

Entropy pair realization

VILLE SALO 

*Department of Mathematics and Statistics, University of Turku,
FI-20014 Turun yliopisto, Finland
(email: [vosalo@utu.fi](mailto:vosal@utu.fi))*

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Abstract. We show that the complete positive entropy (CPE) class α of Barbieri and García-Ramos contains a one-dimensional subshift for all countable ordinals α , that is, the process of alternating topological and transitive closure on the entropy pairs relation of a subshift can end on an arbitrary ordinal. This is the composition of three constructions. We first realize every ordinal as the length of an abstract ‘close-up’ process on a countable compact space. Next, we realize any abstract process on a compact zero-dimensional metrizable space as the process started from a shift-invariant relation on a subshift, the crucial construction being the implementation of every compact metrizable zero-dimensional space as an open invariant quotient of a subshift. Finally, we realize any shift-invariant relation E on a subshift X as the entropy pair relation of a supershift $Y \supset X$, and under strong technical assumptions, we can make the CPE process on Y end on the same ordinal as the close-up process of E .

Key words: entropy, entropy pair, complete positive entropy, CPE class

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1. Introduction

The *entropy* of a subshift X is the limit of $(\log |\mathcal{L}_n(X)|)/n$ where $\mathcal{L}_n(X)$ is the set of words in X of length n , as $n \rightarrow \infty$. It measures the amount of information in orbits of this dynamical system. There are several ways to quantify how this information is produced. One is the entropy pairs relation, due to Blanchard [3, 4], which intuitively measures the amount of information in orbits when we obtain information only when the point is very close to one of two points (the elements of the entropy pair).

The entropy pairs relation (plus the diagonal) is symmetric, reflexive, and topologically closed but not necessarily transitive, and a subshift turns out to have only positive entropy (non-trivial) factors if and only if the smallest closed equivalence relation containing the entropy pairs relation is the full relation [4]. The hierarchy generated by alternately closing the relation topologically and transitively is studied in [2], and it is shown that the full

relation can arise at any countable ordinal. We refer to [2] for the state-of-the-art and further context.

Our main contribution is that such examples can be made expansive. We prove this using relatively general subshift constructions, in particular, Lemma 5 and Proposition 6 construct a large class of relations as entropy pair relations, which may be of independent interest.

By alternating topological and transitive closure on a symmetric reflexive relation E , one obtains an ordinal-indexed tower of equivalence relations. The union of these equivalence relations is, of course, just the smallest closed equivalence relation containing E .

Definition 1. Let X be a topological space and $E \subset X^2$ a symmetric reflexive relation. If E is closed, let $\zeta = 1$, otherwise $\zeta = 0$. Define $E^{(\zeta)} = E$ and write any ordinal greater than ζ in the unique way as $\lambda + 2n + b$, λ a limit ordinal (including 0), $n \in \mathbb{N}$, $b \in \{0, 1\}$, and define

$$\begin{aligned}
 E^{(\lambda+2n+1)} &= \overline{E^{(\lambda+2n)}} && \text{(topological closure),} \\
 E^{(\lambda+2(n+1))} &= (E^{(\lambda+2n+1)})^* && \text{(transitive closure),} \\
 E^{(\lim_i \lambda_i)} &= \bigcup_i E^{(\lambda_i)}.
 \end{aligned}$$

We say the *close-up rank* of E is the least λ such that $E^{(\lambda)} = E^{(\lambda+1)}$. We say E is *equalizing* if $E^{(\lambda)} = X^2$ for some λ .

Ordinals $\lambda + 2n + b$ where $b = 0$ are called *even*, and others *odd*.

The check for topological closedness is made in the beginning so that $E^{(\lambda)}$ is always topologically closed at odd ordinals.

LEMMA 2. *Let $\lambda \geq 1$ be a countable ordinal. Then there exists a closed countable set $X \subset [0, 1]$ and an equalizing closed symmetric reflexive relation $E \subset X^2$ with rank λ .*

For this, we arrange an ordinal of the form ω^α or $\omega^\alpha + 1$ on the circle and join each point with its successor. The calculation of the close-up rank roughly then corresponds to the computation of the Hausdorff or Cantor–Bendixson rank of the ordinal.

In [2], Barbieri and García-Ramos study the close-up process for the closed equivalence relation $EP(X) = E \cup \Delta$ where Δ is the diagonal and E is the family of *entropy pairs* of the compact topological dynamical system (G, X) (G an amenable discrete group acting on X), and where (x, y) is an *entropy pair* if the open cover (U^c, V^c) has positive entropy [5, Definition 14.5] for all disjoint closed neighborhoods $U \ni x, V \ni y$.

In [2], a dynamical system is defined to be of *complete positive entropy (CPE) class α* if the equivalence relation $E \cup \Delta$ is equalizing with rank α . One of the results of [2] is that among abstract dynamical systems, the CPE class α is non-empty for every countable amenable group G .

Nishant Chandgotia asked in personal communication if there is a subshift realization of their process, that is, whether there is a subshift in the CPE class α . We prove a positive answer for $G = \mathbb{Z}$.

THEOREM 3. *Let $\alpha \geq 1$ be a countable ordinal. Then there exists a subshift $X \subset A^{\mathbb{Z}}$ in CPE class α .*

We prove this by first implementing closed relations on topological spaces as shift-invariant relations on subshifts, and then implementing such relations as CPE pairs.

A point $x \in A^{\mathbb{Z}}$ is *Toeplitz* if for all $i \in \mathbb{Z}$: there exists $n > 0$: for all $m \in \mathbb{Z}$: $x_{i+mn} = x_i$. (Often one excludes the periodic points; indeed in our application, the points are never periodic, but we omit this requirement so we need not address it explicitly.) A *Toeplitz subshift* is a subshift generated by a Toeplitz point. A *pointwise zero-entropy Toeplitz subshift* is one where every point generates a Toeplitz subshift with zero entropy. By the variational principle, a pointwise zero-entropy Toeplitz subshift has zero-entropy, since any generic point for an ergodic measure of positive entropy would generate a Toeplitz subshift with positive entropy (see e.g. §18 and equation (13.3) in [5]).

If X is a topological space, the *hit-or-miss topology* on its closed subsets has subsbasis

$$\mathcal{U}_{U,K} = \{Y \subset X \text{ closed} \mid Y \cap K = \emptyset, Y \cap U \neq \emptyset\},$$

where U ranges over open sets of X and K over compact sets of X . For the space of closed subsets of a subshift, this is equal to the topology given by Hausdorff distance [11], and compares the set of words up to length n . This is the topology we use for the space of subshifts of a subshift X . The minimal subshifts form a subspace, and we give it the induced topology. (This subspace need not be closed, consider for example the space of minimal subshifts of the Grand Sturmian subshift [12], where obviously no sequence of Sturmian subshifts with density of 1s properly decreasing to zero has a converging subsequence.)

LEMMA 4. *For any compact metrizable zero-dimensional space Z , there exists a pointwise zero-entropy Toeplitz subshift X and an open quotient map $\phi : X \rightarrow Z$ such that $\phi^{-1}(z)$ is a minimal subshift for every $z \in Z$, and ϕ induces a homeomorphism between Z and the space of minimal subshifts of X .*

The construction is not new (see e.g. [1] for a more general construction in the recursive category), although we do not know a reference that checks the stated topological dynamical properties. The important word in the above statement is ‘open’—intuitively, because taking preimages of relations in an open quotient map induces a ‘close-up process homomorphism’, in particular, the close-up rank is preserved.

We show that any close-up process on a subshift, satisfying some strong technical assumptions, can be realized as the entropy pairs process. We say a relation $E \subset X^2$ is *doubly shift-invariant* if it is invariant under the action of \mathbb{Z}^2 that shifts the two components separately. If λ is a successor ordinal, write $\lambda - 1$ for its predecessor.

If $E \subset X^2$ is a relation, write $E^{[k]} = \{(x_1, x_2, \dots, x_k) \mid \text{for all } i, j : (x_i, x_j) \in E\}$ (so $E = E^{[2]}$ if and only if E is symmetric).

