CORRELATED RANDOM WALK

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ABSTRACT. Random walk on a d-dimensional lattice is investigated such that, at any stage, the probabilities of the step being in the various possible directions depend upon the direction of the previous step. The motion may be characterized by a generating function which is here derived. The generating function is then used to obtain some general properties of the walk. Certain special cases are considered in greater detail. The existence of recurrent points is investigated in particular, and the probability of returning to the origin after $2n$ steps. This latter function is evaluated asymptotically for the cases $d = 1$ and $d =$ an even integer.

1. *Introduction*. We shall consider random walk on a d-dimensional rectangular lattice with a correlation between the directions of successive steps. Let $\mathbf{m}$ denote the general lattice point $(m_1, m_2, \ldots, m_d)$ and $\mathbf{E}_i$ the unit vector parallel to the positive direction of the ith axis. Then a step in the walk is to be strictly of the type $\mathbf{m} \to \mathbf{m} \pm \mathbf{E}_i$. The random walk process is defined as follows.

The walker starts from the origin and, for the first step, all of the $2d$ possible directions are equally likely. At any later stage the probability distribution is to be as follows:

$p =$ the probability that the walker will continue in the same direction and sense as in the previous step,

$q =$ the probability that he will move in exactly the opposite sense, and

$r =$ the probability that he will take any one of the directions orthogonal to his previous step.

Then $p+q+2(d-1)r = 1$.

Processes of this sort do not seem to have been studied except for the case $d = 1$ (cf. (3) and (4)). Goldstein (3) has considered the distribution in the one-dimensional case and has obtained formulae for the moments as well as an asymptotic estimate of the distribution function in terms of hypergeometric functions. He has, moreover, computed some interesting numerical tables in which the accuracy of this asymptotic formula is checked for a number of values of $p-q$.

We shall denote by $P_n(\mathbf{m})$ the probability of arriving at the lattice point $\mathbf{m}$ on the nth step and shall begin by deriving a generating function for this probability. Subsequently our chief interest will be in $P_n(\mathbf{O})$, the probability of returning to the origin (not necessarily for the first time) at the nth step. We shall study it, for simplicity, only for the special case $p = r$. This greatly simplifies the form of the generating function for $P_n(\mathbf{m})$, though it is fairly clear intuitively that the limitation will not affect the general character of the motion very fundamentally. Various aspects of the motion in other special cases—some of which appear to have some biological interest—will be discussed in a subsequent paper. We shall write $p-q = \delta$, and, in the special case referred to, shall denote $P_n(\mathbf{O})$ by $R_{n,d}(\delta)$ to make clear exactly on what it depends. The quantity $R_{n,d}(0)$ was studied by Polya (5), who showed that, for large n,

$$R_{n,d}(0) = O(n^{-\frac{1}{2}d}). \qquad (1\cdot1)$$

It followed that (cf. (1), p. 244) for $d = 1, 2$, every lattice point is a *recurrent point*, i.e. a point through which the walker will, with probability 1, pass an infinite number of times, while for $d > 2$ no recurrent points exist at all. We shall show that, when $d = 1$ or an even integer, Polya's result remains valid (except for the trivial case $\delta = -1$, i.e. $p = r = 0, q = 1$). Indeed, we shall prove that, for these values of d,

$$R_{n,d}(\delta) \sim \left(\frac{1-\delta}{1+\delta}\right)^{\frac{1}{2}d} R_{n,d}(0). \tag{1·2}$$

We shall derive the generating function for $P_n(\mathbf{m})$ in §2, and this will be used in §4 to derive information about the function $R_{n,d}(\delta)$. In §3 we digress a little from this main line of development to consider in more detail the motion in one dimension. There seems to be little doubt that (1·2) holds for every d, but the method of proof used for $d > 1$ (cf. §§4 and 5) depends on a $\frac{1}{2}d$-fold integration and does not apply to odd d.

