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Measure equivalence rigidity via s-malleable deformations

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Abstract

We single out a large class of groups $\mathcal{M}$ for which the following unique prime factorization result holds: if $\Gamma_1, \ldots, \Gamma_n \in \mathcal{M}$ and $\Gamma_1 \times \cdots \times \Gamma_n$ is measure equivalent to a product $\Lambda_1 \times \cdots \times \Lambda_m$ of infinite icc groups, then $n \geq m$, and if $n = m$, then, after permutation of the indices, Γ_i is measure equivalent to Λ_i, for all $1 \leq i \leq n$. This provides an analogue of Monod and Shalom's theorem [*Orbit equivalence rigidity and bounded cohomology*, Ann. of Math. **164** (2006), 825–878] for groups that belong to $\mathcal{M}$. Class $\mathcal{M}$ is constructed using groups whose von Neumann algebras admit an s-malleable deformation in the sense of Sorin Popa and it contains all icc non-amenable groups Γ for which either (i) Γ is an arbitrary wreath product group with amenable base or (ii) Γ admits an unbounded 1-cocycle into its left regular representation. Consequently, we derive several orbit equivalence rigidity results for actions of product groups that belong to $\mathcal{M}$. Finally, for groups Γ satisfying condition (ii), we show that all embeddings of group von Neumann algebras of non-amenable inner amenable groups into $L(\Gamma)$ are 'rigid'. In particular, we provide an alternative solution to a question of Popa that was recently answered by Ding, Kunnawalkam Elayavalli, and Peterson [*Properly Proximal von Neumann Algebras*, Preprint (2022), arXiv:2204.00517].

1. Introduction

Classifying countable groups up to *measure equivalence* is a central topic in measured group theory that has witnessed an explosion of activity for the last 25 years, see the surveys [Sha05, Fur11, Gab10] and the introduction of [HHI21]. The notion of measure equivalence has been introduced by Gromov [Gro93] as a measurable analogue to the geometric notion of quasi-isometry between finitely generated groups. Specifically, two countable groups Γ and Λ are called *measure equivalent* if there exist commuting free measure-preserving actions of Γ and Λ on a standard measure space (Ω, m) such that the actions of Γ and Λ on (Ω, m) each admit a finite measure fundamental domain. Natural examples of measure equivalent groups are two lattices in a locally compact second countable group.

Measure equivalence can be studied through the lenses of orbit equivalence due to the fundamental result that two countable groups are measure equivalent if and only if they admit free ergodic probability measure-preserving (pmp) actions that are stably orbit

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equivalent [Fur99]. Recall that two pmp actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are called *stably orbit equivalent* if there exist non-null subsets $A \subset X, B \subset Y$ and a measure space isomorphism $\theta : A \to B$ such that $\theta(\Gamma x \cap A) = \Lambda\theta(x) \cap B$, for almost every $x \in A$. If $\mu(A) = \nu(B) = 1$, then the two actions are called *orbit equivalent* (OE).

The celebrated work of Ornstein and Weiss [OW80] (see also [Dye59, CFW81]) proves that any two free ergodic pmp actions of infinite amenable groups are OE and, consequently, any two infinite amenable groups are measure equivalent. In sharp contrast, classifying non-amenable groups up to measure equivalence is a much more challenging task and it reveals a very strong rigidity phenomenon. By building on Zimmer's work [Zim84], Furman showed that any countable group which is measure equivalent to a lattice in a higher rank simple Lie group is essentially a lattice in the same Lie group [Fur99]. Then Kida showed that most mapping class groups $\mathrm{Mod}(S)$ are *measure-equivalent superrigid* which means that any countable group that is measure equivalent to $\mathrm{Mod}(S)$, must be virtually isomorphic to it [Kid10]. Subsequently, other such measure-equivalent superrigid groups have been found and we refer the reader to the introduction of [HH22] for more details.

There have been discovered several other remarkable instances where various aspects of the group Γ can be recovered from its measure equivalence class or certain properties of the group action $\Gamma \curvearrowright (X,\mu)$ are remembered by its associated orbit equivalence relation. We only highlight the following developments in this direction and refer the reader to the surveys [Sha05, Fur11] for more information. Gaboriau used the notion of cost to show that the rank of a free group $\mathbb{F}_n$ is an invariant of the orbit equivalence relation of any of its free, ergodic, pmp actions [Gab00]. Then his discovery that measure equivalent groups have proportional ℓ^2-Betti numbers [Gab02] led to significant new progress in the classification problem of pmp actions up to OE, see the survey [Gab10]. Using a completely different conceptual framework, Popa's deformation rigidity/theory [Pop07a] led to an unprecedented development in the theory of von Neumann algebras and provided many other spectacular rigidity results in orbit equivalence, see the surveys [Vae10, Ioa14, Ioa18].

In their breakthrough work [MS06], Monod and Shalom employed techniques from bounded cohomology theory to obtain a series of OE rigidity results, including the following unique prime factorization result: if $\Gamma_1 \times \cdots \times \Gamma_n$ is a product of non-elementary torsion-free hyperbolic groups (more generally, of groups belonging to class $\mathcal{C}_{\text{reg}}$; see [MS06, Notation 1.2]) that is measure equivalent to a product $\Lambda_1 \times \cdots \times \Lambda_m$ of torsion-free groups, then $n \geq m$, and if $n = m$, then, after permutation of the indices, Γ_i is measure equivalent to Λ_i, for all $1 \leq i \leq n$. By building upon C*-algebraic methods from [Oza04, BO08], the above unique prime factorization result has been extended by Sako [Sak09] to products of non-amenable bi-exact groups (see also [CS13]).

In our first main result of the paper, we use the powerful framework of Popa's deformation/rigidity theory to establish a general analogue of Monod and Shalom's unique prime factorization theorem, which applies, in particular, to product of groups with positive first ℓ^2-Betti number. More generally, we obtain such a result for product of groups for which their von Neumann algebras belong to a certain class $\mathcal{M}$ of II$_1$ factors that admit an s-malleable deformation in the sense of Popa [Pop06a, Pop06b] (see Definition 3.1). For simplicity, we say that a countable group Γ belongs to $\mathcal{M}$ if its associated von Neumann algebra $L(\Gamma)$ belongs to $\mathcal{M}$. We refer the reader to Definition 3.3 for the description of class $\mathcal{M}$ and to Example 1.1 for more concrete examples of groups that belong to this class.

Theorem A. *Let $\Gamma_1, \ldots, \Gamma_n$ be groups that belong to $\mathcal{M}$. If $\Gamma_1 \times \cdots \times \Gamma_n$ is measure equivalent to a product $\Lambda_1 \times \cdots \times \Lambda_m$ of infinite icc groups, then $n \geq m$, and if $n = m$, then after permutation of indices, Γ_i is measure equivalent to Λ_i, for any $1 \leq i \leq n$.*

Example 1.1. A countable group Γ belongs to $\mathcal{M}$ whenever Γ is a non-amenable icc group that satisfies one of the following conditions (see Proposition 3.5):

(1) $\Gamma = \Sigma \wr_{G/H} G$ is a generalized wreath product group with Σ amenable, G non-amenable and $H < G$ is an amenable almost malnormal subgroup;
(2) Γ admits an unbounded cocycle for some mixing representation $\pi : \Gamma \to \mathcal{O}(H_{\mathbb{R}})$ such that π is weakly contained in the left regular representation of Γ;
(3) $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ is an amalgamated free product group satisfying $[\Gamma_1 : \Sigma] \geq 2$ and $[\Gamma_2 : \Sigma] \geq 3$, where $\Sigma < \Gamma$ is an amenable almost malnormal[1] subgroup.

We continue by making several remarks about Theorem A. First, note that the class $\mathcal{C}_{\text{reg}}$ considered by Monod and Shalom in their unique prime factorization result [MS06, Theorem 1.16] does not contain groups that have infinite amenable normal subgroups [MS06, Corollary 1.19] and, hence, the subclass of wreath product groups considered in Example 1.1(1) is disjoint from $\mathcal{C}_{\text{reg}}$. Moreover, Example 1.1 provides a large class of groups that are not bi-exact [Sak09] since any bi-exact group cannot contain an infinite subgroup with non-amenable centralizer.

Next, we contrast our result with the following corollary of Gaboriau's work [Gab02]: if a product $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ of n groups with positive first ℓ^2-Betti number is measure equivalent to a product $\Lambda = \Lambda_1 \times \cdots \times \Lambda_m$ of m infinite groups, then $n \geq m$. Indeed, by [Gab02, Théorème 6.3.] we have that the nth ℓ^2-Betti number of Γ vanishes if and only if the nth ℓ^2-Betti number of Λ vanishes. On the other hand, the Künneth formula [Gab02, Propriétés 1.5] implies that the nth ℓ^2-Betti number of Γ is positive, while if $n < m$, then the nth ℓ^2-Betti number of Λ equals to 0. Theorem B strengthens this conclusion in two ways in the case Γ, Λ are icc. First, if $n = m$ we are able to recover the measure equivalence class of each Γ_i. Second, since the groups with positive first ℓ^2-Betti number are precisely the non-amenable groups that admit an unbounded cocycle into the left regular representation [PT11], Example 1.1(2) extends the previous result of Gaboriau to the larger class of groups that admit an unbounded cocycle for some mixing representation that is weakly contained in the left regular representation.

Remark 1.2. Popa's deformation/rigidity theory gave rise to a plethora of striking rigidity results for von Neumann algebras of wreath product groups. Popa's pioneering work [Pop06b, Pop06c] allowed one to distinguish between the group von Neumann algebras of $\mathbb{Z}/2\mathbb{Z} \wr \Gamma$, as Γ is an infinite property (T) group, while Ioana, Popa, and Vaes used a wreath product construction to obtain the first class of groups that are entirely remembered by their von Neumann algebras [IPV13]. Subsequently, several other rigidity results have been obtained for von Neumann algebras of wreath products including primeness, relative solidity, and product rigidity, see [Ioa07, Pop08, CI10, Ioa11, IPV13, SW13, CPS12, BV14, IM19, Dri21, CDD21]. Theorem A establishes a new general rigidity result for wreath product groups by showing that products of arbitrary non-amenable wreath product groups with amenable base satisfy an analogue of Monod and Shalom's unique prime factorization result.

Theorem A follows from the following more general result in which we classify all tensor product decompositions of $L(\Lambda)$, whenever Λ is an icc group that is measure equivalent to a finite product of groups that belong to $\mathcal{M}$.

THEOREM B. *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a product of groups that belong to $\mathcal{M}$ and let Λ be an icc group that is measure equivalent to Γ. Assume $L(\Lambda) = P_1 \bar{\otimes} \cdots \bar{\otimes} P_m$ admits a tensor product decompositions into II$_1$ factors. Then $n \geq m$ and there exists a decomposition $\Lambda = \Lambda_1 \times \cdots \times \Lambda_m$ into infinite groups.*

[1] A subgroup $H < G$ is called almost malnormal if $gHg^{-1} \cap H$ is finite for any $g \in G\backslash$H.

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Moreover, there exist a partition $S_1 \sqcup \cdots \sqcup S_m = \{1, \ldots, n\}$*, a decomposition* $L(\Lambda) = P_1^{t_1} \bar{\otimes} \cdots \bar{\otimes} P_m^{t_m}$*, for some* $t_1, \ldots, t_m > 0$ *with* $t_1 \ldots t_m = 1$*, and a unitary* $u \in L(\Lambda)$ *such that for any* $1 \leq j \leq m$*:*

(1) $\times_{k \in S_j} \Gamma_k$ *is measure equivalent to* Λ_j*;*
(2) $P_j^{t_j} = uL(\Lambda_j)u^*$.

In particular, if $n = m$*, then after permutation of indices,* Γ_i *is measure equivalent to* Λ_i*, for any* $1 \leq i \leq n$*.*

We note that Theorem B provides a complement to [DHI16, Theorem C] where such a classification result has been obtained by Hoff, Ioana, and the present author in the case the groups Γ_i are hyperbolic. Although the proof of Theorem B is inspired by the strategy of the proof of [DHI16, Theorem C], we implement quite differently some of the steps. In order to effectively work with groups from $\mathcal{M}$, which are defined by a property of their von Neumann algebras, we are making use in an essential way of newer techniques from [BMO20, IM19, Dri21]. In particular, our proof uses a relative version of the flip automorphism method introduced by Isono and Marrakchi in [IM19].

Another application of Theorem B is to the study of tensor product decompositions of von Neumann algebras by providing new classes of prime II_1 factors. Recall that a II_1 factor is called *prime* if it does not admit a tensor product decomposition into II_1 factors. Popa discovered in [Pop83] the first examples of prime II_1 factors by showing that the free group factors $L(\mathbb{F}_S)$, with S uncountable, are prime. Then Ge showed in [Ge98] that the free group factors $L(\mathbb{F}_n), 2 \leq n \leq \infty$, are prime, thus providing the first examples of separable prime II_1 factors. Subsequently, a large number of prime II_1 factors have been discovered; see, for instance, the introduction of [CDI22]. As a corollary of Theorem B, we obtain that if Γ is a countable group that belongs to $\mathcal{M}$ and $G = (\times_{i=1}^n \Gamma) \rtimes \mathbb{Z}/n\mathbb{Z}$ is the semidirect product group of the natural translation action $\mathbb{Z}/n\mathbb{Z} \curvearrowright \times_{i=1}^n \Gamma$, then $L(G)$ is a prime II_1 factor. In fact, a more general result holds and for properly formulating it, we give the following notation. Let n be a positive integer, denote by S_n the group of permutations of $\{1, \ldots, n\}$ and consider the permutation action of S_n on $\{1, \ldots, n\}$. For any subset $J \subset \{1, \ldots, n\}$ and subgroup $K < S_n$, we denote $\text{Fix}_K(J) = \{g \in K \mid g \cdot j = j, \text{ for any } j \in J\}$.

Corollary C. *Let* Γ *be a countable group that belongs to* $\mathcal{M}$*. Let* n *be a positive integer and let* K *be any subgroup of* S_n*. Consider the permutation action* $K \curvearrowright \times_{i=1}^n \Gamma$ *and denote* $G = (\times_{i=1}^n \Gamma) \rtimes K$*.*

Then $L(G)$ *is a prime* II_1 *factor if and only if there exists no partition* $J_1 \sqcup J_2 = \{1, \ldots, n\}$ *for which* $K = \text{Fix}_K(J_1) \times \text{Fix}_K(J_2)$*.*

Note that Corollary C provides a large class of prime II_1 factors which admit finite index subfactors that are not prime. Additional such prime II_1 factors have been obtained previously in [DHI16, CD19] by replacing Γ in the statement of Corollary C by any non-elementary hyperbolic group, see also [CDI22, § 5].

We continue by discussing some OE rigidity results for actions of product groups that belong to class $\mathcal{M}$. Furman discovered in [Fur99] the first class of group action $\Gamma \curvearrowright (X, \mu)$ that are *OE superrigid*, that is, any free, ergodic, pmp action that is OE to $\Gamma \curvearrowright (X, \mu)$ must be virtually conjugate[2] to it. Subsequently, a large number of OE superrigidity results have been obtained,

[2] Two pmp actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are *virtually conjugate* if there exist some finite normal subgroups $A < \Gamma$ and $B < \Lambda$ such that the associated actions $\Gamma/A \curvearrowright X/A$ and $\Lambda/B \curvearrowright Y/B$ are induced from conjugate actions.

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see the introduction of [DIP19]. By using part of the proof of Theorem B together with results from measured group theory [HHI21], we derive the following OE superrigidity result within the class of mildly mixing actions. Before stating the result, we recall some notions. A pmp action $\Gamma_1 \times \ldots \times \Gamma_n \curvearrowright (X, \mu)$ is called *irreducible* if its restriction to any subgroup Γ_i is ergodic. A pmp action $\Lambda \curvearrowright (Y, \nu)$ is called *mildly mixing* if whenever $A \subset Y$ is measurable subset satisfying $\liminf_{g \to \infty} \nu(gA \Delta A) = 0$, then $\nu(A) \in \{0, 1\}$.

THEOREM D. *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a product of $n \geq 2$ groups that belong to $\mathcal{M}$. Let $\Gamma \curvearrowright (X, \mu)$ be a free, irreducible, pmp action that is OE to a free, mildly mixing, pmp action $\Lambda \curvearrowright (Y, \nu)$.*

Then $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are virtually conjugate.

Note that this type of superrigidity has been obtained by Monod and Shalom [MS06, Theorem 1.9] for groups Γ_i that are torsion-free hyperbolic groups (more generally, groups that belong to $\mathcal{C}_{\text{reg}}$). In Theorem D we extend this result to groups from $\mathcal{M}$ which are purely defined by a property of their von Neumann algebra.

