} - 1}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-n-1}v^{-n}a^{-(n+1)^2} \prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-n}v^{-n}a^{-n^2} \prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}.$$

THEOREM 9. *For all $k \geq 1$,*

$$D_{3k-1} = \frac{(-1)^{k-1} \sum_{n=0}^{\infty}(-1)^n(uv)^{-n-k}a^{-n^2-4kn-k(2k+1)} \prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-n}v^{-n}a^{-n^2} \prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}},$$

$$D_{3k} = \frac{(-1)^k \sum_{n=0}^{\infty} u^{-n-k-1}v^{-n-k}a^{-n^2-(4k+2)n-(k+1)(2k+1)} \prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-n}v^{-n}a^{-n^2} \prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}.$$

REMARK 10. The recurrence relation $D_{3k-2} = D_{3k} - D_{3k-1}$ gives a formula for D_{3k-2}.

Let p_n/q_n be the nth convergent of the Tasoev continued fraction (8):

$$[0; ua, \overline{va^{2k} - 1, 1, ua^{2k+1} - 1}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty}(-1)^n u^{-n-1}v^{-n}a^{-(n+1)^2} \prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^{\infty} u^{-n}v^{-n}a^{-n^2} \prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}.$$

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THEOREM 11. *For all $k \geq 1$,*

$$D_{3k} = \frac{(-1)^k \sum_{n=0}^{\infty}(-1)^n u^{-n-k-1} v^{-n-k} a^{-n^2-(4k+2)n-(k+1)(2k+1)} \prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^{\infty} u^{-n} v^{-n} a^{-n^2} \prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}},$$

$$D_{3k-2} = \frac{(-1)^k \sum_{n=0}^{\infty}(uv)^{-n-k} a^{-n^2-4kn-k(2k+1)} \prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^{\infty} u^{-n} v^{-n} a^{-n^2} \prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}.$$

REMARK 12. The recurrence relation $D_{3k-1} = D_{3k} - D_{3k-2}$ gives a formula for D_{3k-1}.

LEMMA 13. *For all $n \geq 0$,*

$$p_{3n} = \sum_{\nu=0}^{n-1}(-1)^{n-\nu-1} u^{\nu} v^{\nu+1} a^{(\nu+2)(2\nu+1)} \prod_{i=1}^{n-\nu-1} \frac{a^{2(2\nu+i+1)}+(-1)^i}{a^{2i}-(-1)^i},$$

$$p_{3n+1} = \sum_{\nu=0}^{n}(-1)^{n-\nu}(uv)^{\nu} a^{\nu(2\nu+3)} \prod_{i=1}^{n-\nu} \frac{a^{2(2\nu+i)}-(-1)^i}{a^{2i}-(-1)^i},$$

$$q_{3n} = \sum_{\nu=0}^{n}(uv)^{\nu} a^{\nu(2\nu+1)} \prod_{i=1}^{n-\nu} \frac{a^{2(2\nu+i)}-(-1)^i}{a^{2i}-(-1)^i},$$

$$q_{3n+1} = \sum_{\nu=0}^{n} u^{\nu+1} v^{\nu} a^{(\nu+1)(2\nu+1)} \prod_{i=1}^{n-\nu} \frac{a^{2(2\nu+i+1)}+(-1)^i}{a^{2i}-(-1)^i}.$$

4. Proof of Theorem 1

We prove Theorem 1 by induction. The other theorems can be proven similarly. Let

$$\alpha = [0; \overline{ua^{2k-1}, va^{2k}}\,]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-n-1} v^{-n} a^{-(n+1)^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(uv)^{-n} a^{-n^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}.$$

