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BOOK REVIEW

COMPUTATIONAL METHODS FOR THE STUDY OF DYNAMIC ECONOMIES

Edited by Ramon Marimon and Andrew Scott
Oxford University Press, 1999

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This book, according to its introduction, aims "to help researchers who know how to formulate an economic model but need computational techniques to obtain quantitative solutions" (p. 6). In this it succeeds admirably, but it also does more. The final three chapters give good surveys of specific economic issues that have been successfully studied using computational methods.

The book is very much a collection of papers. The editors have succeeded in getting some good chapters into the book, but they seem to have put no serious effort into making the book a coherent whole. Nevertheless, since nearly all of the chapters are well worth reading separately, I recommend the book as a whole.

Who should read this book? One obvious answer is: graduate students working on dynamic macroeconomics. The book grew out of a summer school for European graduate students organized by the European Economic Association at the European University Institute in Florence, Italy, in 1996. Presumably because of its origins, most of the material in the book is pitched at just about the right level for a second- or third-year graduate student. However, it also should serve as a useful reference book for the rest of us. Indeed, I am still learning from it.

How does it relate to other books? Perhaps its closest competitors are those by Cooley (1995) and Judd (1998). In a way, it strikes a balance between those two books. Whereas Judd's book covers about the same ground as a textbook on numerical analysis and Cooley's book focuses on economics rather than on computational mathematics, the Marimon-Scott book contains some rather sophisticated numerical techniques and yet rarely loses sight of useful economic applications.

Part I of the book deals with solution methods based on approximating a model by one with linear first-order conditions. This is, unfortunately, the weakest part of

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the book. Díaz-Giménez's chapter on linear-quadratic dynamic optimization does not improve on the work of Hansen and Prescott (1995) in any important way. The chapter by Novales, Domínguez, Pérez, and Ruiz has almost 30 pages just on solving difference equations.

Uhlig's chapter, though, comes close to redeeming Part I. Rather than just solving linear difference equations, Uhlig takes the reader through the whole project of writing down a model, deriving the sufficient conditions for equilibrium, calibrating, log-linearizing, and finally calculating the moments of the HP-filtered predicted time series using spectral analysis (thus avoiding simulation). At each stage, Uhlig has a distinctive and attractive approach, and his simulation-free approach to calculating moments is ingenious. The style of presentation is eminently clear and readable, and the attention to detail is impressive. The chapter ends with an excellent discussion of the phenomenon of indeterminacy and an appendix describing the extensively documented MATLAB code that Uhlig has written, which can be downloaded from his Web site.

My only complaint about Uhlig's chapter is that it, too, wastes space on just solving linear difference equations. To see the minimal amount of effort required, note that the problem boils down to finding a bounded solution to the following equation:

$$A\begin{bmatrix} x_{t+1} \\ \lambda_{t+1} \end{bmatrix} = B\begin{bmatrix} x_t \\ \lambda_t \end{bmatrix}, \tag{1}$$

where $t = 0, 1, \ldots$ and x_0 is given exogenously.[1] To find a feedback representation of a bounded solution to (1), just a few quick steps are needed. The first is to dig up a good numerical package that can deliver unitary matrices Q and Z and upper triangular matrices S and T such that $QA = SZ^H$ and $QB = TZ^H$, where $S = (s_{ij})$ and $T = (t_{ij})$ are so organized that there is an m such that $|s_{ii}| > |t_{ii}|$ for precisely those i with $i \leq m$. Then, we can check whether the number of elements in x_t is equal to m. If so, we proceed to partition the matrices Z, S, and T so that the upper-left blocks are $m \times m$. Also, define the auxiliary variables y_t and μ_t via

$$\begin{bmatrix} y_t \\ \mu_t \end{bmatrix} = Z^H\begin{bmatrix} x_t \\ \lambda_t \end{bmatrix}. \tag{2}$$

Premultiplying (1) by Q, we get

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}\begin{bmatrix} y_{t+1} \\ \mu_{t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}\begin{bmatrix} y_t \\ \mu_t \end{bmatrix}. \tag{3}$$

Because of the way S and T are organized, boundedness of the solution implies that $\mu_t = 0$ for all $t = 0, 1, \ldots$. But then (3) says that $y_{t+1} = S_{11}^{-1}T_{11}y_t$ and (2) says that $\lambda_t = Z_{21}y_t$ and $x_t = Z_{11}y_t$, but then, a feedback representation of the solution can be written as follows:

$$\begin{cases} x_{t+1} = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1} x_t \\ \lambda_t = Z_{21} Z_{11}^{-1} x_t. \end{cases} \tag{4}$$

Many pages in the book could have been saved by using this approach.

Part II of the book is devoted to solution methods that allow for nonlinearity of the decision rules. Together, Burnside's chapter on discrete state-space methods, Marcet and Lorenzoni's on parameterized expectations, McGrattan's on weighted residual methods, and Candler's on finite-difference methods for continuous-time models form a wide-ranging survey of available methods.