2. *The generating function.* Let $Q_{\pm i}(\mathbf{m}, n)$ denote the probabilities that the nth step will bring the walker from $\mathbf{m} \mp \mathbf{E}_i$ to $\mathbf{m}$, respectively. Then clearly

$$P_n(\mathbf{m}) = \sum_{i=1}^{d} [Q_i(\mathbf{m}, n) + Q_{-i}(\mathbf{m}, n)]. \tag{2·1}$$

For all i $(1 \leqslant i \leqslant d)$ we define

$$\left.\begin{aligned} Q_{\pm i}(\mathbf{m}, 0) &= 1 \text{ if } \mathbf{m} = \mathbf{O} \quad (\text{i.e. if } m_1 = m_2 = \ldots = m_d = 0), \\ &= 0 \text{ otherwise.} \end{aligned}\right\} \tag{2·2}$$

Also, from the conditions of the walk,

$$\left.\begin{aligned} Q_{\pm i}(\mathbf{m}, 1) &= \frac{1}{2d} \quad \text{if} \quad \mathbf{m} = \pm \mathbf{E}_i \text{ respectively,} \\ &= 0 \text{ otherwise} \end{aligned}\right\} \tag{2·3}$$

We can now write down the following recurrence relations ($i = 1, 2, \ldots, d$)

$$\begin{aligned} Q_i(\mathbf{m}, n) = pQ_i(\mathbf{m} - \mathbf{E}_i, n-1) + qQ_{-i}(\mathbf{m} - \mathbf{E}_i, n-1) \\ + r \sum_{\substack{j=1 \\ j \neq i}}^{d} [Q_j(\mathbf{m} - \mathbf{E}_i, n-1) + Q_{-j}(\mathbf{m} - \mathbf{E}_i, n-1)], \end{aligned} \tag{2·4}$$

and an analogous relation for $Q_{-i}(\mathbf{m}, n)$. We write

$$B_i(\mathbf{x}, z) = \sum_{m_1, m_2, \ldots = -\infty}^{\infty} \sum_{n=0}^{\infty} Q_i(\mathbf{m}, n)\, x_1^{m_1} \ldots x_d^{m_d} z^n, \tag{2·5}$$

and a similar definition for $B_{-i}(\mathbf{x}, z)$ where $\mathbf{x}$ represents the set $(x_1, x_2, \ldots, x_d)$.

Multiply equation (2·4) by $x_1^{m_1} x_2^{m_2} \ldots x_d^{m_d} z^n$ and sum over the entire range

$$-\infty < m_i < \infty, n = 0, 1, \ldots,$$

bearing in mind conditions (2·2) and (2·3). This leads to the equations

$$B_i - \frac{1}{2d} = zx_i\{pB_i + qB_{-i} + r \sum_{j \neq i} (B_j + B_{-j})\}, \tag{2·6}$$

$$B_{-i} - \frac{1}{2d} = \frac{z}{x_i}\{pB_{-i} + qB_i + r \sum_{j \neq i} (B_j + B_{-j})\}, \tag{2·7}$$

where, for simplicity of notation, we have omitted the arguments of the functions. We shall write

$$A(\mathbf{x},z) = \sum_{i=1}^{d} [B_i(\mathbf{x},z)+B_{-i}(\mathbf{x},z)], \tag{2·8}$$

and it is clear from (2·1) that the coefficients in the expansion of $A(\mathbf{x},z)$ will give us the distribution $P(\mathbf{m},n)$. To solve (2·6) and (2·7) we begin by rewriting them:

$$B_1(1-px_1z)-qx_1zB_{-1}-rx_1zB_2-rx_1zB_{-2}-\dots = \frac{1}{2d},$$
$$-\frac{qz}{x_1}B_1+\left(1-\frac{pz}{x_1}\right)B_{-1}-\frac{rz}{x_1}B_2-\frac{rz}{x_1}B_{-2}-\dots = \frac{1}{2d}, \tag{2·9}$$

etc.

Let Δd denote the determinant of order $2d$ whose elements are the coefficients of the $B_{\pm i}$'s in (2·9), and let $D_{\pm i}$ denote the determinant obtained from Δ_d by replacing the column of coefficients of $B_{\pm i}$, respectively, by 1's. Then, by Cramer's rule,

$$B_{\pm i} = D_{\pm i}/(2d\Delta_d), \tag{2·10}$$

and so

$$A(x,z) = \frac{1}{2d\Delta_d}\sum_{i=1}^{d}(D_i+D_{-i}) = \frac{1}{2d\Delta_d}\sum_{i=1}^{d}C_i \quad \text{(say)}. \tag{2·11}$$