The identities hold for D_0 and D_1 because

$$\begin{aligned} D_0 &= q_0\alpha - p_0 = \alpha \\ &= \frac{\sum_{n=0}^{\infty} u^{-n-1} v^{-n} a^{-(n+1)^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(uv)^{-n} a^{-n^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}} \end{aligned}$$

and

$$\begin{aligned} D_1 &= q_1\alpha - p_1 = ua\alpha - 1 \\ &= \frac{(ua \sum_{n=0}^{\infty} u^{-n-1} v^{-n} a^{-(n+1)^2} - \sum_{n=0}^{\infty}(uv)^{-n} a^{-(n+1)^2}) \prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(uv)^{-n} a^{-n^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}} \\ &= \frac{\sum_{n=0}^{\infty}(uv)^{-n} a^{-n^2-2n}(1-a^{2n}) \prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(uv)^{-n} a^{-n^2} \prod_{i=1}^{n}(a^{2i}-1)^{-1}} \end{aligned}$$

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$$= -\frac{\sum_{n=1}^{\infty}(uv)^{-n}a^{-n^2-2n}\prod_{i=1}^{n-1}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(uv)^{-n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}$$

$$= -\frac{\sum_{n=0}^{\infty}(uv)^{-n-1}a^{-n^2-4n-3}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(uv)^{-n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}.$$

Assume that the identities hold for D_{2k-1} and D_{2k}. Since $D_{2k+1} = ua^{2k+1}D_{2k} + D_{2k-1}$,

$$\begin{aligned}
&D_{2k+1}\sum_{n=0}^{\infty}(uv)^{-n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}\\
&\quad= \sum_{n=0}^{\infty}(uv)^{-n-k}a^{-n^2-(4k+2)n-k(2k+1)}\prod_{i=1}^{n}(a^{2i}-1)^{-1}\\
&\qquad - \sum_{n=0}^{\infty}(uv)^{-n-k}a^{-n^2-4kn-k(2k+1)}\prod_{i=1}^{n}(a^{2i}-1)^{-1}\\
&\quad= \sum_{n=0}^{\infty}(uv)^{-n-k}a^{-n^2-(4k+2)n-k(2k+1)}(1-a^{2n})\prod_{i=1}^{n}(a^{2i}-1)^{-1}\\
&\quad= -\sum_{n=1}^{\infty}(uv)^{-n-k}a^{-n^2-(4k+2)n-k(2k+1)}\prod_{i=1}^{n-1}(a^{2i}-1)^{-1}\\
&\quad= -\sum_{n=0}^{\infty}(uv)^{-n-k-1}a^{-n^2-4(k+1)n-(k+1)(2k+3)}\prod_{i=1}^{n}(a^{2i}-1)^{-1}.
\end{aligned}$$

Since $D_{2k+2} = va^{2k+2}D_{2k+1} + D_{2k}$,

$$\begin{aligned}
&D_{2k+2}\sum_{n=0}^{\infty}(uv)^{-n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}\\
&\quad= -\sum_{n=0}^{\infty}u^{-n-k-1}v^{-n-k}a^{-n^2-4(k+1)n-(k+1)(2k+3)+2k+2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}\\
&\qquad + \sum_{n=0}^{\infty}u^{-n-k-1}v^{-n-k}a^{-n^2-(4k+2)n-(k+1)(2k+1)}\prod_{i=1}^{n}(a^{2i}-1)^{-1}\\
&\quad= \sum_{n=0}^{\infty}u^{-n-k-1}v^{-n-k}a^{-n^2-4(k+1)n-(k+1)(2k+1)}(-1+a^{2n})\prod_{i=1}^{n}(a^{2i}-1)^{-1}\\
&\quad= \sum_{n=0}^{\infty}u^{-n-k-2}v^{-n-k-1}a^{-(n+1)^2-4(k+1)(n+1)-(k+1)(2k+1)}\prod_{i=1}^{n-1}(a^{2i}-1)^{-1}\\
&\quad= \sum_{n=0}^{\infty}u^{-n-k-2}v^{-n-k-1}a^{-n^2-(4k+6)n-(k+2)(2k+3)}\prod_{i=1}^{n}(a^{2i}-1)^{-1}.
\end{aligned}$$

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5. Diophantine approximations of Hurwitz continued fractions