LEMMA 5. *Let $X \subset A^{\mathbb{Z}}$ be a subshift with zero entropy and let $E \subset X^2$ be a closed doubly shift-invariant symmetric reflexive relation. Then there exists a subshift Y with*

$X \subset Y \subset \hat{A}^{\mathbb{Z}}$ (over a possibly larger alphabet $\hat{A} \supset A$) such that $\text{EP}(Y)^{(\lambda)} \cap X^2 = E^{(\lambda)}$ for all λ . Furthermore, if $E^{(\lambda)} \cap X^2 = X^2$, then $\text{EP}(Y)^{(\lambda+1)} = Y^2$, and we have $\text{EP}(Y)^{(\lambda)} = Y^2$ if:

- λ is even; or
- λ is odd and $(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2}$.

The intuitive description is that we add an entropy-generating supersystem on top of X , by allowing alternation between words from the two points from X (separated by a special symbol) if and only if they are in the relation E . We want that X originally has no entropy pairs, so we require it has zero entropy.

For $A \in X^k$ and $B \in X^k$, we can think of the pair (A, B) as an element $(A, B) \in X^{2k}$ in a natural way. For some intuition behind the *odd condition* in Lemma 5 for odd $\lambda > 2$, first note that $(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2}$ means precisely that if $A, B \in E^{[k]}$, then either $(a, b) \in E^{(\lambda-1)}$ for some $a \in A$ and $b \in B$ (equivalently $(A, B) \in (E^{(\lambda-1)})^{[2k]}$ by the transitivity of $E^{(\lambda-1)}$), or we can find $A_i, B_i \in E^{[k]}$ such that $(A_i, B_i) \in (E^{(\lambda-1)})^{[2k]}$ and $(A_i, B_i) \rightarrow (A, B)$.

This condition arises from the construction method: In the subshift Y , points encode finite tuples of points from X , any pair of which is in the relation. Elements of Y thus roughly correspond to elements of $E^{[k]}$, and we want that at the final limit step, any pair of points in Y (roughly corresponding to an element of $(E^{[k]})^2$) can be approximated by a pair of points from Y which are in the relation at the previous step, which in terms of the close-up process of E translates to being in $(E^{(\lambda-1)})^{[2k]}$. Not every relation satisfies the odd condition, see Example 1.

This odd condition holds in the construction proving Lemma 2 (see Lemma 7 for the precise statement), and is preserved under open quotients, so it passes to the subshift implementations of abstract processes given by Lemma 4.

We do not know to what extent double shift-invariance, the zero-entropy assumption, and the odd condition can be weakened in the previous lemma. Out of general interest, for implementing any relation as entropy pairs of a supersystem (without precise control on the entropy pairs involving the new points), we provide a simpler construction.

PROPOSITION 6. *Suppose X has zero entropy and $E \subset X^2$ is a closed symmetric reflexive shift-invariant relation. Then there exists a subshift Y with $X \subset Y \subset \hat{A}^{\mathbb{Z}}$ (over a possibly larger alphabet $\hat{A} \supset A$) such that $\text{EP}(Y) \cap X^2 = E$.*

This is optimal in the sense that for positive entropy X , it fails for E the diagonal relation [2, 4, 10], and for X zero-entropy and for any $Y \supset X$, $\text{EP}(Y) \cap X^2$ is closed symmetric reflexive shift-invariant. One could, however, ask for the aesthetically more pleasing $\text{EP}(Y) = E \cup \Delta_Y$ instead of just $\text{EP}(Y) \cap X^2 = E$, and we conjecture that this can be done. Even further, one could consider X with positive entropy, and restrict to relations containing $\text{EP}(X)$.

The constructions are made for \mathbb{Z} , and of course one can ask whether they can be generalized to general amenable groups. We conjecture that all CPE classes are inhabited by subshifts on all countable amenable groups.

Particularly in the case of \mathbb{Z}^d , one can also ask if this can be done by subshifts of finite type (SFTs). We conjecture that \mathbb{Z}^2 -SFTs inhabit exactly the CPE classes α where α is recursive. (This has been proved by Westrick [14] (independently of our conjecture).)

2. Definitions and conventions

A *topological dynamical system* is (X, f) where X is a compact metrizable space and $f : X \rightarrow X$ a homeomorphism. The *full shift* is the dynamical system $(A^{\mathbb{Z}}, \sigma)$, where A is a finite set and $A^{\mathbb{Z}}$ has the (Cantor) product topology, and $\sigma(x)_i = x_{i+1}$ is the *left shift*. A *subshift* is a closed σ -invariant subsystem of the full shift. A subshift is *minimal* if it does not contain non-empty proper subshifts. Topological dynamical systems form a category with morphisms the dynamics-commuting continuous maps. A *cover* of a dynamical system Y refers to a surjective morphism $g : X \rightarrow Y$, or its domain X . A cover is *almost one-to-one* if the set of points with a unique preimage is dense. See any standard reference for the basic theory and definitions related to topological entropy [5].

We denote the real interval as $[0, 1]$, but other than this, our intervals $[a, b]$ are usually interpreted as discrete, which should not cause confusion. Words—finite or infinite—are functions from contiguous subsets of \mathbb{Z} to a (by default finite) alphabet. We typically call infinite words *points*, as we think of them as points of the full shift dynamical system (or another subshift). Positioning conventions (precise domains of words) should always be clear from context, or are explained when used. By default, we identify words whose domains are translates of each other and the letters are correspondingly translated, so finite words can be seen as functions $w : [0, n - 1] \rightarrow A$. Concatenation of words is presented by $u \cdot v$ or uv . A point $x \in A^{\mathbb{N}}$ is *eventually constant* if there exists n : for all $i \geq n : x_i = x_{i+1}$.

A (*deterministic*) *substitution* is a map $\tau : A \rightarrow B^*$ where A, B are (possibly infinite) alphabets, and B^* denotes the set of finite words over B , including the empty word. We can apply a substitution to a word or a point by replacing the individual symbols $a \in A$ by words $\tau(a)$. A *non-deterministic substitution* is a function $\tau : A \rightarrow \mathcal{P}(B^*)$ where \mathcal{P} denotes the power set. Applying a non-deterministic substitution to a word or point (or a set of words or points) results in the set of all words or points obtainable by replacing each symbol $a \in A$ by one of its images $w \in \tau(a)$ (independently for each occurrence of the symbol).

3. The proofs

3.1. *Close-up processes of arbitrary length.* See any standard reference for definitions and basic properties of ordinal arithmetic (e.g. [9]). The following lemma is roughly just the computation of the Hausdorff rank of a (cyclic) order corresponding to an ordinal. Since the definition of the close-up process involves parity and the restriction to compact spaces leads to a small issue with non-compact ordinals, it seems easier to give a direct proof than to find a reference.

LEMMA 7. *Let $\lambda \geq 1$ be a countable ordinal. Then there exists a closed countable set $X \subset [0, 1]$ and an equalizing closed symmetric reflexive relation $E \subset X^2$ with rank λ . Furthermore, if $\lambda > 2$ is an odd ordinal, then $(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]}} \cap \overline{(E^{[k]})^2}$.*

Proof. To each $\alpha = \omega^\zeta$ or $\alpha = \omega^\zeta + 1$ for $\zeta \geq 1$, we associate a countable $X \subset [0, 1]$, such that for $\alpha = \omega^{\lambda+n}$, the close-up rank of E is $\lambda + 2n$, and for $\alpha = \omega^{\lambda+n} + 1$, it is $\lambda + 2n + 1$.

Let α be as above and let $A = \{0, 1, \dots, \alpha\}$ with the order topology. As a Von Neumann ordinal, $A = \alpha + 1$, but the intended reading of $\phi(A)$ is $\phi(A) = \{\phi(a) \mid a \in A\}$. Take an injective order homomorphism $\phi : A \rightarrow [0, 1]$, so that $\phi(0) = 0, \phi(\alpha) = 1$. Identify $0 \equiv 1$ to obtain a continuous function $\phi : A \rightarrow \phi(A) \subset \mathbb{S}^1 = [0, 1]/\equiv$, and let $\phi(A) = X$. The reason for arranging the ordinal on a sphere is simply to compactify the non-compact ordinals, and the reason for wanting non-compact ordinals is to allow the last step of the close-up process to be a transitive closure step.

While we do the construction on $\mathbb{S}^1 = [0, 1]/\equiv$ rather than $[0, 1]$, we note that any non-full closed subset of \mathbb{S}^1 is homeomorphic to a closed subset of $[0, 1]$.