It remains to compute the various determinants involved. For this purpose it is convenient to make the following definitions:

$$\xi_i = \tfrac{1}{2}(x_i+x_i^{-1}), \tag{2·12}$$

$$\delta = p-q, \tag{2·13}$$

$$F_i = \begin{vmatrix} 1+x_i^2-2\delta x_i z & -x_i^2+\delta x_i z \\ -1+\dfrac{\delta z}{x_i} & 1-\dfrac{pz}{x_i} \end{vmatrix} = 1-2p\xi_i z+\delta(p+q)z^2, \tag{2·14}$$

$$G_i = \begin{vmatrix} 1+x_i^2-2\delta x_i z & 0 \\ -1+\dfrac{\delta z}{x_i} & -\dfrac{rz}{x_i} \end{vmatrix} = 2rz(\delta z-\xi_i), \tag{2·15}$$

$$H_i = \begin{vmatrix} 1+x_i^2-2\delta z x_i & 1-x_i^2 \\ -1+\dfrac{\delta z}{x_i} & 1 \end{vmatrix} = 2(1-\delta\xi_i z), \tag{2·16}$$

$$\lambda_i = F_i-G_i = 1-2(p-r)\xi_i z+\delta z^2(p+q-2r), \tag{2·17}$$

$$L_i = \sum_{j=1}^{i}\lambda_j^{-1}, \tag{2·18}$$

$$K_i = \sum_{j=1}^{i}\lambda_j^{-1}G_j. \tag{2·19}$$

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To compute Δ_d we first multiply each $(2i)$th row by x_i^2 and subtract from the previous row. We then subtract each column of even order from the previous column. The resulting determinant is

$$\Delta_d = \begin{vmatrix} 1+x_1^2-2\delta x_1 z & -x_1^2+\delta x_1 z & 0 & 0 & \dots & 0 \\ -1+\dfrac{\delta z}{x_1} & 1-\dfrac{pz}{x_1} & 0 & -\dfrac{rz}{x_1} & \dots & \\ 0 & 0 & 1+x_2^2-2\delta z x_2 & -x_2^2+\delta x_2 z & \dots & \\ 0 & -\dfrac{rz}{x_2} & -1+\dfrac{\delta z}{x_2} & 1-\dfrac{pz}{x_2} & \dots & \\ \vdots & \vdots & \vdots & \vdots & & \end{vmatrix}. \tag{2·20}$$

We expand this determinant by the Laplace rule in terms of the minors of order 2 obtained by selecting all pairs of columns in each row-pair $2i-1$, $2i$. When allowance is made for the zero terms and the signs of the remaining terms are allocated appropriately it is seen that Δ_d reduces to the d-rowed determinant

$$\Delta_d = \begin{vmatrix} F_1 & G_1 & G_1 & \dots & G_1 \\ G_2 & F_2 & G_2 & \dots & G_2 \\ \vdots & & & & \\ G_d & \dots & \dots & \dots & F_d \end{vmatrix}$$

$$= \begin{vmatrix} G_1+\lambda_1 & G_1 & \dots & G_1 \\ G_2 & G_2+\lambda_2 & & G_2 \\ G_3 & G_3 & & \vdots \\ \vdots & \vdots & & \vdots \\ G_d & G_d & & G_d+\lambda_d \end{vmatrix}$$

by (2·17),

$$= \begin{vmatrix} G_1+\lambda_1 & -\lambda_1 & 0 & \dots & 0 \\ G_2 & \lambda_2 & -\lambda_2 & & 0 \\ G_3 & 0 & \lambda_3 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ G_{d-1} & 0 & 0 & & -\lambda_{d-1} \\ G_d & 0 & 0 & & \lambda_d \end{vmatrix}, \tag{2·21}$$

by subtracting successive columns. To evaluate the determinant in (2·21) we add successively $(\lambda_2/\lambda_1)\times$ the first row to the second, $(\lambda_3/\lambda_2)\times$ the resulting second row to the third, etc. This leads to

$$\Delta_d = \begin{vmatrix} \lambda_1(1+K_1) & -\lambda_1 & 0 & \dots & 0 \\ \lambda_2(1+K_2) & 0 & -\lambda_2 & & \vdots \\ \vdots & \vdots & \vdots & & -\lambda_{d-1} \\ \lambda_d(1+K_d) & 0 & 0 & & 0 \end{vmatrix}$$

$$= (1+K_d)\prod_{i=1}^{d}\lambda_i. \tag{2·22}$$

We now proceed to compute C_i. Consider, to fix the ideas, the case $i = 1$:

$$D_{-1} = \begin{vmatrix} 1-px_1z & 1 & -rx_1z & \cdots \\ -\frac{qz}{x_1} & 1 & -\frac{rz}{x_1} & \cdots \\ -rx_2z & 1 & 1-px_2z & \cdots \\ -\frac{rz}{x_2} & 1 & -\frac{qz}{x_2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{rz}{x_d} & 1 & -\frac{rz}{x_d} & \cdots \end{vmatrix},$$