Let p_n/q_n be the nth convergent of the tanh-type Hurwitz continued fraction [4, Theorem 5], [8, (1)]:

$$\begin{aligned}\alpha &= [0; ua, v(a+b), u(a+2b), v(a+3b), u(a+4b), v(a+5b), \dots] \\ &= \frac{{}_0F_1(; \frac{a}{b}+1; \frac{1}{uvb^2})}{ua\,{}_0F_1(; \frac{a}{b}; \frac{1}{uvb^2})} = \frac{\sum_{n=0}^{\infty}(n!)^{-1}u^{-n-1}(vb)^{-n}\prod_{i=1}^{n}(a+bi)^{-1}}{\sum_{n=0}^{\infty}(n!)^{-1}(uvb)^{-n}\prod_{i=1}^{n-1}(a+bi)^{-1}},\end{aligned}$$

where

$${}_0F_1(; c; z) = \sum_{n=0}^{\infty}\frac{1}{(c)_n}\frac{z^n}{n!}$$

is the confluent hypergeometric limit function. Here $(c)_n = c(c+1)\dots(c+n-1)$ if $n \ge 1$ and $(c)_0 = 1$.

Theorem 14. *For all $k \ge 1$,*

$$\begin{aligned}D_{2k-1} &= -\frac{{}_0F_1(; \frac{a}{b}+2k; \frac{1}{uvb^2})}{(uvb^2)^k(\frac{a}{b})_{2k}\,{}_0F_1(; \frac{a}{b}; \frac{1}{uvb^2})} \\ &= -\frac{\sum_{n=0}^{\infty}(n!)^{-1}(uv)^{-n-k}b^{-n}\prod_{i=1}^{n+2k-1}(a+bi)^{-1}}{\sum_{n=0}^{\infty}(n!)^{-1}(uvb)^{-n}\prod_{i=1}^{n-1}(a+bi)^{-1}}, \\ D_{2k} &= \frac{{}_0F_1(; \frac{a}{b}+2k+1; \frac{1}{uvb^2})}{ua(uvb^2)^k(\frac{a}{b}+1)_{2k}\,{}_0F_1(; \frac{a}{b}; \frac{1}{uvb^2})} \\ &= \frac{\sum_{n=0}^{\infty}(n!)^{-1}u^{-n-k-1}v^{-n-k}b^{-n}\prod_{i=1}^{n+2k}(a+bi)^{-1}}{\sum_{n=0}^{\infty}(n!)^{-1}(uvb)^{-n}\prod_{i=1}^{n-1}(a+bi)^{-1}}.\end{aligned}$$

If we put $a = 1$ and $b = 2$ in Theorem 14, we immediately get the following corollary.

Corollary 15. *Let p_n/q_n be the nth convergent of the continued fraction*

$$\sqrt{\frac{v}{u}}\tanh\frac{1}{\sqrt{uv}} = [0; \overline{(4k-3)u, (4k-1)v}]_{k=1}^{\infty}.$$

Then for all $k \ge 1$,

$$\begin{aligned}D_{2k-1} &= -\frac{{}_0F_1(; \frac{1}{2}+2k; \frac{1}{4uv})}{(uv)^k(4k-1)!!\,{}_0F_1(; \frac{1}{2}; \frac{1}{4uv})}, \\ D_{2k} &= \frac{{}_0F_1(; \frac{3}{2}+2k; \frac{1}{4uv})}{u^{k+1}v^k(4k+1)!!\,{}_0F_1(; \frac{1}{2}; \frac{1}{4uv})},\end{aligned}$$

where $(4k-1)!! = (4k-1)(4k-3)\cdots 3\cdot 1$.

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Let p_n/q_n be the nth convergent of the tan-type Hurwitz continued fraction [4, Theorem 6], [8, (2)]:

$$\alpha = [0; ua-1, 1, v(a+b)-2, 1, u(a+2b)-2, 1, v(a+3b)-2, 1, \dots]$$
$$= \frac{{}_0F_1(;\frac{a}{b}+1;\frac{-1}{uvb^2})}{ua\,{}_0F_1(;\frac{a}{b};\frac{-1}{uvb^2})} = \frac{\sum_{n=0}^{\infty}(-1)^n(n!)^{-1}u^{-n-1}(vb)^{-n}\prod_{i=1}^{n}(a+bi)^{-1}}{\sum_{n=0}^{\infty}(-1)^n(n!)^{-1}(uvb)^{-n}\prod_{i=1}^{n-1}(a+bi)^{-1}}.$$

