Ergod. Th. & Dynam. Sys. (2021), **41**, 48–65
doi:10.1017/etds.2019.49

The C^0 integrability of symplectic twist maps without conjugate points

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(*Received* 11 *March* 2019 *and accepted in revised form* 17 *June* 2019)

Abstract. It is proved that a symplectic twist map of the cotangent bundle $T^*\mathbb{T}^d$ of the d-dimensional torus that is without conjugate points is C^0-integrable, that is $T^*\mathbb{T}^d$ is foliated by a family of invariant C^0 Lagrangian graphs.

Key words: symplectic twist map, complete integrability, weak KAM theory
2010 Mathematics Subject Classification: 37C25, 37E40, 37J35 (Primary); 37J45 (Secondary)

Let $d \geq 1$ be an integer, $\mathbb{T}^d$ the d-dimensional torus, and

$$F : T^*\mathbb{T}^d \longrightarrow T^*\mathbb{T}^d$$

a twist map. These maps will be defined precisely in part 1; they are examples of symplectic diffeomorphisms of $T^*\mathbb{T}^d$. They are important because they represent (via a symplectic change of coordinates) the dynamics of a generic symplectic diffeomorphism of $\mathbb{R}^{2d}$ near its elliptic periodic points (see for example [**Go**, p. 93]). In this article, we assume that F is without conjugate points, which means that

$$\text{for all } (x, p) \in T^*\mathbb{T}^d, \text{ for all } n \in \mathbb{Z}^*, \quad V(F^n(x, p)) \cap DF^n(x, p) \cdot V(x, p) = \{0\},$$

where $V(x, p)$ denotes the vertical space at point (x, p). This is a strong assumption, and we study here its consequences on the dynamics of F.

We first describe the periodic orbits of F. To state our result precisely, let

$$\overline{F} : T^*\mathbb{R}^d \longrightarrow T^*\mathbb{R}^d$$

be a lift of F to $T^*\mathbb{R}^d$ (identified with $\mathbb{R}^d \times (\mathbb{R}^d)^*$). If $\omega \in T^*\mathbb{T}^d$ is a periodic point of F with period $N \in \mathbb{N}^*$ and $\overline{\omega} = (x, p) \in \mathbb{R}^d \times (\mathbb{R}^d)^*$ a lift of ω, then for some $r \in \mathbb{Z}^d$ we have

$$\overline{F}^N(x, p) = (x + r, p).$$

Reciprocally, if this equality holds and ω is the projection of (x, p) on $T^*\mathbb{T}^d$, then $F^N(\omega) = \omega$, which means that ω is a periodic point of F, and N is a period of the orbit. So we may see the following result as a way to describe the periodic orbits of F.

THEOREM 1. *Let F be a twist map without conjugate points. For every $N \in \mathbb{N}^*$, for every $r \in \mathbb{Z}^d$, for every $x \in \mathbb{R}^d$, there is a unique $p \in (\mathbb{R}^d)^*$ such that $\overline{F}^N(x, p) = (x + r, p)$.*

Let x be a point on the torus $\mathbb{T}^d$. As a consequence of Theorem 1, F has a countable number of periodic orbits intersecting $T_x^*\mathbb{T}^d$. Each of them is determined by an integer $N \geq 1$ (which is a period of the orbit) and a vector $r \in \mathbb{Z}^d$ (we may call it the homotopy class of the orbit).

We prove that if we fix N and r and let x vary in $\mathbb{T}^d$, the set

$$\overline{\mathcal{G}}_{N,r} = \{(x, p) \in \mathbb{T}^d \times (\mathbb{R}^d)^* \text{ such that } \overline{F}^N(x, p) = (x + r, p)\}$$

is a lift to $T^*\mathbb{R}^d$ of an invariant Lagrangian submanifold $\mathcal{G}_{N,r}$ of $T^*\mathbb{T}^d$. So this gives rise to a sequence of Lagrangian submanifolds, each of them being a union of periodic orbits of F. It is natural to wonder if we can find other invariant Lagrangian submanifolds of $T^*\mathbb{T}^d$. It was suggested by Bialy in [**Bi**] that F is without conjugate points if and only $T^*\mathbb{T}^d$ may be written as the union of F-invariant Lagrangian graphs.

Cheng and Sun showed that this holds true when $d = 1$: they proved in [**Ch-Su**] that F is without conjugate points if and only if $T^*\mathbb{T}^1$ is foliated by continuous, closed, invariant curves that are not null-homotopic (a standard result due to Birkhoff states that these curves must be graphs over $\mathbb{T}^1$). Here we generalize their result in any dimension as follows:

THEOREM 2. *Let $F : T^*\mathbb{T}^d \longrightarrow T^*\mathbb{T}^d$ be a twist map. Then F is without conjugate points if and only if there is a continuous foliation of $T^*\mathbb{T}^d$ by Lipschitz, Lagrangian invariant graphs.*

The techniques used by Cheng and Sun do not carry over to the higher dimensional case. Our proof uses ideas coming from weak KAM and Aubry-Mather theory: each leaf of the foliation is a dual Aubry set associated to some cohomology class. This strategy was already used in [**AABZ**], where it is shown that a similar result holds for a class of Hamiltonian flows. Note that in the case of a geodesic flow of a Riemannian metric on the torus, this result was proved in 1994 by Heber (see [**He**]). However, many arguments used in the continuous setting have no analogue in the discrete case. For example, there is no useful numerical quantity (as the Hamiltonian in the continuous case) which is constant along the orbit of F.

This article is organized as follows: we recall some basic facts in §1. The proof of Theorem 1 is given in §2. In the four following sections, we show that the sets $\mathcal{G}_{N,r}$ are invariant Lagrangian graphs of $T^*\mathbb{T}^d$ as well as dual Aubry sets associated to a cohomology class. Finally, we give a proof of Theorem 2 in the last two sections.

1. *Twist maps without conjugate points*

Let $d \geq 1$ be an integer. Denote by $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ the d-dimensional torus. Let $\mathcal{T} = \mathbb{R}^d \times \mathbb{R}^d$ and let $\mathcal{T}^* = \mathbb{R}^d \times (\mathbb{R}^d)^*$ be the cotangent space of $\mathbb{R}^d$. Consider a generating function, that is a map $S : \mathcal{T} \longrightarrow \mathbb{R}$ of class C^2 which satisfies the following two conditions:

https://doi.org/10.1017/etds.2019.49 Published online by Cambridge University Press

(C1) for all $r \in \mathbb{Z}^d$, for all $(x, y) \in \mathcal{T}$, $S(x + r, y + r) = S(x, y)$;
(C2) ('uniform twist condition', see [**Bi-McK**]) There is a real number $A > 0$ for which

$$\text{for all } (x, y) \in \mathcal{T}, \text{ for all } \xi \in \mathbb{R}^d, \quad \sum_{i,j} \frac{\partial^2 S(x, y)}{\partial x_i \partial y_j}(x, y)\xi_i \xi_j \leq -A\|\xi\|^2.$$

We may then define various notions of action. For example, the action of a finite sequence $\gamma = (x_0, x_1, \ldots, x_n)$ with values in $\mathbb{R}^d$ is

$$S(\gamma) = S(x_0, x_1, \ldots, x_n) = \sum_{k=0}^{n-1} S(x_k, x_{k+1}).$$

If we fix an integer $n \geq 2$ and two points x_0 and x_n in $\mathbb{R}^d$, we can define the 'action with fixed endpoints' as the map

$$S_{x_0,x_n,n} : (x_1, \ldots x_{n-1}) \in (\mathbb{R}^d)^{n-1} \longmapsto S(x_0, x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}.$$

Its critical points are the finite sequences $(x_1, \ldots x_{n-1})$ for which

$$\text{for all } k \in \{1, \ldots, n-1\}, \quad \partial_2 S(x_{k-1}, x_k) + \partial_1 S(x_k, x_{k+1}) = 0.$$

The sequence $(x_0, x_1, \ldots, x_n)$ will be called a finite extremal sequence. An (infinite) sequence $(x_n)_{n\in\mathbb{Z}}$ with values in $\mathbb{R}^d$ is said to be extremal if it satisfies

$$\text{for all } n \in \mathbb{Z}, \quad \partial_2 S(x_{n-1}, x_n) + \partial_1 S(x_n, x_{n+1}) = 0.$$

Condition (C2) implies (see [**Go**], Ch. 4) that for every $x_0 \in \mathbb{R}^d$ and every $y_0 \in \mathbb{R}^d$, the maps

$$x \longmapsto \partial_2 S(x, y_0) \quad \text{and} \quad y \longmapsto \partial_1 S(x_0, y)$$

are diffeomorphisms. As an immediate consequence, every finite extremal sequence may be uniquely extended to an infinite extremal sequence. In particular, for every $(x, y) \in \mathcal{T}$, there is a unique extremal sequence $(x_n)_{n\in\mathbb{Z}}$ for which $x_0 = x$ and $x_1 = y$. We shall denote by

$$\sigma : (x, y) = (x_0, x_1) \in \mathcal{T} \longmapsto (x_1, x_2) \in \mathcal{T}$$

the corresponding shift diffeomorphism.