while

$$D_1 = \begin{vmatrix} 1 & -qx_1z & \cdots \\ 1 & 1-\frac{pz}{x_j} & \cdots \\ \vdots & -rx_2z & \cdots \\ \vdots & \vdots & \vdots \\ 1 & -\frac{rz}{r_d} & \cdots \end{vmatrix}$$

nge the first two columns of D_1 and then combine with D_{-1} to give

$$C_1 = \begin{vmatrix} 1-\delta x_1z & 1 & -rx_1z & \cdots \\ -1+\frac{\delta z}{x_1} & 1 & -\frac{rz}{x_1} & \cdots \\ 0 & \vdots & 1-px_2z & \cdots \\ \vdots & \vdots & -\frac{qz}{x_2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \vdots & \cdots \end{vmatrix}. \tag{2·23}$$

We now perform on C_1 the same operations as were used on Δ_d and obtain, after a little reduction

$$C_1 = \begin{vmatrix} H_1 & G_1 & \cdots & G_1 \\ H_2 & F_2 & \cdots & G_2 \\ H_3 & G_3 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ H_d & G_d & \cdots & F_d \end{vmatrix}. \tag{2·24}$$

Now, by (2·15) and (2·16), $\quad H_i = (\delta/r)\,G_i + 2(1-\delta^2),$ (2·25)

and so $C_1 = (\delta/r)\,X_1 + 2(1-\delta^2)\,Y_1$ (say), where (2·26)

$$X_1 = \begin{vmatrix} G_1 & G_1 & \cdots & G_1 \\ G_2 & G_2+\lambda_2 & \cdots & G_2 \\ \vdots & \vdots & \vdots & \vdots \\ G_d & G_d & \cdots & G_d+\lambda_d \end{vmatrix} = G_1 \prod_{i=2}^{d} \lambda_i, \tag{2·27}$$

and

$$Y_1 = \begin{vmatrix} 1 & G_1 & \cdots & G_1 \\ 1 & G_2+\lambda_2 & \cdots & G_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & G_d & \cdots & G_d+\lambda_d \end{vmatrix}. \tag{2·28}$$

In Y_1 we subtract the $(d-1)$th column from the dth, the $(d-2)$th from the $(d-1)$th, ..., and finally the 2nd column from the 3rd. We operate on the resulting determinant as before by adding $(\lambda_2/\lambda_1)\times$ the first row to the second, $(\lambda_3/\lambda_2)\times$ the resulting second row to the third, etc. This leads finally to

$$Y_1 = (1+K_d-\lambda_d G_1)\prod_{i=2}^{d}\lambda_i. \tag{2·29}$$

From (2·26), (2·27) and (2·29) we get

$$C_1\Big/\prod_{i=1}^{d}\lambda_i = (\delta G_1)/(r\lambda_1)+2(1-\delta^2z^2)/\lambda_1+2(1-\delta^2z^2)\,[K_d/\lambda_1-L_dG_1/\lambda_1], \tag{2·30}$$

It is clear from the symmetry that a similar formula must hold for the other C_i and so

$$\sum_{i=1}^{d}C_i\Big/\prod_{i=1}^{d}\lambda_i = (\delta/r)\,K_d+2(1-\delta^2z^2)\,L_d. \tag{2·31}$$

It follows that

$$\begin{aligned}A(\mathbf{x},z) &= \frac{1}{2d}[(\delta/r)\,K_d+2(1-\delta^2z^2)\,L_d]/(1+K_d)\\ &= \frac{1}{d}\,\frac{\sum_{i=1}^{d}(1-\delta\xi_i z)/[1-2(p-r)\,\xi_i z+\delta(p+q-2r)\,z^2]}{1+\sum_{i=1}^{d}2rz(\delta z-\xi_i)/[1-2(p-r)\,\xi_i z+\delta(p+q-2r)\,z^2]}.\end{aligned} \tag{2·32}$$

In the special case $p=r$, $\qquad G_i = 1-\delta^2z^2\,(i=1,2,\ldots,d),$ (2·33)

and so

$$A(\mathbf{x},z) = \frac{1-\dfrac{\delta z}{d}\sum_{i=1}^{d}\xi_i}{1-2pz\sum_{i=1}^{d}\xi_i+\delta z^2}. \tag{2·34}$$