Theorem 16. *For all $k \ge 1$,*

$$D_{4k-2} = \frac{{}_0F_1(;\frac{a}{b}+2k;\frac{-1}{uvb^2})}{(uvb^2)^k(\frac{a}{b})_{2k}\,{}_0F_1(;\frac{a}{b};\frac{-1}{uvb^2})}$$
$$= \frac{\sum_{n=0}^{\infty}(-1)^n(n!)^{-1}(uv)^{-n-k}b^{-n}\prod_{i=1}^{n+2k-1}(a+bi)^{-1}}{\sum_{n=0}^{\infty}(-1)^n(n!)^{-1}(uvb)^{-n}\prod_{i=1}^{n-1}(a+bi)^{-1}},$$
$$D_{4k} = \frac{{}_0F_1(;\frac{a}{b}+2k+1;\frac{-1}{uvb^2})}{ua(uvb^2)^k(\frac{a}{b}+1)_{2k}\,{}_0F_1(;\frac{a}{b};\frac{-1}{uvb^2})}$$
$$= \frac{\sum_{n=0}^{\infty}(-1)^n(n!)^{-1}u^{-n-k-1}v^{-n-k}b^{-n}\prod_{i=1}^{n+2k}(a+bi)^{-1}}{\sum_{n=0}^{\infty}(-1)^n(n!)^{-1}(uvb)^{-n}\prod_{i=1}^{n-1}(a+bi)^{-1}}.$$

Remark 17. For odd k, the recurrence relation $D_k = D_{k+1} - D_{k-1}$ gives a formula for D_k.

If we put $a = 1$ and $b = 2$ in Theorem 16, we immediately get the following corollary.

Corollary 18. *Let p_n/q_n be the nth convergent of the continued fraction*

$$\sqrt{\frac{v}{u}}\tan\frac{1}{\sqrt{uv}} = [0; u-1, \overline{1, (4k-3)v-2, 1, (4k+1)u-2, 1}\,]_{k=1}^{\infty}.$$

Then for all $k \ge 1$,

$$D_{4k-2} = \frac{{}_0F_1(;\frac{1}{2}+2k;\frac{-1}{4uv})}{(uv)^k(4k-1)!!\,{}_0F_1(;\frac{1}{2};\frac{-1}{4uv})},$$
$$D_{4k} = \frac{{}_0F_1(;\frac{3}{2}+2k;\frac{-1}{4uv})}{u^{k+1}v^k(4k+1)!!\,{}_0F_1(;\frac{1}{2};\frac{-1}{4uv})},$$

where $(4k-1)!! = (4k-1)(4k-3)\cdots 3\cdot 1$.

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Let p_n/q_n be the nth convergent of the e-type Hurwitz continued fraction (see [6, Theorem 1], [8, (3)]):

$$\begin{aligned}\alpha &= [0; \overline{u(a+bk)-1, 1, v-1}]_{k=1}^{\infty}\\ &= \frac{v\,{}_0F_1(;\frac{a}{b}+2;\frac{1}{u^2v^2b^2})}{uv(a+b)\,{}_0F_1(;\frac{a}{b}+1;\frac{1}{u^2v^2b^2}) - {}_0F_1(;\frac{a}{b}+2;\frac{1}{u^2v^2b^2})}\\ &= \frac{\sum_{n=0}^{\infty} u^{-2n-1}v^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n+1}(a+bi)^{-1}}{\sum_{n=0}^{\infty} b^{-n}(n!)^{-1}((uv)^{-2n}\prod_{i=1}^{n}(a+bi)^{-1} - (uv)^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1})}.\end{aligned}$$