Let $E \subset (\mathbb{S}^1)^2$ be the union of the diagonal relation Δ and the successor and predecessor relations, that is,

$$E = \{(\phi(\beta), \phi(\beta)), (\phi(\beta), \phi(\beta + 1)), (\phi(\beta + 1), \phi(\beta)) \mid \beta + 1 < \alpha\}.$$

Observe that if α is a successor, we do not join $\phi(\alpha) = \phi(0)$ with its ‘predecessor’ $\phi(\alpha - 1)$. It is easy to see that X and E are closed. Consider the close-up process for this relation, starting from $E^{(1)} = E$.

Note that at all times, $E^{(\lambda)}$ for $\lambda \geq 1$ is symmetric and reflexive, so we simply have to figure out which pairs $(\phi(\beta), \phi(\beta + \gamma))$ are contained in $E^{(\lambda)}$ for each ordinal $\lambda \geq 1$. What happens is that only the length of the jumps γ matters. Alternately, we are allowed to first take less than ω steps with the current relation (a transitive closure step), and then to close up these jumps into a single superjump consisting of ω many previous jumps (a topological closure step).

Concretely, we prove an exact characterization of $E^{(\lambda)}$ by transfinite induction: every ordinal can be written in a unique way as $\lambda + 2n$ (even ordinals) or $\lambda + 2n + 1$ (odd ordinals) where $n \in \mathbb{N}$ and λ is a limit ordinal (interpreting 0 as a limit ordinal). For all λ, n , we prove by induction for all ordinals $\lambda + 2n + b$ greater than 0 ($n \in \mathbb{N}, b \in \{0, 1\}$) that

$$E^{(\lambda+2n)} = \{(\phi(\beta), \phi(\beta + \gamma)), (\phi(\beta + \gamma), \phi(\beta)) \mid \gamma < \omega^{\lambda+n}, \beta + \gamma < \alpha\},$$

$$E^{(\lambda+2n+1)} = \{(\phi(\beta), \phi(\beta + \gamma)), (\phi(\beta + \gamma), \phi(\beta)) \mid \gamma \leq \omega^{\lambda+n}, \beta + \gamma < \alpha\}.$$

The claim for $E^{(1)}$ is (setting $\lambda = 0, n = 0, b = 1$) equivalent to having (up to symmetry) $E^{(1)} \ni (\beta, \beta + \gamma)$ if and only if $\gamma < 2, \beta + \gamma < \alpha$. This is precisely the definition of $E^{(1)} = E$.

For the induction steps, suppose $E^{(\lambda+2n)}$ is of the claimed form, and consider $E^{(\lambda+2n+1)}$. This is a topological closure step, that is, $E^{(\lambda+2n+1)} = \overline{E^{(\lambda+2n)}}$. Taking the closure of the pairs given by the inductive assumption, we clearly have

$$\{(\phi(\beta), \phi(\beta + \gamma)) \mid \gamma \leq \omega^{\lambda+n}, \beta + \gamma < \alpha\} \subset E^{(\lambda+2n+1)}.$$

If $\beta > 0$, then $(\phi(\beta), \phi(\beta + \gamma)) \notin E^{(\lambda+2n+1)}$ for $\gamma \geq \omega^{\lambda+n} + 1, \beta + \gamma < \alpha$ since otherwise $(\phi(\theta), \phi(\beta + \gamma)) \in E^{(\lambda+2n)}$ for $\theta \leq \beta$ arbitrarily close to β and

$\gamma \geq \omega^{\lambda+n} + 1$ (note that $\beta + \omega^{\lambda+n} + 1$ is isolated), contradicting the inductive assumption.

If $\beta = 0$, then there may be additional limits on the right end of $[0, 1]/\equiv$. In this case, suppose we have $(\phi(\beta), \phi(\beta + \gamma)) = (0, \phi(\gamma)) \in E^{(\lambda+2n+1)}$ for $\omega^{\lambda+n} + 1 \leq \gamma < \alpha$. To obtain such pairs, we must have $(\phi(\delta), \phi(\theta)) \in E^{(\lambda+2n)}$ for $\theta < \alpha$ arbitrarily close to α and $\delta \leq \gamma$ arbitrarily close to γ . If $\alpha = \omega^\zeta + 1$, then α cannot be approximated from below, so this is impossible.

If $\alpha = \omega^\zeta$, then observe that since $\gamma < \alpha$, we have $\omega^{\lambda+n} + 1 \leq \gamma < \alpha$, so since $\alpha = \omega^\zeta$, we have $\omega^{\lambda+n} + 1 \leq \gamma \leq \omega^{\zeta'} \cdot m$ for some $\zeta' < \zeta, m \in \mathbb{N}$ (here we simplify the successor and limit case of ζ into one). Observe that we even have $\omega^{\zeta'} \cdot \omega \leq \alpha$. Now, for any $\delta \leq \gamma$ and $\omega^{\zeta'} \cdot 2m \leq \theta < \alpha$, we have

$$\delta + \omega^{\lambda+n} + 1 \leq \omega^{\zeta'} \cdot 2m \leq \theta$$

by left distributivity and monotonicity of ordinal multiplication, which means $(\phi(\delta), \phi(\theta)) \notin E^{(\lambda+2n)}$, and thus $(\phi(\delta), \phi(\theta)) \notin E^{(\lambda+2n)}$. This concludes the analysis of the step from $\lambda + 2n$ to $\lambda + 2n + 1$.

Suppose then that $E^{(\lambda+2n+1)}$ is of the claimed form and consider $E^{(\lambda+2n+2)}$. Since $(\phi(\beta), \phi(\beta + \gamma)) \in E^{(\lambda+2n+1)}$ for all $\beta, \beta + \gamma < \alpha$ with $\gamma \leq \omega^{\lambda+n}$, taking the transitive closure, we have $(\phi(\beta), \phi(\beta + \gamma)) \in E^{(\lambda+2n+2)}$ for any $m \in \mathbb{N}$ and $\gamma \leq \omega^{\lambda+n} \cdot m$ with $\beta + \gamma < \alpha$. This is indeed equivalent to having $(\phi(\beta), \phi(\beta + \gamma)) \in E^{(\lambda+2n+2)}$ for any $\gamma < \omega^{\lambda+n} \cdot \omega = \omega^{\lambda+n+1}$ as required.

To see that $E^{(\lambda+2n+2)}$ does not contain any pairs $(\phi(\beta), \phi(\beta + \gamma))$ with $\gamma > \omega^{\lambda+n} \cdot \omega, \beta + \gamma < \alpha$, consider any tuple $(\phi(\beta_1), \phi(\beta_2), \dots, \phi(\beta_k))$ with $(\phi(\beta_i), \phi(\beta_{i+1})) \in E^{(\lambda+2n+1)}, \beta_i \neq \beta_{i+1}$ for all i , and $\beta_1 < \beta_k$. If β_j is minimal among of the β_i , then it easily follows from the inductive assumption that

$$\beta_k \leq \beta_j + \omega^{\lambda+n} \cdot k \leq \beta_1 + \omega^{\lambda+n} \cdot k < \beta_1 + \omega^{\lambda+n} \cdot \omega.$$

This concludes the analysis of the step from $\lambda + 2n + 1$ to $\lambda + 2n + 2$.

Finally, we look at the case where λ is an infinite limit ordinal. By definition, $E^{(\lambda)} = \bigcup_{\delta < \lambda} E^{(\delta)}$. We need to show that $E^{(\lambda)}$ contains $(\phi(\beta), \phi(\beta + \gamma))$ for $\beta + \gamma < \alpha$ if and only if $\gamma < \omega^\lambda$. Since $\gamma < \omega^\lambda$ is equivalent to there exists $\delta < \lambda : \gamma < \omega^\delta$, this follows from the inductive assumption on the relations $E^{(\delta)}$. This concludes the analysis of the limit steps.

Now observe that by the above characterization, $E^{(\lambda+2n+b)} = X^2$ if and only if $(0, \gamma) \in E^{(\lambda+2n+b)}$ for all $\gamma < \alpha$. This happens if and only if $b = 0$ and $\gamma < \alpha \implies \gamma < \omega^{\lambda+n}$, that is, $\alpha \leq \omega^{\lambda+n}$; or $b = 1$ and $\gamma < \alpha \implies \gamma \leq \omega^{\lambda+n}$, that is, $\alpha \leq \omega^{\lambda+n} + 1$.