3. *One-dimensional motion.* (*a*) The motion is given by

$$A(x,z) = \frac{1-\frac{1}{2}\delta z(x+x^{-1})}{1-\frac{1}{2}(1+\delta)\,z(x+x^{-1})+\delta z^2} = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}P_n(m)\,x^m z^n. \tag{3·1}$$

When $\delta=0$ (uncorrelated walk), $P_n(m)$ is simply the coefficient of x^m in $\frac{1}{2^n}\left(x+\frac{1}{x}\right)^n$, and $A(x,z)$ reduces to

$$A_0(x,z) = \frac{1}{1-\frac{1}{2}z(x+x-1)}. \tag{3·2}$$

It is easily verified, by considering $[\partial A/\partial\delta]_{\delta=0}$, that

$$\left[\frac{\partial P_n(m)}{\partial\delta}\right]_{\delta=0} = \frac{m^2-n}{n}\,[P_n(m)]_{\delta=0}. \tag{3·3}$$

It follows that, for small $|\,\delta_1\,|$,

$$[P_n(m)]_{\delta=\delta_1}-[P_n(m)]_{\delta=0}$$

has the same sign as $(m^2-n)\,\delta_1$, i.e. the introduction of a small positive correlation increases the probability that the walker will be at distances from the origin between

$n^{\frac{1}{2}}$ and n and decreases the probability that he will be nearer than $n^{\frac{1}{2}}$ to the origin. The effect of a small negative correlation is in the exactly opposite direction.

(b) *Moments of distribution.* Let

$$\mu_i^{(n)} = \sum_{m=-n}^{n} P_n(m)\, m^i$$

and
$$\mu_{[i]}^{(n)} = \sum_{m=-n}^{n} P_n(m)\{m(m-1)\dots(m-i+1)\}, \tag{3·4}$$

then $\mu_1^{(n)} = \mu_{[1]}^{(n)} = 0$, and $\mu_i^{(n)} = 0$ for all odd i.

It follows from (2·1) that

$$\sum_{m=-n}^{n}\sum_{n=0}^{\infty} P_n(m)\{m(m-1)\dots(m-i+1)\}\, x^{m-i}z^n = \frac{\partial^i}{\partial x^i}[A(x,z)]. \tag{3·5}$$

Put $x = 1$. Then
$$\sum_{n=0}^{\infty} \mu_{[i]}^{(n)} z^n = M_i(z) \quad \text{(say)} \quad = \left[\frac{\partial^i A}{\partial x^i}\right]_{x=1}. \tag{3·6}$$

It is easily seen, after some algebraic manipulation, that

$$M_i(z) = \frac{i!}{2^{i-1}} \frac{(1+\delta)^{i-2} z^{i-1}(1+\delta z)}{(1-z)^i (1-\delta z)^{i-1}} \frac{a^{i-1}-b^{i-1}}{a-b}, \tag{3·7}$$

where a, b are the roots of the quadratic equation

$$\phi^2 + \frac{2(1-z)(1-\delta z)}{z(1+\delta)}\phi - \frac{2(1-z)(1-\delta z)}{z(1+\delta)} = 0. \tag{3·8}$$

In particular
$$M_2(z) = \frac{z(1+\delta z)}{(1-z)^2(1-\delta z)}$$

$$= \frac{\delta^2-2\delta-1}{(1-\delta)^2}\frac{1}{1-z} + \frac{1+\delta}{1-\delta}\frac{1}{(1-z)^2} + \frac{2\delta}{(1-\delta)^2}\frac{1}{1-\delta z}, \tag{3·9}$$

and so
$$\mu_{[2]}^{(n)} = \frac{1+\delta}{1-\delta} n - \frac{2\delta(1-\delta^n)}{(1-\delta)^2}. \tag{3·10}$$

Since $\mu_1^{(n)} = \mu_{[1]}^{(n)} = 0$, it follows that the scatter σ_n^2 is given by

$$\sigma_n^2 = \mu_{[2]}^{(n)} = \frac{1+\delta}{1-\delta} n - \frac{2\delta(1-\delta^n)}{(1-\delta)^2}. \tag{3·11}$$

Again it follows from (3·7) that, for large n and even i,

$$\mu_i^{(n)} \sim \mu_{[i]}^{(n)} \sim \frac{i!}{2^{\frac{1}{2}i}(\frac{1}{2}i)!}\left(\frac{1+\delta}{1-\delta}\right)^{\frac{1}{2}i} n^{\frac{1}{2}i}. \tag{3·12}$$

Hence by a well-known theorem originally due to Markoff (cf. (2)) the distribution for large n is *asymptotically* Gaussian with zero mean and standard deviation (cf. (3))

$$\left(\frac{1+\delta}{1-\delta}\right)^{\frac{1}{2}} n.$$

One immediate application is the estimate, for large n, of the value of m at which

$$[P_n(m)]_{\delta=\delta_1} = [P_n(m)]_{\delta=0}.$$

It turns out that
$$m^2 \sim n\frac{1+\delta_1}{2\delta_1}\log_e\frac{1+\delta_1}{1-\delta_1}. \tag{3·13}$$

This agrees with the result obtained above for small δ_1.