Finally, in the last part of this paper we discuss some structural results for II_1 factors that belong to a subclass of $\mathcal{M}$. We say that a non-amenable tracial von Neumann algebra M belongs to class $\mathcal{M}_0$ if there exists an s-malleable deformation $(\tilde{M}, (\alpha_t)_{t \in \mathbb{R}})$ of M satisfying:

- $L^2(\tilde{M}) \ominus L^2(M)$ is a mixing M-M-bimodule relative to $\mathbb{C}1$;
- $L^2(\tilde{M}) \ominus L^2(M)$ is weakly contained in the coarse bimodule $L^2(M) \otimes L^2(M)$ as M-M-bimodules.

We refer the reader to §§ 2.3 and 3.1 for the terminology used in defining the class $\mathcal{M}_0$ and we note that any II_1 factor from $\mathcal{M}_0$ belongs to $\mathcal{M}$, see Definition 3.3.

In Theorem E we show that all embeddings of group von Neumann algebras of non-amenable inner amenable groups in any II_1 factor that belongs to $\mathcal{M}_0$ are *rigid*. A countable group Γ is *inner amenable* if there exists an atomless mean on Γ which is invariant by the action of Γ on itself by conjugation. Effros made in [Eff75] a connection of this group theoretic notion to von Neumann algebras by showing that an icc group Γ is inner amenable whenever its group von Neumann algebra has property Gamma. The converse is false as was shown by Vaes [Vae12].

THEOREM E. *Let M be a von Neumann algebra in $\mathcal{M}_0$ and let $(\tilde{M}, (\alpha_t)_{t \in \mathbb{R}})$ be the associated s-malleable deformation of M. Let Γ be a non-amenable inner amenable group satisfying $L(\Gamma) \subset M$.*

Then $L(\Gamma)$ is α-rigid, i.e. $\alpha_t \to \text{id}$ uniformly on the unit ball of $L(\Gamma)$.

Note that von Neumann algebras with property Gamma exhibit strong structural results (see, for instance, [Pet09, HU16, IS19]) that are enough for obtaining various rigidity results via Popa's deformation/rigidity theory. In order to work with the more general class of inner amenable groups, we use an idea from [Tuc14, Theorem 11] on how to use Popa's spectral gap principle. An additional obstacle that arises here is the fact that $E_{L(\Gamma)}(\alpha_t(u_g))$ is not necessarily a scalar multiple of u_g, where $g \in \Gamma$; here, we denoted by $\{u_g\}_{g \in \Gamma}$ the canonical unitaries that generate $L(\Gamma)$ and by $E_{L(\Gamma)} : \tilde{M} \to L(\Gamma)$ the canonical conditional expectation. We overcome this difficulty by using an augmentation technique based on the comultiplication map associated to $L(\Gamma)$ (see [PV10]).

We continue by discussing several applications of Theorem E. Chifan and Sinclair proved in [CS13] that any countable group Γ for which $\beta_1^{(2)}(\Gamma) > 0$ is not inner amenable. Theorem E recovers and strengthens this fact in the following way. While it is unknown that the non-vanishing of the first ℓ^2-Betti number is a group von Neumann algebra invariant, we derive from Theorem E that any group that has isomorphic von Neumann algebra to $L(\Gamma)$ is not inner amenable as well.

Corollary E. *Let Γ be any countable group for which $\beta_1^{(2)}(\Gamma) > 0$. If Λ is any countable group for which $L(\Gamma) \cong L(\Lambda)$, then Λ is not inner amenable.*

To put Theorem E into a better perspective, we note that it provides an alternative solution to a question of Popa. Since any non-amenable property Gamma von Neumann algebra cannot embed into the free group factor $L(\mathbb{F}_n)$ (see [Oza04]), Popa asked in [Pop21] if it still true that the group von Neumann algebra of a non-amenable inner amenable group cannot embed into $L(\mathbb{F}_n)$. Recently, inspired by the notion of properly proximal groups [BIP21] (see also [IPR19]), Ding, Kunnawalkam Elayavalli, and Peterson developed in [DKP22] subtle boundary techniques to define a notion of proper proximality for tracial von Neumann algebras, and as a consequence, they answered Popa's question in a positive way. As a particular case of Theorem E, we give a new proof for Popa's question by using methods from Popa's deformation/rigidity theory.

Moreover, as a corollary of Theorem E we completely classify all embeddings of group von Neumann algebras $L(G)$ of non-amenable inner amenable groups in any free product $M = M_1 * M_2$ of tracial von Neumann algebras by showing that $L(G) \prec_M M_i$, for some i. Here, $\prec_M$ refers to Popa's intertwining-by-bimodules technique, see § 2.2. Consequently, we obtain a new class of examples for which the Kurosh-type rigidity results discovered in [Oza06] for free products von Neumann algebras hold. Namely, Ozawa proved using C*-algebraic techniques that if there is an isomorphism $\theta : M_1 * \cdots * M_m \to N_1 * \cdots * N_n$, where all von Neumann algebras M_i and N_j are non-amenable, semiexact, non-prime II_1 factors, then $m = n$, and after a permutation of indices, $\theta(M_i)$ is unitarily conjugate to N_i, for any $i \in \overline{1, n}$. By using Popa's deformation/rigidity theory, Ioana, Popa, and Peterson obtained the previous Kurosh-type rigidity result for property (T) II_1 factors [IPP08]. Shortly after, by developing a new approach rooted on closable derivations, Peterson unified and generalized these Kurosh-type rigidity results by covering L^2-rigid II_1 factors, which include all non-amenable non-prime, property (T), and property Gamma II_1 factors [Pet09]. By classifying certain amenable subalgebras of amalgamated free product von Neumann algebras, Ioana then extended the previous Kurosh-type rigidity result by covering non-amenable II_1 factors that admit a Cartan subalgebra [Ioa15]. We also refer the reader to [HU16] for certain Kurosh-type rigidity results for type III factors. As a corollary of Theorem E, we extend the previous Kurosh-type rigidity results to the class of II_1 factors of non-amenable inner amenable groups, see Corollary 8.1.

2. Preliminaries

2.1 Terminology

Throughout the paper we consider *tracial von Neumann algebras* (M, τ), i.e. von Neumann algebras M equipped with a faithful normal tracial state $\tau : M \to \mathbb{C}$. This induces a norm on M by the formula $\|x\|_2 = \tau(x^*x)^{1/2}$, for any $x \in M$. We will always assume that M is *separable*, i.e. the $\|\cdot\|_2$-completion of M denoted by $L^2(M)$ is separable as a Hilbert space. We denote by $\mathcal{Z}(M)$ the *center* of M and by $\mathcal{U}(M)$ its *unitary group*. For two von Neumann subalgebras $P_1, P_2 \subset M$, we denote by $P_1 \vee P_2 = W^*(P_1 \cup P_2)$ the von Neumann algebra generated by P_1 and P_2.

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All inclusions $P \subset M$ of von Neumann algebras are assumed unital. We denote by $E_P : M \to P$ the unique τ-preserving *conditional expectation* from M onto P, by $e_P : L^2(M) \to L^2(P)$ the orthogonal projection onto $L^2(P)$ and by $\langle M, e_P\rangle$ the Jones' basic construction of $P \subset M$. We also denote by $P' \cap M = \{x \in M \mid xy = yx, \text{ for all } y \in P\}$ the *relative commutant* of P in M and by $\mathcal{N}_M(P) = \{u \in \mathcal{U}(M) \mid uPu^* = P\}$ the *normalizer* of P in M.

The *amplification* of a II_1 factor (M,τ) by a number $t > 0$ is defined to be $M^t = p(\mathbb{B}(\ell^2(\mathbb{Z}))\bar{\otimes} M)p$, for a projection $p \in \mathbb{B}(\ell^2(\mathbb{Z}))\bar{\otimes} M$ satisfying $(\mathrm{Tr}\otimes\tau)(p) = t$. Here Tr denotes the usual trace on $\mathbb{B}(\ell^2(\mathbb{Z}))$. Since M is a II_1 factor, M^t is well defined. Note that if $M = P_1\bar{\otimes}P_2$, for some II_1 factors P_1 and P_2, then there is a natural isomorphism $M = P_1^t\bar{\otimes}P_2^{1/t}$, for any $t > 0$.

Finally, for a positive integer n, we denote by $\overline{1,n}$ the set $\{1,\dots,n\}$. If $S \subset \overline{1,n}$ we denote its complement by $\widehat{S} = \overline{1,n} \setminus S$. In the case that $S = \{i\}$, we will simply write $\hat{i}$ instead of $\widehat{\{i\}}$. In addition, given any product group $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$, we will denote their subproduct supported on S by $\Gamma_S = \times_{i\in S}\Gamma_i$.

2.2 Intertwining-by-bimodules

We next recall from [Pop06b, Theorem 2.1 and Corollary 2.3] the powerful *intertwining-by-bimodules* technique of Popa.

THEOREM 2.1 [Pop06b]. *Let (M,τ) be a tracial von Neumann algebra and $P \subset pMp, Q \subset qMq$ be von Neumann subalgebras. Let $\mathcal{U} \subset \mathcal{U}(P)$ be a subgroup such that $\mathcal{U}'' = P$.*

Then the following are equivalent.

(1) *There exist projections $p_0 \in P, q_0 \in Q$, a $*$-homomorphism $\theta : p_0Pp_0 \to q_0Qq_0$ and a non-zero partial isometry $v \in q_0Mp_0$ such that $\theta(x)v = vx$, for all $x \in p_0Pp_0$.*
(2) *There is no sequence $(u_n)_n \subset \mathcal{U}$ satisfying $\|E_Q(xu_ny)\|_2 \to 0$, for all $x, y \in M$.*

If one of the equivalent conditions of Theorem 2.1 holds true, we write $P \prec_M Q$, and say that *a corner of P embeds into Q inside M*. If $Pp' \prec_M Q$ for any non-zero projection $p' \in P' \cap pMp$, then we write $P \prec_M^s Q$.

LEMMA 2.2. *Let $\Lambda \curvearrowright B$ be a trace-preserving action and denote $M = B \rtimes \Lambda$. Let $p \in B$ be a non-zero projection and let $A \subset pBp$ be a von Neumann subalgebra such that $A' \cap pMp \subset A$.*

Let $\Lambda_0 < \Lambda$ be a subgroup and $\mathcal{G} \subset \mathcal{N}_{pMp}(A)$ a group of unitaries. If there is a projection $e \in \mathcal{G}' \cap pMp$ satisfying $\mathcal{G}''e \prec_M^s B \times \Lambda_0$, then there is a projection $f \in (A \cup \mathcal{G})' \cap pMp$ with $e \le f$ satisfying $(A\cup\mathcal{G})''f \prec_M^s B \rtimes \Lambda_0$.

Proof. Throughout the proof we use the terminology that a set $F \subset \Lambda$ is said to be small relative to $\{\Lambda_0\}$ if it is contained into a finite union of $s\Lambda_0t$, where $s, t \in \Lambda$. For any $F \subset \Lambda$, let $\mathcal{H}_F \subset L^2(M)$ be the $\|\cdot\|_2$-closed linear span of $\{Bv_\lambda \mid \lambda \in F\}$ and denote by $P_F : L^2(M) \to \mathcal{H}_F$ the orthogonal projection onto $\mathcal{H}_F$. Let $\epsilon > 0$ and denote $T = E_{A'\cap pMp}(e)$. Note that $T \in (A\cup\mathcal{G})' \cap pMp \subset A$ and that T belongs to the $\|\cdot\|_2$-closed convex hull of $\{aea^* \mid a \in \mathcal{U}(A)\}$. Thus, we can take $a_1,\dots,a_n \in \mathcal{U}(A)$ and $\alpha_1,\dots,\alpha_n \in [0,1]$ such that if we denote $T_0 = \sum_{i=1}^n \alpha_i a_i e a_i^*$, then $\|T - T_0\|_2 \le \epsilon$.

Since $\mathcal{G}''e \prec_M^s B \times \Lambda_0$, it follows from [Vae13, Lemma 2.5] that there exists $F \subset \Lambda$ that is small relative to $\{\Lambda_0\}$ such that $\|we - P_F(we)\|_2 \le \epsilon/n$, for all $w \in \mathcal{G}$. Hence, for all $a \in \mathcal{U}(A), w \in \mathcal{G}$, we have

$$\left\|awT - \alpha_i\sum_{i=1}^n a(wa_iw^*)P_F(we)a_i^*\right\|_2 \le \epsilon + \sum_{i=1}^n \alpha_i\|aw(a_iea_i^*) - a(wa_iw^*)P_F(we)a_i^*\|_2 \le 2\epsilon. \tag{2.1}$$

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Since $A \subset pBp$, we have $a(wa_iw^*)P_F(we)a_i^* \in \mathcal{H}_F$ and, hence, $\|awT - P_F(awT)\|_2 \leq 2\epsilon$, for all $a \in \mathcal{U}(A), w \in \mathcal{G}$. Therefore, there exists a sequence $\{F_n\}_{n\geq 1}$ of subsets of Λ that are small relative to $\{\Lambda_0\}$ such that $\|awT - P_{F_n}(awT)\|_2 \to 0$ uniformly in $a \in \mathcal{U}(A), w \in \mathcal{G}$.

For every $\delta > 0$ define the spectral projection $q_\delta = \chi_{(\delta,\infty)}(T) \in A$ and let $T_\delta \in A$ satisfying $TT_\delta = q_\delta$. If we denote by q_0 the support projection of T, then $\|q_\delta - q_0\|_2 \to 0$ as $\delta \to 0$. These altogether imply that $\|awq_\delta - P_{F_n}(awq_\delta)\|_2 \to 0$ uniformly in $a \in \mathcal{U}(A), w \in \mathcal{G}$ and, hence, $\|awq_0 - P_{F_n}(awq_0)\|_2 \to 0$ uniformly in $a \in \mathcal{U}(A),\ w \in \mathcal{G}$. Finally, note that $q_0 \in (A \cup \mathcal{G})' \cap pMp$ and $q_0 \geq e$. This concludes the proof. □

The following proposition follows from [BMO20, IM19] and it is essentially contained in the proof of [Dri21, Theorem 4.2]. We record it here for the convenience of the reader.

Proposition 2.3. *Let $M = P\bar{\otimes}Q$ be a tensor product of II$_1$ factors. Let $Q_n, n \geq 1$, be a decreasing sequence of von Neumann subalgebras such that $P \prec_M \bigvee_{n\geq 1}(Q_n' \cap M)$.*

If P does not have property Gamma, then there exists $m \geq 1$ such that $P \prec_M Q_m' \cap M$.

Recall that a II$_1$ factor (M,τ) has property Gamma if it admits a central sequence $(x_n)_n \subset \mathcal{U}(M)$ for which $\inf_n \|x_n - \tau(x_n)1\|_2 > 0$.

2.3 Bimodules

Let M, N be tracial von Neumann algebras. An M-N bimodule ${}_M\mathcal{H}_N$ is a Hilbert space $\mathcal{H}$ together with a $*$-homomorphism $\pi_\mathcal{H} : M \odot N^{\text{op}} \to \mathbb{B}(\mathcal{H})$ that is normal on M and N^{op}, where $M \odot N^{\text{op}}$ is the algebraic tensor product between M and the opposite von Neumann algebra N^{op} of N. Examples of bimodules include the trivial M-bimodule ${}_ML^2(M)_M$ and the coarse M-N-bimodule ${}_ML^2(M) \otimes L^2(N)_N$. For two M-N-bimodules ${}_M\mathcal{H}_N$ and ${}_M\mathcal{K}_N$, we say that ${}_M\mathcal{H}_N$ is *weakly contained* in ${}_M\mathcal{K}_N$ if $\|\pi_\mathcal{H}(x)\| \leq \|\pi_\mathcal{K}(x)\|$, for any $x \in M \odot N^{\text{op}}$.

Let $A \subset M$ be an inclusion of tracial von Neumann algebras and let ${}_M\mathcal{H}_M$ be an M-bimodule. We say that ${}_M\mathcal{H}_M$ is *mixing relative* to A if for any sequence $(x_n)_n \subset (M)_1$ satisfying $\|E_A(xu_ny)\|_2 \to 0$, for all $x, y \in M$, we have

$$\lim_{n\to\infty} \sup_{y\in(M)_1} \langle x_n\xi y, \eta\rangle, \quad \text{for all } \xi, \eta \in \mathcal{H}.$$

2.4 Relative amenability

A tracial von Neumann algebra (M,τ) is *amenable* if there is a positive linear functional $\Phi : \mathbb{B}(L^2(M)) \to \mathbb{C}$ such that $\Phi_{|M} = \tau$ and Φ is *M-central*, meaning $\Phi(xT) = \Phi(Tx)$, for all $x \in M$ and $T \in \mathbb{B}(L^2(M))$. By Connes' classification of amenable factors [Con76], it follows that M is amenable if and only if M is approximately finite dimensional.