The generating function also gives rise to a symplectic diffeomorphism F of $T^*\mathbb{T}^d$, the cotangent bundle of $\mathbb{T}^d$. Let $\overline{F} : \mathcal{T}^* \longrightarrow \mathcal{T}^*$ be the diffeomorphism (twist map) implicitly defined by

$$\overline{F}(x, p) = (x', p') \Longleftrightarrow p = -\partial_1 S(x, x') \quad \text{and} \quad p' = \partial_2 S(x, x').$$

It turns out that $\overline{F}$ and σ are conjugated: the map

$$\mathcal{L} : (x, y) \in \mathcal{T} \longmapsto (x, -\partial_1 S(x, y)) \in \mathcal{T}^*$$

is a diffeomorphism for which $\overline{F} = \mathcal{L} \circ \sigma \circ \mathcal{L}^{-1}$. The diffeomorphism $\overline{F}$ is exact symplectic, which means that $\overline{F}^*\alpha - \alpha = dS$, where $\alpha = \sum_{i=1}^d x_i \, dq_i$ is the Liouville 1-form on $T^*\mathbb{R}^d$. Note that condition (C1) implies that $\overline{F}$ is the lift to $\mathcal{T}^*$ of a symplectic diffeomorphism F of $T^*\mathbb{T}^d$. In this article, we are interested in the dynamics of F, and we will use S as a useful tool for our study.

As a matter of fact, condition (C2) has strong consequences on the behaviour of S. For example, the following result may be shown (see [**Go**, p. 105] or [**McK-Me-St**, p. 568] for a proof).

LEMMA 1.1. *There exists $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $\gamma > 0$ such that*

$$\textit{for all } (x, y) \in \mathcal{T}, \quad S(x, y) \geq \alpha + \beta\|x - y\| + \gamma\|x - y\|^2.$$

As an immediate consequence, we can construct extremal sequences going through two arbitrary points in $\mathbb{R}^d$.

LEMMA 1.2. *For every $(x, y) \in \mathcal{T}$, for every integer $N \geq 1$, there exists an extremal sequence $(x_n)_{n\in\mathbb{Z}}$ for which $x_0 = x$ and $x_N = y$.*

Proof. We already know that this is the case when $N = 1$. When $N \geq 2$, it suffices to show that the map $S_{x,y,N}$ has a critical point. In view of Lemma 1.1, $S_{x,y,N}$ is coercive and therefore achieves its minimum at a point $(x_1, \ldots, x_{N-1})$. We then extend the finite extremal sequence $(x_0 = x, x_1, \ldots, x_{N-1}, x_N = y)$ to an infinite extremal sequence $(x_n)_{n\in\mathbb{Z}}$. □

Let us show that this extremal sequence is unique if we assume that F is without conjugate points. Let $\pi : (x, p) \in T^*\mathbb{T}^d \longmapsto x \in \mathbb{T}^d$ be the canonical projection. For every $(x, p) \in T^*\mathbb{T}^d$, the vertical space at (x, p) is

$$V(x, p) = \mathrm{Ker}(D\pi_{|T_{(x,p)}T^*\mathbb{T}^d}).$$

Definition 1.3. F is without conjugate points if

$$\text{for all } (x, p) \in T^*\mathbb{T}^d, \text{ for all } n \in \mathbb{Z}^*, \quad V(F^n(x, p)) \cap DF^n(x, p) \cdot V(x, p) = \{0\}.$$

PROPOSITION 1.4. *If F is without conjugate points, then for every $(x, y) \in \mathcal{T}$ and for every integer $N \geq 2$, the map $S_{x,y,N}$ has a unique critical point; and at that point, $S_{x,y,N}$ achieves its minimum.*

Proof. For the 'existence' part, we refer to the proof of Lemma 1.2. Now assume by contradiction that $S_{x,y,N}$ has (at least) two distinct critical points. It is shown in [**Bi-McK**] that if F is without conjugate points, then every critical point of $S_{x,y,N}$ is in fact a strict local minimum. $S_{x,y,N}$ is then a coercive C^2 map with two distinct strict local minima. We can apply an existence theorem for saddle points in finite dimension (see [**St**, Theorem 1.1, p. 74]). It says that $S_{x,y,N}$ possesses a third critical point which is not a local minimum of $S_{x,y,N}$. This is a contradiction. □

COROLLARY 1.5. *If F is without conjugate points, then for every $(x, y) \in \mathcal{T}$ and every integer $N \geq 1$, there is a unique extremal sequence $(x_n)_{n\in\mathbb{Z}}$ with $x_0 = x$ and $x_N = y$.*

Remark 1.6. Assume that F is without conjugate points. Let $(x_n)_{n\in\mathbb{Z}}$ be an extremal sequence, and let k and l be two integers with $l - k \geq 2$. It follows from Proposition 1.4 that

$$\text{for all } (y_{k+1}, \ldots, y_{l-1}) \in (\mathbb{R}^d)^{l-k-1},$$
$$S(x_k, x_{k+1}, \ldots, x_{l-1}, x_l) \leq S(x_k, y_{k+1}, \ldots, x_{k+1}, x_l).$$

Equality holds if and only if $y_i = x_i$ for every $i \in \{k + 1, \ldots, l - 1\}$. This means that an extremal sequence minimizes the action with fixed endpoints between any two of its points.

2. *Construction of periodic orbits*

In this section, we prove Theorem 1. Let us fix $r \in \mathbb{Z}^d$ and $N \in \mathbb{N}^*$. We have to show that for every $x \in \mathbb{R}^d$, there is a unique $p \in (\mathbb{R}^d)^*$ for which $\overline{F}^N(x, p) = (x + r, p)$. The change of variable $(x, y) = \mathcal{L}^{-1}(x, p)$ and the relation $\overline{F} = \mathcal{L} \circ \sigma \circ \mathcal{L}^{-1}$ lead to

$$\overline{F}^N(x, p) = (x + r, p) \Longleftrightarrow \sigma^N(x, y) = (x + r, y + r).$$

According to the results of the previous part, the only $y \in \mathbb{R}^d$ for which we may have $\sigma^N(x, y) = (x + r, y + r)$ is $y = x_1$, where $(x_n)_{n\in\mathbb{Z}}$ is the unique extremal sequence satisfying $x_0 = x$ and $x_N = x + r$. And for this y, we have $\sigma^N(x, y) = (x + r, y + r)$ if and only if $x_{N+1} = x_1 + r$. It turns out that this equality holds, as it is a special case of the following general result.

PROPOSITION 2.1. *Let $r \in \mathbb{Z}^d$ and $N \in \mathbb{N}^*$. If $(x_n)_{n\in\mathbb{Z}}$ is an extremal sequence such that $x_N = x_0 + r$, then $x_{n+N} = x_n + r$ for all $n \in \mathbb{Z}$.*

For the proof, we shall use a technique of metric geometry introduced by Busemann (see [**Bu**, §32]) when he was studying G-spaces without conjugate points.

For every $(x, y) \in \mathcal{T}$ and for every integer $N \in \mathbb{N}^*$, we denote by $\mathcal{A}_N(x, y)$ the minimum of the function $S_{x,y,N}$ when $N \geq 2$, and $S(x, y)$ if $N = 1$. As the minimum is attained at a single point, $\mathcal{A}_N$ is a continuous function. We clearly have $\mathcal{A}_N(x + r, y + r) = \mathcal{A}_N(x, y)$ for every $(x, y) \in \mathcal{T}$ and every $r \in \mathbb{Z}^d$.

LEMMA 2.2. *For every x, y, z in $\mathbb{R}^d$, for every N, N' in $\mathbb{N}^*$, the following triangular inequality holds:*

$$\mathcal{A}_{N+N'}(x, z) \leq \mathcal{A}_N(x, y) + \mathcal{A}_{N'}(y, z).$$

Moreover, one has equality if and only if $y = w_N$, where (w_n) is the extremal sequence for which $w_0 = x$ and $w_{N+N'} = z$.

Proof. Let (x_n) be the extremal sequence with $x_0 = x$ and $x_N = y$, and (y_n) the extremal sequence with $y_N = y$ and $y_{N+N'} = z$. So we have

$$\mathcal{A}_N(x, y) = S(x_0, x_1, \ldots, x_N) \quad \text{and} \quad \mathcal{A}_{N'}(y, z) = S(y_N, y_{N+1}, \ldots, y_{N+N'}).$$

Let (z_n) be the sequence defined by

$$z_n = \begin{cases} x_n & \text{for } n \leq N, \\ y_n & \text{for } n \geq N. \end{cases}$$

As we have $z_0 = x_0 = x$ and $z_{N+N'} = y_{N+N'} = z$, the definition of $\mathcal{A}_{N+N'}(x, z)$ implies that

$$\mathcal{A}_{N+N'}(x, z) \leq S(z_0, z_1, \ldots, z_{N+N'}) = S(x_0, x_1, \ldots, x_N) + S(y_N, y_{N+1}, \ldots, y_{N+N'}),$$

whence the inequality $\mathcal{A}_{N+N'}(x, z) \leq \mathcal{A}_N(x, y) + \mathcal{A}_{N'}(y, z)$.

If equality holds, then $S_{x,z,N+N'}$ achieves its minimum at $(z_1, z_2, \ldots, z_{N+N'-1})$. But $S_{x,z,N+N'}$ achieves its minimum at a unique point, namely $(w_1, w_2, \ldots, w_{N+N'-1})$. So we must have $z_N = w_N$, and therefore $y = w_N$. □

Consider the function

$$f : x \in \mathbb{R}^d \longmapsto \mathcal{A}_N(x, x + r) \in \mathbb{R}.$$

As f is continuous and $\mathbb{Z}^d$-periodic, there exist two points a and b in $\mathbb{R}^d$ with $f(a) = \min_{\mathbb{R}^d} f$ and $f(b) = \max_{\mathbb{R}^d} f$. We first establish Proposition 2.1 for a particular extremal sequence.