THEOREM 19. *For all $k \ge 1$,*

$$\begin{aligned}D_{3k-1} &= \frac{(-1)^k}{(uvb)^k(\frac{a}{b}+2)_k}\frac{{}_0F_1(;\frac{a}{b}+k+2;\frac{1}{u^2v^2b^2}) - uv(a+b(k+1))_0F_1(;\frac{a}{b}+k+1;\frac{1}{u^2v^2b^2})}{uv(a+b)\,{}_0F_1(;\frac{a}{b}+1;\frac{1}{u^2v^2b^2}) - {}_0F_1(;\frac{a}{b}+2;\frac{1}{u^2v^2b^2})}\\ &= \frac{(-1)^k\sum_{n=0}^{\infty} b^{-n}(n!)^{-1}((uv)^{-2n-k-1}\prod_{i=1}^{n+k+1}(a+bi)^{-1} - (uv)^{-2n-k}\prod_{i=1}^{n+k}(a+bi)^{-1})}{\sum_{n=0}^{\infty} b^{-n}(n!)^{-1}((uv)^{-2n}\prod_{i=1}^{n}(a+bi)^{-1} - (uv)^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1})},\\ D_{3k} &= \frac{(-1)^k v}{(uvb)^k(\frac{a}{b}+2)_k}\frac{{}_0F_1(;\frac{a}{b}+k+2;\frac{1}{u^2v^2b^2})}{uv(a+b)\,{}_0F_1(;\frac{a}{b}+1;\frac{1}{u^2v^2b^2}) - {}_0F_1(;\frac{a}{b}+2;\frac{1}{u^2v^2b^2})}\\ &= \frac{(-1)^k\sum_{n=0}^{\infty} u^{-2n-k-1}v^{-2n-k}b^{-n}(n!)^{-1}\prod_{i=1}^{n+k+1}(a+bi)^{-1}}{\sum_{n=0}^{\infty} b^{-n}(n!)^{-1}((uv)^{-2n}\prod_{i=1}^{n}(a+bi)^{-1} - (uv)^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1})}.\end{aligned}$$

REMARK 20. The recurrence relation $D_{3k-2} = D_{3k-1} - D_{3k-3}$ gives a formula for D_{3k-2}.

COROLLARY 21. *Let p_n/q_n be the nth convergent of the continued fraction*

$$e^{1/s} - 1 = [0; \overline{(2k-1)s-1, 1, 1}\,]_{k=1}^{\infty}.$$

Then for all $k \ge 1$,

$$D_{3k-1} = \frac{(-1)^k e^{1/(2s)}}{s^k}\sum_{n=0}^{\infty}\frac{(n+k)!}{n!}\Big(\frac{1}{(2s)^{2n+1}(2n+2k+1)!} - \frac{1}{(2s)^{2n}(2n+2k)!}\Big),$$

$$D_{3k} = \frac{(-1)^k e^{1/(2s)}}{s^{k+1}}\sum_{n=0}^{\infty}\frac{(n+k)!}{(2s)^{2n}n!(2n+2k+1)!}.$$

Let p_n/q_n be the nth convergent of another e-type Hurwitz continued fraction (see [6, Theorem 2], [8, (4)]):

$$\begin{aligned}\alpha &= [0; \overline{v-1, 1, u(a+bk)-1}]_{k=1}^{\infty}\\ &= \frac{{}_0F_1(;\frac{a}{b}+2;\frac{1}{u^2v^2b^2})}{uv^2(a+b)\,{}_0F_1(;\frac{a}{b}+1;\frac{1}{u^2v^2b^2})} + \frac{1}{v}.\end{aligned}$$

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THEOREM 22. *For all* $k \geq 1$,

$$D_{3k-1} = \frac{(-1)^{k-1} \sum_{n=0}^{\infty} (uv)^{-2n-k} b^{-n} (n!)^{-1} \prod_{i=1}^{n+k} (a+bi)^{-1}}{\sum_{n=0}^{\infty} (uv)^{-2n} b^{-n} (n!)^{-1} \prod_{i=1}^{n} (a+bi)^{-1}},$$

$$D_{3k} = \frac{(-1)^k \sum_{n=0}^{\infty} b^{-n}(n!)^{-1}(u^{-2n-k} v^{-2n-k-1} \prod_{i=1}^{n+k}(a+bi)^{-1} + u^{-2n-k-1} v^{-2n-k-2} \prod_{i=1}^{n+k+1}(a+bi)^{-1})}{\sum_{n=0}^{\infty} (uv)^{-2n} b^{-n} (n!)^{-1} \prod_{i=1}^{n} (a+bi)^{-1}}.$$

REMARK 23. The recurrence relation $D_{3k-2} = D_{3k-1} - D_{3k-3}$ gives a formula for D_{3k-2}.