We continue by recalling the notion of relative amenability which is due to Ozawa and Popa [OP10]. Fix a tracial von Neumann algebra (M,τ). Let $p \in M$ be a projection and $P \subset pMp, Q \subset M$ be von Neumann subalgebras. Following [OP10, Definition 2.2], we say that P is *amenable relative to Q inside M* if there is a positive linear functional $\Phi : p\langle M, e_Q\rangle p \to \mathbb{C}$ such that $\Phi_{|pMp} = \tau$ and Φ is P-central. We say that P is *strongly non-amenable relative to Q* if Pp' is non-amenable relative to Q for any non-zero projection $p' \in P' \cap pMp$ (equivalently, for any non-zero projection $p' \in \mathcal{N}_M(P)' \cap pMp$ by [DHI16, Lemma 2.6]).

Note that if $P \subset pMp$ and $Q \subset M$ are tracial von Neumann algebras, then $P \subset pMp$ is amenable relative to Q if and only if ${}_PL^2(pM)_M$ is weakly contained in ${}_PL^2(p\langle M, e_Q\rangle)_M$. We also recall that ${}_ML^2(\langle M, e_Q\rangle)_M \cong_M (L^2(M) \otimes_Q L^2(M))_M$. It is clear that P is amenable relative to $\mathbb{C}1$ inside M if and only if P is amenable. The following lemma generalizes this fact and it is inspired by the proof of [DHI16, Lemma 5.6]. For completeness, we provide all the details.

LEMMA 2.4. *Let* M_0 *and* $M \subset \tilde{M}$ *be some tracial von Neumann algebras and let* $Q \subset q(M_0 \bar{\otimes} M)q$ *be a von Neumann subalgebra. The following hold.*

(1) *Assume that* $Q\tilde{z}$ *is amenable relative to* M_0 *inside* $M_0 \bar{\otimes} \tilde{M}$, *for a non-zero projection* $\tilde{z} \in Q' \cap q(M_0 \bar{\otimes} \tilde{M})q$. *Then* Qz *is amenable relative to* M_0 *inside* $M_0 \bar{\otimes} M$, *where* $z \in Q' \cap q(M_0 \bar{\otimes} M)q$ *is the support projection of* $E_{\mathcal{M}}(\tilde{z})$.
(2) *If* $Q \prec_{M_0 \bar{\otimes} \tilde{M}} M_0$, *then* $Q \prec_{M_0 \bar{\otimes} M} M_0$.

Proof. (1) Let $\mathcal{M} = M_0 \bar{\otimes} M$ and $\tilde{\mathcal{M}} = M_0 \bar{\otimes} \tilde{M}$. The assumption implies that the bimodule ${}_{Q\tilde{z}}L^2(\tilde{z}\tilde{\mathcal{M}})_{\tilde{\mathcal{M}}}$ is weakly contained in ${}_{Q\tilde{z}}L^2(\tilde{z}\langle \tilde{\mathcal{M}}, e_{M_0}\rangle)_{\tilde{\mathcal{M}}}$. If we denote by $z \in Q' \cap q(M_0 \bar{\otimes} M)q)$ the support projection of $E_{\mathcal{M}}(\tilde{z})$, we obtain that

$$ {}_{Qz}L^2(z\mathcal{M})_{\mathcal{M}} \text{ is weakly contained in } {}_{Qz}L^2(z\langle \tilde{\mathcal{M}}, e_{M_0}\rangle)_{\mathcal{M}}. \tag{2.2} $$

Note that ${}_{\mathcal{M}}L^2(\langle \tilde{\mathcal{M}}, e_{M_0}\rangle)_{\mathcal{M}} \cong_{\mathcal{M}} L^2(\tilde{\mathcal{M}}) \otimes L^2(\tilde{M})_{\mathcal{M}}$. Note also that ${}_{\mathcal{M}}L^2(\tilde{\mathcal{M}})_{M_0}$ is weakly contained in ${}_{\mathcal{M}}L^2(\mathcal{M})_{M_0}$ and ${}_{\mathbb{C}}L^2(\tilde{M})_M$ is weakly contained in ${}_{\mathbb{C}}L^2(M)_M$.

These altogether imply that ${}_{Qz}L^2(\langle \tilde{\mathcal{M}}, e_{M_0}\rangle)_{\mathcal{M}}$ is weakly contained in ${}_{Qz}L^2(\langle \mathcal{M}, e_{M_0}\rangle)_{\mathcal{M}}$. Using (2.2) we deduce that ${}_{Qz}L^2(z\mathcal{M})_{\mathcal{M}}$ is weakly contained in ${}_{Qz}L^2(z\langle \mathcal{M}, e_{M_0}\rangle)_{\mathcal{M}}$, which shows that Qz is amenable relative to M_0 inside $\mathcal{M}$.

(2) By assuming the contrary, there exists a sequence $u_n \in \mathcal{U}(Q)$ such that

$$ \|E_{M_0}(xu_ny)\|_2 \to 0, \quad \text{for all } x, y \in M_0 \bar{\otimes} M. \tag{2.3} $$

We want to show that (2.3) holds for all $x, y \in M_0 \bar{\otimes} \tilde{M}$, which will contradict the assumption. Note that it is enough to consider $x = 1$ and $y \in \tilde{M}$. In this case, by using (2.3) we obtain $E_{M_0}(u_ny) = E_{M_0}(E_{M_0 \bar{\otimes} M}(u_ny)) = E_{M_0}(u_nE_{M_0 \bar{\otimes} M}(y))$, which goes to 0 in the $\|\cdot\|_2$-norm. This finishes the proof. □

3. Malleable deformations for von Neumann algebras: class $\mathfrak{M}$

3.1 Malleable deformations

Popa introduced in [Pop06a, Pop06b] the notion of an s-malleable deformation of a von Neumann algebra. This notion has been successfully used in the framework of his deformation/rigidity theory and led to a plethora of remarkable results in the theory of von Neumann algebras, see the surveys [Pop07a, Vae10, Ioa14, Ioa18]. We also refer the reader to [dSHHS20] for recent developments on s-malleable deformations.

DEFINITION 3.1. Let (M, τ) be a tracial von Neumannn algebra. A pair $(\tilde{M}, (\alpha_t)_{t\in\mathbb{R}})$ is called an *s-malleable deformation* of M if the following conditions hold:

- $(\tilde{M}, \tilde{\tau})$ is a tracial von Neumann algebra such that $M \subset \tilde{M}$ and $\tau = \tilde{\tau}_{|M}$;
- $(\alpha_t)_{t\in\mathbb{R}} \subset \mathrm{A}ut(\tilde{M}, \tilde{\tau})$ is a 1-parameter group with $\lim_{t\to 0} \|\alpha_t(x) - x\|_2 = 0$, for any $x \in \tilde{M}$;
- there is $\beta \in \mathrm{A}ut(\tilde{M}, \tilde{\tau})$ satisfying $\beta_{|M} = \mathrm{Id}_M$, $\beta^2 = \mathrm{Id}_{\tilde{M}}$ and $\beta\alpha_t = \alpha_{-t}\beta$, for any $t \in \mathbb{R}$;
- α_t does not converge uniformly to the identity on $(M)_1$ as $t \to 0$.

For a subalgebra $Q \subset qMq$, we say that Q is *α-rigid* if α_t converges uniformly to the identity on $(Q)_1$ as $t \to 0$. We will repeatedly use the following stability result for s-malleable deformations.

PROPOSITION 3.2 [Vae13, Proposition 3.4]. *Let* $(\tilde{M}, (\alpha_t)_{t\in\mathbb{R}})$ *be an s-malleable deformation of a tracial von Neumann algebra* M. *Let* $P \subset pMp$ *be a subalgebra that is generated by a group of unitaries* $\mathcal{G} \subset \mathcal{U}(P)$. *Assume that* $\alpha_t \to \mathrm{id}$ *uniformly on* $r\mathcal{G}r$ *for a projection* $r \in pMp$.

Then there is a projection $z \in \mathcal{N}_{pMp}(P)' \cap pMp$ *with* $r \leq z$ *such that* Pz *is* α*-rigid.*

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3.2 Definition of class $\mathcal{M}$

We are now ready to define the class of II_1 factors that is used in our main results stated in the introduction.

Definition 3.3. We say that a non-amenable II_1 factor M belongs to class $\mathcal{M}$ if there exists an s-malleable deformation $(\tilde{M}, (\alpha_t)_{t\in\mathbb{R}})$ of M and an amenable subalgebra $A \subset M$ satisfying $L^2(\tilde{M}) \ominus L^2(M)$ is a mixing M-M-bimodule relative to A, $L^2(\tilde{M}) \ominus L^2(M)$ is weakly contained in the coarse bimodule $L^2(M) \otimes L^2(M)$ as M-M-bimodules and one of the following holds:

(1) $A = \mathbb{C}1$;
(2) if N is a tracial von Neumann algebra and $P \subset p(M\bar{\otimes}N)p$ a subalgebra such that $P \prec_{M\otimes N} A \otimes N$ and $P' \cap p(M\bar{\otimes}N)p$ is strongly non-amenable relative to $1 \otimes N$, then $P \prec_{M\otimes N} 1 \otimes N$.

As a consequence of Popa's spectral gap principle [Pop07b], we continue with the following remark.

Remark 3.4. Let $M \in \mathcal{M}$ be a II_1 factor, denote by $(\tilde{M}, (\alpha_t)_{t\in\mathbb{R}})$ the associated s-malleable deformation of M and let N be any tracial von Neumann algebra. If $Q \subset q(M\bar{\otimes}N)q$ is a von Neumann subalgebra which is strongly non-amenable relative to N, then $Q' \cap q(M\bar{\otimes}N)q$ is $(\alpha \otimes \mathrm{id})$-rigid. This essentially follows from [Pop07b] (see, for instance, the proofs of [Ioa12, Lemma 2.2] or [Dri21, Lemma 3.5]).

The following proposition provides concrete examples of group von Neumann algebras that belong to $\mathcal{M}$. The result is a consequence of Remark 3.4 and [Dri21, Proposition 3.4]).

Proposition 3.5. *If Γ belongs to one of the three classes of groups mentioned in Example 1.1, then $L(\Gamma)$ belongs to $\mathcal{M}$.*

Next, we present a useful result for group von Neumann algebras $L(\Gamma)$ that belong to $\mathcal{M}$ in order to understand structural results of trace-preserving actions of Γ. In fact, this is a direct consequence of Popa's spectral gap principle [Pop07b].

Lemma 3.6. *Let $\Gamma \curvearrowright B$ be a trace-preserving action and denote $\mathcal{M} = B \rtimes \Gamma$. We denote by $\Psi : \mathcal{M} \to \mathcal{M}\bar{\otimes}L(\Gamma)$ the $*$-homomorphism given by $\Psi(bu_g) = bu_g \otimes u_g$, for all $b \in B$ and $g \in \Gamma$. Assume that $L(\Gamma)$ belongs to $\mathcal{M}$ and let $(\tilde{M}, (\alpha_t)_{t\in\mathbb{R}})$ be the associated s-malleable deformation.*

If $P \subset p\mathcal{M}p$ is a von Neumann subalgebra that is strongly non-amenable relative to B inside $\mathcal{M}$, then $\Psi(P' \cap p\mathcal{M}p)$ is $(\mathrm{id} \otimes \alpha)$-rigid. Moreover, if we also assume that $P' \cap p\mathcal{M}p \nprec_{\mathcal{M}} B$, then $\Psi(P \vee (P' \cap p\mathcal{M}p))$ is $(\mathrm{id} \otimes \alpha)$-rigid.

Proof. Denote $M = L(\Gamma)$. By applying [Dri20a, Lemma 2.10], we get that $\Psi(P)$ is strongly non-amenable relative to $\mathcal{M}\bar{\otimes}1$ inside $\mathcal{M}\bar{\otimes}M$. Then Remark 3.4 implies that $\Psi(P' \cap p\mathcal{M}p)$ is $(\mathrm{id} \otimes \alpha)$-rigid. For proving the second part, assume in addition that $P' \cap p\mathcal{M}p \nprec_{\mathcal{M}} B$ and let $A \subset M$ be an amenable subalgebra as given by the assumption that M belongs to $\mathcal{M}$. By using [Ioa12, Lemma 9.2(1)] we get $\Psi(P' \cap p\mathcal{M}p) \nprec_{\mathcal{M}\bar{\otimes}M} \mathcal{M} \otimes 1$, and by assumption we must have $\Psi(P' \cap p\mathcal{M}p) \nprec_{\mathcal{M}\bar{\otimes}M} \mathcal{M} \otimes A$. Hence, in combination with $\Psi(P' \cap p\mathcal{M}p)$ being $(\mathrm{id} \otimes \alpha)$-rigid and the fact that $L^2(\mathcal{M}\bar{\otimes}\tilde{M}) \ominus L^2(\mathcal{M}\bar{\otimes}M)$ is a mixing $\mathcal{M}\bar{\otimes}M$-bimodule relative to $\mathcal{M}\bar{\otimes}A$, we get from [dSHHS20, Corollary 6.7] that $\Psi(P \vee (P' \cap p\mathcal{M}p))$ is $(\mathrm{id} \otimes \alpha)$-rigid. □

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3.3 Measure equivalence and non-property Gamma for class $\mathcal{M}$

In this subsection we show that the lack of property Gamma is preserved under measure equivalence for finite products of groups whose von Neumann algebras belong to $\mathcal{M}$, see Proposition 3.8. For proving this result, we first establish the following notation that will be assumed for Proposition 3.8, but will also be useful for the following sections.

Notation 3.7. Let Λ be a countable icc group that is measure equivalent to a product $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ of $n \geq 1$ groups. By using [Fur99, Lemma 3.2], there exist $d \geq 1$, free ergodic pmp actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ such that

$$\mathcal{R}(\Lambda \curvearrowright Y) = \mathcal{R}(\Gamma \times \mathbb{Z}/d\mathbb{Z} \curvearrowright X \times \mathbb{Z}/d\mathbb{Z}) \cap (Y \times Y).$$

Here, we considered that $\mathbb{Z}/d\mathbb{Z} \curvearrowright (\mathbb{Z}/d\mathbb{Z}, c)$ acts by addition and c is the counting measure. We also identified Y as a measurable subset of $X \times \mathbb{Z}/d\mathbb{Z}$ and denote $p = 1_Y \in L^\infty(X \times \mathbb{Z}/d\mathbb{Z})$. Note that $L^\infty(\mathbb{Z}/d\mathbb{Z}) \rtimes \mathbb{Z}/d\mathbb{Z} = \mathbb{M}_d(\mathbb{C})$. Hence, by letting $B = L^\infty(Y)$, $A = L^\infty(X) \otimes M_d(\mathbb{C})$, and $M = A \rtimes \Gamma$, we have $pMp = B \rtimes \Lambda$ and $B \subset pAp$. Denote by $\{u_g\}_{g\in\Gamma}$ and $\{v_\lambda\}_{\lambda\in\Lambda}$ the canonical unitaries implementing the actions $\Gamma \curvearrowright A$ and $\Lambda \curvearrowright B$, respectively.

Following [PV10] we define the $*$-homomorphism $\Delta : pMp \to pMp \bar{\otimes} L(\Lambda)$ by $\Delta(bv_\lambda) = bv_\lambda \otimes v_\lambda$, for all $b \in B, \lambda \in \Lambda$. One can extend Δ to a $*$-homomorphism $\Delta : M \to M \bar{\otimes} L(\Lambda)$ and verify that $\Delta(M)' \cap M \bar{\otimes} L(\Lambda) = \mathbb{C}1$ since Λ is icc (see the first part of [DHI16, § 5] for more details).

For any $i \in \overline{1,n}$, let $\Psi^i : M \to M \bar{\otimes} L(\Gamma_i)$ be the $*$-homomorphism given by $\Psi^i(xu_g) = xu_g \otimes u_g$, for all $x \in A \rtimes \Gamma_{\widehat{i}}, g \in \Gamma_i$.

Proposition 3.8. *Assume that $L(\Gamma_i)$ belongs to $\mathcal{M}$, for any $1 \leq i \leq n$.*

Then $L(\Lambda)$ does not have property Gamma.

Proof. Let $(\tilde{M}_i, (\alpha_t^i)_{t\in\mathbb{R}})$ be the associated s-malleable deformation of $L(\Gamma_i) \in \mathcal{M}$. By assuming that $L(\Lambda)$ has property Gamma, we can use [HU16, Theorem 3.1] to obtain a decreasing sequence of diffuse abelian von Neumann subalgebras $Q_n \subset L(\Lambda)$ with $n \geq 1$ such that $L(\Lambda) = \bigvee_{n\geq 1}(Q_n' \cap L(\Lambda))$. Since Q_1 is abelian, it follows that $(Q_1' \cap pMp)' \cap pMp \subset Q_1' \cap pMp$. Hence, for all k, $n \geq 1$, we have

$$\mathcal{Z}(Q_n' \cap pMp) \subset Q_1' \cap pMp \subset Q_k' \cap pMp. \tag{3.1}$$

Using Zorn's lemma and a maximality argument, one can show that for any $m \geq 1$, there exist maximal projections $r_m^1, \ldots, r_m^n \in Q_m' \cap pMp$ satisfying $Q_m r_m^i \nprec_M A \rtimes \Gamma_{\widehat{i}}$, for any $i \in \overline{1,n}$. One can check that $r_m^i \in \mathcal{Z}(Q_m' \cap pMp)$ and $Q_m(p - r_m^i) \prec_M^s A \rtimes \Gamma_{\widehat{i}}$, for any $i \in \overline{1,n}$ (see the proof of [DHI16, Lemma 2.4]).