LEMMA 2.3. *The extremal sequence (x_n) for which $x_0 = b$ and $x_N = x_0 + r$ satisfies*

$$\text{for all } n \in \mathbb{Z}, \quad x_{n+N} = x_n + r.$$

Proof. Using the periodicity of $\mathcal{A}_{2N}$ and the triangular inequality, we get

$$\mathcal{A}_{2N}(x_0, x_{2N}) = \mathcal{A}_{2N}(x_0 + r, x_{2N} + r) \leq \mathcal{A}_N(x_0 + r, x_{2N}) + \mathcal{A}_N(x_{2N}, x_{2N} + r),$$

so that

$$\mathcal{A}_{2N}(x_0, x_{2N}) \leq \mathcal{A}_N(x_N, x_{2N}) + f(x_{2N}).$$

As extremal sequences are action-minimizing (see Remark 1.6), we also have

$$\mathcal{A}_{2N}(x_0, x_{2N}) = \mathcal{A}_N(x_0, x_N) + \mathcal{A}_N(x_N, x_{2N}) = f(b) + \mathcal{A}_N(x_N, x_{2N}),$$

so that the last inequality yields

$$\mathcal{A}_{2N}(x_0, x_{2N}) \leq \mathcal{A}_{2N}(x_0, x_{2N}) - f(b) + f(x_{2N}) \leq \mathcal{A}_{2N}(x_0, x_{2N}),$$

because f achieves its maximum at b. This implies that equality holds in all the previous inequalities. Lemma 2.2 then tells us that $x_{2N} = y_N$, where (y_n) is the unique extremal sequence with $y_0 = x_0 + r$ and $y_{2N} = x_{2N} + r$.

As the extremal sequences (y_n) and $(x_n + r)$ are equal at $n = 0$ and $n = 2N$, Corollary 1.5 implies that they are equal for all n. So we have $y_N = x_N + r$, and therefore $x_{2N} = y_N = x_N + r$. Now the two extremal sequences (x_{n+N}) and $(x_n + r)$ are equal at $n = 0$ and $n = N$, so they are equal. □

LEMMA 2.4. *The function f is constant.*

Proof. We only need to show that $\max_{\mathbb{R}^d} f = f(b) \leq f(a) = \min_{\mathbb{R}^d} f$. From the preceding lemma, we have $x_{nN} = x_0 + nr = b + nr$ for all integer n, so that

$$\text{for all } n \geq 1, \quad \mathcal{A}_{nN}(b, b + nr) = n\mathcal{A}_N(b, b + r) = nf(b).$$

On the other hand, the triangular inequality implies that for every $n \geq 3$,

$$\mathcal{A}_{nN}(b, b + nr) \leq \mathcal{A}_N(b, a + r) + \sum_{i=1}^{n-2} \mathcal{A}_N(a + ir, a + (i + 1)r) + \mathcal{A}_N(a + (n - 1)r, b + nr).$$

These two relations and the fact that $\mathcal{A}_N$ is $\mathbb{Z}^d$-invariant lead to

$$nf(b) \leq \mathcal{A}_N(b, a + r) + (n - 2)f(a) + \mathcal{A}_N(a, b + r).$$

When we divide by n and let n go to infinity, we obtain $f(b) \leq f(a)$. □

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As the function f achieves its maximum at every point, the conclusion of Lemma 2.3 holds for every $b \in \mathbb{R}^d$. This ends the proof of Proposition 2.1 and the proof of Theorem 1.

COROLLARY 2.5. *If F is without conjugate points, then we have:*
(i) *every constant sequence is an extremal sequence;*
(ii) *every extremal sequence is either injective or constant;*
(iii) *for every $r \in \mathbb{Z}^d$, the quantity $S(x, x + r)$ does not depend on x.*

Proof. Let $x \in \mathbb{R}^d$, and $(x_n)_{n\in\mathbb{Z}}$ the extremal sequence for which $x_0 = x_1 = x$. Using Proposition 2.1 with $N = 1$ and $r = 0$, we may conclude that (x_n) is a constant sequence, which proves (i). Let $(x_n)_{n\in\mathbb{Z}}$ be extremal and not injective. We may assume that $x_0 = x_N$ with $N \in \mathbb{N}^*$. The constant sequence equal to x_0 is extremal, so Corollary 1.5 tells us that (x_n) is a constant sequence, which proves (ii). For all $x \in \mathbb{R}^d$ and $r \in \mathbb{Z}^d$, we have $S(x, x + r) = \mathcal{A}_1(x, x + r) = f(x)$, and according to Lemma 2.4 this quantity does not depend on x, which proves (iii). □

3. *The Green bundles*

In this section, we introduce two Lagrangian bundles that we will make use of later. In the context of twist maps, they were first studied in [**Bi-McK**]. We fix some set of symplectic coordinates $(x_1, \ldots, x_d, p_1, \ldots, p_d)$ on $T^*\mathbb{T}^d$. Every Lagrangian vector space $\mathcal{L} \subset T_{(x,p)}T^*\mathbb{T}^d$ transverse to $V(x, p)$ is then the graph of a symmetric matrix S, with $\mathcal{L} = \{v \text{ such that } dp(v) = S dx(v)\}$. So there is a partial ordering between these vector spaces: if $\mathcal{L}_1$ and $\mathcal{L}_2$ are two of them, we say that $\mathcal{L}_2$ is above $\mathcal{L}_1$ ($\mathcal{L}_1 \le \mathcal{L}_2$) if the symmetric matrix $S_2 - S_1$ is non-negative; and $\mathcal{L}_2$ is strictly above $\mathcal{L}_1$ ($\mathcal{L}_1 < \mathcal{L}_2$) if $S_2 - S_1$ is positive definite.

For every $(x, p) \in T^*\mathbb{T}^d$ and for every integer $n \neq 0$, we define

$$G_n(x, p) = DF^n(V(F^{-n}(x, p))) \subset T_{(x,p)}T^*\mathbb{T}^d.$$

This is a family of Lagrangian vector spaces, all of them being transverse to the vertical $V(x, p)$ since F does not have conjugate points. It is proved in [**Bi-McK**] (see [**Ar**], Proposition 7 for a slightly different proof) that for all integers $n \ge 2$ and $k \ge 2$, one has

$$G_{-1}(x, p) < G_{-k}(x, p) < G_{-(k+1)}(x, p) < G_{n+1}(x, p) < G_n(x, p) < G_1(x, p).$$

This implies that the sequences $(G_n(x, p))_{n\ge1}$ and $(G_{-n}(x, p))_{n\ge1}$ converge. Their limits

$$G_+(x, p) = \lim_{n\to\infty} G_n(x, p) \quad \text{and} \quad G_-(x, p) = \lim_{n\to-\infty} G_n(x, p)$$

are two Lagrangian vector spaces, called the positive and negative Green subspaces at (x, p). They are transverse to the vertical, with $G_-(x, p) \le G_+(x, p)$. By letting (x, p) vary over $T^*\mathbb{T}^d$, we obtain two (measurable) Lagrangian bundles. Both of them are clearly F-invariant.

The following dynamical criterion will be used in the next section to establish that some vectors tangent to $T^*\mathbb{T}^d$ belong to the Green bundles. Both the statement and the proof are due to Marie-Claude Arnaud.

PROPOSITION 3.1. *Endow $T^*\mathbb{T}^d$ with a Riemannian metric. Assume that the orbit of (x, p) is relatively compact. Then, for every $v \in T_{(x,p)}T^*\mathbb{T}^d$,*

$$v \notin G_-(x, p) \implies \lim_{n\to-\infty} \|D_{(x,p)}(\pi \circ F^n)(v)\| = +\infty$$

and

$$v \notin G_+(x, p) \implies \lim_{n\to\infty} \|D_{(x,p)}(\pi \circ F^n)(v)\| = +\infty.$$

Proof. For any integer k, let $(x_k, p_k) = F^k(x, p)$. Equip each $T_{(x_k,p_k)}T^*\mathbb{T}^d$ with the standard symplectic basis $\mathcal{B} = (e_1, \ldots, e_d, f_1, \ldots, f_d)$ induced by the coordinates. The vector space spanned by $(e_1, \ldots, e_d)$ is called the horizontal space. For any integer k, the matrix representation of $D_{(x,p)}F^n$ with respect to the basis $\mathcal{B}$ is

$$M_n(x, p) = \begin{pmatrix} A_n(x, p) & B_n(x, p) \\ C_n(x, p) & D_n(x, p) \end{pmatrix},$$

where A_n, B_n, C_n and D_n are square matrices of order d. Let $S_n(x_k, p_k)$ be the symmetric matrix associated to the Lagrangian vector space $G_n(x_k, p_k)$. Since $DF^n(x, p)(V(x, p)) = G_n(x_n, p_n)$ and $DF^n(x, p)(G_{-n}(x, p)) = V(x_n, p_n)$, we get

$$D_n(x, p) = S_n(x_n, p_n)B_n(x, p) \quad \text{and} \quad A_n(x, p) = -B_n(x, p)S_{-n}(x, p)$$

and therefore

$$M_n(x, p) = \begin{pmatrix} -B_n(x, p)S_{-n}(x, p) & B_n(x, p) \\ C_n(x, p) & S_n(x_n, p_n)B_n(x, p) \end{pmatrix}.$$