Now, to prove the original claim, for $\lambda = 1$, we can realize close-up rank 1 by setting $E = X^2$ for any choice of X . For larger countable ordinals, in the above construction, we obtained for the choice $\alpha = \omega^{\lambda+n}$ that the close-up rank of E is $\lambda + 2n$, and that for the choice $\alpha = \omega^{\lambda+n} + 1$, it is $\lambda + 2n + 1$.

For the last sentence, observe that in the case of odd $\lambda + 2n + 1$, the pairs added at the last step are just those joining some $\phi(\beta)$ for $\beta < \alpha - 1$ with $\phi(\alpha - 1)$. We can realize any such limit through pairs $(\phi(\beta), \phi(\beta_i)) \in E^{(\lambda+2n)}$, where $\beta_i \nearrow \alpha - 1$.

Now consider any $(a_1, \dots, a_k, b_1, \dots, b_k) \in (E^{[k]})^2$, where we can take the a_i to be sorted in increasing order of $\phi^{-1}(a_i)$, and similarly for the b_i . If $(a_1, b_1) \in E^{(\lambda+2n)}$, then

$$(a_1, \dots, a_k, b_1, \dots, b_k) \in (E^{(\lambda+2n)})^{[2k]} \cap (E^{[k]})^2 \subset \overline{(E^{(\lambda+2n)})^{[2k]} \cap (E^{[k]})^2}$$

follows from transitivity of $E^{(\lambda+2n)}$. Otherwise, by the above paragraph and up to symmetry, we can assume (a_1, b_1) is the limit of $(\phi(\beta), \phi(\beta_i))$. We have $b_i = b_1$ for all i since $\phi(\alpha - 1)$ is not in E -relation with any other point. We have $a_i \in \{a_1, \phi(\phi^{-1}(a_1) + 1)\}$ for all i (recall that the original relation E is just the successor relation, symmetrized and reflexivized), and by the characterization of $E^{(\lambda+2n)}$, we then have $(a_i, \phi(\beta_i)) \in E^{(\lambda+2n)}$ for any a_i , as required. □

3.2. *Spaces as Toeplitz subshifts.* See e.g. [12] for basic information on Toeplitz subshifts. We recall a basic lemma about Toeplitz subshifts. This is a special case of [7, Theorem 5.23]. (Their term for the formula below is ‘isochronous’, and their term for Toeplitz is ‘regularly almost periodic’.)

LEMMA 8. *Suppose $x \in A^{\mathbb{Z}}$ satisfies*

$$\begin{aligned} &\text{for all } i \in \mathbb{Z}, k > 0 : \text{there exists } j \in \mathbb{Z} : \text{there exists } n > 0 : \text{for all } \ell \in [-k, k], \\ &\quad m \in \mathbb{Z} : x_{j+\ell+mn} = x_{i+\ell}. \end{aligned}$$

Then $\overline{O(x)}$ is Toeplitz.

In words, the assumption is that every subword appearing in x at i appears in some arithmetic progression in x (but not necessarily one going through position i).

Proof. We need to show that the orbit-closure of x contains a Toeplitz point with the same language as x . Let $w_1 = x|_{[-1,1]}$ and let j_1 be such that an arithmetic progression of w_1 s with difference n_1 begins at j_1 . Replace x by $x^1 = \sigma^{j_1}(x)$, so that we still have $w_1 = x^1|_{[-1,1]}$, but now the central three coordinates are in an arithmetic progression. From now on, restrict to shifting by multiples of n_1 .

By the original assumption, the central word $x^1|_{[-n_1-|j_1|-2, n_1+|j_1|+2]}$ is in an arithmetic progression beginning at some j_2 with difference n_2 , we may take $n_1 \mid n_2$. Let $x^2 = \sigma^{h_2 n_1}(x^1)$, where $h_2 = j_2 \parallel n_1 \in \mathbb{Z}$ and where \parallel denotes integer division discarding remainder. Since we shifted by a multiple of n_1 , in x^2 , the word at $[-1, 1]$ still lies in an arithmetic progression with difference n_1 . However, since we shifted by a number with distance at most n_1 from j_2 , now the subword $x^2|_{[-2-|j_1|, 2+|j_1|]}$ lies in an arithmetic progression with difference n_2 . Observe that this contains the word $x|_{[-2,2]}$.

In x^2 , we have inserted Toeplitz periods for more coordinates, without modifying the periods we already introduced for x^1 , and also introduced larger subwords of x into the periodic area. An obvious induction gives a Toeplitz point with the same language as that of x in the limit. □

The *positioned words* over A are $A^{**} = \{w : [a, b] \rightarrow A \mid a, b \in \mathbb{Z}\}$, formally the same as finite words, but we do not identify them up to translations. For

$w \in A^{**}$, $w : [a, b] \rightarrow A$, and $X \subset A^{\mathbb{Z}}$ a subshift (deduced from context), write $[w] = \{x \in X \mid x|_{[a,b]} = w\}$.

LEMMA 4. *For any compact metrizable zero-dimensional space Z , there exists a pointwise zero-entropy Toeplitz subshift X and an open quotient map $\phi : X \rightarrow Z$ such that $\phi^{-1}(z)$ is a minimal subshift for every $z \in Z$, and ϕ induces a homeomorphism between Z and the space of minimal subshifts of X .*

Proof. Define the subshift $X' \subset \{0, 1, \#\}^{\mathbb{Z}}$ by requiring that in every point, there is an arithmetic progression $\{3k + i \mid k \in \mathbb{Z}\}$ containing only #, such that $\{3k + i + 1 \mid k \in \mathbb{Z}\}$ is a constant sequence containing only 0 or only 1, and recursively $\{3k + i + 2 \mid k \in \mathbb{Z}\}$ is a point of X' .

This indeed describes a unique subshift whose typical points look like

$$\dots \#z_0\#\#z_0z_1\#z_0\#\#z_0\#\#z_0z_1\#z_0z_2\#z_0\#\#z_0z_1\#z_0\#\#\dots,$$

where $z \in \{0, 1\}^{\mathbb{N}}$, and we associate a sequence of bits to every point in X' by locating the unique infinite 3-progression of #s (which can be deduced from any word of length 3), outputting the bit to the right of one (thus any) of them, and recursively extracting the rest of the bits from the third 3-progression. Let $\phi : X' \rightarrow \{0, 1\}^{\mathbb{N}}$ extract this sequence of bits, so ϕ is continuous, and $\phi(x) = \phi(\sigma(x))$ for all $x \in X'$.

Since Z is compact, metrizable, and zero-dimensional, we can take (up to homeomorphism) $Z \subset \{0, 1\}^{\mathbb{N}}$ a closed set such that no point in Z is eventually constant, equivalently, 01 appears infinitely many times in every $z \in Z$. Define $X = \phi^{-1}(Z)$, which is clearly a subshift. A simple computation shows that the number of words of length n in X (even in X') is $O(n^{\log_3 6})$, so X has zero entropy.

The map ϕ is surjective and continuous by definition, and it is closed since X is compact and Z Hausdorff, so it is a quotient map from X to Z .

Let us now analyze the dynamical structure of Z . Call the unique infinite 3-progression of #s in a point of X the 0-skeleton. Recursively, we refer to the 0-skeleton of the point of X' obtained after i iterations of the recursive extraction process as the i -skeleton. The skeleton refers collectively to the family of all the i -skeletons, more precisely, the skeleton of $x \in X$ is $\pi(x)$, where $\pi(\#) = \#$, $\pi(0) = \pi(1) = 0$.

Dynamically, extraction of the skeletons gives an almost-one-to-one subshift cover of the 3-odometer $(\mathbb{Z}_3, (a \mapsto a + 1))$ (where \mathbb{Z}_3 is the space of 3-adic integers), more precisely, $\pi(X)$ is a two-to-one, and almost one-to-one, cover of the 3-odometer, by the obvious map $\psi : \pi(X) \rightarrow \mathbb{Z}_3$ that records the offsets of the #-progressions in an intertwining way. What happens is that a typical $\psi(\pi(x))$ determines the skeleton $\pi(x)$ entirely. If it does not, then there is a single ‘hole’ left after filling in the #s. If there is a hole left, then $|\psi^{-1}(\psi(\pi(x)))| = 2$.