Since $Q_m \nprec_M A$, [DHI16, Lemma 2.8(2)] implies that $\bigwedge_{i=1}^n (p - r_m^i) = 0$, which proves that $\bigvee_{i=1}^n r_m^i = p$. Hence, for any $m \geq 1$ there is $i_m \in \overline{1,n}$ such that $\tau(r_m^{i_m}) \geq \tau(p)/n$. Up to passing to a subsequence, we can assume that there is $j \in \overline{1,n}$ such that $i_m = j$, for all $m \geq 1$. Next, note that (3.1) gives that $r_m^j \in \mathcal{Z}(Q_m' \cap pMp) \subset Q_{m-1}' \cap pMp$. Since $Q_{m-1} r_m^j \nprec_M A \rtimes \Gamma_{\widehat{j}}$, it follows from the choice of all the r_m^j's that $\{r_m^j\}_{m\geq 1}$ is a decreasing sequence of projections. If we let $r^j = \bigwedge_{m\geq 1} r_m^j$, we deduce that r^j is a non-zero projection since $\tau(r^j) \geq \tau(p)/n$. For all $m \geq k \geq 1$, since $Q_m \subset Q_k$ we have that $r_m^j \in (Q_k' \cap pMp)' \cap pMp$. Consequently, by letting $m \to \infty$, we deduce that $r^j \in (Q_k' \cap pMp)' \cap pMp$, for all $k \geq 1$, which implies that $r^j \in L(\Lambda)' \cap pMp = \mathbb{C}p$. Since $r^j \neq 0$, we derive that $r^j = p$ and, therefore, we must have $r_m^j = p$, for any $m \geq 1$. This implies that $Q_m \nprec_M A \rtimes \Gamma_{\widehat{j}}$, for any $m \geq 1$.

Since Γ_j is non-amenable, it follows from [DHI16, Lemma 2.9] that $L(\Lambda)$ is non-amenable relative to $A \rtimes \Gamma_{\widehat{j}}$ inside M. Since relative amenability is closed under inductive limits (see [DHI16, Lemma 2.7]), there exists $k \geq 1$ such that $Q_k' \cap pMp$ is non-amenable relative to

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$A \rtimes \Gamma_{\widehat{j}}$ inside M. Using [DHI16, Lemma 2.6] there is a non-zero projection $z^j \in \mathcal{Z}(Q_k' \cap pMp)$ such that

$$(Q_k' \cap pMp)z^j \text{ is strongly non-amenable relative to } A \rtimes \Gamma_{\widehat{j}} \text{ inside } M. \tag{3.2}$$

This implies by Lemma 3.6 that $\Psi^j(Q_k z^j)$ is $(\mathrm{id} \otimes \alpha^j)$-rigid. Fix an arbitrary $m \geq k$. Since $Q_m \subset Q_k$, we have $z^j \in \mathcal{Z}(Q_k' \cap pMp) \subset Q_m' \cap pMp$ and

$$\Psi^j(Q_m z^j) \text{ is } (\mathrm{id} \otimes \alpha^j)\text{-rigid and } Q_m z^j \nprec_M A \rtimes \Gamma_{\widehat{j}}. \tag{3.3}$$

Equation (3.2) also implies that

$$z_j(Q_m' \cap pMp)z^j \text{ is strongly non-amenable relative to } A \rtimes \Gamma_{\widehat{j}} \text{ inside } M. \tag{3.4}$$

By combining (3.3) and (3.4), it follows from the second part of Lemma 3.6 that $\Psi^j(z^j(Q_m' \cap pMp)z^j)$ is $(\mathrm{id} \otimes \alpha^j)$-rigid, for any $m \geq k$. Note that (3.3) gives, in particular, that $z^j(Q_m' \cap pMp)z^j \nprec_M A \rtimes \Gamma_{\widehat{j}}$, for any $m \geq k$. Therefore, we may apply [dSHHS20, Theorem 3.5] (see also [Dri21, Theorem 3.2]) to deduce that $\Psi^j(z^j \bigvee_{m\geq k}(Q_m' \cap pMp)z^j)$ is $(\mathrm{id} \otimes \alpha^j)$-rigid. Using [dSHHS20, Proposition 5.6] there exists a non-zero projection $\tilde{z}^j \in \mathcal{Z}(\bigvee_{m\geq n}(Q_m' \cap pMp))$ such that $\Psi^j(\bigvee_{m\geq k}(Q_m' \cap pMp)\tilde{z}^j)$ is $(\mathrm{id} \otimes \alpha^j)$-rigid. Note, however, that $\bigvee_{m\geq k}(Q_m' \cap pMp)$ is a factor since $\bigvee_{m\geq k}(Q_k' \cap L(\Lambda)) = L(\Lambda)$ and Λ is icc. In particular, $\Psi^j(L(\Lambda))$ is $(\mathrm{id} \otimes \alpha^j)$-rigid. Since $\Psi^j(B) \subset M \otimes 1$, it follows that $\Psi^j(M)$ is $(\mathrm{id} \otimes \alpha^j)$-rigid, which gives that $L(\Gamma_j)$ is α^j-rigid, contradiction. This ends the proof of the proposition. □

4. Measure equivalence and tensor product decompositions for class $\mathcal{M}$

In this section we establish the main ingredients needed for the proof of Theorem B by building upon methods from [DHI16, IM19]. Throughout this section, we will use Notation 3.7 and the following assumption.

Assumption 4.1. For any $i \in \overline{1,n}$, assume that $L(\Gamma_i)$ belongs to $\mathcal{M}$ and denote by $(\tilde{M}_i, (\alpha_t^i)_{t\in\mathbb{R}})$ the associated s-malleable deformation of $L(\Gamma_i)$.

4.1 Step 1

The main goal of this subsection is to prove the following theorem.

Theorem 4.2. *Let $L(\Lambda) = P_1 \bar{\otimes} P_2$ be a tensor product decomposition into II$_1$ factors.*

Then there is a partition $S_1 \sqcup S_2 = \{1, \ldots, n\}$ into non-empty sets such that $\Delta(A \rtimes \Gamma_{S_i}) \prec^s_{M\bar{\otimes}L(\Lambda)} M\bar{\otimes}P_i$, for all $i \in \{1, 2\}$.

Before proceeding to the proof of Theorem 4.2, we make the following remark and prove two lemmas.

Remark 4.3. In this remark we explain why the proof of Theorem 4.2 uses a relative version of the flip automorphism method introduced by Isono and Marrakchi [IM19]. The conclusions (C1) and (C2) of Theorems 4.2 and 4.6, respectively, assert that:

(C1) $\exists$ a partition $S_1 \sqcup S_2 = \{1, \ldots, n\}$ such that $\Delta(A \rtimes \Gamma_{S_i}) \prec^s_{M\bar{\otimes}L(\Lambda)} M\bar{\otimes}P_i$, for all $i \in \{1, 2\}$;
(C2) $\exists$ a partition $T_1 \sqcup T_2 = \{1, \ldots, n\}$ such that $P_i \prec^s_M A \rtimes \Gamma_{T_i}$, for all $i \in \{1, 2\}$.

Note that conclusion (C2) cannot be directly obtained by using Popa's spectral gap arguments (Lemma 3.6) since if P_1 and P_2 are both amenable relative to $A \rtimes \Gamma_{\widehat{i}}$ for some $i \in \overline{1,n}$, one cannot immediately derive a contradiction. To overcome this difficulty, we first show conclusion

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(C1) and use this result in order to prove conclusion (C2). Finally, note that since P_1 and P_2 do not hold any 'relative solidity properties', Lemma 3.6 cannot be directly applied for proving conclusion (C1). Hence, we proceed by using the flip automorphism method [IM19] in order to obtain a situation where Lemma 3.6 can actually be applied.

LEMMA 4.4. *Let $L(\Lambda) = P_1\bar{\otimes}P_2$ be a tensor product decomposition into II_1 factors and denote $\mathcal{M} = M\bar{\otimes}L(\Lambda)$.*

Then there is a partition $S_1 \sqcup S_2 = \{1,\dots,n\}$ and a projection $0 \neq z \in \Delta(L(\Gamma))' \cap \mathcal{M}$ such that:

- *$\Delta(L(\Gamma_i))z$ is strongly non-amenable relative to $M\bar{\otimes}P_1$ inside $\mathcal{M}$ for all $i \in S_2$;*
- *$\Delta(L(\Gamma_i))z$ is strongly non-amenable relative to $M\bar{\otimes}P_2$ inside $\mathcal{M}$ for all $i \in S_1$.*

Proof. Let $i \in \{1,\dots,n\}$. Since Γ_i is non-amenable, by [KV15, Proposition 2.4] we get that $\Delta(L(\Gamma_i))$ is strongly non-amenable relative to $M \otimes 1$ inside $\mathcal{M}$. It follows that for every non-zero projection $z \in \Delta(L(\Gamma))' \cap \mathcal{M}$, there exist $f(i,z) \in \{1,2\}$ and a non-zero projection $p(i,z) \in \Delta(L(\Gamma))' \cap \mathcal{M}$ with $p(i,z) \leq z$ such that

$$\Delta(L(\Gamma_i))p(i,z) \text{ is strongly non-amenable relative to } M\bar{\otimes}P_{f(i,z)} \text{ inside } \mathcal{M}. \tag{4.1}$$

Indeed, otherwise there exists a non-zero projection $z \in \Delta(L(\Gamma))' \cap \mathcal{M}$ such that for any $k \in \{1,2\}$ and non-zero projection $z_0 \in \Delta(L(\Gamma))' \cap \mathcal{M}$ with $z_0 \leq z$, there exists a non-zero projection $\tilde{z}_0 \in \Delta(L(\Gamma))' \cap \mathcal{M}$ with $\tilde{z}_0 \leq z_0$ for which $\Delta(L(\Gamma_i))\tilde{z}_0$ is amenable relative to $M\bar{\otimes}P_k$ inside $\mathcal{M}$. By using [PV14, Proposition 2.7] we derive that there exists a non-zero projection $\tilde{z}_1 \in \Delta(L(\Gamma))' \cap \mathcal{M}$ with $\tilde{z}_1 \leq z$ for which $\Delta(L(\Gamma_i))\tilde{z}_1$ is amenable relative to $M\bar{\otimes}1$ inside $\mathcal{M}$, contradiction.

By applying (4.1) finitely many times, the proof will be obtained as follows. Define $z_1 = p(1,1)$ and $f(1) = f(1,1)$. For any $i \in \{2,\dots,n\}$ we recursively define $z_i = p(i, z_{i-1})$ and $f(i) = f(i, z_{i-1})$. Note that $z_1 \geq z_2 \geq \cdots \geq z_n$ are non-zero projections in $\Delta(L(\Gamma))' \cap \mathcal{M}$. Hence, the lemma follows by letting $S_1 = f^{-1}(2)$, $S_2 = f^{-1}(1)$, and $z = z_n$. □

We continue with the following notation that will be used in the following lemma, but also in the proof of Theorem 4.2. For any $1 \leq j \leq n$, denote $\Psi^{j,4} = \mathrm{id}_M \otimes \mathrm{id}_M \otimes \mathrm{id}_M \otimes \Psi^j$ and $\alpha^{j,5} = \mathrm{id}_M \otimes \mathrm{id}_M \otimes \mathrm{id}_M \otimes \mathrm{id}_M \otimes \alpha^j$. By letting $\mathcal{M} = M\bar{\otimes}L(\Lambda)$, note that $\Psi^j(p) = p \otimes 1$ and $\Psi^{j,4}(\mathcal{M}\bar{\otimes}\mathcal{M}) \subset M\bar{\otimes}L(\Lambda)\bar{\otimes}M\bar{\otimes}pMp\bar{\otimes}L(\Gamma_j)$.

LEMMA 4.5. *Let $\sigma \in \mathrm{Aut}(\mathcal{M}\bar{\otimes}\mathcal{M})$ be an automorphism for which $\sigma_{|(M\otimes 1)\bar{\otimes}(M\otimes 1)} = \mathrm{id}_{(M\otimes 1)\bar{\otimes}(M\otimes 1)}$ and $(1 \otimes L(\Lambda))\bar{\otimes}(1 \otimes L(\Lambda))$ is σ-invariant. Then $\Psi^{j,4}(\sigma(\Delta(M)\bar{\otimes}\Delta(M))z$ is not $\alpha^{j,5}$-rigid, for all non-zero projections $z \in \Psi^{j,4}(\sigma(\Delta(M)\bar{\otimes}\Delta(M))' \cap M\bar{\otimes}L(\Lambda)\bar{\otimes}M\bar{\otimes}pMp\bar{\otimes}L(\Gamma_j)$ and $j \in \overline{1,n}$.*

Proof. By assuming the contrary, there exist $j \in \{1,\dots,n\}$ and a projection z as in the statement such that $\Psi^{j,4}(\sigma(\Delta(M)\bar{\otimes}\Delta(M))z$ is $\alpha^{j,5}$-rigid. Hence, for any $\epsilon > 0$, there is $t_0 > 0$ such that

$$\|\Psi^{j,4}(\sigma(v_g \otimes v_g \otimes v_h \otimes v_h))z - \alpha_t^{j,5}(\Psi^{j,4}(\sigma(v_g \otimes v_g \otimes v_h \otimes v_h))z)\|_2 \leq \epsilon,$$

for all $g, h \in \Lambda$ and $|t| \leq t_0$. Since σ acts trivially on $(M \otimes 1)\bar{\otimes}(M \otimes 1)$, we obtain that

$$\|\Psi^{j,4}(\sigma(1 \otimes v_g \otimes 1 \otimes v_h))z - \alpha_t^{j,5}(\Psi^{j,4}(\sigma(1 \otimes v_g \otimes 1 \otimes v_h))z)\|_2 \leq \epsilon,$$

for all $g, h \in \Lambda$ and $|t| \leq t_0$. If we let $\mathcal{G} = \{\Psi^{j,4}(\sigma(1 \otimes v_g \otimes 1 \otimes v_h)) |\ g, h \in \Lambda\}$, we get that $\mathcal{G}'' = \Psi^{j,4}(1 \otimes L(\Lambda) \otimes 1\bar{\otimes}L(\Lambda)) = 1 \otimes L(\Lambda) \otimes 1\bar{\otimes}\Psi^j(L(\Lambda))$ since $(1 \otimes L(\Lambda))\bar{\otimes}(1 \otimes L(\Lambda))$ is σ-invariant. Note that $\mathcal{N}_{M\bar{\otimes}L(\Lambda)\bar{\otimes}M\bar{\otimes}pMp\bar{\otimes}L(\Gamma_j)}(\mathcal{G}'') \subset 1 \otimes 1 \otimes 1 \otimes (\Psi^j(L(\Lambda))' \cap (pMp\bar{\otimes}L(\Gamma_j)))$.

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By applying Proposition 3.2 we obtain a non-zero projection $z_0 \in \Psi^j(L(\Lambda))' \cap (pMp\bar{\otimes}L(\Gamma_j))$ such that $\Psi^j(L(\Lambda))z_0$ is $(\mathrm{id}\otimes\alpha^j)$-rigid. Since $\Psi^j(B) = B\otimes 1$ and $\Psi^j(pMp)' \cap (pMp\bar{\otimes}L(\Gamma_j)) = \mathbb{C}(p\otimes 1)$, it follows from Proposition 3.2 that $\Psi^j(pMp)$ is $(\mathrm{id}\otimes\alpha^j)$-rigid, and hence, $\Psi^j(M)$ is $(\mathrm{id}\otimes\alpha^j)$-rigid. This shows that $L(\Gamma_j)$ is α^j-rigid: contradiction. □

Proof of Theorem 4.2. Denote $\mathcal{M} = M\bar{\otimes}L(\Lambda)$ and $\tilde{\mathcal{M}} = M\bar{\otimes}M$. For proving this theorem, we use the following variation of the flip automorphism method from [IM19]. Namely, since $L(\Lambda) = P_1\bar{\otimes}P_2$, we define $\sigma_{P_1} \in \mathrm{Aut}(\mathcal{M}\bar{\otimes}\mathcal{M})$ by letting $\sigma_{P_1}(m\otimes p_1\otimes p_2\otimes m'\otimes p_1'\otimes p_2') = m\otimes p_1'\otimes p_2\otimes m'\otimes p_1\otimes p_2'$, for all $m, m' \in M, p_1, p_1' \in P_1, p_2, p_2' \in P_2$.