Let us consider another symplectic basis: $\mathcal{B}' = (e'_1, \ldots, e'_d, f_1, \ldots, f_d)$ in $T_{(x_k,p_k)}T^*\mathbb{T}^d$, with $e'_i = e_i + S_-(x_k, p_k)e_i$ for all i, so that the horizontal space spanned by $(e'_1, \ldots, e'_d)$ is now $G_-(x_k, p_k)$. In this new basis, the matrix of $D_{(x,p)}F^n$ becomes

$$M'_n(x, p) = \begin{pmatrix} I_d & O_d \\ -S_-(x_n, p_n) & I_d \end{pmatrix} \times \begin{pmatrix} -B_n(x_n, p_n)S_{-n}(x, p) & B_n(x, p) \\ C_n(x, p) & S_n(x_n, p_n)B_n(x, p) \end{pmatrix} \times \begin{pmatrix} I_d & O_d \\ S_-(x, p) & I_d \end{pmatrix}$$

and therefore

$$M'_n(x, p) = \begin{pmatrix} B_n(x, p)[S_-(x, p) - S_{-n}(x, p)] & B_n(x, p) \\ O_d & [S_n(x_n, p_n) - S_-(x_n, p_n)]B_n(x, p) \end{pmatrix}.$$

Take $v \in T_{(x,p)}T^*\mathbb{T}^d$ and write $v = v_1 + v_2$ with $v_1 \in G_-(x, p)$ and $v_2 \in V(x, p)$. We assume $v \notin G_-(x, p)$, so that $v_2 \neq 0$. Let $v'_n = DF^n(x, p) \cdot v$ and write $v'_n = v'_{n,1} + v'_{n,2}$, with $v'_{n,1} \in G_-(x_n, p_n)$ and $v'_{n,2} \in V(x_n, p_n)$. Then $D\pi(v'_n) = D\pi(v'_{n,1})$, with

$$v'_{n,1} = B_n(x, p)[(S_-(x, p) - S_{-n}(x, p)) \cdot v_1 + v_2].$$

Since $G_{-1}(x_n, p_n) \le G_-(x_n, p_n) \le G_1(x_n, p_n)$ for all n and the (x_n, p_n) remain in a compact set, there exists a constant $C > 0$ such that $\|D\pi(v'_{n,1})\| \ge C\|v'_{n,1}\|$ for all n. Hence

$$\|D\pi(v'_{n,1})\| \ge C\|v'_{n,1}\| \ge C\, N_n(x, p)\, \|v_2 + \varepsilon_n\|,$$

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where $\varepsilon_n = (S_-(x, p) - S_{-n}(x, p)) \cdot v_1$ is a vector whose norm converges to 0 and $N_n(x, p)$ is the conorm of $B_n(x, p)$. So it only remains to show that $N_n(x, p) \to \infty$ when $n \to \infty$.

The matrix of $DF^n(x, p)$ being symplectic, we have

$$ {}^t[(S_k(x_k) - S_-(x_k))B_k]B_k(S_- - S_{-k}) = I_d, $$

and therefore

$$ (S_- - S_{-k})^t B_k(S_k(x_k) - S_-(x_k))B_k = I_d, $$

so that

$$ {}^tB_k(S_k(x_k) - S_-(x_k))B_k = (S_- - S_{-k})^{-1}. $$

It follows that for every $v \in \mathbb{R}^d$, one has

$$ \langle (S_k(x_k) - S_-(x_k))B_k v, B_k v\rangle = \langle (S_- - S_{-k})^{-1} v, v\rangle. \qquad (*) $$

Let M_k be the smallest eigenvalue of $(S_- - S_{-k})^{-1}$. It goes to infinity with k because $S_- - S_{-k}$ is positive definite and converges to O_d. So the right-hand side of $(*)$ is $\geq M_k\|v\|^2$. Now using that $S_k(x_k) - S_-(x_k) \leq S_1(x_k) - S_{-1}(x_k)$, the left-hand side of $(*)$ is $\leq \langle [S_1(x_k) - S_{-1}(x_k)]B_k(v), B_k(v)\rangle$. As the orbit is precompact, $S_1(x_k) - S_{-1}(x_k) \leq S$ for some fixed symmetric positive definite matrix S, and therefore there is an $M > 0$ (independent of k) such that the left-hand side is $\leq M\|B_k(v)\|^2$. Combining the two inequalities we obtain $\|B_k(v)\|^2 \geq (M_k/M)\|v\|^2$, with $M_k \to \infty$. This implies the result on the conorm. □

4. *Some invariant Lagrangian submanifolds of $T^*\mathbb{T}^d$*

In this section, we shall see how the translation-invariant orbits of $\overline{F}$ may be used to construct invariant Lagrangian graphs in $T^*\mathbb{T}^d$. We first introduce some notations. For every $r \in \mathbb{Z}^d$ and every $N \in \mathbb{N}^*$, we consider the following sets:

$$ \overline{\mathcal{G}}_{N,r} = \{(x, y) \in \mathcal{T},\ x \in \mathbb{R}^d \text{ and } \sigma^N(x, y) = (x + r, y + r)\} \quad \text{and} \quad \overline{\mathcal{G}}^*_{N,r} = \mathcal{L}(\overline{\mathcal{G}}_{N,r}). $$

They are closely related to the extremal sequences studied in the preceding section. As a matter of fact, if $(x, y) \in \overline{\mathcal{G}}_{N,r}$, then the extremal sequence $(x_n)_{n\in\mathbb{Z}}$ for which $x_0 = x$ and $x_1 = y$ satisfies $x_N = x_0 + r$ (and hence $x_{n+N} = x_n + r$ for every n by Proposition 2.1). Reciprocally, if (x_n) is an extremal sequence for which $x_N = x_0 + r$, then $(x_0, x_1) \in \overline{\mathcal{G}}_{N,r}$. As for $\overline{\mathcal{G}}^*_{N,r}$, it contains all the $(x, p) \in \mathbb{T}^d \times (\mathbb{R}^d)^*$ given by Theorem 1 if we fix N and r and let x vary in $\mathbb{R}^d$.

According to the results of the last section, there exists for every $x \in \mathbb{R}^d$ a unique $y \in \mathbb{R}^d$ for which $(x, y) \in \overline{\mathcal{G}}_{N,r}$. This implies that $\overline{\mathcal{G}}_{N,r}$ (and hence $\overline{\mathcal{G}}^*_{N,r}$ as well) is a graph. Moreover $\overline{\mathcal{G}}_{N,r}$ is clearly invariant by σ, whereas $\overline{\mathcal{G}}^*_{N,r}$ is invariant by $\overline{F}$. Note that as a consequence of Corollary 2.5, $\overline{\mathcal{G}}_{N,0} = \{(x, x), x \in \mathbb{R}^d\}$.

Now consider $\mathcal{G}^*_{N,r}$, the projection of $\overline{\mathcal{G}}^*_{N,r}$ on $T^*\mathbb{T}^d$. It turns out that this set has many interesting properties:

PROPOSITION 4.1. *The set $\mathcal{G}^*_{N,r}$ satisfies:*

(i) *it is a graph which is F-invariant;*

(ii) *for all $\omega \in \mathcal{G}^*_{N,r}$, $F^N(\omega) = \omega$;*

(iii) *it is a Lagrangian submanifold of $T^*\mathbb{T}^d$.*

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Proof. Let $x \in \mathbb{R}^d$ and let be p the unique element of $(\mathbb{R}^d)^*$ for which $(x, p) \in \overline{\mathcal{G}}^*_{N,r}$. Condition (C1) implies that if $\overline{F}(x, p) = (x', p')$, then $\overline{F}(x + s, p) = (x' + s, p')$ for every $s \in \mathbb{Z}^d$. Therefore $(x + s, p) \in \overline{\mathcal{G}}^*_{N,r}$ for every $s \in \mathbb{Z}^d$. They all have the same projection on $T^*\mathbb{T}^d$, so $\mathcal{G}^*_{N,r}$ is a graph. It is F-invariant as $\overline{\mathcal{G}}^*_{N,r}$ is $\overline{F}$-invariant. This proves (i). It follows from the definitions that if $(x, p) \in \overline{\mathcal{G}}^*_{N,r}$, then $\overline{F}^N(x, p) = (x + r, p)$. This readily implies property (ii).

We now prove that $\overline{\mathcal{G}}^*_{N,r}$ (and therefore $\mathcal{G}^*_{N,r}$ as well) is a smooth manifold. The main difficulty is to check that we can apply the implicit function theorem to

$$\mathcal{F} : (x, p) \in \mathcal{T}^* \longmapsto \pi(\overline{F}^N(x, p)) - (x + r) \in \mathbb{R}^d.$$

This will imply that the map sending $x \in \mathbb{R}^d$ to the unique $p \in (\mathbb{R}^d)^*$ for which $(x, p) \in \mathcal{T}^*$ is smooth, and hence the smoothness of $\overline{\mathcal{G}}^*_{N,r}$.

So all we need to do is to verify that at every point in $\mathcal{T}^*$, the differential of $\mathcal{F}$ with respect to p is invertible. Let $(x_0, p_0) \in \mathcal{T}^*$, $(x_1, p_1) = \overline{F}^N(x_0, p_0)$, and $x_2 = \mathcal{F}(x_0, p_0)$. Let

$$i : p \in (\mathbb{R}^d)^* \longmapsto (x_0, p) \in \mathcal{T}^*$$

be the canonical injection. The differential of $\mathcal{F}$ with respect to p at the point (x_0, p_0) is

$$D_p\mathcal{F}(x_0, p_0) : v \in T_{p_0}(\mathbb{R}^d)^* \longmapsto D\pi(x_1, p_1) \circ D\overline{F}^N(x_0, p_0) \circ Di(p_0) \cdot v \in T_{x_2}\mathbb{R}^d.$$

Let v belong to the kernel of $D_p\mathcal{F}(x_0, p_0)$. Then $D\overline{F}^N(x_0, p_0) \circ Di(p_0) \cdot v \in V(x_1, p_1)$. As $Di(p_0) \cdot v \in V(x_0, p_0)$, we have $Di(p_0) \cdot v = 0$ (because F is without conjugate points), hence $v = 0$.