(c) *Probability of return.* We write

$$\rho_{1,\delta}(z) = \sum_{n=0}^{\infty} R_{n,1}(\delta)\, z^n \tag{3·14}$$

$$= \text{the part of } A(x,z) \text{ independent of } x.$$

We define
$$R_{0,1}(\delta) = 1. \tag{3·15}$$

The function $\rho_{1,\delta}(z)$ is, as we have said, simply the absolute term in the Laurent expansion of $A(x,z)$ as a series of positive and negative powers of x. To evaluate this we expand $A(x,z)$ in positive powers of $\left(x+\frac{1}{x}\right)$, remembering that the absolute term in $\left(x+\frac{1}{x}\right)^i$ is zero if i is odd, while if i is even it is

$$i!/(\tfrac{1}{2}i!)^2 = (-1)^{\frac{1}{2}i}\, 2^i \binom{-\frac{1}{2}}{\frac{1}{2}i}. \tag{3·16}$$

After a little manipulation this finally gives us

$$\rho_{1,\delta}(z) = (1+\delta)^{-1}\{\delta + (1-\delta^2 z^2)^{\frac{1}{2}}(1-z^2)^{-\frac{1}{2}}\}. \tag{3·17}$$

Now $R_{n,1}(\delta)$ is given by $\frac{1}{2\pi i}\int \rho_{1,\delta}(z)\, z^{-n-1}\, dz$, where the complex integral is to be taken round any simple contour encircling the origin once positively and excluding the branch points at $z = \pm 1,\ \pm\delta^{-1}$. For $n \geqslant 1$, we have

$$R_{2n,1}(\delta) = \frac{1}{2\pi i(1+\delta)}\int (1-\delta^2 z^2)^{\frac{1}{2}}(1-z^2)^{-\frac{1}{2}} z^{-2n-1}\, dz, \tag{3·18}$$

while $R_{i,1}(\delta)$ is clearly zero for odd i. The integral in (3·18) is a well-known integral representation of a hypergeometric function of δ^2 (cf. (7), p. 292). In fact, it may be verified by direct differentiation that it satisfies the differential equation for

$$F(-\tfrac{1}{2}, -n;\ \tfrac{1}{2}-n;\ \delta^2).$$

Being also a polynomial in δ^2 it must therefore be proportional to this hypergeometric function.

The constant of proportionality is determined from the case $\delta = 0$, and we get finally

$$R_{2n,1}(\delta) = R_{2n,1}(0)\,(1+\delta)^{-1} F(-\tfrac{1}{2}, -n;\ \tfrac{1}{2}-n;\ \delta^2). \tag{3·19}$$

It can be deduced from the general properties of the hypergeometric function (cf. (6), p. 298), or may be verified even more easily in this case by expansion and inspection of the series, that, as n tends to infinity, the hypergeometric function in (3·19) tends to $(1-\delta^2)^{\frac{1}{2}}$. It follows that, for large n,

$$R_{2n,1}(\delta) \sim (1-\delta)^{\frac{1}{2}}(1+\delta)^{-\frac{1}{2}} R_{2n,1}(0). \tag{3·20}$$

4. *Two-dimensional lattice.* We shall consider the case $p = r$ and so can describe the walk in terms of the single parameter δ where

$$p = r = \tfrac{1}{4}(1+\delta), \quad q = \tfrac{1}{4}(1-3\delta), \tag{4·1}$$

and (2·34) may be written

$$A(\mathbf{x}, z) = \frac{1 - \frac{1}{2}(\xi_1 + \xi_2)\,\delta z}{1 - \frac{1}{2}(1+\delta)(\xi_1+\xi_2)\,z + \delta z^2}. \tag{4·2}$$

We define

$$\rho_{2,\delta}(z) = \sum_{n=0}^{\infty} R_{n,2}(\delta)\, z^n, \tag{4·3}$$

with

$$R_{0,2}(\delta) = 1. \tag{4·4}$$

To derive $\rho_{2,\delta}(z)$ from $A(\mathbf{x}, z)$ we have to expand (4·2) and isolate the terms independent of x_1, x_2. For this, as well as for later applications, we shall need the following lemma.