It is easy to see that if $\psi^{-1}(\psi(\pi(x)))$ is a singleton, then for any $z \in Z$, there is exactly one point $x' \in X$ with $\phi(x') = z$ and $\pi(x') = \pi(x)$, since after filling in the skeleton and the bits to the right of it, there is no hole left to insert a #, thus no hole to insert a bit either. If there is a hole left, then we can insert any of 0, 1, #, but all other cells are determined by $\psi(\pi(x))$ and $\phi(x)$.

It follows that the subshift X itself is a three-to-one and almost-one-to-one subshift cover of the system $\mathbb{Z}_3 \times Z$ under $(\alpha, z) \mapsto (\alpha + 1, z)$, where \mathbb{Z}_3 denotes the 3-odometer.

We now show the openness of ϕ . Suppose $x \in [u]$ and $\phi(x) = z$. Pick $x' \in [u]$ such that $\phi(x') = z$ and

$$\text{for all } y, y' : \pi(y') = \pi(y) = \pi(x') \wedge \phi(y) = \phi(y') \implies y = y'.$$

This is possible because $[u]$ determines only part of the skeleton of x , and we may modify the rest to find x' such that $\pi(x')$ is a singleton ψ -fiber. The bits left in u after determining part of the skeleton do not constrain this since there are no eventually constant points in Z ; that is, if there is a hole left in x after determining the skeleton and filling in the bits of z , and this hole is filled with $a \in \{0, 1, \#\}$, then we can pick x' so that this hole is filled with a , at the point of the process where a suitable bit emerges.

Now pick $w \in \{0, 1\}^*$, with $z \in [w]$, long enough so the bits in u are determined by those in w (when using the skeleton of x'), so that for all $z' \in [w]$, there exists a (unique) point $y \in [u]$ with $\pi(y) = \pi(x') \wedge \phi(y) = z'$. This means $\phi([u]) \supset [w] \ni z$. Thus ϕ is open.

To see that every point generates a Toeplitz subshift, let $x \in X$ be arbitrary and suppose $x \in [u]$ for some $u \in A^{**}$, $u : [a, b] \rightarrow A$. If $\psi^{-1}(\psi(\pi(x)))$ is a singleton, that is, $\pi(x)$ and $\phi(x)$ determine x uniquely, then clearly there exists a period $3^n > 0$ such that $\sigma^{i3^n}(x) \in [u]$ for all $i \in \mathbb{Z}$, that is, x is already Toeplitz. In general, it suffices to show the condition from the previous lemma, so we show that the word u appears in some arithmetic progression in x .

For this, note that since $\pi(X)$ is an almost-one-to-one cover of \mathbb{Z}_3 and \mathbb{Z}_3 is minimal, we can find j such that the set $\psi^{-1}(\psi(\pi(\sigma^j(x))))$ has very small diameter. In particular, we can ensure that the symbols in $\pi(x)_{[a,b]}$ are uniquely determined by $\psi(\pi(\sigma^j(x)))$. It follows (as discussed in the proof) that the skeleton $\psi(\pi(\sigma^j(x)))$ and finitely many bits in $\phi(x)$ actually determine the word $\sigma^j(x)_{[a,b]}$ uniquely. Since $\phi(x)$ is not eventually constant, we can pick j so that together with the bits of z , the forced word is $\sigma^j(x)_{[a,b]} = u$. The subword $[j + a, j + b]$ of x then appears in an arithmetic progression because of the equicontinuity properties of the 3-odometer, as required.

Observe now that (since every Toeplitz subshift is minimal) every point generates a minimal subshift, and this subshift maps to a single point in ϕ . Conversely, the language of a minimal subshift is determined by the bits in its image in a continuous way (again use that no point in Z is eventually constant). Thus the fibers of ϕ are precisely the minimal subshifts of X and ϕ induces a homeomorphism between them and Z . □

3.3. Relations as entropy pairs. Next, we realize relations as entropy pairs. We first deal with the easy case where E is not doubly shift-invariant, but we only realize the relation, not the whole process. We prove this statement as it sounds nicer than the technical result we actually need, and illustrates some of the ideas used in the following section. However, it is not used in the construction of CPE ranks.

For $y \in A^{\mathbb{Z}}$, write $w \sqsubset y \iff$ there exists $i : y_{[i, i+|w|-1]} = w$, and for $Y \subset A^{\mathbb{Z}}$, write $w \sqsubset Y \iff$ there exists $y \in Y : w \sqsubset y$.

PROPOSITION 6. *Suppose X has zero entropy and $E \subset X^2$ is a closed symmetric reflexive shift-invariant relation. Then there exists a subshift Y with $X \subset Y \subset \hat{A}^{\mathbb{Z}}$ (over a possibly larger alphabet $\hat{A} \supset A$) such that $EP(Y) \cap X^2 = E$.*

Proof. Let $\hat{A} = A \cup \{\#\}$, fix a sequence $(x^n, y^n) \in E$ such that each pair $(x, y) \in E$ is a limit point of the sequence, and define

$$Y_n = \{ \dots \#u_i\#u_{i+1}\#u_{i+2}\# \dots \mid \text{for all } i : u_i \in \{x^n_{[-n,n]}, y^n_{[-n,n]}\} \}$$

(that is, Y_n is the shift-invariant set of all points of the stated form). Let $Y = \overline{\bigcup_n Y_n}$.

If $(x, y) \in E \setminus \Delta$, then obviously by picking larger and larger n such that (x^n, y^n) is very close to (x, y) , $Y_n \subset Y$ implies that $(x, y) \in EP(Y)$. (This is obvious from the intuition stated in the introduction: even if we only obtain information when the orbit is very close to x and y , we obtain an exponential amount of information from points of Y_n if $d((x_n, y_n), (x, y))$ is very small. The concrete computation is also straightforward.)

Suppose then that $(x, y) \in X^2 \setminus E$. Since E is closed, for some n , we have $([x_{[-k,k]}] \times [y_{[-k,k]}]) \cap E = \emptyset$. We claim that for $U = [x_{[-k,k]}]$, $V = [y_{[-k,k]}]$, the cover $(C_0, C_1) = (U^c, V^c)$ has zero topological entropy. By definition, to show this, we must give for each m , a set of binary words $W \subset \{0, 1\}^m$, whose size grows subexponentially in m , such that for any $z \in Y$, there exists $w \in W$ such that $\sigma^i(z) \in C_{w_i}$ for all $i \in \{0, 1, 2, \dots, m-1\}$. In other words, we must cover all partial orbits of length m with a subexponential (in m) number of sets of the form $\bigcap_{i=0}^{m-1} \sigma^{-i}(C_{w_i})$.

Since the sets C_i are clopen, we can equivalently consider a covering problem for words, that is, it is sufficient to find a set of binary words $W \subset \{0, 1\}^m$ such that for each $t \in \mathcal{L}_m(Y)$, there exists $w \in W$ such that $(U^c, V^c)^w$ intersects $[t]$, where $(U^c, V^c)^u$ is defined inductively on word length by

$$(U^c, V^c)^0 = U^c, (U^c, V^c)^1 = V^c, (U^c, V^c)^{a \cdot w} = (U^c, V^c)^a \cap \sigma^{-1}((U^c, V^c)^w)$$

for $a \in \{0, 1\}$, $w \in \{0, 1\}^m$. In fact, we need at most $(|A| + 1)^{2k}$ times more sets to cover the partial orbits than for this covering problem for words, and this constant factor is subsumed by subexponential growth. Let us say $w \in \{0, 1\}^m$ covers $t \sqsubset \mathcal{L}_m(Y)$ if $(U^c, V^c)^w \cap [t] \neq \emptyset$.

We first note that the words t containing at most one $\#$ are easy to cover. For such $t \sqsubset Y$, we always have $t \sqsubset X$ or $t = u\#v$, where u and v are words in X . All such words appear in a morphic image of the zero entropy subshift $X^2 \times \overline{O(\dots 000111 \dots)}$, so they cannot generate entropy with respect to any cover. More concretely, letting $f(m)$ be the number of words of length m in X , the number of words of length m of the forms $t \sqsubset X$ or $u\#v$, where $u, v \sqsubset X$, is at most $f(m) + \sum_{i=0}^m f(i)f(m-i-1)$. If f grows subexponentially, its self-convolution grows subexponentially as well. Thus for each word of these two forms, we can simply include an arbitrary covering word in W .