We finally show that $\mathcal{G}^*_{N,r}$ is Lagrangian. Let $(x, p) \in \mathcal{G}^*_{N,r}$ and $v \in T_{(x,p)}\mathcal{G}^*_{N,r}$. As a consequence of (ii), the restriction of F^N to $\mathcal{G}^*_{N,r}$ is the identity map, and the same is true for all F^{-nN} if $n \in \mathbb{Z}$. Passing to the differential, we get $DF^{-nN}(x, p) \cdot v = v$, and hence $D(\pi \circ F^{-nN})(x, p) \cdot v = D\pi(x, p) \cdot v$ is of constant norm. Using Proposition 3.1, this implies that $v \in G_{(x,p)}$. Hence $T_{(x,p)}\mathcal{G}^*_{N,r} \subset G_{(x,p)}$, and these two vector spaces have the same dimension, so they coincide. □

5. *Some results in discrete weak KAM theory*

Weak KAM theory was initially developed by Mather, Mané and Fathi to study the dynamics of some special Hamiltonian flows. This theory was adapted to the twist maps by Garibaldi and Thieullen. We briefly recall the facts we shall make use of in the rest of this paper. We refer to [**Ga-Th**] for the proofs.

To every generating function S one can associate a real $\widetilde{S}$ (called 'minimizing holonomic value'). It is defined as

$$\widetilde{S} = \operatorname{Inf}\left\{\liminf_{n\to+\infty} \frac{1}{n} S(x_0, \ldots, x_n)\right\},$$

with the infimum taken over all sequences $(x_n)_{n\in\mathbb{N}}$ with values in $\mathbb{R}^d$. One also has

$$\widetilde{S} = \operatorname{Inf}_{n\geq 1}\left\{\frac{1}{n} S(x_0, \ldots, x_n), x_0, \ldots, x_n \in \mathbb{R}^d \text{ with } x_n - x_0 \in \mathbb{Z}^d\right\}.$$

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One usually normalizes the generating function, using $S - \widetilde{S}$ instead of S. The action of a finite sequence $(x_0, \ldots x_n)$ is then

$$\widetilde{S}(x_0, \ldots x_n) = S(x_0, \ldots, x_n) - n\widetilde{S}.$$

Let us note that we now have $\widetilde{S}(x_0, \ldots, x_n) \geq 0$ as soon as $x_n - x_0 \in \mathbb{Z}^d$, and $\widetilde{S}$ is the smallest real number with this property.

The Mané potential is a function $\phi : \mathcal{T} \longrightarrow \mathbb{R}$ defined as follows: for every $(x, y) \in \mathcal{T}$,

$$\phi(x, y) = \mathrm{Inf}_{n\geq 1}\{\widetilde{S}(x_0, \ldots, x_n),\ x_0, \ldots, x_n \in \mathbb{R}^d \text{ with } x_0 = x \text{ and } x_n - y \in \mathbb{Z}^d\}.$$

It is a continuous function. It is $\mathbb{Z}^d$-periodic with respect to each variable. It satisfies the triangular inequality $\phi(x, z) \leq \phi(x, y) + \phi(y, z)$.

A function $u : \mathbb{R}^d \longrightarrow \mathbb{R}$ is called a sub-action if it is $\mathbb{Z}^d$-periodic and if

$$\text{for all } x \in \mathbb{R}^d, \text{ for all } y \in \mathbb{R}^d, \quad u(y) - u(x) \leq \phi(x, y).$$

As a consequence of the triangular inequality for ϕ, the maps $\phi(x_0, \cdot)$ and $-\phi(\cdot, x_0)$ are sub-actions for every $x_0 \in \mathbb{R}^d$.

One can associate to S a subset $\overline{\mathcal{A}}$ of $\mathcal{T}$ called the Aubry set : $(x, y) \in \mathcal{T}$ belongs to $\overline{\mathcal{A}}$ if for every $\varepsilon > 0$ there exists an integer $n \geq 1$ and a finite sequence $(x_0, x_1, \ldots, x_n)$ with values in $\mathbb{R}^d$ for which

$$x_n - x_0 \in \mathbb{Z}^d, \quad \|x - x_0\| < \varepsilon, \quad \|y - x_1\| < \varepsilon, \quad \text{and} \quad \widetilde{S}(x_0, x_1 \ldots, x_n) < \varepsilon.$$

The Aubry set is non-empty and closed. It is invariant by the action of $\mathbb{Z}^d$: if $(x, y) \in \overline{\mathcal{A}}$, then $(x + r, y + r) \in \overline{\mathcal{A}}$ for all $r \in \mathbb{Z}^d$. It is also invariant by σ. An important property of $\overline{\mathcal{A}}$ is that it is a Lipschitz graph. This means that the projection on the first factor $\mathrm{pr}_1 : \overline{\mathcal{A}} \longrightarrow \mathbb{R}^d$ is injective (hence for every $x \in \mathrm{pr}_1(\overline{\mathcal{A}})$, there exists a unique $y \in \mathbb{R}^d$ for which $(x, y) \in \overline{\mathcal{A}}$), and that the map $x \in \mathrm{pr}_1(\overline{\mathcal{A}}) \longmapsto y \in \mathbb{R}^d$ is Lipschitz.

There is a simple link between $\mathrm{pr}_1(\overline{\mathcal{A}})$ and the Mané potential ϕ: a point $x \in \mathbb{R}^d$ belongs to $\mathrm{pr}_1(\overline{\mathcal{A}})$ if and only if $\phi(x, x) = 0$. If this is the case, the unique element $y \in \mathbb{R}^d$ for which $(x, y) \in \overline{\mathcal{A}}$ is characterized by the relations

$$\phi(x, y) = \widetilde{S}(x, y) = S(x, y) - \widetilde{S} \quad \text{and} \quad \phi(x, y) + \phi(y, x) = 0.$$

We also consider the dual Aubry set $\overline{\mathcal{A}}^* = \mathcal{L}(\overline{\mathcal{A}}) \subset \mathcal{T}^*$. It is a Lipschitz graph, invariant by $\overline{F}$. It can be interpreted as the set of differentials of sub-actions, thanks to the following result: every sub-action $u : \mathbb{R}^d \longrightarrow \mathbb{R}$ is differentiable at every point $x \in \mathrm{pr}_1(\overline{\mathcal{A}})$, the differential being $D_x u = \mathcal{L}(x, y) \in \overline{\mathcal{A}}^*$, where $y \in \mathbb{R}^d$ is the unique element for which $(x, y) \in \overline{\mathcal{A}}$. Finally, if $(x, p) \in \overline{\mathcal{A}}^*$, then $(x + s, p) \in \overline{\mathcal{A}}^*$ for every $s \in \mathbb{Z}^d$, so that we can project $\overline{\mathcal{A}}^*$ on $T^*\mathbb{T}^d$; the result is an F-invariant Lipschitz graph denoted by $\mathcal{A}^*$.

In order to construct the foliation alluded to in Theorem 2, we shall consider a family of Aubry sets, parameterized by a cohomology class $c \in H^1(\mathbb{T}^d, \mathbb{R})$. This is how they are defined: let ω be a closed 1-form and $\tilde{\omega}$ a lift to $\mathbb{R}^d$. Let us denote by $u : \mathbb{R}^d \longrightarrow \mathbb{R}$ a primitive of the exact 1-form $\tilde{\omega}$. It is easy to check that the map

$$S_u : (x, y) \in \mathcal{T} \longmapsto S(x, y) + u(x) - u(y) \in \mathbb{R}$$

is a generating function.

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When we replace S with S_u, some mathematical objects associated to S will be altered, while others remain unchanged. For example, S and S_u clearly have the same extremal sequences, so that $\sigma_u = \sigma$. On the other hand, $\mathcal{L}$ becomes $\mathcal{L}_u = T_u^{-1} \circ \mathcal{L}$, where T_u is the translation

$$T_u : (x, p) \in \mathcal{T}^* \longmapsto (x, p + Du(x)) \in \mathcal{T}^*.$$

As for $\overline{F}$, it is changed into $\overline{F}_u = T_u^{-1} \circ \overline{F} \circ T_u$. So if F is without conjugate points, the same is true for F_u. One may check that the real $\widetilde{S}_u$ only depends on the cohomology class c of ω, so that it can be denoted by $\widetilde{S}_c$. This gives rise to the α-Mather function $\alpha : c \in H^1(\mathbb{T}^d, \mathbb{R}) \longmapsto -\widetilde{S}_c \in \mathbb{R}$, which is both convex and superlinear.

As a matter of fact, the Aubry set $\overline{\mathcal{A}}(S_u)$ also only depends on c, so it will be denoted by $\overline{\mathcal{A}}_c$. Its dual counterpart $\mathcal{L}_u(\overline{\mathcal{A}}_c) = T_u^{-1}(\mathcal{L}(\mathcal{A}_c))$ is then invariant by $\overline{F}_u = T_u^{-1} \circ \overline{F} \circ T_u$. As we are more interested in $\overline{F}$-invariant subsets of $\mathcal{T}^*$, it is natural to define the dual Aubry set associated to the cohomology class c as $\overline{\mathcal{A}}_c^* = \mathcal{L}(\overline{\mathcal{A}}_c)$. This is an $\overline{F}$-invariant Lipschitz graph. Its projection $\mathcal{A}_c^*$ on $T^*\mathbb{T}^d$ is an F-invariant Lipschitz graph of $T^*\mathbb{T}^d$.