LEMMA. (*a*) $R_{n,d}(0)$ *is the term independent of* $x_1, x_2, \ldots, x_d$ *in the multinomial expansion of* $(2d)^{-n}(x_1 + x_1^{-1} + x_2 + x_2^{-1} \ldots + x_d + x_d^{-1})^n$.

(*b*) $R_{n,d}(0) = 0$ *for all odd n.*

$$(c)\quad R_{2n,2}(0) = [R_{2n,1}(0)]^2 = \frac{1}{2^{4n}}\frac{[(2n)!]^2}{(n!)^4}. \tag{4·5}$$

$$(d)\ \textit{For large } n,\qquad R_{2n,d}(0) \sim 2\left(\frac{d}{4\pi n}\right)^{\frac{1}{2}d}\left[1 - \frac{d}{8n} + O(n^{-2})\right]. \tag{4·6}$$

$$(e)\ \textit{For } d \leqslant 4,\qquad R_{2n,d}(0) \sim 2\left[\frac{d}{\pi(4n+1)}\right]^{\frac{1}{2}d} \tag{4·7}$$

gives a good approximation for all $n \geqslant 1$.

Proof. (*a*) and (*b*) are obvious. (*c*) and the leading term in (4·6) are due to Polya (5). The approximation (4·7) clearly agrees with (4·6) for large n and its validity for small n can be verified directly by computation. It is, in fact, not needed for what follows but has been included as having an interest of its own. It remains to prove the expansion (4·6). Now, by (*a*),

$$\begin{aligned} R_{2n,d}(0) &= (2d)^{-n} \sum_{f_1+f_2\ldots+f_d=n} \frac{(2n)!}{\prod_{i=1}^{d}(2f_i)!} \prod_{i=1}^{d}\binom{2f_i}{f_i} \\ &= \frac{(2n)!}{(2d)^n}\sum_{f_1+\ldots+f_d=n}\prod_{i=1}^{d}(f_i!)^{-2} \\ &= (2n)!/d^n \times \text{the coefficient of } z^{2n} \text{ in the expansion of } [I_0(z)]^d \\ &= \frac{(2n)!}{2\pi i d^n}\int_C [I_0(z)]^d z^{-2n-1}\,dz, \end{aligned} \tag{4·8}$$

where C is any appropriate contour encircling the origin once in the positive direction.

The integrand in (4·8) has maxima at $z = 0$ and $z = \infty$ on the positive real axis and a unique minimum along that axis. To obtain an asymptotic evaluation of the integral for large n we write it

$$\int_C \exp\{\phi(z)\,dz\},$$

where

$$\phi(z) = d\log I_0(z) - (2n+1)\log z, \tag{4·9}$$

$$\phi'(z) = dI_1(z)/I_0(z) - (2n+1)/z, \tag{4·10}$$

and so the minimum is given by the root of the equation

$$zI_1(z)/I_0(z) = (2n+1)/d. \tag{4·11}$$

For large z the left-hand side of $(4\cdot11)\sim z$, and so we expect, for large n, that the root will lie in the neighbourhood of $(2n+1)/d$. Since the root x_0 (say) is, therefore, large, we estimate it by the use of an asymptotic expansion of $I_1(z)/I_0(z)$. This can be obtained directly from the known asymptotic series for I_0, I_1 (cf. (7), p. 373) or, more simply, by noting that $\psi = I_1/I_0$ satisfies the Riccati equation

$$\frac{d\psi}{dz} = 1 - \psi/z - \psi^2, \tag{4·12}$$

and

$$\lim_{z\to\infty} \psi = 1. \tag{4·13}$$

If we substitute $1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$ for ψ in (4·12) the a_i's can be computed successively.

This leads to
$$\frac{I_1(z)}{I_0(z)} \sim 1 - \frac{1}{2z} - \frac{1}{8z^2} - \frac{1}{8z^3} - \frac{25}{128z^4} + O(z^{-5}). \tag{4·14}$$

The asymptotic evaluation of (4·8) is now completely straightforward except that we have to bear in mind that the saddle-point is duplicated at $-x_0$ because, by (4·10), $\phi'(z)$ is an odd function of z. The estimate (4·6) follows by a standard argument.