By applying Lemmas 4.4 and 2.4 we obtain a partition $S_1 \sqcup S_2 = \{1,\dots,n\}$ and a non-zero projection $z \in \Delta(L(\Gamma))' \cap \mathcal{M}$ such that

$$\begin{aligned}&\Delta(L(\Gamma_i))z\otimes 1 \text{ is strongly non-amenable relative to } (M\bar{\otimes}P_2)\bar{\otimes}(M\bar{\otimes}P_1) \text{ inside } \mathcal{M}\bar{\otimes}\mathcal{M},\\ &1\otimes\Delta(L(\Gamma_j))z \text{ is strongly non-amenable relative to } (M\bar{\otimes}P_2)\bar{\otimes}(M\bar{\otimes}P_1) \text{ inside } \mathcal{M}\bar{\otimes}\mathcal{M},\end{aligned} \tag{4.2}$$

for all $i \in S_1$ and $j \in S_2$. By applying the flip automorphism σ_{P_1} to (4.2), we derive that

$$\begin{aligned}&\sigma_{P_1}(\Delta(L(\Gamma_i))z\otimes 1) \text{ is strongly non-amenable relative to } \mathcal{M}\bar{\otimes}(M\otimes 1) \text{ inside } \mathcal{M}\bar{\otimes}\mathcal{M},\\ &\sigma_{P_1}(1\otimes\Delta(L(\Gamma_j))z) \text{ is strongly non-amenable relative to } \mathcal{M}\bar{\otimes}(M\otimes 1) \text{ inside } \mathcal{M}\bar{\otimes}\mathcal{M},\end{aligned}$$

for all $i \in S_1$ and $j \in S_2$. By using Lemma 2.4, we further deduce that

$$\begin{aligned}&\sigma_{P_1}(\Delta(L(\Gamma_i))z\otimes 1) \text{ is strongly non-amenable relative to } \mathcal{M}\bar{\otimes}(M\otimes 1) \text{ inside } \mathcal{M}\bar{\otimes}\tilde{\mathcal{M}},\\ &\sigma_{P_1}(1\otimes\Delta(L(\Gamma_j))z) \text{ is strongly non-amenable relative to } \mathcal{M}\bar{\otimes}(M\otimes 1) \text{ inside } \mathcal{M}\bar{\otimes}\tilde{\mathcal{M}},\end{aligned} \tag{4.3}$$

for all $i \in S_1$ and $j \in S_2$. Denote $\widehat{z} = \sigma_{P_1}(z\otimes z) \in \sigma_{P_1}(\Delta(L(\Gamma))\bar{\otimes}\Delta(L(\Gamma)))' \cap (\mathcal{M}\bar{\otimes}\mathcal{M})$ and note that $\widehat{z} \le \sigma_{P_1}(z\otimes 1)$, $\widehat{z} \le \sigma_{P_1}(1\otimes z)$. For ease of notation, we denote $Q_i = \Delta(L(\Gamma_i))\otimes 1$ and $R_i = \Delta(L(\Gamma_{\widehat{i}})\bar{\otimes}\Delta(L(\Gamma))$, for all $i \in S_1$. Similarly, denote $Q_j = 1\otimes\Delta(L(\Gamma_j))$ and $R_j = \Delta(L(\Gamma))\bar{\otimes}\Delta(L(\Gamma_{\widehat{j}})$, for all $j \in S_2$. Note that $Q_i \vee R_i = \Delta(L(\Gamma))\bar{\otimes}\Delta(L(\Gamma))$, for any $i \in \{1,\dots,n\}$.

By applying a similar argument to that used in the proof of Lemma 4.4, we deduce from (4.3) that there exist a non-zero projection $\tilde{z} \in \sigma_{P_1}(\Delta(L(\Gamma))\bar{\otimes}\Delta(L(\Gamma)))' \cap (\mathcal{M}\bar{\otimes}\tilde{\mathcal{M}})$ and a function $\varphi : \{1,\dots,n\} \to \{1,\dots,n\}$ such that

$$\sigma_{P_1}(Q_i)\tilde{z} \text{ is strongly non-amenable relative to } \mathcal{M}\bar{\otimes}M\bar{\otimes}(A\rtimes\Gamma_{\widehat{\varphi(i)}}), \tag{4.4}$$

for all $i \in \{1,\dots,n\}$. By Lemma 3.6, we get that for any $i \in \{1,\dots,n\}$,

$$\Psi^{\varphi(i),4}(\sigma_{P_1}(R_i)\tilde{z}) \text{ is } \alpha^{\varphi(i),5}\text{-rigid}. \tag{4.5}$$

Next, we claim that the map φ is bijective. If this does not hold, it is easy to see that we can deduce from (4.4) that there exists $j \in \{1,\dots,n\}$ such that

$$\Psi^{j,4}(\sigma_{P_1}(\Delta(L(\Gamma))\bar{\otimes}\Delta(L(\Gamma)))\tilde{z}) \text{ is } \alpha^{j,5}\text{-rigid}. \tag{4.6}$$

Using the position of $B \subset A$, and $\sigma_{P_1}(\Delta(B)\bar{\otimes}\Delta(B)) \subset \mathcal{M}\bar{\otimes}(M\otimes 1)$, we obtain that

$$\Psi^{j,4}(\sigma_{P_1}(\Delta(A)\bar{\otimes}\Delta(A)) \text{ is } \alpha^{j,5}\text{-rigid}. \tag{4.7}$$

Relations (4.6) and (4.7) in combination with Proposition 3.2 gives a contradiction to Lemma 4.5. This shows that φ is indeed bijective. Next, we claim that for all $i \in \{1,\dots,n\}$,

$$\sigma_{P_1}(R_i)\tilde{z} \prec^s_{\mathcal{M}\bar{\otimes}\tilde{\mathcal{M}}} \mathcal{M}\bar{\otimes}M\bar{\otimes}(A\rtimes\Gamma_{\widehat{g(i)}}), \tag{4.8}$$

Assume by contradiction that there is $i \in \{1,\dots,n\}$ for which (4.8) does not hold. Then by using [DHI16, Lemma 2.4(2)], it follows that, up to replacing $\tilde{z}$ by a smaller non-zero projection, we

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have $\sigma_{P_1}(R_i)\tilde{z} \nprec_{\mathcal{M}\bar{\otimes}\tilde{\mathcal{M}}} \mathcal{M}\bar{\otimes}M\bar{\otimes}(A \rtimes \Gamma_{\widehat{g(i)}})$. Using (4.4) and (4.5) we may apply Lemma 3.6 to deduce that $\Psi^{j,4}(\sigma_{P_1}(\Delta(L(\Gamma))\bar{\otimes}\Delta(L(\Gamma)))\tilde{z})$ is $\alpha^{j,5}$-rigid. As before, Proposition 3.2 leads to a contradiction.

Finally, by applying [DHI16, Lemma 2.8(2)] finitely many times, we deduce from (4.8) that

$$\sigma_{P_1}(\Delta(L(\Gamma_{\widehat{S_1}}))\bar{\otimes}\Delta(L(\Gamma_{\widehat{S_2}}))) \prec_{\mathcal{M}\bar{\otimes}\tilde{\mathcal{M}}} \mathcal{M}\bar{\otimes}(M \otimes 1).$$

By applying Lemma 2.4 we further obtain that $\sigma_{P_1}(\Delta(L(\Gamma_{\widehat{S_1}}))\bar{\otimes}\Delta(L(\Gamma_{\widehat{S_2}}))) \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\bar{\otimes}(M \otimes 1)$. By applying the flip automorphism σ_{P_1} to the previous intertwining relation, we deduce that $\Delta(L(\Gamma_{S_i})) \prec_{\mathcal{M}} M\bar{\otimes}P_i$, for all $i \in \{1,2\}$. Since $\Delta(A) \prec^s_{\mathcal{M}} B \otimes 1$, we may use [BV14, Lemma 2.3] to get that $\Delta(A \rtimes \Gamma_{S_i}) \prec_{\mathcal{M}} M\bar{\otimes}P_i$, for all $i \in \{1,2\}$. Since $\mathcal{N}_{\mathcal{M}}(\Delta(A \rtimes \Gamma_{S_i}))' \cap \mathcal{M} \subset \Delta(M)' \cap \mathcal{M} = \mathbb{C}1$, we obtain $\Delta(A \rtimes \Gamma_{S_i}) \prec^s_{\mathcal{M}} M\bar{\otimes}P_i$, for all $i \in \{1,2\}$.

For showing that S_1 and S_2 are non-empty sets, we suppose the contrary. Hence, without loss of generality, assume that S_2 is empty. This shows that $\Delta(M) \prec_{\mathcal{M}} M\bar{\otimes}P_1$, which implies from [Ioa11, Lemma 9.2] that $L(\Lambda) \prec_{L(\Lambda)} P_1$. This shows that P_2 is not diffuse: a contradiction. ☐

4.2 Step 2

By using Step 1, we obtain the following intertwining result. Recall that we are using Notation 3.7 and Assumption 4.1.

Theorem 4.6. *Let $L(\Lambda) = P_1\bar{\otimes}P_2$ be a tensor product decomposition into II$_1$ factors.*

Then there is a partition $T_1 \sqcup T_2 = \{1, \dots, n\}$ such that $P_i \prec^s_M A \rtimes \Gamma_{T_i}$, for all $i \in \{1,2\}$.

Throughout the proof we are using the following notation: if N is a tracial von Neumann algebra and $P \subset pNp$ and $Q \subset qNq$ are von Neumann subalgebras, we denote $P \prec^{s'}_N Q$ if $P \prec_N Qq'$, for any non-zero projection $q' \in Q' \cap qNq$.

Proof of Theorem 4.6. Theorem 4.2 implies that there exist projections $r_1 \in \Delta(L(\Gamma_{S_1}))$, $q_1 \in M\bar{\otimes}P_1$, a non-zero partial isometry $w_1 \in q_1(M\bar{\otimes}M)r_1$ and a $*$-homomorphism $\varphi_1 : r_1\Delta(L(\Gamma_{S_1}))r_1 \to q_1(M\bar{\otimes}P_1)q_1$ such that $\varphi_1(x)w_1 = w_1x$, for all $x \in r_1\Delta(L(\Gamma_{S_1}))r_1$. Fix an arbitrary $j_0 \in S_1$. Since $L(\Gamma_{j_0})$ is a II$_1$ factor we can apply [CdSS18, Lemma 4.5] and therefore assume without loss of generality that $r_1 \in L(\Gamma_{j_0})$. In addition, we can assume that the support projection of $E_{M\bar{\otimes}P_1}(w_1w_1^*)$ equals q_1. For any $j \in S_1$, denote $Q_1^j = \varphi_1(r_1\Delta(L(\Gamma_j))r_1) \subset q_1(M\bar{\otimes}P_1)q_1$ and let $Q_1 = \bigvee_{j\in S_1} Q_1^j$. Note that for any subset $S \subset S_1$, we have $\Delta(L(\Gamma_S)) \prec^{s'}_{M\bar{\otimes}P_1} \bigvee_{j\in S} Q_1^j$. Indeed, let $S \subset S_1$ and consider a non-zero projection $z \in Q_1' \cap q_1(M\bar{\otimes}P_1)q_1$. Note that $\tilde{w}_1 := zw_1 \neq 0$ since otherwise $zE_{M\bar{\otimes}P_1}(w_1w_1^*) = 0$, which implies that $z = 0$, false. This shows that the $*$-homomorphism $\tilde{\varphi}_1 : r_1\Delta(L(\Gamma_S))r_1 \to \bigvee_{j\in S} Q_1^j z$ satisfies $\tilde{\varphi}_1(x)\tilde{w}_1 = \tilde{w}_1x$, for all $x \in r_1\Delta(L(\Gamma_S))r_1$. By replacing $\tilde{w}_1$ by the partial isometry from its polar decomposition, we derive that $\Delta(L(\Gamma_S)) \prec_{M\bar{\otimes}P_1} \bigvee_{j\in S} Q_1^j z$. By using [DHI16, Lemma 2.4] it follows that $\Delta(L(\Gamma_S)) \prec^{s'}_{M\bar{\otimes}P_1} \bigvee_{j\in S} Q_1^j$. By applying [Dri20b, Lemma 2.3], we derive that for any subset $S \subset S_1$,

$$\Delta(L(\Gamma_S)) \prec^{s'}_{M\bar{\otimes}M} \bigvee_{j\in S} Q_1^j. \tag{4.9}$$

The rest of the proof is divided between three claims. ☐

Claim 1. *For any $j \in S_1$ and non-zero projection $z \in Q_1' \cap q_1(M\bar{\otimes}P_1)q_1$, there exist $k \in \{1, \dots, n\}$ and a non-zero projection $z_0 \in Q_1' \cap q_1(M\bar{\otimes}P_1)q_1$ with $z_0 \le z$ such that $Q_1^j z_0$ is strongly non-amenable relative to $M\bar{\otimes}(A \rtimes \Gamma_{\widehat{k}})$ inside $M\bar{\otimes}M$.*

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Proof of Claim 1. We assume by contradiction that there exist $j \in S_1$ and a non-zero projection $z \in Q_1' \cap q_1(M\bar{\otimes}P_1)q_1$ such that $Q_1^j z$ is amenable relative to $M\bar{\otimes}(A \rtimes \Gamma_{\widehat{k}})$, for all $k \in \{1,\dots,n\}$. By applying [PV14, Proposition 2.7] we get that $Q_1^j z$ is amenable relative to $M \otimes 1$ inside $M\bar{\otimes}M$. Relation (4.9) implies that $\Delta(L(\Gamma_j)) \prec_{M\bar{\otimes}M} Q_1^j z$. We can apply [DHI16, Lemma 2.4(3) and Lemma 2.6(3)] and derive that there exists a non-zero projection $r' \in \Delta(L(\Gamma_j))' \cap M\bar{\otimes}M$ such that $\Delta(L(\Gamma_j))r'$ is amenable relative to $Q_1^j z \oplus \mathbb{C}(1-z)$. Using [OP10, Proposition 2.4(3)] we derive that $\Delta(L(\Gamma_j))r'$ is amenable relative to $M \otimes 1$. By using [IPV13, Lemma 10.2(5)] we deduce that Γ_j is amenable: a contradiction. Thus, there exist $k \in \{1,\dots,n\}$ such that $Q_1^j z$ is non-amenable relative to $M\bar{\otimes}(A \rtimes \Gamma_{\widehat{k}})$. By [DHI16, Lemma 2.6], there exists a non-zero projection $z_0 \in \mathcal{N}_{q_1(M\bar{\otimes}M)q_1}(Q_1^j)' \cap q_1(M\bar{\otimes}M_1)q_1 \subset Q_1' \cap q_1(M\bar{\otimes}P_1)q_1$ with $z_0 \leq z$ such that $Q_1^j z_0$ is strongly non-amenable relative to $M\bar{\otimes}(A \rtimes \Gamma_{\widehat{k}})$. □

By applying Claim 1 finitely many times and proceeding as in the proof of Lemma 4.4, there exist a non-zero projection $z \in Q_1' \cap q_1(M\bar{\otimes}P_1)q_1$ and a map $\overline{1,n} \ni j \to k_j \in \overline{1,n}$ such that

$$Q_1^j z \text{ is strongly non-amenable relative to } M\bar{\otimes}(A \rtimes \Gamma_{\widehat{k_j}}), \text{ for any } j \in \overline{1,n}. \tag{4.10}$$

Claim 2. *We claim that $P_2 \prec_M^s A \rtimes \Gamma_{\widehat{k_j}}$, for all $j \in S_1$.*

Proof of Claim 2. Fix an arbitrary $j \in S_1$. We are in one of the following situations. First, if we assume that $(1 \otimes P_2)z \prec_{M\bar{\otimes}M} M\bar{\otimes}(A \rtimes \Gamma_{\widehat{k_j}})$, we get $P_2 \prec_M A \rtimes \Gamma_{\widehat{k_j}}$. Since $\mathcal{N}_{pMp}(P_2)' \cap pMp \subset L(\Gamma)' \cap pMp = \mathbb{C}p$, the claim follows from [DHI16, Lemma 2.4(3)]. Second, assume that $(1 \otimes P_2)z \nprec_{M\bar{\otimes}M} M\bar{\otimes}(A \rtimes \Gamma_{\widehat{k_j}})$. Since $Q_1^j z \subset ((1 \otimes P_2)z)' \cap z(M\bar{\otimes}M)z$, (4.10) implies that $((1 \otimes P_2)z)' \cap z(M\bar{\otimes}M)z$ is strongly non-amenable relative to $M\bar{\otimes}(A \rtimes \Gamma_{\widehat{k_j}})$. Altogether, we can apply Lemma 3.6 to deduce that $(1 \otimes \Psi^{k_j})(z(M\bar{\otimes}L(\Lambda))z)$ is $(\mathrm{id} \otimes \mathrm{id} \otimes \alpha^{k_j})$-rigid. Since $M\bar{\otimes}L(\Lambda)$ is a II_1 factor and $(1 \otimes \Psi^{k_j})(1 \otimes B) \subset 1 \otimes B \otimes 1$, it follows that $(1 \otimes \Psi^{k_j})(M\bar{\otimes}M)$ is $(\mathrm{id} \otimes \mathrm{id} \otimes \alpha^{k_j})$-rigid, which implies that $L(\Gamma_{k_j})$ is α^{k_j}-rigid: a contradiction. This completes the proof of the claim. □

Note that Q_1 and $(1 \otimes P_2)q_1$ are commuting subalgebras of $q_1(M\bar{\otimes}M)q_1$ Thus, (4.10) together with Lemma 3.6 imply that for any $j \in S_1$ we have

$$(1 \otimes \Psi^{k_j})\Big(\bigvee_{i \in S_1 \setminus \{j\}} Q_1^i z \vee (1 \otimes P_2)z\Big) \text{ is } (\mathrm{id} \otimes \mathrm{id} \otimes \alpha^{k_j})\text{-rigid}. \tag{4.11}$$

We now ready to prove the following.