We shall make use of the following notations: if $c \in (\mathbb{R}^d)^*$ is a cohomology class, then $S_c : (x, y) \in \mathcal{T} \longmapsto S(x, y) + c(x - y) \in \mathbb{R}$ is its associated generating function and ϕ_c the corresponding Mané potential.

6. *From periodic orbits to Aubry sets*

In this section, we show that if F is without conjugate points, then each of the Lagrangian submanifolds $\mathcal{G}_{N,r}^*$ defined in §3 is in fact a dual Aubry set $\mathcal{A}_c^*$ for a suitable cohomology class c. This is the content of the following result:

PROPOSITION 6.1. *Let $N \geq 1$, $r \in \mathbb{Z}^d$ and $u : \mathbb{R}^d \longrightarrow \mathbb{R}$ a smooth map such that $\overline{\mathcal{G}}_{N,r}^*$ is the graph of Du. Then $\mathcal{G}_{N,r}^* = \mathcal{A}_c^*$, c being the cohomology class of the closed 1-form induced by Du on $\mathbb{T}^d$.*

We first establish some special properties of the sets $\mathcal{A}_c^*$ and the Mané potential ϕ_c when F is without conjugate points. As remarked earlier, the symplectic diffeomorphism $\overline{F}_u = T_u^{-1} \circ \overline{F} \circ T_u$ is then free of conjugate points as well, so that we may use the results obtained in §2, using S_u instead of S.

LEMMA 6.2. *If F is without conjugate points, then* $\mathrm{pr}_1(\overline{\mathcal{A}}_c) = \mathbb{R}^d$ *for every cohomology class c.*

Proof. We pick $y \in \mathbb{R}^d$, and show that $y \in \mathrm{pr}_1(\overline{\mathcal{A}}_c)$, i.e. $\phi_c(y, y) = 0$. Let $x \in \mathrm{pr}_1(\overline{\mathcal{A}}_c)$. As $\phi_c(x, x) = 0$, there exists for every $\varepsilon > 0$ a finite sequence $(x_0, \ldots, x_n)$ with $x_0 = x$, $x_n = x_0 + r$ and $r \in \mathbb{Z}^d$, and $\widetilde{S}_c(x_0, \ldots, x_n) \leq \varepsilon$. We may assume that $(x_0, \ldots, x_n)$ is an extremal sequence (see Remark 1.6). Then we have (with the notations introduced in part 2) $S_c(x_0, \ldots, x_n) = \mathcal{A}_n(x, x + r) = f(x)$. Lemma 2.4 tells us that the extremal sequence $(y_n)_{n\in\mathbb{Z}}$ with $y_0 = y$ and $y_n = y_0 + r$ satisfies $S(y_0, \ldots, y_n) = S(x_0, \ldots, x_n)$. Hence

$$\begin{aligned} S_c(y_0, \ldots, y_n) &= S(y_0, \ldots, y_n) + c(y_0 - y_n) \\ &= S(x_0, \ldots, x_n) + c(x_0 - x_n) = S_c(x_0, \ldots, x_n) \end{aligned}$$

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and therefore $\widetilde{S}_c(y_0, \ldots, y_n) = \widetilde{S}_c(x_0, \ldots, x_n) \le \varepsilon$. It follows that $\phi_c(y, y) \le \varepsilon$. This holds for every $\varepsilon > 0$, so that $\phi_c(y, y) = 0$. □

LEMMA 6.3. *If F is without conjugate points, then ϕ_c is additive and antisymmetric for every cohomology class c:*

$$\text{for all } (x, y, z) \in (\mathbb{R}^d)^3, \quad \phi_c(x, z) = \phi_c(x, y) + \phi_c(y, z) \quad \text{and}$$
$$\phi_c(x, y) + \phi_c(y, x) = 0.$$

Proof. Let us fix x et y in $\mathbb{R}^d$. As explained in §4, the maps $\phi_c(x, \cdot)$ and $\phi_c(y, \cdot)$ are two sub-actions, and are therefore differentiable at every $z \in \mathrm{pr}_1(\overline{\mathcal{A}}_c)$, both differentials being equal to $\mathcal{L}_c(z, z')$, with $(z, z') \in \overline{\mathcal{A}}_c$. As we know that $\mathrm{pr}_1(\overline{\mathcal{A}}_c) = \mathbb{R}^d$, we may conclude that these two maps are differentiable everywhere, with the same differential. Hence they are equal up to a constant:

$$\text{there exists } C \in \mathbb{R} \text{ such that for all } z \in \mathbb{R}^d, \quad \phi_c(x, z) = \phi_c(y, z) + C.$$

Choosing $z = y$ and then $z = x$, we get $C = \phi_c(x, y) = -\phi_c(y, x)$. This yields the two relations $\phi_c(x, y) + \phi_c(y, x) = 0$ and $\phi_c(x, z) = \phi_c(x, y) + \phi_c(y, z)$. □

Remark 6.4. When F is without conjugate points, the dual Aubry set $\overline{\mathcal{A}}^*$ is then the graph of the differential of the maps $\phi_c(x_0, \cdot)$, and the same is true for its projection on $T^*\mathbb{T}^d$. As for every $c \in H^1(\mathbb{T}^d, \mathbb{R})$, we have $\overline{\mathcal{A}}_c^* = \mathcal{L}(\overline{\mathcal{A}}_c) = T_u \circ \mathcal{L}_u(\overline{\mathcal{A}}_c)$, the set $\mathcal{A}_c^*$ is the graph of a closed 1-form whose cohomology class is c.

We are now able to prove Proposition 6.1. Let us fix $N \in \mathbb{N}^*$, $r \in \mathbb{Z}^d$, and $u : \mathbb{R}^d \longrightarrow \mathbb{R}$ a smooth map for which $\overline{\mathcal{G}}_{N,r}^*$ is the graph of Du. The projection of Du on $\mathbb{T}^d$ is then a closed 1-form with cohomology class c. We want to show that $\mathcal{G}_{N,r}^* = \mathcal{A}_c^*$.

We first handle the case where $c = 0$, so that u is a $\mathbb{Z}^d$-periodic function. For every $x \in \mathbb{R}^d$, $(x, Du(x)) \in \overline{\mathcal{G}}_{N,r}^*$ and this set is invariant by $\overline{F}$, so we have $\overline{F}(x, Du(x)) = (y, Du(y))$ for a (unique) $y \in \mathbb{R}^d$, denoted by $y = y(x)$. We shall make use of the following result (see **[Go]** (Theorem 35.2, p. 128) or **[McK-Me-St]** (Theorem 1, p. 569) for a proof):

LEMMA 6.5. *Let $u : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a C^2 map and $\mathcal{G}^* \subset \mathcal{T}^*$ the graph of Du. Assume $\mathcal{G}^*$ is invariant by $\overline{F}$ and define $\mathcal{G} = \mathcal{L}^{-1}(\mathcal{G}^*)$. Then there exists a real number C for which*

$$\text{for all } (x, y) \in \mathcal{G}, \quad S(x, y) + u(x) - u(y) = C. \tag{$*$}$$

More precisely, we have

$$\text{for all } (x, y) \in \mathcal{T}, \quad u(y) - u(x) \le S(x, y) - C, \tag{$**$}$$

and equality holds if and only if $(x, y) \in \mathcal{G}$.

Remark 6.6. It is straightforward from the proof of Lemma 6.5 that its conclusion holds if u is assumed to be C^1 with bounded differential. We shall make use of this later.

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Let $x_0 \in \mathbb{R}^d$, and (x_k) the sequence defined by $x_{k+1} = y(x_k)$. Then (x_k) is an extremal sequence. As $(x_0, Du(x_0)) \in \overline{\mathcal{G}}^*_{N,r}$, we have $(x_0, x_1) \in \overline{\mathcal{G}}_{N,r}$ and hence $x_N = x_0 + r$. As a consequence of $(*)$, $u(x_{k+1}) - u(x_k) = S(x_k, x_{k+1}) - C$ for every integer k. Summing up these equalities, we get

$$\sum_{k=0}^{N-1} S(x_k, x_{k+1}) - N \times C = u(x_N) - u(x_0),$$

and the right-hand side vanishes because u is $\mathbb{Z}^d$-periodic. So we have $C = S(x_0, \ldots, x_N)/N$, and this implies $C \geq \widetilde{S}$ by definition of $\widetilde{S}$. Applying inequality $(**)$, we obtain

$$\text{for all } x \in \mathbb{R}^d, \ \text{for all } y \in \mathbb{R}^d, \quad u(y) - u(x) \leq S(x, y) - \widetilde{S} = \widetilde{S}(x, y),$$

and this means that u is a sub-action. As explained in §5, the differential of u at every point of $\mathrm{pr}_1(\overline{\mathcal{A}})$ belongs to $\overline{\mathcal{A}}^*$. Since $\mathrm{pr}_1(\overline{\mathcal{A}}) = \mathbb{R}^d$, the graph of Du (that is, $\overline{\mathcal{G}}^*_{N,r}$) is then included in $\overline{\mathcal{A}}^*$; as $\overline{\mathcal{A}}^*$ is also a graph, these two sets are the same.