To cover words with at least two $\#$ -symbols, observe that in Y , all such words have an arithmetic progression of $\#$ s. Consider a progression $\alpha = \{i(2n+2) + c \mid i \in \mathbb{Z}\} \cap \{0, 1, 2, \dots, m-1\}$, where $n \geq 0$, $c \in [0, 2n+1]$ and $|\alpha| \geq 2$. Then the set of words where $\#$ s appear exactly in this arithmetic progression consists precisely of words obtained

by inserting any combination of $x^n_{[-n,n]}$ or $y^n_{[-n,n]}$ between $i(2n + 2) + c$ and $(i + 1)(2n + 2) + c$ for each i (and we have up to four choices in total for the prefix and suffix).

Since $(x^n, y^n) \in E$, by shift-invariance also $(\sigma^\ell(x^n), \sigma^\ell(y^n)) \in E$ for all $\ell \in \mathbb{Z}$ and thus from $([x_{[-k,k]}] \times [y_{[-k,k]}]) \cap E = \emptyset$, we obtain that for all ℓ ,

$$\{x^n_{[\ell-k,\ell+k]}, y^n_{[\ell-k,\ell+k]}\} \neq \{x_{[-k,k]}, y_{[-k,k]}\},$$

and thus see that there is a single word $w_{n,c}$ of length m that covers every word t with progression α . This gives a polynomial bound on the number of words w needed to cover those t with at least two #s, all in all, we have obtained a subexponential bound. \square

3.4. Doubly invariant relations as entropy pairs

LEMMA 5. *Let $X \subset A^{\mathbb{Z}}$ be a subshift with zero entropy and let $E \subset X^2$ be a closed doubly shift-invariant symmetric reflexive relation. Then there exists a subshift Y with $X \subset Y \subset \hat{A}^{\mathbb{Z}}$ (over a possibly larger alphabet $\hat{A} \supset A$) such that $\text{EP}(Y)^{(\lambda)} \cap X^2 = E^{(\lambda)}$ for all λ . Furthermore, if $E^{(\lambda)} \cap X^2 = X^2$, then $\text{EP}(Y)^{(\lambda+1)} = Y^2$, and we have $\text{EP}(Y)^{(\lambda)} = Y^2$ if:*

- λ is even; or
- λ is odd and $(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2}$.

Proof. Define $w_0 = 1$ and inductively $w_{i+1} = w_i 0^{i+1} w_i$. Define

$$w_\omega = \lim_i w_i \in \{0, 1\}^{\mathbb{N}} = 1010010100010100101000010100101000010100101 \dots$$

Another description is that w_ω is obtained from the ruler sequence A007814 in OEIS [13] or universal counterexample [6]

$$01020103010201040102010301020105 \dots$$

by applying the substitution $n \mapsto 10^{n+1}$ (or $0 \mapsto 1, n \mapsto 0^n$). We refer to the words w_i and their obvious positioned variants as *full words* and refer to the process of extending a positioned $w_i \in A^{**}$ to $w_{i+1} \in A^{**}$ (which can be done in exactly two ways) as the *left or right full extension*.

By induction $n_i = |w_i| = 3 \cdot 2^i - i - 2$ and $\sum_j (w_i)_j = 2^i$. A short computation shows that for every $k \geq 1$, there exists a subword of w_ω of length k (for example its prefix), where the symbol 1 appears at least $k/3$ times (any k/c works just as well for what follows).

We now perform some substitutions to obtain a subshift. Let $\tau(0) = \{0\}$, $\tau(1) = \{1, 11\}$ be a non-deterministic substitution and define $Y''' = \tau(w_\omega) \subset \{0, 1\}^{\mathbb{N}}$, that is, Y''' is the set of all possible points obtained by replacing 0s in w_ω by 0s, and 1s by either 1 or 11. Let $\hat{A} = A \cup \{\#\}$ where $\# \notin A$, and let $\tau' : \hat{A} \rightarrow \{0, 1\}$ be the deterministic substitution mapping $\# \mapsto 1$ and $a \mapsto 0$ for $a \in A$. Let $Y'' = (\tau')^{-1}(Y''')$. Let Y' be the \mathbb{Z} -subshift whose language consists of the left-extendable words in the language of Y'' . Let Y be the subshift of Y' with additional forbidden words

$$\{u_0 \# u_1 \# \dots \# u_k \mid \text{there exists } x_0, x_1, \dots, x_k \in X : \text{for all } i, j : x_i \in [u_i] \wedge (x_i, x_j) \in E\}$$

(where the u_i are quantified over A^*). The double occurrences of # fit this pattern by picking empty words u_i .

Say that a set of words $U \subset A^*$ is a *friendship* if for any finite subset $V \subset U$ there exist points $(b_v)_v \in X^V$ such that for all $u, v \in V : b_u \in [u] \wedge (b_u, b_v) \in E$, so since E is doubly shift-invariant, the forbidden words of Y precisely require that the set of words $U \subset A^*$ that appear in y is a friendship. (We remark that even for a closed doubly shift-invariant equivalence relation, for a finite set U , in general, being a friendship is stronger than all pairs $\{u, v\}$ being friendships for $u, v \in U$.)

For a concrete picture of points in Y' , consider the sequence $w_\omega \in \{0, 1\}^{\mathbb{N}}$. Its two-sided orbit closure contains points of the form

... 10100101000010100101000010100101000010100101 ...

We allow any of the 1s to be duplicated, e.g.

... 101100101000110110010110000101001101000101001011 ... ,

and then replace the 1s by #, that is,

... #0##00#0#000##0##00#0##0000#0#00##0#000#0#00#0## ... ,

finally replace maximal 0-sequences by words over A of the same length, e.g.

... #0##01#0#011##1##10#1##0100#1#01##1#101#1#11#1## ... ,

if $A = \{0, 1\}$. Such are the points of Y' . In points of Y , the finite words should additionally be from the language of X , and should form a friendship.

If $U \subset A^*$ is a friendship, then by compactness to every word $u \in U$, we can associate a point $b_u \in [u]$, so that $(b_u, b_v) \in E$ for all $u, v \in U$. Fix such points for each set U and call b_u the (*friendship*) *bracelet of u (with respect to U)*. Of particular importance in what follows are (sets of) bracelets for sets of words $U \subset A^*$ that appear in some point $y \in Y$, or the union of such sets for two points y, z . (It is worth noting that in such a situation, the bracelets are not enforced to be in the orbit closure of y or z .)

Call the binary word $\tau'(w)$ recording positions of # by 1 in a word w the *skeleton*, similarly for points $x \in Y$, and the word where each maximal occurrence of 11 is further replaced by 1 the *preskeleton*. We call a (positioned or not) word $w \in \hat{A}^* \cup \hat{A}^{\mathbb{Z}}$ *full* if it is equal to (a shift of) $(\tau')^{-1}(w_n)$ for some n , or if (more generally) its preskeleton is full.

We now explain the technical trick for characterizing the entropy pairs $EP(Y)$, and show that $(y, z) \notin \Delta_Y$ is an entropy pair if and only if the set

$$U = \{u \in A^* \mid u \sqsubset y \vee u \sqsubset z\},$$

containing all non-# subwords from y and z , is a friendship.

Suppose $u', v' \in \hat{A}^{**}$ are any two positioned words that appear in some points of Y , with # as their first and last symbols (note that words beginning and ending with # are dense in Y in the natural topology of $\hat{A}^{**} \cup \hat{A}^{\mathbb{Z}}$), and let U be the set of words containing $u \in A^*$ if and only if $\#u\# \sqsubset u'$ or $\#u\# \sqsubset v'$. Suppose U is a friendship. Extend u' and v' to full positioned words whose preskeletons are the same length, in an arbitrary way, though so that their finite subwords over A^* remain a friendship (safe continuations for words can

always be found in the bracelets proving the friendship). The left and right ends of the resulting words u'' and v'' have some left and right offsets n_1, n_2 , respectively, in the sense that $u'' \in A^{[a,b]}$ and $v'' \in A^{[a+n_1,b+n_2]}$.

Now, continue extending the words by alternately turning them into full positioned words by adding a left and a right full extension, so that no new occurrence of $\#\#$ appears, and all finite subwords over A^* remain a friendship. Note that the left and right ends continue to differ by exactly n_1 and n_2 , since the length of the left or right continuation is always the same if no occurrence of $\#\#$ appears. Extend the words so many times that at least $\max(|n_1|, |n_2|)$ undoubled $\#$ s are obtained on both sides. Now finally double some of the $\#$ to get two positioned full words u, v which have u' and v' at the center, have the same preskeleton, and have the same left and right length, that is, $u, v \in A^{[c,d]}$ for some $c, d \in \mathbb{Z}$.