Claim 3. *The map $S_1 \ni j \to k_j \in \{1,\dots,n\}$ is injective.*

Proof of Claim 3. Assume by contradiction that there exist two distinct elements $j_1, j_2 \in S_1$ such that $k := k_{j_1} = k_{j_2}$. Thus, $(S_1 \setminus \{j_1\}) \cup (S_1 \setminus \{j_2\}) = S_1$. Since the algebras $Q_1^j z, j \in S_1$, are commuting, we deduce from (4.11) that $(1 \otimes \Psi^k)(Q_1 z)$ is $(\mathrm{id} \otimes \mathrm{id} \otimes \alpha^k)$-rigid. As in the proof of Claim 1, we note that $zw_1 \neq 0$. Note also that $Q_1 z w_1 = z w_1 r_1 \Delta(L(\Gamma_{S_1}))r_1$. By applying Proposition 3.2 we obtain a non-zero projection $e_1 \in (1 \otimes \Psi^k)(\Delta(L(\Gamma)))' \cap M\bar{\otimes}M\bar{\otimes}L(\Gamma)$ such that

$$(1 \otimes \Psi^k)(\Delta(L(\Gamma_{S_1})))e_1 \text{ is } (\mathrm{id} \otimes \mathrm{id} \otimes \alpha^k)\text{-rigid}. \tag{4.12}$$

Since $z \in q_1(M\bar{\otimes}P_1)q_1$ and $M\bar{\otimes}P_1$ is a II_1 factor, one can check that (4.11) implies

$$\Psi^k(P_2) \text{ is } (\mathrm{id} \otimes \alpha^k) \text{ -rigid}. \tag{4.13}$$

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Next, since $\Delta(L(\Gamma_{S_2})) \prec^s_{M\bar{\otimes}L(\Lambda)} M\bar{\otimes}P_2$, we obtain from [DHI16, Remark 2.2] that

$$(1\otimes\Psi^k)(\Delta(L(\Gamma_{S_2}))) \prec^s_{M\bar{\otimes}M\bar{\otimes}L(\Gamma_k)} M\bar{\otimes}\Psi^k(P_2).$$

Therefore, $(1\otimes\Psi^k)(\Delta(L(\Gamma_{S_2})))e_1 \prec_{M\bar{\otimes}M\bar{\otimes}L(\Gamma_k)} M\bar{\otimes}\Psi(P_2)$, which implies by (4.13) that there is a projection $0\neq e_2 \in (1\otimes\Psi^k)(\Delta(L(\Gamma_{S_2})))'\cap(M\bar{\otimes}M\bar{\otimes}L(\Gamma_k))$ with $e_2\leq e_1$ such that

$$(1\otimes\Psi^k)(\Delta(L(\Gamma_{S_2})))e_2 \text{ is } (\mathrm{id}\otimes\mathrm{id}\otimes\alpha^k)\text{-rigid.} \tag{4.14}$$

Note that (4.12) implies that $e_2(1\otimes\Psi^k)(\Delta(L(\Gamma_{S_1})))e_2$ is $(\mathrm{id}\otimes\mathrm{id}\otimes\alpha^k)$-rigid. Together with (4.14) and the fact that $\Psi^k(A)\subset A\otimes 1$, we deduce from Proposition 3.2 that there exists a non-zero projection $e_3\in(1\otimes\Psi^k)(\Delta(M))'\cap M\bar{\otimes}M\bar{\otimes}L(\Gamma_k)$ such that

$$(1\otimes\Psi^k)(\Delta(M))e_3 \text{ is } (\mathrm{id}\otimes\mathrm{id}\otimes\alpha^k)\text{-rigid.} \tag{4.15}$$

This implies that for any $\epsilon>0$, there exists $t_0>0$ such that for all $|t|\leq t_0$ and $g\in\Lambda$,

$$\|(1\otimes\Psi^k)(v_g\otimes v_g)e_3 - (\mathrm{id}\otimes\mathrm{id}\otimes\alpha_t^k)((1\otimes\Psi^k)(v_g\otimes v_g)e_3)\|_2\leq\epsilon,$$

and, therefore,

$$\|(1\otimes\Psi^k)(1\otimes v_g)e_3 - (\mathrm{id}\otimes\mathrm{id}\otimes\alpha_t^k)((1\otimes\Psi)(1\otimes v_g)e_3)\|_2\leq\epsilon.$$

Note that $\Psi^k(B)\subset B\otimes 1$. By applying Proposition 3.2 we get that $\Psi^k(M)e_0$ is $(\mathrm{id}\otimes\alpha^k)$-rigid for a projection $0\neq e_0\in\Psi^k(M)'\cap(M\bar{\otimes}L(\Gamma_k))$. Since Γ_k is icc, we get $\Psi^k(M)'\cap(M\bar{\otimes}L(\Gamma_k))=\mathbb{C}1$. Thus, we obtain that $L(\Gamma_k)$ is α^k-rigid, contradiction.

Denote $R_1=\{k_j\mid j\in S_1\}\subset\{1,\ldots,n\}$. Claim 3 implies that $|S_1|=|R_1|$ while Claim 2 together with [DHI16, Lemma 2.8(2)] gives that $P_1\prec^s_M B\rtimes\Lambda_{\widehat{R_1}}$. In a similar way, there exists a subset $R_2\subset\{1,\ldots,n\}$ with $|S_2|=|R_2|$ such that $P_2\prec^s_M A\rtimes\Gamma_{\widehat{R_2}}$. By using [CDD21, Proposition 4.4] we deduce that $L(\Gamma)\prec^s_M A\rtimes\Gamma_{\widehat{R_1}\cup\widehat{R_2}}$. Using [BV14, Lemma 2.3] we get that $M\prec_M A\rtimes\Gamma_{\widehat{R_1}\cup\widehat{R_2}}$, which implies that $\widehat{R_1}\cup\widehat{R_2}=\{1,\ldots,n\}$.

Finally, we let $T_1=\widehat{R_1}$ and $T_2=\widehat{R_2}$. Since $S_1\sqcup S_2=\{1,\ldots,n\}$ is a partition, it follows that $T_1\sqcup T_2=\{1,\ldots,n\}$ is a partition as well. This ends the proof. □

5. From unitary conjugacy of subalgebras to cohomologous cocycles

In this section we prove Proposition 5.1 which provides sufficient conditions at the von Neumann algebra level for untwisting the underlying cocycle of an orbit equivalence of irreducible actions.

Throughout this section we will use the well-known fact that if $\Gamma\curvearrowright(X,\mu)$ and $\Lambda\curvearrowright(Y,\nu)$ are free ergodic pmp actions such that there is a measure space isomorphism $\theta:X\to Y$ with $\theta(\Gamma x)=\Lambda\theta(x)$, for almost every $x\in X$, then the induced isomorphism of von Neumann algebras $\pi:L^\infty(X)\to L^\infty(Y)$ given by $\pi(a)=a\circ\theta^{-1}$ extends to an isomorphism $\pi:L^\infty(X)\rtimes\Gamma\to L^\infty(Y)\rtimes\Lambda$ satisfying $\pi(u_g)=v_{\theta\circ g\circ\theta^{-1}}$, for any $g\in\Gamma$. Here and throughout the section, we denote by $v_\phi\in\mathcal{U}(L^\infty(Y)\rtimes\Lambda)$ the associated unitary of $\phi\in[\mathcal{R}(\Lambda\curvearrowright Y)]$; see [AP10, § 1.5.2] for more details.

Proposition 5.1. *Let $\Gamma=\Gamma_1\times\cdots\times\Gamma_n\curvearrowright(X,\mu)$ and $\Lambda=\Lambda_1\times\cdots\times\Lambda_n\curvearrowright(Y,\nu)$ be free, irreducible, pmp actions such that are OE via a map $\theta:X\to Y$. Denote by $\pi:L^\infty(X)\rtimes\Gamma\to L^\infty(Y)\rtimes\Lambda$ the $*$-isomorphism associated to θ and let $c:\Gamma\times X\to\Lambda$ be the Zimmer cocycle associated to θ.*

If there exist $u_1,\ldots,u_n\in\mathcal{U}(L^\infty(Y)\rtimes\Lambda)$ such that $\pi(L^\infty(X)\rtimes\Gamma_{\widehat{i}})=u_i(L^\infty(Y)\rtimes\Lambda_{\widehat{i}})u_i^*$, for any $i\in\{1,\ldots,n\}$, then c is cohomologous to a group isomorphism $\delta:\Gamma\to\Lambda$.

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We first need the following elementary result. For completeness, we provide a proof.

Lemma 5.2. *Let $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ be free, ergodic, pmp actions. For any $1 \le i \le 2$, assume that there exist a $*$-isomorphism $\pi_i : L^\infty(X) \rtimes \Gamma \to L^\infty(Y) \rtimes \Lambda$ such that $\pi_i(L^\infty(X)) = L^\infty(Y)$, let $\theta_i : X \to Y$ be the measure space isomorphism defined by $\pi_i(a) = a \circ \theta_i^{-1}$, for any $a \in L^\infty(X)$, and let $c_i : \Gamma \times X \to \Lambda$ be the Zimmer cocycle associated to θ_i.*

If there exists $\omega \in \mathcal{U}(L^\infty(Y) \rtimes \Lambda)$ such that $\pi_2 = \mathrm{Ad}(\omega) \circ \pi_1$, then the cocycles c_1 and c_2 are cohomologous.

Proof. Since $\omega \in \mathcal{N}_{L^\infty(Y)\rtimes\Lambda}(L^\infty(Y))$, we can write $\omega = bv_\varphi$, for some $b \in \mathcal{U}(L^\infty(Y))$ and $\varphi \in [\mathcal{R}(\Lambda \curvearrowright Y)]$ (see, for instance, [AP10, Lemma 12.1.16]). Take a measurable map $\psi : Y \to \Lambda$ such that $\varphi^{-1}(y) = \psi(y)y$, for almost every $y \in Y$. For any $a \in L^\infty(X)$, we have $a \circ \theta_2^{-1} = \omega(a \circ \theta_1^{-1})\omega^* = a \circ \theta_1^{-1} \circ \varphi^{-1}$. This shows that $\theta_2 = \varphi \circ \theta_1$. We will prove the lemma by showing that

$$c_1(g,x)\psi(\theta_2(x)) = \psi(\theta_2(gx))c_2(g,x), \text{ for all } g \in \Gamma \text{ and almost every } x \in X. \tag{5.1}$$

To this end, fix an arbitrary $g \in \Gamma$. Define $\tilde{\psi} = \psi \circ \theta_2$. Since for almost every $y \in Y$ and $i \in \{1,2\}$, we have $(\theta_i \circ g \circ \theta_i^{-1})(y) = c_i(g, \theta_i^{-1}(y))y$, it follows that

$$\begin{aligned} c_1(g^{-1}, \theta_2^{-1}(y))\tilde{\psi}(\theta_2^{-1}(y))y &= c_1(g^{-1}, \theta_1^{-1}(\psi(y)y))\psi(y)y \\ &= (\theta_1 \circ g \circ \theta_1^{-1})(\psi(y)y)) = (\theta_1 \circ g \circ \theta_1^{-1} \circ \varphi^{-1})(y) \\ &= (\varphi^{-1} \circ \theta_2 \circ g \circ \theta_2^{-1})(y) = \psi((\theta_2 \circ g \circ \theta_2^{-1})(y))(\theta_2 \circ g \circ \theta_2^{-1})(y) \\ &= \tilde{\psi}(g\theta_2^{-1}(y))c_2(g, \theta_2^{-1}(y))y. \end{aligned}$$

Since $\Lambda \curvearrowright Y$ is free, we obtain that (5.1) holds, thus proving the lemma. □

The following lemma is a particular case of [HHI21, Lemma 3.1] and it goes back to [MS06, § 5]. For the convenience of the reader, we provide a short proof for it using von Neumann algebras.

Lemma 5.3 [HHI21]. *Let $\Gamma = \Gamma_1 \times \Gamma_2 \overset{\sigma}{\curvearrowright} (X,\mu)$ and $\Lambda = \Lambda_1 \times \Lambda_2 \overset{\rho}{\curvearrowright} (Y,\nu)$ be free, ergodic, pmp actions with Γ_1 and Λ_1 acting ergodically. Assume that there exists a measure space isomorphism $\theta : X \to Y$ such that $\theta(\Gamma \cdot x) = \Lambda \cdot \theta(x)$ and $\theta(\Gamma_1 \cdot x) = \Lambda_1 \cdot \theta(x)$ for almost every $x \in X$. Let c be the Zimmer cocycle associated to θ.*

Then there exists a group isomorphism $\delta_2 : \Gamma_2 \to \Lambda_2$ such that $c(g,x) \in \Lambda_1\delta_2(g_2)$ for every $g = (g_1, g_2) \in \Gamma$ and almost every $x \in X$.

Proof. Denote by $\pi : L^\infty(X) \rtimes \Gamma \to L^\infty(Y) \rtimes \Lambda$ the $*$-isomorphism associated to θ. For ease of notation, we suppress π. Recall that for each $g \in \Gamma$ we can decompose

$$u_g = \sum_{\lambda\in\Lambda} 1_{Y_{g,\lambda}} v_\lambda, \tag{5.2}$$

where $Y_{g,\lambda} = \{y \in Y \mid c(g^{-1}, \theta^{-1}(y)) = \lambda^{-1}\}$, as $\lambda \in \Lambda$. By assumption, $c(g,x) \in \Lambda_1$, for any $g \in \Gamma_1$ and almost every $x \in X$. Hence, we deduce $N := L^\infty(X) \rtimes \Gamma_1 = L^\infty(Y) \rtimes \Lambda_1$.

Next, we fix $g \in \Gamma_2$. Note that the actions $\sigma_{|\Gamma_2}$ and $\rho_{|\Lambda_2}$ extend in a natural way to actions on N. We can write $u_g = \sum_{\lambda\in\Lambda_2} b^g_\lambda v_\lambda$, with $b^g_\lambda \in N$, for all $\lambda \in \Lambda_2$. Note that for any $a \in N$ we have $b^g_\lambda \rho_\lambda(a) = \sigma_g(a)b^g_\lambda$, for all $\lambda \in \Lambda_2$. Thus, for any $\lambda \in \Lambda_2$, we get $(b^g_\lambda)^* b^g_\lambda \in N' \cap M = \mathbb{C}1$. Assume by contradiction that there exist $\lambda_1 \neq \lambda_2 \in \Lambda_2$ such that $b^g_{\lambda_1}$ and $b^g_{\lambda_2}$ are non-zero. Thus, there exist $\lambda_0 \in \Lambda_2 \setminus \{e\}$ and a unitary $c \in N$ such that $\rho_{\lambda_0}(a)c = ca$, for all $a \in N$. By writing $c = \sum_{\lambda\in\Lambda_1} c_\lambda v_\lambda$, we have $\rho_{\lambda_0\lambda^{-1}}(a)\rho_{\lambda^{-1}}(c_\lambda) = a\rho_{\lambda^{-1}}(c_\lambda)$, for all $a \in L^\infty(Y)$ and $\lambda \in \Lambda_1$. Since $\lambda_0\lambda^{-1}$ acts freely, we get that $c = 0$: a contradiction. Thus, we have shown that there exist a

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map $\delta_2 : \Gamma_2 \to \Lambda_2$ and a unitary $b_g \in N$, as $g \in \Gamma_2$ satisfying

$$u_g = b_g v_{\delta_2(g)}. \tag{5.3}$$

One immediately obtains that $\delta_2 : \Gamma_2 \to \Lambda_2$ is a group homomorphism. In a similar way, we can write $v_\lambda = \tilde{b}_\lambda u_{\eta_2(\lambda)}$ for some $\tilde{b}_\lambda \in N$ and a group homomorphism $\eta_2 : \Lambda_2 \to \Gamma_2$. It follows that $\eta_2 \circ \delta_2 = \mathrm{Id}$ and $\delta_2 \circ \eta_2 = \mathrm{Id}$, hence showing that δ_2 is a group isomorphism.