Assume now that $c \neq 0$. Let $S_u : (x, y) \longmapsto S(x, y) - u(x) + u(y)$ be the generating function and

$$T_u : (x, p) \in \mathcal{T}^* \longmapsto (x, p + Du(x)) \in \mathcal{T}^*$$

the translation. As S and S_u have the same extremal sequences, the sets $\overline{\mathcal{G}}_{N,r}(S)$ and $\overline{\mathcal{G}}_{N,r}(S_u)$ are equal. Using this and the fact that $\mathcal{L}_u = T_u^{-1} \circ \mathcal{L}$, we get

$$\overline{\mathcal{G}}^*_{N,r}(S_u) = \mathcal{L}_u(\overline{\mathcal{G}}_{N,r}(S_u)) = T_u^{-1} \circ \mathcal{L}(\overline{\mathcal{G}}_{N,r}(S)) = T_u^{-1}(\overline{\mathcal{G}}^*_{N,r}(S)).$$

The very definition of u implies that this set is the null section. We may then apply the preceding case: the null section is in fact the dual Aubry set associated to S_u, and this means that $\overline{\mathcal{G}}^*_{N,r} = \overline{\mathcal{A}}^*_c$.

7. *Some supplementary results on Aubry sets*

In this section, we establish some technical properties concerning Aubry sets. They will be needed for the proof of Theorem 2. The main problem is the following: if (c_n) is a sequence of cohomological classes that converges to c, what can be said about the Aubry sets $\mathcal{A}^*_{c_n}$ and the Mané potentials ϕ_{c_n}? Do they converge in some sense to $\mathcal{A}^*_c$ and ϕ_c? In the Hamiltonian case, every Aubry set is contained in a level set of the Hamiltonian, so that the $\mathcal{A}^*_{c_n}$ may not explode as n goes to infinity. There is no such easy argument in the discrete case, and therefore some new techniques are required. We state and prove four results; only the last one requires F to be without conjugate points.

LEMMA 7.1. *Let c be a cohomology class, let $(x, y) \in \mathcal{A}_c$ and $(y, z) = \sigma(x, y)$. Then*

$$\widetilde{S}_c \geq S(x, y) + S(y, z) - S(x, z).$$

Proof. As $\overline{\mathcal{A}}_c$ is invariant by σ, both (x, y) and (y, z) belong to $\overline{\mathcal{A}}_c$, so that

$$\phi_c(x, y) = S_c(x, y) - \widetilde{S}_c \quad \text{and} \quad -\phi_c(z, y) = \phi_c(y, z) = S_c(y, z) - \widetilde{S}_c.$$

Summing up these two equalities and using the triangular inequality for ϕ_c, we get

$$S_c(x, y) + S_c(y, z) - 2\widetilde{S}_c = \phi_c(x, y) - \phi_c(z, y) \leq \phi_c(x, z).$$

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By definition of ϕ_c, one has $\phi_c(x, z) \le S_c(x, z) - \widetilde{S}_c$, whence

$$S_c(x, y) + S_c(y, z) - 2\widetilde{S}_c \le S_c(x, z) - \widetilde{S}_c,$$

and therefore

$$\widetilde{S}_c \ge S_c(x, y) + S_c(y, z) - S_c(x, z) = S(x, y) + S(y, z) - S(x, z). \qquad \square$$

LEMMA 7.2. *Let (c_n) be a convergent sequence of cohomology classes, with $c_n \longrightarrow c$. Then for every $\varepsilon > 0$, one has*

$$\text{for all } (x, y) \in \mathcal{T}, \text{ there exists } N_0 \in \mathbb{N} \text{ such that } n \ge N_0 \Longrightarrow \phi_{c_n}(x, y) \le \varepsilon + \phi_c(x, y).$$

Proof. Let $\varepsilon > 0$ and $(x, y) \in \mathcal{T}$. By definition of $\phi_c(x, y)$, there exists an integer $N \ge 1$ and a finite sequence $\gamma = (x_0, \ldots, x_N)$ with $x_0 = x$, $y - x_N \in \mathbb{Z}^d$, and $\widetilde{S}_c(\gamma) \le \phi_c(x, y) + \varepsilon$. As

$$\widetilde{S}_c(\gamma) = S(\gamma) + c \cdot (x_0 - x_N) - N\widetilde{S}_c \quad \text{and} \quad \widetilde{S}_{c_n}(\gamma) = S(\gamma) + c_n \cdot (x_0 - x_N) - N\widetilde{S}_{c_n},$$

this implies, as $c_n \longrightarrow c$, that

$$\lim_{n\to+\infty} \widetilde{S}_{c_n}(\gamma) = \lim_{n\to+\infty} (S(\gamma) + c_n \cdot (x_0 - x_N) - N\widetilde{S}_{c_n}) = \widetilde{S}_c(\gamma).$$

So if n is large enough, one has $\widetilde{S}_{c_n}(\gamma) \le \widetilde{S}_c(\gamma) + \varepsilon$, and hence $\widetilde{S}_{c_n}(\gamma) \le \phi_c(x, y) + 2\varepsilon$. Since $\phi_{c_n}(x, y) \le \widetilde{S}_{c_n}(\gamma)$, we finally get $\phi_{c_n}(x, y) \le \phi_c(x, y) + 2\varepsilon$. $\square$

LEMMA 7.3. *Let K be a compact set in $H^1(\mathbb{T}^d, \mathbb{R})$. There exists a constant $M \ge 0$ such that*

$$\text{for all } c \in K, \text{ for all } (x, y) \in \overline{\mathcal{A}}_c, \quad \|y - x\| \le M.$$

Proof. Here we use a proof by contradiction. If the conclusion was not true, we could find a sequence (c_n) in K and a sequence (x_n, y_n) in $\mathcal{T}$ with $(x_n, y_n) \in \overline{\mathcal{A}}_{c_n}$ for every n and $\|y_n - x_n\| \longrightarrow +\infty$. Since $(x_n, y_n) \in \overline{\mathcal{A}}_{c_n}$, one has

$$\text{for all } n, \quad \phi_{c_n}(x_n, y_n) = \widetilde{S}_{c_n}(x_n, y_n) = S(x_n, y_n) - c_n \cdot (x_n - y_n) - \widetilde{S}_{c_n}. \qquad (*)$$

As the sequence (c_n) is bounded, there exists a constant $C \ge 0$ for which

$$\text{for all } n \in \mathbb{N}, \ |\widetilde{S}_{c_n}| \le C \quad \text{and} \quad |c_n \cdot (x_n - y_n)| \le C\|x_n - y_n\|.$$

Since $S(x_n, y_n) \ge \alpha + \beta\|x_n - y_n\| + \gamma\|x_n - y_n\|^2$ (according to Lemma 1.1), the right-hand side of $(*)$ is unbounded as n goes to infinity. We will now check that the left-hand side remains bounded, and thus get a contradiction. Since ϕ_{c_n} is $\mathbb{Z}^d$-periodic with respect to each variable, one has

$$\phi_{c_n}(x_n, y_n) \le \mathrm{Max}\{\phi_{c_n}(x, y), (x, y) \in [0, 1]^d \times [0, 1]^d\}.$$

Now $\phi_{c_n}(x, y) \le \widetilde{S}_{c_n}(x, y) = S(x, y) - c_n \cdot (x - y) - \widetilde{S}_{c_n}$, and this quantity is bounded since the three variables x, y and c_n belong to compact sets. $\square$

LEMMA 7.4. *Assume that F is without conjugate points. Let K be a compact set in $H^1(\mathbb{T}^d, \mathbb{R})$, and let $x \in \mathbb{R}^d$. Then the maps $\phi_c(x, \cdot)$ are uniformly Lipschitz.*

Proof. The maps $\phi_c(x, \cdot)$ are $\mathbb{Z}^d$-periodic, and everywhere differentiable since F is without conjugate points. So all we need to do is to check that the differentials $D_y\phi_c(x, y)$ are uniformly bounded when $c \in K$ and $y \in [0, 1]^d$. For every such y, let $y' \in \mathbb{R}^d$ with $(y, y') \in \overline{\mathcal{A}}_c$. We then have $D_y\phi_c(x, y) = \mathcal{L}_c(y, y') = T_c^{-1} \circ \mathcal{L}(y, y')$. To conclude the proof, simply note that c is bounded, as well as $\|y - y'\|$, according Lemma 7.3. $\square$

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8. *A continuous foliation of $T^*\mathbb{T}^d$*

In this part, we give a proof of Theorem 2. We begin with the easy implication: we assume that $T^*\mathbb{T}^d$ is a disjoint union of Lipschitz, Lagrangian F-invariant graphs and establish that F is without conjugate points. First note that every orbit $(x_n, p_n)_{n\in\mathbb{Z}}$ of $\overline{F}$ is minimizing. For it is contained in the graph of a Du, with $u : \mathbb{R}^d \longrightarrow \mathbb{R}$ of class C^1 with Du bounded. We apply Remark 6.6: there is a constant C such that

$$\text{for all } k \in \mathbb{Z}, \quad u(x_{k+1}) - u(x_k) = S(x_k, x_{k+1}) - C,$$

and therefore $S(x_0, \ldots, x_n) = nC + u(x_n) - u(x_0)$ for every integer $n \geq 1$. Moreover, if $(y_0, \ldots, y_n)$ is any finite sequence in $\mathbb{R}^d$ with $x_0 = y_0$ and $x_n = y_n$, then by Lemma 6.5, $u(y_{k+1}) - u(y_k) \leq S(y_k, y_{k+1}) - C$ for all k, so that

$$S(y_0, \ldots, y_n) \geq nC + u(y_n) - u(y_0) = S(x_0, \ldots, x_n).$$

As every minimizing orbit is free of conjugate points (see [**Bi-McK**] or [**Ar**] for a proof), we may conclude that $\overline{F}$ (and F as well) does not have conjugate points.

We now assume that F is without conjugate points and check that the dual Aubry sets $\mathcal{A}^*_c$, with c in $H^1(\mathbb{T}^d, \mathbb{R})$, are the leaves of a continuous foliation of $T^*\mathbb{T}^d$. Let us establish that the sets $\mathcal{A}^*_c$ realize a partition of $T^*\mathbb{T}^d$. We first prove that these sets are disjoint.