Now suppose u and v have preskeleton w_i . By the analysis of the density of 1s in the ruler sequence, for each k , we can find a word $w \in (\hat{A} \cup \{_\})^*$ of length at most $2k$ such that:

- the $_$ -symbols appear in segments of length exactly $|u| = |v|$;
- there are at least $k/(3 \cdot 2^i) - 2$ such segments; and
- replacing the $_$ ^{$|u|$} -subwords individually by u or v always leads to a valid word of Y .

To see this, recall that for all k , we can find a word of length k in the ruler sequence with at least $k/3$ many 1-symbols. Observing that all 1s appear in disjoint w_i -words, we can have at least $k/(3 \cdot 2^i) - 2$ occurrences of w_i in a word w' of length k in the ruler sequence. Replacing these occurrences of w_i by $_$ ^{$|u|$} in w' , and then filling the remaining gaps with arbitrary words from the bracelets of words in U , we obtain a word w with the desired properties.

The interchangeability, together with the lower bound on maximal density, shows that the cover $([u]^c, [v]^c)$ generates entropy in Y , and thus we have shown that any points whose non- $\#$ subwords form a friendship are entropy pairs. However, if the union $U \subset A^*$ of the sets of subwords of two words u, v is not a friendship, then these words do not even appear together in any point of Y , by definition, so neither can u and v , and it follows that the cover $([u]^c, [v]^c)$ has zero entropy. This concludes the characterization of $EP(Y)$.

An important corollary of this characterization is that $EP(Y)$ is doubly shift-invariant, and therefore so is any $EP(Y)^{(\lambda)}$ by an obvious transfinite induction. If $(x, y) \in F$ and F is a doubly shift-invariant closed relation, then $(x', y) \in F$ for any $x' \in O(x)$. Let us call a doubly shift-invariant equivalence relation F with this property *hereditary*. It is easy to show by transfinite induction that if a doubly shift-invariant equivalence relation F is hereditary, then so is $F^{(\lambda)}$, for any ordinal λ . In particular, the relations $EP(Y)^{(\lambda)}$ are all hereditary since $EP(Y)$ is doubly shift-invariant and closed.

It is clear from the characterization that $EP(Y) \cap X^2 = E$ as required. We need to show $EP(Y)^{(\lambda)} \cap X^2 = E^{(\lambda)}$ for all λ , and we do this by transfinite induction. On limit ordinal steps, this obviously continues to hold. On even successor steps, that is, transitive closure steps, we use the fact that points of Y can be replaced by any point in their orbit closure, and the orbit-closure of every point contains a point of X . This allows turning any transitivity sequence $(x = y_0, y_1, \dots, y_k = x')$, where $x, x' \in X, y_i \in Y, (y_i, y_{i+1}) \in E^{(\lambda-1)}$, into

one where $y_i \in X$ for all i . This shows $(EP(Y)^{(\lambda-1)})^* \cap X^2 = (EP(Y)^{(\lambda-1)} \cap X^2)^*$ as required.

Now, note that if $EP(Y)^{(\lambda)}$ is transitively closed, $y \in Y \setminus X$ and b is any bracelet of y , or more generally, any point in X whose set of A^* -subwords together with A^* -subwords of Y forms a friendship, then we have $(y, z) \in EP(Y)^{(\lambda)}$ if and only if $(b, z) \in EP(Y)^{(\lambda)}$. This is because $(b, y) \in EP(Y)$ by the characterization of entropy pairs of Y . In other words, on every transitive step, every $y \in Y$ satisfies the same relations as any of its bracelets or any point that could be chosen as a bracelet for it.

On a topological closure step, that is, for an odd ordinal λ , suppose $\lim_i(x_i, y_i) = (x, y) \in X^2$ where $(x_i, y_i) \in EP(Y)^{(\lambda-1)}$ for all i . It is enough to show that the x_i can be replaced by points from X , as then (also applying this idea to the right component) we have $\overline{EP(Y)^{(\lambda-1)} \cap X^2} = \overline{EP(Y)^{\lambda-1} \cap X^2}$. If there is a subsequence of the x_i in X , then we can directly restrict to that. Otherwise, restrict to a subsequence such that $x_i \notin X$ for all i .

In x_i , take a maximal central positioned word $u_i : A^{[-a,a]}$ not containing $\#$ (where necessarily $a \rightarrow \infty$ as $i \rightarrow \infty$) and replace x_i by the bracelet b_{u_i} with respect to the language of x_i , centered to have u_i in the same place as in x_i . We have $(b_{u_i}, y_i) \in EP(Y)^{(\lambda-1)}$ since $EP(Y)^{(\lambda-1)}$ is transitively closed and $(b_{u_i}, x_i) \in EP(Y)$, and clearly $(b_{u_i}, y_i) \rightarrow (x, y)$, as required.

Finally, we verify the last claim. Suppose thus that $E^{(\lambda)} = X^2$ for some λ . If λ is even, then $EP(Y)^{(\lambda)}$ is transitively closed so since any $y \in Y \setminus X$ appears in the same relations as any bracelet of its, $EP(Y)^{(\lambda)} = Y^2$.

Suppose then that λ is odd. Let $(y, z) \in Y^2$, and without loss of generality suppose $y, z \in Y \setminus X$ (this suffices since $EP(Y)^{(\lambda)}$ is closed and $X \subset \overline{Y \setminus X}$). Let $(b_u)_u$ and $(c_v)_v$ be the respective bracelets of their subwords with respect to their languages. If any two of these bracelets, one from y and one from z , are in relation on step $\lambda - 1$, then already $(y, z) \in EP(Y)^{(\lambda-1)}$ since $EP(Y)^{(\lambda-1)}$ is transitive.

Otherwise, let m be arbitrary and let (up to swapping y and z) $n \geq m$ and k be such that $y_{[-m,m]} = u_1\#u_2\#\dots\#u_k$, $U = (u_1, \dots, u_k) \in (A^*)^k$ and $z_{[-m,n]} = v_1\#v_2\#\dots\#v_k$, $V = (v_1, \dots, v_k) \in (A^*)^k$. Let $B = (b_u)_{u \in U}$ and $C = (c_v)_{v \in V}$ be bracelets for these words with respect to the subwords of y and z , respectively.

Now, recall that the bracelets of a point are always (by definition) in the E -relation together, and apply the odd condition to $(B, C) \in (E^{(k)})^2$. Since $(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2}$, we can find in $(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2$ arbitrarily good approximations $B' = (b'_u)_{u \in U}$ and $C' = (c'_v)_{v \in V}$ to B and C , respectively. In particular, we can find good enough approximations to ensure that b'_u contains u and c'_v contains v , for each $u \in U, v \in V$. Then since $EP(Y)^{(\lambda-1)}$ is transitive and $(b'_u, c'_v) \in E^{(\lambda-1)}$, any pair of points (y_k, z_k) in Y , for which we can take b'_u and c'_v as one of their bracelets, satisfies $(y_k, z_k) \in EP(Y)^{(\lambda-1)}$.

We construct the pair y_k, z_k so that $(y_k, z_k)_{[-k,k]} = (y, z)_{[-k,k]}$. For this, directly let $(y_k)_{[-m,m]} = u_1\#u_2\#\dots\#u_k$ and $(z_k)_{[-m,n]} = v_1\#v_2\#\dots\#v_k$. Extend y_k to a point of Y by copying the skeleton from y and filling all the remaining gaps arbitrarily so that all words between two $\#$ s (and possible infinite tails without an occurrence of $\#$) are subwords of the words in B' . Extend z_k similarly. These are indeed valid points: the skeletons are correct since we copied them from valid points, and the set of A^* -subwords of each point

is a friendship, since $B', C' \in E^{[k]}$. Clearly we can take any b'_u and c'_v as one of their bracelets, so $(y_k, z_k) \in EP(Y)^{(\lambda-1)}$ gives an approximation to (y, z) that is correct in the interval $[-m, m]$. □

3.5. *Implementation of arbitrary CPE ranks.* For the main proof, we make some simple topological observations.