6. Proof of Theorem 14 and Corollaries 21 and 26

We shall prove Theorem 14 by induction. Other theorems can be proven similarly. Let

$$\begin{aligned}
\alpha &= [0; ua, v(a+b), u(a+2b), v(a+3b), u(a+4b), v(a+5b), \ldots] \\
&= \frac{\sum_{n=0}^{\infty} (n!)^{-1} u^{-n-1} (vb)^{-n} \prod_{i=1}^{n} (a+bi)^{-1}}{\sum_{n=0}^{\infty} (n!)^{-1} (uvb)^{-n} \prod_{i=1}^{n-1} (a+bi)^{-1}} \\
&= \frac{\sum_{n=0}^{\infty} (n!)^{-1} u^{-n-1} (vb)^{-n} \prod_{i=1}^{n} (a+bi)^{-1}}{\sum_{n=0}^{\infty} (n!)^{-1} (uvb)^{-n} \prod_{i=1}^{n-1} (a+bi)^{-1}}.
\end{aligned}$$

The identities hold for D_0 and D_1 because

$$\begin{aligned}
D_0 &= q_0\alpha - p_0 = \alpha \\
&= \frac{\sum_{n=0}^{\infty} (n!)^{-1} u^{-n-1} (vb)^{-n} \prod_{i=1}^{n} (a+bi)^{-1}}{\sum_{n=0}^{\infty} (n!)^{-1} (uvb)^{-n} \prod_{i=1}^{n-1} (a+bi)^{-1}}
\end{aligned}$$

and

$$\begin{aligned}
D_1 &= q_1\alpha - p_1 = ua\alpha - 1 \\
&= \frac{(ua \sum_{n=0}^{\infty} (n!)^{-1} u^{-n-1} (vb)^{-n} - \sum_{n=0}^{\infty} (n!)^{-1} (uvb)^{-n} (a+bn)) \prod_{i=1}^{n-1} (a+bi)^{-1}}{\sum_{n=0}^{\infty} (n!)^{-1} (uvb)^{-n} \prod_{i=1}^{n-1} (a+bi)^{-1}} \\
&= -\frac{\sum_{n=1}^{\infty} ((n-1)!)^{-1} (uv)^{-n} b^{-n+1} \prod_{i=1}^{n} (a+bi)^{-1}}{\sum_{n=0}^{\infty} (n!)^{-1} (uvb)^{-n} \prod_{i=1}^{n-1} (a+bi)^{-1}} \\
&= -\frac{\sum_{n=0}^{\infty} (n!)^{-1} (uv)^{-n-1} b^{-n} \prod_{i=1}^{n+1} (a+bi)^{-1}}{\sum_{n=0}^{\infty} (n!)^{-1} (uvb)^{-n} \prod_{i=1}^{n-1} (a+bi)^{-1}}.
\end{aligned}$$

Next, we shall prove Corollary 21. Put $a = -1$, $b = 2$, $u = s$ and $v = 2$ in Theorem 19. Since

$$\prod_{i=1}^{n} (a+bi)^{-1} = \prod_{i=1}^{n} (2i-1)^{-1} = \frac{2^n n!}{(2n)!},$$

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we obtain

$$\sum_{n=0}^{\infty} b^{-n}(n!)^{-1}\Big((uv)^{-2n}\prod_{i=1}^{n}(a+bi)^{-1}-(uv)^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1}\Big)$$
$$=\sum_{n=0}^{\infty}\frac{1}{(2s)^{2n}(2n)!}-\sum_{n=0}^{\infty}\frac{1}{(2s)^{2n+1}(2n+1)!}$$
$$=\cosh\Big(\frac{1}{2s}\Big)-\sinh\Big(\frac{1}{2s}\Big)=e^{-1/(2s)}.$$