Therefore, by combining (5.2) and (5.3), we deduce that for any $g \in \Gamma_2$, we have that $\mu(Y_{g,\lambda}) = 0$, for any $\lambda \notin \Lambda_1\delta_2(g)$. This implies that $c(g,x) \in \Lambda_1\delta_2(g)$, for any $g \in \Gamma_2$. Finally, if $g = (g_1, g_2) \in \Gamma$, we get that $c(g,x) = c(g_1, g_2x)c(g_2,x) \in \Lambda_1\delta_2(g_2)$. This ends the proof of the lemma. □

5.1 Proof of Proposition 5.1

Fix an arbitrary $i \in \overline{1,n}$. By Lemma 5.2 we get that the underlying Zimmer cocycle $c_i : \Gamma \times X \to \Lambda$ of the orbit equivalence $\theta_i : X \to Y$ associated to $\pi_i := \mathrm{Ad}(u_i^*) \circ \pi : L^\infty(X) \rtimes \Gamma \to L^\infty(Y) \rtimes \Lambda$ is cohomologous to c. Hence, there is a map $\varphi_i : X \to \Lambda$ such that $c(g,x) = \varphi_i(gx)^{-1}c_i(g,x)\varphi_i(x)$, for all $g \in \Gamma$ and almost every $x \in X$. Note that $\Gamma_{\hat{i}} \curvearrowright X$ and $\Lambda_{\hat{i}} \curvearrowright Y$ are ergodic. Since $\pi_i(L^\infty(X) \rtimes \Gamma_{\hat{i}}) = L^\infty(Y) \rtimes \Lambda_{\hat{i}}$, we get that $\theta_i(\Gamma_{\hat{i}} \cdot x) = \Lambda_{\hat{i}} \cdot \theta(x)$, for almost every $x \in X$ and, hence, we obtain from Lemma 5.3 that there is a group isomorphism $\delta_i : \Gamma_i \to \Lambda_i$ such that $c_i(g,x) \in \Lambda_{\hat{i}}\delta_i(g_i)$, for every $g = (g_{\hat{i}}, g_i) \in \Gamma = \Gamma_{\hat{i}} \times \Gamma_i$ and almost every $x \in X$.

Next, since $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ we decompose $\varphi_i = \varphi_i^1 \ldots \varphi_i^n$ and $c_i = c_i^1 \ldots c_i^n$ where φ_i^j and c_i^j are valued to Λ_j, for any $j \in \overline{1,n}$. By letting $\varphi = \varphi_1^1 \ldots \varphi_n^n : X \to \Lambda$ and $\tilde{c} : \Gamma \times X \to \Lambda$ defined by $\tilde{c}(g,x) = \varphi(gx)c(g,x)\varphi(x)^{-1}$, we get $\tilde{c}(g,x) = \phi(gx)\phi_i(gx)^{-1}c_i(g,x)\phi_i(x)\phi(x)^{-1}$, for all $i \in \overline{1,n}$, $g \in \Gamma$ and almost every $x \in X$. Consequently, we obtain that for every $g = (g_{\hat{i}}, g_i) \in \Gamma = \Gamma_{\hat{i}} \times \Gamma_i$ and almost every $x \in X$, we have $\tilde{c}^i(g,x) = c_i^i(g,x) = \delta_i(g_i)$; here, we denoted by $\tilde{c} = \tilde{c}^1 \ldots \tilde{c}^n$ the decomposition along $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$. We define the group isomorphism $\delta : \Gamma \to \Lambda$ by letting $\delta(g_1 \ldots g_n) = \delta_1(g_1) \ldots \delta_n(g_n)$, for all $g_1 \in \Gamma_1, \ldots, g_n \in \Gamma_n$. This shows that $\tilde{c}(g,x) = \delta(g)$, for all $g \in \Gamma$ and almost every $x \in X$, which entails to c is cohomologous to the group isomorphism δ.

6. Proofs of Theorems A and B and Corollary C

In this section we prove the first three main results stated in the introduction. Towards this, we first prove an abstract version of [DHI16, Lemma 5.10] in the sense that we only require the lack of property Gamma instead of relative solidity assumptions. In order to properly state and prove the result, we assume the terminology introduced in Notation 3.7.

Lemma 6.1. *Let $L(\Lambda) = P_1 \bar{\otimes} P_2$ be a tensor product decomposition into II$_1$ factors.*

Assume there exist two partitions $T_1 \sqcup T_2 = S_1 \sqcup S_2 = \{1, \ldots, n\}$ such that for any $i \in \{1,2\}$ we have $P_i \prec^s_M A \rtimes \Gamma_{T_i}$ and $\Delta(A \rtimes \Gamma_{S_i}) \prec^s_{M \bar{\otimes} L(\Lambda)} M \bar{\otimes} P_i$.

If $L(\Lambda)$ does not have property Gamma, then $T_i = S_i$, for any $i \in \{1,2\}$. Moreover, there exist subgroups $\Sigma_1, \Sigma_2 < \Lambda$ such that for all $i \in \{1,2\}$ we have:

1. *$B \rtimes \Sigma_i \prec^s_M A \rtimes \Gamma_{T_i}$ and $A \rtimes \Gamma_{S_i} \prec^s_M B \rtimes \Sigma_i$;*
2. *$P_i \prec^s_{L(\Lambda)} L(\Sigma_i)$ and $L(\Sigma_i) \prec^s_{L(\Lambda)} P_i$.*

Proof. (1) The assumption $\Delta(A \rtimes \Gamma_{S_1}) \prec^s_{M \bar{\otimes} L(\Lambda)} M \bar{\otimes} P_1$ implies from [DHI16, Theorem 4.1] (see also [Ioa12, Theorem 3.1] and [CdSS16, Theorem 3.3]) that there exists a decreasing sequence of subgroups $\Omega_k < \Lambda$, $k \geq 1$, such that $A \rtimes \Gamma_{S_1} \prec^s_M B \rtimes \Omega_k$, for any $k \geq 1$ and $P_2 \prec_{L(\Lambda)} L(\cup_{k \geq 1} C_\Lambda(\Omega_k))$. Using Proposition 2.3, there is $k \geq 1$ such that $P_2 \prec_{L(\Lambda)} L(\Omega_k)' \cap L(\Lambda)$ and using [Vae08, Lemma 3.5] we further derive that $L(\Omega_k) \prec_{L(\Lambda)} P_1$. By letting $\Sigma_1 = \Omega_k$,

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we get

$$A \rtimes \Gamma_{S_1} \prec^s_M B \rtimes \Sigma_1 \text{ and } L(\Sigma_1) \prec_{L(\Lambda)} P_1. \tag{6.1}$$

We continue by showing that $B \rtimes \Sigma_1 \prec^s_M A \rtimes \Gamma_{T_1}$. By applying [DHI16, Lemma 2.4], we get from (6.1) a non-zero projection $e \in L(\Sigma_1)' \cap L(\Lambda)$ such that $L(\Sigma_1)e \prec^s_{L(\Lambda)} P_1$. Since $P_1 \prec^s_M A \rtimes \Gamma_{T_1}$, we obtain from [Vae08, Lemma 3.7] that $L(\Sigma_1)e \prec^s_M A \rtimes \Gamma_{T_1}$. By applying Lemma 2.2 there exists a projection $f \in (B \rtimes \Sigma_1)' \cap pMp \subset B$ with $f \geq e$ such that $(B \rtimes \Sigma_1)f \prec^s_M A \rtimes \Gamma_{T_1}$. Since $f \in B$, $e \in L(\Lambda)$ and $f \geq e$, we deduce that $f = 1$. Thus, $B \rtimes \Sigma_1 \prec^s_M A \rtimes \Gamma_{T_1}$. Similarly, there exists a subgroup $\Sigma_2 < \Lambda$ satisfying conclusion (1).

(2) This follows verbatim the proofs of Claims 2 and 3 from [DHI16, Lemma 5.10]. □

6.1 Proof of Theorem A

Assume Notation 3.7. Fix an arbitrary $i \in \overline{1,m}$. By applying Theorem 4.2 there is a partition $S_1^i \sqcup S_2^i = \overline{1,n}$ such that $\Delta(A \rtimes \Gamma_{S_1^i}) \prec_{M\bar{\otimes}L(\Lambda)} M\bar{\otimes}L(\Lambda_{\hat{i}})$ and $\Delta(A \rtimes \Gamma_{S_2^i}) \prec_{M\bar{\otimes}L(\Lambda)} M\bar{\otimes}L(\Lambda_i)$. Standard arguments (see, for instance, the proof of [Ioa11, Lemma 9.2(1)]) imply that

$$A \rtimes \Gamma_{S_1^i} \prec^s_M B \rtimes \Lambda_{\hat{i}} \quad \text{and} \quad A \rtimes \Gamma_{S_2^i} \prec^s_M B \rtimes \Lambda_i. \tag{6.2}$$

Hence, Theorem 4.6 combined with [BV14, Lemma 2.3] gives that there is a partition $T_1^i \sqcup T_2^i = \overline{1,n}$ such that

$$B \rtimes \Lambda_{\hat{i}} \prec^s_M A \rtimes \Gamma_{T_1^i} \quad \text{and} \quad B \rtimes \Lambda_i \prec^s_M A \rtimes \Gamma_{T_2^i}. \tag{6.3}$$

By applying [Vae08, Lemma 3.7] we derive that $S_1^i = T_1^i$, $S_2^i = T_2^i$. Consequently, by using relations (6.2) and (6.3), [DHI16, Proposition 3.1] implies that $\Gamma_{T_1^i}$ and $\Lambda_{\hat{i}}$ are measure equivalent and $\Gamma_{T_2^i}$ and Λ_i are measure equivalent as well, for any $i \in \overline{1,m}$. The conclusion now follows by a simple induction argument.

6.2 Proof of Theorem B

We first obtain the following classification of tensor product decompositions in the spirit of [DHI16, Theorem C]. Theorem B will then follow by applying this result together with an induction argument.

Theorem 6.2. *Let Γ and Λ be countable icc groups that are measure equivalent. Assume $L(\Lambda) = P_1\bar{\otimes}P_2$ and $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ is a product into icc groups such that $L(\Gamma_i)$ belongs to $\mathfrak{M}$ for any $i \in \{1,\ldots,n\}$.*

Then there exist a direct product decomposition $\Lambda = \Lambda_1 \times \Lambda_2$, a partition $T_1 \sqcup T_2 = \{1,\ldots,n\}$, a decomposition $L(\Lambda) = P_1^{t_1}\bar{\otimes}P_2^{t_2}$, for some $t_1, t_2 > 0$ with $t_1t_2 = 1$, and a unitary $u \in L(\Lambda)$ such that:

1. *$P_1^{t_1} = uL(\Lambda_1)u^*$ and $P_2^{t_2} = uL(\Lambda_2)u^*$;*
2. *Λ_1 is measure equivalent to $\times_{j\in T_1}\Gamma_j$ and Λ_2 is measure equivalent to $\times_{j\in T_2}\Gamma_j$.*

Proof. For the proof, we assume Notation 3.7. Using Proposition 3.8, we get that $L(\Gamma)$ does not have property Gamma. Next, by applying Theorems 4.2 and 4.6 and Lemma 6.1, we obtain a partition $T_1 \sqcup T_2 = \{1,\ldots,n\}$ and some subgroups $\Sigma_1, \Sigma_2 < \Lambda$ such that for all $i \in \{1,2\}$ we have:

1. $B \rtimes \Sigma_i \prec^s_M A \rtimes \Gamma_{T_i}$ and $A \rtimes \Gamma_{S_i} \prec^s_M B \rtimes \Sigma_i$;
2. $P_i \prec^s_{L(\Lambda)} L(\Sigma_i)$ and $L(\Sigma_i) \prec^s_{L(\Lambda)} P_i$.

Part (2) together with [DHI16, Theorem 6.1] give a product decomposition $\Lambda = \Lambda_1 \times \Lambda_2$, a decomposition $L(\Lambda) = P_1^{t_1}\bar{\otimes}P_2^{t_2}$, for some $t_1, t_2 > 0$ with $t_1t_2 = 1$, and a unitary $u \in L(\Lambda)$ such

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that $P_1^{t_1} = uL(\Lambda_1)u^*$ and $P_2^{t_2} = uL(\Lambda_2)u^*$. In addition, we have that Λ_i is measure equivalent to Σ_i, for any $i \in \{1, 2\}$.

Part (1) together with [DHI16, Proposition 3.1] implies that for any $i \in \{1, 2\}$, Γ_{T_i} is measure equivalent to Σ_i and, hence, to Λ_i. □

6.3 Proof of Corollary C

Assume first that there exists a partition $J_1 \sqcup J_2 = \{1, \dots, n\}$ for which $K = \text{Fix}_K(J_1) \times \text{Fix}_K(J_2)$. By letting $G_1 = (\times_{i\in J_1}\Gamma) \rtimes \text{Fix}_K(J_2)$ and $G_2 = (\times_{i\in J_2}\Gamma) \rtimes \text{Fix}_K(J_1)$, it follows that $G = G_1 \times G_2$. This clearly shows that $L(G)$ is not prime.

For proving the other implication, assume that $L(G) = P_1 \bar{\otimes} P_2$ can be written as a tensor product of diffuse factors. Using Theorem B and its proof, it follows that there exist a direct product decomposition $G = G_1 \times G_2$ into infinite groups and a partition $J_1 \sqcup J_2 = \overline{1, n}$ such that $L(G_i) \prec_{L(G)} L(\Gamma_{J_i})$, for any $i \in \overline{1, 2}$. By [CI18, Lemma 2.2] we get a finite index subgroup $G_i^0 < G_i$ such that $G_i^0 < \Gamma_{J_i}$, for any $i \in \overline{1, 2}$. By passing to relative commutants, we get that $\Gamma_{J_2} < G_2$ since G_1 is icc. By passing again to relative commutants, we deduce that $G_1 < C_G(\Gamma_{J_2}) = (\times_{i\in J_1}\Gamma) \rtimes \text{Fix}_K(J_2)$. Similarly, we get $G_2 < (\times_{i\in J_2}\Gamma) \rtimes \text{Fix}_K(J_1)$. This proves $K = \text{Fix}_K(J_1) \times \text{Fix}_K(J_2)$ which ends the proof.

7. Proof of Theorem D

7.1 OE rigidity for irreducible actions

An important ingredient for proving Theorem D is the following OE rigidity result for irreducible actions of product group that belong to $\mathcal{M}$.

THEOREM 7.1. *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a product of $n \geq 2$ groups that belong to $\mathcal{M}$. Let $\Lambda = \Lambda_1 \times \cdots \times \Lambda_m$ be a product of $m \geq 2$ infinite icc groups. Assume $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are OE free, irreducible, pmp actions.*

If $m \geq n$, then $m = n$ and $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are conjugate.

Proof. Theorem A implies that $m \geq n$. For the remaining part of the proof, assume that $m = n$. By assumption, we have the identification $M := L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda$ with $A := L^\infty(X) = L^\infty(Y)$. By proceeding as in the proof of Theorem A, it follows that for any i there is a partition $S_1^i \sqcup S_2^i = \overline{1, n}$ such that $A \rtimes \Gamma_{S_1^i} \prec_M^s A \rtimes \Lambda_{\widehat{i}}$ and $A \rtimes \Lambda_{\widehat{i}} \prec_M^s A \rtimes \Gamma_{S_1^i}$. Hence, by using [IPP08, Lemma 8.4] there is $u_i \in \mathcal{U}(M)$ such that $u_i(A \rtimes \Lambda_{\widehat{i}})u_i^* = A \rtimes \Gamma_{S_1^i}$. Theorem A implies that S_1^i has $n - 1$ elements. It is easy to see that there is a bijection φ of the set $\overline{1, n}$ such that $S_1^i = \widehat{\varphi(i)}$ for any i. Thus, we can apply Proposition 5.1 and derive that the Zimmer cocycle associated to the orbit equivalence between $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ is cohomologous to a group isomorphism. Hence, by applying [Vae07, Lemma 4.7] we get that $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are conjugate. □

7.2 Strongly cocycle rigidity

We start this subsection by recording the following particular case of [HHI21, Theorem 7.1] which is inspired by several works [Fur99, MS06, Kid08]. For properly formulating the result we introduce the following definition (see also [HHI21, § 7]). We say that a product group $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ is *strongly cocycle rigid* if given any two free, irreducible, pmp actions $\Gamma \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ that are OE, the underlying Zimmer cocycle is cohomologous to a group isomorphism.