Marcet and Lorenzoni's chapter clearly highlights the great advantage of the basic version of the parameterized expectations approach (PEA): it requires almost no knowledge of the standard tools of numerical analysis. In this sense, the method is ideal for impatient graduate students who want to progress quickly on their thesis work without having to spend much time studying numerical analysis textbooks. The chapter also discusses some practical issues that anyone is bound to run into when using PEA and suggests how to deal with them efficiently. The chapter is deliberately kept at an elementary level, avoiding the theoretical aspects of PEA (Marcet and Marshall, 1992) as well as the sophisticated numerical methods recommended by Christiano and Fisher (1997) that can make PEA more efficient.

Of course, shunning sophisticated numerical methods has a price in terms of computational efficiency. This makes McGrattan's chapter particularly welcome. Rather than avoid the standard tools of numerical analysis, she briskly surveys and makes good use of Gaussian quadrature, piecewise linear and quadratic interpolation, orthogonal polynomials, Newton–Rhapson iterations, and methods for sparse systems of linear equations. In just 29 densely packed pages, she manages to convey what weighted residual methods are (including the many varieties and how to classify them) as well as what basic ideas are needed to apply them to economic models. This is a remarkable pedagogical feat. It must be said that some readers may occasionally want to consult a textbook on numerical analysis to look up the details of some of the concepts, and most will need to try some of the methods on a computer to get a sense of the practical problems involved. However, the chapter gives such an excellent overview that filling in these gaps is not hard.

My only complaint about Part II of the book is that, perhaps inevitably, it is incomplete. For example, none of the authors say much about the problems that arise with multidimensional state spaces. Marcet and Lorenzoni mention in passing that PEA suffers less than the alternatives from the curse of dimensionality. McGrattan mentions using complete sets of polynomials rather than tensor products in order to make the growth of the number of coefficients polynomial rather than exponential in the number of dimensions. However, nowhere is there any detailed discussion of piecewise multilinear functions on $\mathbb{R}^n$ or multidimensional Gaussian quadrature. Nor is there any detailed discussion of the curse of dimensionality and how to deal with it.

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Also, a very popular method for solving dynamic economic problems [e.g., Krusell and Smith (1998)] is not mentioned at all in Part II (although Ríos-Rull mentions it briefly in Part III): fixed-point iteration with interpolation rather than discretization of the state space. In her chapter, McGrattan sets up the following extremely general functional equation:

$$F(f) = 0 \tag{5}$$

where $f\colon \Omega \to \mathbb{R}^m$ is an unknown function and $F\colon C_1 \to C_2$ is a known operator from one function space to another function space. Note that, in dynamic economics, this functional equation typically can be written as

$$F(f)(x) = h(x, f(x), f(f(x))) \tag{6}$$

for some known function h. This immediately suggests defining an operator T so that $T(f)(x)$ is that $y \in \mathbb{R}^m$such that $h(x, y, f(y)) = 0$. The solution to our functional equation evidently is just a fixed point under this operator.

To find an approximation to this fixed point, one typically goes through three steps: (1) specify an initial guess of the values of f on a finite subset $\hat{\Omega} \in \Omega$; (2) specify an interpolation rule in order to evaluate the function elsewhere; and finally, (3) update the guess by setting $F(f)(x)$ to zero at the finitely many points in $\hat{\Omega}$. Then, step (3) is repeated until a criterion of convergence is satisfied. A chapter outlining this approach with a detailed discussion of interpolation rules (piecewise multilinear functions, cubic splines, and so on) would have been a welcome addition to the book.

The book ends with Part III, which presents a set of successful applications of numerical methods in economics. In this part, computational methods are relegated to the background, and the focus is on the economics. Sargent and Velde discuss optimal fiscal policy, and İmrohoroğlu, İmrohoroğlu, and Joines discuss social security in the context of many-period overlapping-generations models. The final chapter is Ríos-Rull's survey of computational work on economies with heterogeneous agents.

The chapter by Sargent and Velde is a fairly good introduction to optimal policy in dynamic models but, given the focus on linear-quadratic economies, the applicability is rather limited. The lecture that Scott gave in Florence would have been a more appropriate choice for this book. Then, again, since we have an excellent textbook-style survey by Chari and Kehoe (1998), we may not need yet another introduction to optimal fiscal policy in dynamic models.

Ríos-Rull's chapter, however, is a welcome improvement and an update on his chapter in Cooley's (1995) book. Without getting too deeply into the technical details, he manages to convey not just the ideas involved in a set of existing papers, but also the impressive power of a certain approach to understanding inequality. In this, Ríos-Rull's chapter epitomizes the spirit of the book, which is to inspire and enable its readers to do exciting research without the arbitrary shackles imposed by the demand for closed-form solutions.

NOTE

1. Because of certainty equivalence, we can ignore uncertainty during the derivations and stick it on again at the end if we want. To avoid indeterminacy, it is customary to let the one-step-ahead prediction error of x_t (but not of λ_t) be given exogenously. The error does not need to be zero, however; this means, for example, that a technology shock may be included in x_t.

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