On the other hand, the numerator of D_{3k-1} is equal to

$$\frac{(-1)^k}{s^k}\sum_{n=0}^{\infty}\frac{(n+k)!}{n!}\Big(\frac{1}{(2s)^{2n+1}(2n+2k+1)!}-\frac{1}{(2s)^{2n}(2n+2k)!}\Big).$$

The numerator of D_{3k} is equal to

$$\frac{(-1)^k}{s^{k+1}}\sum_{n=0}^{\infty}\frac{(n+k)!}{(2s)^{2n}n!(2n+2k+1)!}.$$

7. Diophantine approximations in terms of integrals

In [12, Theorem 1], for the continued fraction

$$\sqrt{\frac{v}{u}}\tanh\frac{1}{\sqrt{uv}}=[0;\overline{(4k-3)u,(4k-1)v}\,]_{k=1}^{\infty},$$

we have

$$D_{2k-1}=\frac{1}{e^{2/\sqrt{uv}}+1}\Big(\frac{4}{uv}\Big)^k\int_0^1\frac{x^{2k-1}(x-1)^{2k-1}}{(2k-1)!}e^{2x/\sqrt{uv}}\,dx,$$
$$D_{2k}=\frac{2}{(e^{2/\sqrt{uv}}+1)u}\Big(\frac{4}{uv}\Big)^k\int_0^1\frac{x^{2k}(x-1)^{2k}}{(2k)!}e^{2x/\sqrt{uv}}\,dx.$$

Together with the identities in Corollary 15 we get the following corollary.

Corollary 24.

$$(e^{2/\sqrt{uv}}+1)\frac{{}_0F_1(;\frac{1}{2}+2k;\frac{1}{4uv})}{{}_0F_1(;\frac{1}{2};\frac{1}{4uv})}$$
$$=-2^{2k}(4k-1)!!\int_0^1\frac{x^{2k-1}(x-1)^{2k-1}}{(2k-1)!}e^{2x/\sqrt{uv}}\,dx,$$

$$(e^{2/\sqrt{uv}}+1)\frac{{}_0F_1(;\frac{3}{2}+2k;\frac{1}{4uv})}{{}_0F_1(;\frac{1}{2};\frac{1}{4uv})}$$
$$=2^{2k+1}(4k+1)!!\int_0^1\frac{x^{2k}(x-1)^{2k}}{(2k)!}e^{2x/\sqrt{uv}}\,dx.$$

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In [12, Theorem 2], for the continued fraction

$$\sqrt{\frac{v}{u}}\tan\frac{1}{\sqrt{uv}} = [0; u-1, \overline{1, (4k-3)v-2, 1, (4k+1)u-2, 1}]_{k=1}^{\infty},$$

we have

$$D_{4k-2} = -\frac{1}{e^{2\sqrt{-1}/\sqrt{uv}}+1}\Big(\frac{4}{uv}\Big)^k \int_0^1 \frac{x^{2k-1}(x-1)^{2k-1}}{(2k-1)!} e^{2\sqrt{-1}x/\sqrt{uv}}\,dx,$$

$$D_{4k} = \frac{2}{(e^{2\sqrt{-1}/\sqrt{uv}}+1)u}\Big(\frac{4}{uv}\Big)^k \int_0^1 \frac{x^{2k}(x-1)^{2k}}{(2k)!} e^{2\sqrt{-1}x/\sqrt{uv}}\,dx.$$

Together with the identities in Corollary 18 we get the following corollary.