LEMMA 9. *Suppose X, Y are first-countable topological spaces, and $f : X \rightarrow Y$ is a quotient map. Then the following are equivalent:*

- *f is open;*
- *whenever $\lim_i y_i = y \in Y$ and $f(x) = y$, then there exist $x_i \in X$ such that $f(x_i) = y_i$ and $\lim_i x_i = x$.*

Proof. Suppose f is open and $f(x) = y$. Then every neighborhood $U \ni x$ maps to a neighborhood $f(U)$ of y , in particular, if $\lim_i y_i = y$, then for all large enough i , y_i has a preimage x_i in U . Picking open sets U from a decreasing countable neighborhood basis of x , and always picking preimages for y_i from U until they start having preimages in the next neighborhood of x , the claim is proved.

For the other direction, suppose f is not open, and let U be open such that for some $y \in f(U)$, every neighborhood of y contains a point without a preimage in U . Pick $x \in U$ such that $f(x) = y$. By first-countability, there is a sequence $y_i \rightarrow y$ such that the points y_i have no preimage in U . The second condition fails, since we cannot pick $x_i \in U$. □

The following lemmas explain why we want ϕ to be open.

LEMMA 10. *Let X, Y be first-countable topological spaces. Let $f : X \rightarrow Y$ be an open quotient map, $E \subset Y^2$ a symmetric reflexive closed relation, and $F = (f^{-1} \times f^{-1})(E)$. Then $F^{(\lambda)} = (f^{-1} \times f^{-1})(E^{(\lambda)})$ for all ordinals λ .*

Proof. Since X is first-countable, so is X^2 , and thus the closure of a set $C \subset X^2$ is just the set of limits of sequences in C . (In other words, X^2 has the *Fréchet–Urysohn* property. For general topological spaces, even if X is Fréchet–Urysohn, X^2 need not be [8].)

Write $g = f \times f : X^2 \rightarrow Y^2$, and observe this is also an open quotient map. We have $F^{(\zeta)} = g^{-1}(E^{(\zeta)})$ where $\zeta = 0$ if E is not closed, and $\zeta = 1$ otherwise, since E is closed if and only if F is closed since g is a quotient map. We proceed by induction. Suppose λ is an odd ordinal, and $g(x, x') = (y, y')$. If $(x, x') \in F^{(\lambda)}$, then there exist $(x_i, x'_i) \in F^{(\lambda-1)}$ with $(x_i, x'_i) \rightarrow (x, x')$ and then $g(x_i, x'_i) \rightarrow (y, y')$ by continuity showing $E^{(\lambda)} \supset g(F^{(\lambda)})$, which implies $g^{-1}(E^{(\lambda)}) \supset F^{(\lambda)}$.

For the other inclusion, suppose $(y, y') \in E^{(\lambda)}$, so $(y, y') = \lim_i (y_i, y'_i)$ for some $(y_i, y'_i) \in E^{(\lambda-1)}$. By the previous lemma, there exist $(x_i, x'_i) \in X^2$ such that $g(x_i, x'_i) = (y_i, y'_i)$ and $(x_i, x'_i) \rightarrow (x, x')$. Then by induction, we have $(x_i, x'_i) \in F^{(\lambda-1)}$ for all i , so $(x, x') \in F^{(\lambda)}$, showing $g^{-1}(E^{(\lambda)}) \subset F^{(\lambda)}$.

If λ is an even successor ordinal, suppose $g(x, x') = (y, y')$. If $(x, x') \in F^{(\lambda)}$, there exist $x = x_0, x_1, \dots, x_k = x'$ such that $(x_i, x_{i+1}) \in F^{(\lambda-1)}$ for all i , and then

by induction, $g(x_i, x_{i+1}) \in E^{(\lambda-1)}$ and since $y = f(x_0), y' = f(x_k)$, we have $(y, y') \in E^{(\lambda)}$.

If however $(y, y') \in E^{(\lambda)}$, then there exist $y = y_0, y_1, \dots, y_k = y'$ with $(y_i, y_{i+1}) \in E^{(\lambda-1)}$. Pick any preimages $f(x_i) = y_i$ with $x_0 = x, x_k = x'$ by induction $(x_i, x_{i+1}) \in F^{(\lambda-1)}$ for all i , and thus $(x, x') \in F^{(\lambda)}$.

For the limit ordinal case, observe that union commutes with preimage. □

LEMMA 11. *Let X, Y be first-countable. Let $f : X \rightarrow Y$ be an open quotient map, $E \subset Y^2$ a symmetric reflexive relation, and $F = (f^{-1} \times f^{-1})(E)$. Then*

$$(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2}$$

if and only if

$$(F^{[k]})^2 \subset \overline{(F^{(\lambda-1)})^{[2k]} \cap (F^{[k]})^2}.$$

Proof. Suppose $(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2}$ and consider $(A', B') \in (F^{[k]})^2$. Consider $f(A')$ and $f(B')$ (where f is applied diagonally to all elements of the tuple), which are by definition k -tuples $A, B \in E^{[k]}$. In $(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2$, we find arbitrarily good approximations (A_i, B_i) to (A, B) . Take any f -preimages, $f(A'_i) = A_i, f(B'_i) = B_i$ which are close to A', B' elementwise, using Lemma 9, obtaining by definition $A'_i, B'_i \in F^{[k]}$. By the previous lemma, we have $(F^{(\lambda-1)})^{[2k]} = (f \times f)^{-1}(E^{(\lambda-1)})^{[2k]}$ so we have $(A'_i, B'_i) \in (F^{(\lambda-1)})^{[2k]}$. This implies $(F^{[k]})^2 \subset \overline{(F^{(\lambda-1)})^{[2k]} \cap (F^{[k]})^2}$.

The other direction is similar, but using continuity instead of openness. □

THEOREM 3. *Let $\alpha \geq 1$ be a countable ordinal. Then there exists a subshift $X \subset A^{\mathbb{Z}}$ in CPE class α .*

Proof. If $\alpha = 1, X = A^{\mathbb{Z}}$ is an example.

Let $\lambda \geq 2$ be a countable ordinal. By Lemma 7, there exists a closed countable set $Z \subset [0, 1]$ and a closed symmetric reflexive relation $E \subset Z^2$ such that E is equalizing with close-up rank λ . Furthermore, if λ is an odd ordinal, then $(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2}$.

By Lemma 4, since every countable compact subspace of $[0, 1]$ is zero-dimensional, there exists a pointwise zero-entropy Toeplitz subshift X and a doubly shift-invariant closed equivalence relation $F \subset X^2$ such that Z is the open quotient of X , by $\phi : X \rightarrow Z$, and $(\phi(x), \phi(y)) \in E \iff (x, y) \in F$.

Since ϕ is an open quotient map, Lemma 10 shows that $E^{(\lambda)} = Z^2$ if and only if $(\phi^{-1} \times \phi^{-1})(E) = F$ satisfies $F^{(\lambda)} = X^2$. If λ is an odd ordinal, then $(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2}$, and by the previous lemma, we have $(F^{[k]})^2 \subset \overline{(F^{(\lambda-1)})^{[2k]} \cap (F^{[k]})^2}$.

Now, apply Lemma 5 to obtain $Y \supset X$ such that λ is the minimal ordinal satisfying $EP(Y)^{(\lambda)} = Y^2$. This Y is a subshift in CPE class λ . □

3.6. *Not every relation satisfies odd condition.* We show that not every relation satisfies the odd condition, at least for one-step processes.

Example 1. Let $X = [0, 3]$. For $i, j \in \{0, 1, 2\}$, let $X_{i,j} \subset [i, j + 1]$ be a dense set such that $X_{i,j} \cap X_{i',j'} = \emptyset$ for $(i, j) \neq (i', j')$.

Indexing modulo 3, let $R_i = X_{i,i+1}^2$. Now clearly $E = R_0 \cup R_1 \cup R_2 \cup \Delta_X$ is reflexive, symmetric, and transitive but is not closed. The close-up rank is 1: $E^{(0)} = E$ and $E^{(1)} = [0, 3]^2$.

The odd condition $(E^{[2]})^2 \subset \overline{E^{[4]} \cap (E^{[2]})^2}$ fails because $(E^{[2]})^2$ contains quadruples $(x, y, z, w) \in [0, 1] \times [1, 2] \times [1, 2] \times [2, 3]$, while $E^{[4]}$ only contains quadruples contained in $[0, 2]^4 \cup [1, 3]^4 \cup ([0, 1] \cup [2, 3])^2$.

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