PROPOSITION 8.1. *If c and d are two distinct cohomology classes, then $\mathcal{A}^*_c \cap \mathcal{A}^*_d = \emptyset$.*

Proof. We assume that $\mathcal{A}^*_c$ and $\mathcal{A}^*_d$ are not disjoint, so that $\overline{\mathcal{A}}_c$ and $\overline{\mathcal{A}}_c$ intersect at some point $(x, x') \in \mathcal{T}$, and show that we then have $\overline{\mathcal{A}}_c = \overline{\mathcal{A}}_d$. This implies $\mathcal{A}^*_c = \mathcal{A}^*_d$ and therefore $c = d$ (see Remark 6.4).

We first prove that $\widetilde{S}_{(c+d)/2} = \frac{1}{2}(\widetilde{S}_c + \widetilde{S}_d)$. Let (x_n) be the extremal sequence for which $x_0 = x$ and $x_1 = x'$. As the Aubry set $\overline{\mathcal{A}}_c$ is invariant by σ, one has $(x_k, x_{k+1}) \in \overline{\mathcal{A}}_c$ for every integer k, so that $\phi_c(x_k, x_{k+1}) = \widetilde{S}_c(x_k, x_{k+1}) = S_c(x_k, x_{k+1}) - \widetilde{S}_c$ for all k. Summing up these equalities and using the fact that ϕ_c is additive, we get

$$\sum_{k=0}^{n-1} S_c(x_k, x_{k+1}) = n\widetilde{S}_c + \phi_c(x_0, x_n) = n\widetilde{S}_c + O(1),$$

as ϕ_c is bounded. A similar equality holds for the cohomology class d, so that

$$\sum_{k=0}^{n-1} S_{(c+d)/2}(x_k, x_{k+1}) = \frac{1}{2}\sum_{k=0}^{n-1} S_c(x_k, x_{k+1}) + S_c(x_k, x_{k+1}) = n\frac{\widetilde{S}_c + \widetilde{S}_d}{2} + O(1).$$

This implies, by definition of $\widetilde{S}_{(c+d)/2}$, that $\widetilde{S}_{(c+d)/2} \leq \frac{1}{2}(\widetilde{S}_c + \widetilde{S}_d)$. On the other hand, the map $c \longmapsto \widetilde{S}_c$ is concave, hence we have equality: $\widetilde{S}_{(c+d)/2} = \frac{1}{2}(\widetilde{S}_c + \widetilde{S}_d)$.

Let us see how to use this relation to prove that $\overline{\mathcal{A}}_c = \overline{\mathcal{A}}_d$. Pick $(y, y') \in \overline{\mathcal{A}}_{(c+d)/2}$. This means that for every $\varepsilon > 0$ there is a finite sequence $(y_0, \ldots, y_n)$ with $\|y - y_0\| \leq \varepsilon$, $\|y' - y_1\| \leq \varepsilon$, $y_n - y_0 \in \mathbb{Z}^d$, and

$$\widetilde{S}_{(c+d)/2}(y_0, y_1, \ldots, y_n) = \sum_{k=0}^{n-1} S_{(c+d)/2}(y_k, y_{k+1}) - n\widetilde{S}_{(c+d)/2} \leq \varepsilon.$$

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As $\widetilde{S}_{(c+d)/2} = \frac{1}{2}(\widetilde{S}_c + \widetilde{S}_d)$, this may be rewritten as $\frac{1}{2}(\Sigma_c + \Sigma_d) \leq \varepsilon$, where

$$\Sigma_c = \sum_{k=0}^{n-1} S_c(y_k, y_{k+1}) - n\widetilde{S}_c \quad \text{and} \quad \Sigma_d = \sum_{k=0}^{n-1} S_d(y_k, y_{k+1}) - n\widetilde{S}_d.$$

As Σ_c and Σ_d are both non-negative quantities, each of them must be smaller than 2ε. This implies that (x, x') belongs to $\overline{\mathcal{A}}_c$ and to $\overline{\mathcal{A}}_d$. This proves that $\overline{\mathcal{A}}_{(c+d)/2} \subset \overline{\mathcal{A}}_c \cap \overline{\mathcal{A}}_d$. But these three Aubry sets are all graphs and their projections on the first factor is $\mathbb{R}^d$, so they are equal. □

Then we establish that the union of all these dual Aubry sets $\overline{\mathcal{A}}_c^*$ is equal to $\mathbb{T}^d \times (\mathbb{R}^d)^*$, and that they vary continuously with c.

PROPOSITION 8.2. *For every $x \in \mathbb{R}^d$, the map*

$$F_x : c \in H^1(\mathbb{T}^d, \mathbb{R}) \longmapsto p \in (\mathbb{R}^d)^* \quad \text{with } (x, p) \in \overline{\mathcal{A}}_c^*,$$

is a homeomorphism.

Proof. We first establish that F_x is coercive. Let K be a compact set in $(\mathbb{R}^d)^*$, $c \in F_x^{-1}(K)$, $p = F_x(c)$ (so that $(x, p) \in \overline{\mathcal{A}}_c^*$), $(x, x') = \mathcal{L}^{-1}(x, p) \in \overline{\mathcal{A}}_c$ and $(x', x'') = \sigma((x, x')) \in \overline{\mathcal{A}}_c$. According to Lemma 7.1, we then have

$$\widetilde{S}_c \geq S(x, x') + S(x', x'') - S(x, x'').$$

As $p \in K$, x' and x'' remain in compact sets in $\mathbb{R}^d$, so that the right-hand side is bounded below. Since the map $c \longmapsto -\widetilde{S}_c$ is convex and superlinear, one may conclude that c is bounded.

We next show that F is continuous. Let (c_n) be a sequence in $H^1(\mathbb{T}^d, \mathbb{R})$. Assume that it converges to c. We have to prove that $F_x(c_n)$ goes to $F_x(c)$. Let $y_n \in \mathbb{R}^d$ with $(x, y_n) \in \overline{\mathcal{A}}_{c_n}$ for every n. We shall establish that the sequence (y_n) is convergent (the limit being some $y_\infty \in \mathbb{R}^d$) and that $(x, y_\infty) \in \overline{\mathcal{A}}_c$. According to Lemma 7.3, the sequence (y_n) is bounded. So we only need to show that if y_∞ is a cluster point of the sequence (y_n), then $(x, y_\infty) \in \overline{\mathcal{A}}_c$. This then implies that y_∞ is unique (because $\overline{\mathcal{A}}_c$ is a graph), and that the sequence converges to y_∞.

So let us consider a convergent subsequence of (y_n) (it will still be denoted by (y_n) in order to keep notations as simple as possible), with limit $y_\infty \in \mathbb{R}^d$. As $(x, y_n) \in \overline{\mathcal{A}}_{c_n}$, one has

$$\text{for all } n \in \mathbb{N}, \quad \phi_{c_n}(x, y_n) = \widetilde{S}_{c_n}(x, y_n) = S(x, y_n) + c_n \cdot (x - y_n) - \widetilde{S}_{c_n}.$$

When n goes to infinity, the right-hand side n converges to $S_c(x, y_\infty) - \widetilde{S}_c$. The left-hand side may be rewritten as $p_{c_n}(x, y_n) = \phi_c(x, y_\infty) + u_n + v_n$, with

$$u_n = \phi_{c_n}(x, y_n) - \phi_{c_n}(x, y_\infty) \quad \text{and} \quad v_n = \phi_{c_n}(x, y_\infty) - \phi_c(x, y_\infty).$$

According to Lemma 7.4, the maps $\phi_{c_n}(x, \cdot)$ are uniformly Lipschitz, and hence the sequence (u_n) converges to 0. Moreover, we already know that $\phi_{c_n}(x, y_n) \longrightarrow S_c(x, y_\infty) - \widetilde{S}_c \geq \phi_c(x, y_\infty)$, and hence the sequence (v_n) is convergent, its limit ℓ being non-negative. On the other hand, Lemma 7.2 tells us that for every $\varepsilon > 0$, one has

$$\phi_{c_n}(x, y_\infty) \leq \varepsilon + \phi_c(x, y_\infty), \quad \text{so that } v_n \leq \varepsilon,$$

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when n is large enough. This implies that ℓ has to be non-positive, and therefore that $\lim v_n = 0$. So $\lim \phi_{c_n}(x, y_n) = \phi_c(x, y_\infty)$. From this we deduce that $\phi_c(x, y_\infty) = S_c(x, y_\infty) - \widetilde{S}_c$ and hence that $(x, y_\infty) \in \mathcal{A}_c$.

To finish the proof, we use a topological argument: as F_x is a continuous and injective map between two vector spaces of the same dimension, the invariance of domain (see [**Do**] p. 567) states that F_x is an open map. On the other hand, F_x is a closed map, since it is continuous and coercive. Hence $F_x(H^1(\mathbb{T}^d, \mathbb{R}))$ is both open and closed, so it has to be equal to $(\mathbb{R}^d)^*$. Hence F_x is bijective. As it is also continuous and open, it is a homeomorphism. □

Another consequence of this proposition is that the map

$$\mathcal{F} : (x, c) \in \mathbb{T}^d \times H^1(\mathbb{T}^d, \mathbb{R}) \longmapsto F_x(c) \in T^*\mathbb{T}^d$$

is continuous, and therefore the dual Aubry sets are the leaves of a continuous foliation of $T^*\mathbb{T}^d$.

Acknowledgements. The author is grateful to Marie-Claude Arnaud for stimulating conversations and ideas during the preparation of this paper, and to the referee for suggestions.

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