COROLLARY 25.

$$(e^{2\sqrt{-1}/\sqrt{uv}}+1)\frac{{}_0F_1(;\frac{1}{2}+2k;\frac{-1}{4uv})}{{}_0F_1(;\frac{1}{2};\frac{-1}{4uv})} = -2^{2k}(4k-1)!!\int_0^1 \frac{x^{2k-1}(x-1)^{2k-1}}{(2k-1)!} e^{2\sqrt{-1}x/\sqrt{uv}}\,dx,$$

$$(e^{2\sqrt{-1}/\sqrt{uv}}+1)\frac{{}_0F_1(;\frac{3}{2}+2k;\frac{-1}{4uv})}{{}_0F_1(;\frac{1}{2};\frac{-1}{4uv})} = 2^{2k+1}(4k+1)!!\int_0^1 \frac{x^{2k}(x-1)^{2k}}{(2k)!} e^{2\sqrt{-1}x/\sqrt{uv}}\,dx.$$

In [11, 15], for the continued fraction

$$e^{1/s} = [1; \overline{(2k-1)s-1, 1, 1}]_{k=1}^{\infty},$$

we have

$$D_{3k-1} = -\frac{1}{s^k}\int_0^1 \frac{x^{k-1}(x-1)^k}{(k-1)!} e^{x/s}\,dx,$$

$$D_{3k} = \frac{1}{s^{k+1}}\int_0^1 \frac{x^k(x-1)^k}{k!} e^{x/s}\,dx.$$

Together with the identities in Corollary 21 we get the following corollary.

COROLLARY 26.

$$e^{1/(2s)}\sum_{n=0}^{\infty}\frac{(n+k)!}{n!}\Big(\frac{1}{(2s)^{2n+1}(2n+2k+1)!} - \frac{1}{(2s)^{2n}(2n+2k)!}\Big) = (-1)^{k+1}\int_0^1 \frac{x^{k-1}(x-1)^k}{(k-1)!} e^{x/s}\,dx,$$

$$e^{1/(2s)}\sum_{n=0}^{\infty}\frac{(n+k)!}{(2s)^{2n}n!(2n+2k+1)!} = (-1)^k\int_0^1 \frac{x^k(x-1)^k}{k!} e^{x/s}\,dx.$$

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8. Some additional comments

The identities in Theorems 1 to 10 may be obtained in indirect ways. For example, let p_n/q_n be the nth convergent of the Tasoev continued fraction $\alpha = [0; \overline{ua^{2k-1}, va^{2k}}]_{k=1}^{\infty}$ in Theorem 1. Then we know that for all $n \ge 1$,

$$p_{2n-1} = \sum_{\nu=0}^{n-1} (uv)^{\nu} a^{\nu(2\nu+3)} \prod_{i=1}^{n-\nu-1} \frac{a^{2(2\nu+i)}-1}{a^{2i}-1},$$

$$p_{2n} = \sum_{\nu=0}^{n-1} u^{\nu} v^{\nu+1} a^{(\nu+2)(2\nu+1)} \prod_{i=1}^{n-\nu-1} \frac{a^{2(2\nu+i+1)}-1}{a^{2i}-1},$$

$$q_{2n-1} = \sum_{\nu=0}^{n-1} u^{\nu+1} v^{\nu} a^{(\nu+1)(2\nu+1)} \prod_{i=1}^{n-\nu-1} \frac{a^{2(2\nu+i+1)}-1}{a^{2i}-1},$$

$$q_{2n} = \sum_{\nu=0}^{n} (uv)^{\nu} a^{\nu(2\nu+1)} \prod_{i=1}^{n-\nu} \frac{a^{2(2\nu+i)}-1}{a^{2i}-1}.$$

(These identities are also proven by induction.) After some manipulations of series,

$$(q_{2k-1}\alpha - p_{2k-1}) \sum_{n=0}^{\infty} (uv)^{-n} a^{-n^2} \prod_{i=1}^{n} (a^{2i}-1)^{-1}$$
$$= -\sum_{n=0}^{\infty} (uv)^{-n-k} a^{-n^2-4kn-k(2k+1)} \prod_{i=1}^{n} (a^{2i}-1)^{-1}$$

and

$$(q_{2k}\alpha - p_{2k}) \sum_{n=0}^{\infty} (uv)^{-n} a^{-n^2} \prod_{i=1}^{n} (a^{2i}-1)^{-1}$$
$$= \sum_{n=0}^{\infty} u^{-n-k-1} v^{-n-k} a^{-n^2-(4k+2)n-(k+1)(2k+1)} \prod_{i=1}^{n} (a^{2i}-1)^{-1}.$$

But these manipulations are not easy.

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