THEOREM 7.2 [HHI21]. *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be an icc strongly cocycle rigid group. Assume $\Gamma \curvearrowright (X, \mu)$ is a free, irreducible, pmp action that is OE to a free, mildly mixing, pmp action $\Lambda \curvearrowright (Y, \nu)$.*

Then $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are virtually conjugate.

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COROLLARY 7.3. *If $\Gamma_1, \dots, \Gamma_n$ are countable groups with $L(\Gamma_i) \in \mathcal{M}$, for any $i \in \{1, \dots, n\}$, then $\Gamma_1 \times \cdots \times \Gamma_n$ is strongly cocycle rigid.*

Proof. Denote $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ and let $\Gamma \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ be free, irreducible, pmp actions that are OE. The proof of Theorem 7.1 gives, in particular, that the underlying Zimmer cocycle is cohomologous to a group isomorphism. $\square$

7.3 Proof of Theorem D

This is a direct consequence of Corollary 7.3 and Theorem 7.2.

8. Proofs of Theorem E and Corollary F

8.1 Proof of Theorem E

Let $\{u_g\}_{g \in \Gamma}$ be the canonical unitaries that generate $L(\Gamma)$. Denote $\mathcal{M} = M \bar{\otimes} L(\Gamma)$, $\tilde{\mathcal{M}} = \tilde{M} \bar{\otimes} L(\Gamma)$, and $\hat{\alpha}_t = \alpha_t \otimes \text{id} \in \text{Aut}(\tilde{\mathcal{M}})$. Note that the $*$-homomorphism $\Delta : L(\Gamma) \to L(\Gamma) \bar{\otimes} L(\Gamma)$ defined by $\Delta(u_g) = u_g \otimes u_g$, as $g \in \Gamma$ (see [PV10]), naturally extends to a map $\Delta : \ell^2(\Gamma) \to \ell^2(\Gamma) \otimes \ell^2(\Gamma)$. By denoting $\hat{\xi} = \Delta(\xi)$, for any $\xi \in \ell^2(\Gamma)$, it follows that if $\xi = \sum_{g \in \Gamma} \xi_g u_g \in \ell^2(\Gamma)$ and $t \in \mathbb{R}$, then

$$\|\hat{\alpha}_t(\hat{\xi}) - \hat{\xi}\|_2^2 = \sum_{g \in \Gamma} |\xi_g|^2 \|\alpha_t(u_g) - u_g\|_2^2. \tag{8.1}$$

Since Γ is inner amenable, there exists a sequence $(\xi_n)_{n \geq 1} \subset \ell^2(\Gamma)$ of unit vectors satisfying $\|u_g \xi_n - \xi_n u_g\|_2 \to 0$, for all $g \in \Gamma$ and $\xi_n(g) \to 0$, for any $g \in \Gamma$. Let ω be a free ultrafilter on $\mathbb{N}$. The remaining part of the proof is divided between two claims.

CLAIM 1. *We claim that $\lim_{t \to 0}(\lim_{n \to \omega} \|\hat{\alpha}_t(\hat{\xi}_n) - \hat{\xi}_n\|) = 0$.*

Proof of Claim 1. We define the unitary representations $\pi : \Gamma \to \mathcal{U}(L^2(\tilde{M}) \ominus L^2(M))$ by $\pi_g(\xi) = u_g \xi u_g^*$, for all $g \in \Gamma, \xi \in L^2(\tilde{M}) \ominus L^2(M)$ and $d : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ by $d_g(x) = u_g x u_g^*$, for all $g \in \Gamma, x \in \ell^2(\Gamma)$. Since $L^2(\tilde{M}) \ominus L^2(M)$ is weakly contained in the coarse bimodule $L^2(M) \otimes L^2(M)$ as M-bimodules, we derive that $L^2(\tilde{M}) \ominus L^2(M)$ is weakly contained in the coarse bimodule $L^2(L(\Gamma)) \otimes L^2(L(\Gamma))$ as $L(\Gamma)$-bimodules. Therefore, π is weakly contained in the left regular representation λ_Γ. Consequently, by applying [BdlHV08, Corollary E.2.6] we derive that $\pi \otimes d$ is weakly contained in λ_Γ. Note that $\hat{\pi} := \pi \otimes d : \Gamma \to \mathcal{U}(L^2(\tilde{\mathcal{M}}) \ominus L^2(\mathcal{M}))$ is defined by $\hat{\pi}_g(\eta) = \hat{u}_g \eta \hat{u}_g^*$ for all $g \in \Gamma, \eta \in L^2(\tilde{\mathcal{M}}) \ominus L^2(\mathcal{M})$. Since Γ is non-amenable, it follows that the trivial representation 1_Γ is not weakly contained in $\hat{\pi}$. This implies that for any $\epsilon > 0$, there exist $\delta > 0$ and a finite set $F \subset \Gamma$ satisfying that for any unit vector $\eta \in L^2(\tilde{\mathcal{M}})$ for which $\|\hat{\pi}_g(\eta) - \eta\|_2 \leq \delta$, as $g \in F$, we have

$$\|\eta - E_{\mathcal{M}}(\eta)\|_2 \leq \epsilon. \tag{8.2}$$

Since $\tau(\hat{\alpha}_t(\hat{u}_g)\hat{u}_h^*) = \tau(\alpha_t(u_g)u_h^*)\delta_{g,h}$, we obtain that $E_{\Delta(L(\Gamma))}(\hat{\alpha}_t(\hat{u}_g)) = \tau(\alpha_t(u_g)u_g^*)\hat{u}_g$, for any $g \in \Gamma$. This implies that for all $g \in \Gamma$ and $\xi \in \ell^2(\Gamma)$, we have

$$\|\hat{\alpha}_t(\hat{u}_g)\hat{\xi} - \hat{u}_g\hat{\xi}\|_2 = \|\alpha_t(u_g) - u_g\|_2 \|\xi\|_2 \quad \text{and} \quad \|\hat{\xi}\hat{\alpha}_t(\hat{u}_g) - \hat{\xi}\hat{u}_g\|_2 = \|\alpha_t(u_g) - u_g\|_2 \|\xi\|_2. \tag{8.3}$$

Let $t_0 > 0$ such that $\|\alpha_t(u_g) - u_g\|_2 \leq \delta/4$, for all $|t| < t_0$ and $g \in F$. Take also $n_0 \in \mathbb{N}$ such that $\|u_g \xi_n - \xi_n u_g\|_2 \leq \delta/2$, for all $g \in F$ and $n \geq n_0$. Together with (8.3) we obtain

$$\begin{aligned} \|\hat{\alpha}_t(\hat{u}_g)\hat{\xi}_n - \hat{\xi}_n\hat{\alpha}_t(\hat{u}_g)\|_2 &\leq \|\hat{\alpha}_t(\hat{u}_g)\hat{\xi}_n - \hat{u}_g\hat{\xi}_n\|_2 + \|u_g\xi_n - \xi_n u_g\|_2 + \|\hat{\xi}_n\hat{u}_g - \hat{\xi}_n\hat{\alpha}_t(\hat{u}_g)\|_2 \\ &\leq \delta/2 + \delta/4 + \delta/2 = \delta, \end{aligned} \tag{8.4}$$

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for all $g \in F, n \geq n_0$ and $|t| \leq t_0$. By applying $\hat{\alpha}_{-t}$ in (8.4) and by replacing t by $-t$, we get that $\|\hat{\alpha}_t(\hat{\xi}_n)\hat{u}_g - \hat{u}_g\hat{\alpha}_t(\hat{\xi}_n)\|_2 \leq \delta$, for all $g \in F, n \geq n_0$ and $|t| \leq t_0$. Using (8.2), we get $\|\hat{\alpha}_t(\hat{\xi}_n) - E_{\mathcal{M}}(\hat{\alpha}_t(\hat{\xi}_n))\|_2 \leq \epsilon$, and by using Popa's transversality property, see [Pop08, Lemma 2.1], we further derive that $\|\hat{\alpha}_{2t}(\hat{\xi}_n)) - \hat{\xi}_n\|_2 \leq 2\epsilon$, for all $n \geq n_0$ and $|t| \leq t_0$. This ends the proof of the claim. □

For all $t \in \mathbb{R}$ and $r > 0$, we denote $B_r^t = \{g \in \Gamma \mid \|\alpha_t(u_g) - u_g\|_2 \leq r\}$. We are now ready to prove the following claim.

CLAIM 2. *We claim that* $\lim_{t\to 0}(\sup_{g\in\Gamma} \|\alpha_t(u_g) - u_g\|_2) = 0$.

Proof of Claim 2. To this end, fix some arbitrary $\epsilon > 0$. Let $t_1 > 0$ and $n_1 \in \mathbb{N}$ such that $\|\hat{\alpha}_t(\hat{\xi}_n) - \hat{\xi}_n\|_2 \leq \epsilon/4$, for all $|t| \leq t_1$ and $n \geq n_1$. Fix $g \in \Gamma$ and $|t| \leq t_1$. We continue by showing that there exists an unbounded sequence $(k_n)_n \subset \Gamma$ such that $k_n, gk_ng^{-1} \in B^t_{\epsilon/2}$, for any $n \geq 1$. Since $\|\hat{u}_g\hat{\xi}_n - \hat{\xi}_n\hat{u}_g\|_2 \to 0$, we get that there exists $n_2 \in \mathbb{N}$ such that

$$\|\hat{\alpha}_t(\hat{\xi}_n) - \hat{\xi}_n\|_2^2 + \|\hat{\alpha}_t(\hat{u}_g\hat{\xi}_n\hat{u}_g^*) - \hat{u}_g\hat{\xi}_n\hat{u}_g^*\|_2^2 \leq \epsilon^2/4, \text{ for any } n \geq n_2.$$

By writing $\xi_n = \sum_{g\in\Gamma} \xi_{n,g}u_g \in \ell^2(\Gamma)$ and using (8.1) we obtain that

$$\sum_{h\in\Gamma} |\xi_{n,h}|^2(\|\alpha_t(u_h) - u_h\|_2^2 + \|\alpha_t(u_{ghg^{-1}}) - u_{ghg^{-1}}\|_2^2) \leq \epsilon^2/4, \text{ for any } n \geq n_2.$$

For any $n \geq n_2$, since $\sum_{h\in\Gamma} |\xi_{n,h}|^2 = 1$, there exists $k_n \in \Gamma$ with $\xi_{n,k_n} \neq 0$ such that $\|\alpha_t(u_{k_n}) - u_{k_n}\|_2 \leq \epsilon/2$ and $\|\alpha_t(u_{gk_ng^{-1}}) - u_{gk_ng^{-1}}\|_2 \leq \epsilon/2$. Since $\xi_{n,h} \to 0$, for any $h \in \Gamma$, it follows that k_n can be chosen such that $k_n \to \infty$.

Next, we note that for any $n \geq n_2$, we have

$$\|\alpha_t(u_{gk_ng^{-1}})u_g - u_g\alpha_t(u_{k_n})\|_2 \leq \|\alpha_t(u_{k_n}) - u_{k_n}\|_2 + \|\alpha_t(u_{gk_ng^{-1}}) - u_{gk_ng^{-1}}\|_2 \leq \epsilon. \tag{8.5}$$

By letting $e : L^2(\tilde{M}) \to L^2(M)$ be the orthogonal projection, we have $v_{g,t} := \alpha_t(u_g) - e(\alpha_t(u_g)) \in L^2(\tilde{M}) \ominus L^2(M)$. By applying α_{-t} to (8.5) and by projecting onto $L^2(\tilde{M}) \ominus L^2(M)$, we get

$$\|u_{gk_ng^{-1}}v_{-t,g} - v_{-t,g}u_{k_n}\|_2 \leq \epsilon, \text{ for all } n \geq n_2. \tag{8.6}$$

Since the M-bimodule $L(\tilde{M}) \ominus L^2(M)$ is mixing and $k_n \to \infty$, we obtain that

$$\lim_{n\to\infty} \langle v_{-t,g}u_{k_n}, u_{gk_ng^{-1}}v_{-t,g}\rangle = 0. \tag{8.7}$$

By combining (8.6) and (8.7), it follows that $\|v_{-t,g}\|_2 \leq \epsilon/\sqrt{2}$. By using once again Popa's transversality property, see [Pop08, Lemma 2.1], we obtain $\|\alpha_{-t}(u_g) - u_g\| \leq \epsilon\sqrt{2}$. Since t was arbitrary chosen such that $|t| \leq t_1$ and $g \in \Gamma$ arbitrary, we get that $\lim_{t\to 0}(\sup_{g\in\Gamma} \|\alpha_t(u_g) - u_g\|_2) = 0$.

Standard arguments now imply the conclusion. □

8.2 Proof of Corollary F

The proof follows directly from Theorem E.

8.3 Consequence to Kurosh-type rigidity results

We conclude our paper with the following rigidity result for tracial free product factors arising from non-amenable inner amenable groups.

COROLLARY 8.1. *Let* $M = L(\Gamma_1) * \cdots * L(\Gamma_m) = L(\Lambda_1) * \cdots * L(\Lambda_n)$, *where all the groups* Γ_i *and* Λ_j *are non-amenable inner amenable icc groups.*

Then $m=n$, and after a permutation of indices, $L(\Gamma_i)$ is unitarily conjugate to $L(\Lambda_i)$, for any $i\in\overline{1,n}$.

Proof. Fix an arbitrary $i\in\overline{1,m}$. By decomposing $M=L(\Gamma_1*\cdots*\Gamma_{n-1})*L(\Gamma_n)$, we note that M belongs to $\mathfrak{M}$ and let $(\tilde{M},(\alpha_t)_{t\in\mathbb{R}})$ be the associated s-malleable deformation of M. Since Γ_i is non-amenable inner amenable group, Theorem E implies that $L(\Gamma_i)$ is α-rigid. By applying the main technical result of [IPP08] (see also [Ioa15, Theorem 2.11]), we get that $L(\Gamma_i)\prec_M L(\Gamma_1*\cdots*\Gamma_{n-1})$ or $L(\Gamma_i)\prec_M L(\Gamma_n)$. By assuming the latter holds, there exist projections $p\in L(\Gamma_i), q\in L(\Gamma_n)$, a non-zero partial isometry $v\in qMp$ and a $*$-homomorphism $\theta: pL(\Gamma_i)p\to qL(\Gamma_n)q$ satisfying $\theta(x)v=vx$, for all $x\in pL(\Gamma_i)p$. Note that [IPP08, Theorem 1.2.1] gives that $vv^*\in L(\Gamma_n)$ and, hence, $vL(\Gamma_i)v^*\subset L(\Gamma_n)$. Note that since $L(\Gamma_i)'\cap M=\mathbb{C}1$ and $v^*v\in p(L(\Gamma_i)'\cap M)p$, we get that $v^*v=p$. By letting u be a unitary that extends v, we derive that $upL(\Gamma_i)pu^*\subset L(\Gamma_n)$. Since $L(\Gamma_n)$ is a factor, after passing to a new unitary u, one can replace p by its central support in $L(\Gamma_i)$; therefore, we obtain that $uL(\Gamma_i)u^*\subset L(\Gamma_n)$. Similarly, if $L(\Gamma_i)\prec_M L(\Gamma_1*\cdots*\Gamma_{n-1})$ holds, we obtain a unitary $u\in M$ such that $uL(\Gamma_i)u^*\subset L(\Gamma_1*\cdots*\Gamma_{n-1})$. By repeating this argument finitely many times, we conclude that there exists a map $\sigma:\overline{1,m}\to\overline{1,n}$ such that for any $i\in\overline{1,m}$, there is a unitary $u_i\in M$ satisfying $u_iL(\Gamma_i)u_i^*\subset L(\Lambda_{\sigma(i)})$.

In a similar way, we obtain a map $\tau:\overline{1,n}\to\overline{1,m}$ and a unitary $w_j\in M$, for any $j\in\overline{1,n}$, such that $w_jL(\Lambda_j)w_j^*\subset L(\Gamma_{\tau(j)})$, for any $j\in\overline{1,n}$. Thus, $u_{\tau(j)}w_jL(\Lambda_j)w_j^*u_{\tau(j)}^*\subset L(\Lambda_{\sigma(\tau(j))})$, for any $j\in\overline{1,n}$. By applying [IPP08, Theorem 1.2.1] we deduce that $\sigma\circ\tau=\mathrm{Id}$ and $u_{\tau(j)}w_j\in L(\Lambda_j)$, for any $j\in\overline{1,n}$. Similarly, we get $\tau\circ\sigma=\mathrm{Id}$ and $w_{\sigma(i)}u_i\in L(\Gamma_i)$ for any $i\in\overline{1,m}$. In particular, $m=n$ and $u_iL(\Gamma_i)u_i^*=L(\Lambda_{\sigma(i)})$, for any $i\in\overline{1,n}$. □

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Conflicts of Interest

None.

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