We next write $u = \frac{1}{2}(\xi_1 + \xi_2)$ and

$$A(\mathbf{x}, z) = \frac{1 - u\delta z}{1 - (1+\delta)uz + \delta z^2} \tag{4·15}$$

$$= \frac{1}{2i\sin\theta}\left\{\frac{e^{i\theta} - 2\delta v/(1+\delta)}{1 - e^{i\theta}\delta^{\frac{1}{2}}z} - \frac{e^{-i\theta} - 2\delta v/(1+\delta)}{1 - e^{-i\theta}\delta^{\frac{1}{2}}z}\right\}, \tag{4·16}$$

where $\cos\theta = v = \frac{1}{2}\delta^{-\frac{1}{2}}(1+\delta)u$.

If we write $A(\mathbf{x}, z) = \sum_{n=0}^{\infty} A_n(\mathbf{x}) z^n$ we have from (4·15) that

$$A_n(\mathbf{x}) = \delta^{\frac{1}{2}n}\{U_n - 2\delta v/(1+\delta)\, U_{n-1}\}, \tag{4·17}$$

where $U_n = U_n(v) = \sin(n+1)\theta/\sin\theta$, the Tchebycheff polynomial of the second kind.

We can easily deduce from the elementary properties of the polynomials U_n the simpler form

$$A_n(\mathbf{x}) = \frac{\delta^{\frac{1}{2}n}}{1+\delta}(U_n - \delta U_{n-2}), \tag{4·18}$$

and so

$$R_{2n,2}(\delta) = \frac{\delta^n}{1+\delta}(V_{2n} - \delta V_{2n-2}), \tag{4·19}$$

where V_{2i} represents the result of replacing the powers v^{2j} in U_{2i} by the respective quantities

$$\left(\frac{\delta^{\frac{1}{2}} + \delta^{-\frac{1}{2}}}{2}\right)^{2j} R_{2j,2}(0). \tag{4·20}$$

Since U_{2n}, U_{2n-2} are both even functions of v there is no need to consider odd powers. To estimate (4·19) for large n we note that, by (4·6)

$$R_{2n,2}(0) \sim \frac{1}{\pi n}\left(1 - \frac{1}{4n} + \ldots\right)$$

$$\sim \frac{2}{\pi}\left[\frac{1}{2n+1} + \frac{\frac{1}{2}}{(2n+1)(2n+2)} + \frac{a}{(2n+1)(2n+2)(2n+3)} + \ldots\right], \tag{4·21}$$

where $a, \ldots$ are constants and the series converges asymptotically. If now

$$U_{2n}(v) = a_0 v^{2n} + a_1 v^{2n-2} + \ldots \tag{4·22}$$

then

$$V_{2n} \sim \frac{2}{\pi}[L_1^{(n)} + \tfrac{1}{2}L_2^{(n)} + aL_3^{(n)} + \ldots], \tag{4·23}$$

where

$$L_i^{(n)} = \frac{a_0 w^{2n}}{(2n+1)(2n+2)\ldots(2n+i)} + \frac{a_1 w^{2n-2}}{(2n-1)\ldots(2n+i-2)} + \ldots$$

$$= \frac{1}{w^i}\int_{\theta_n=0}^{w}\int_{\theta_{n-1}=0}^{\theta_n}\ldots\int_{\theta_1=0}^{\theta_2} U_{2n}(\theta_1)\,d\theta_1 d\theta_2 \ldots d\theta_n, \tag{4·24}$$

with

$$w = \tfrac{1}{2}(\delta^{\frac{1}{2}} + \delta^{-\frac{1}{2}}). \tag{4·25}$$

The estimation of V_{2n} is thus reduced to that of the successive $L_i^{(n)}$. The repeated integration in (4·24) can be effected by the use of the trigonometric representation of the polynomials involved and it is easily shown by induction that for large n (since $|w| > 1$)

$$L_i^{(n)} \sim \frac{2}{(4nw)^i}\left\{T_{2n+i}(w) - \binom{i-1}{1}T_{2n+i-2}(w) + \binom{i-1}{2}T_{2n+i-4}(w) + \ldots\right\}, \tag{4·26}$$

where $T_j = \cos j\theta$, the Tchebycheff polynomial of the first kind. But it follows from (4·25) and the definition of $T_n(w)$ that

$$T_n(w) = \tfrac{1}{2}(\delta^{\frac{1}{2}n} + \delta^{-\frac{1}{2}n}), \tag{4·27}$$

and so, for large n,

$$T_n(w) \sim \tfrac{1}{2}\delta^{-\frac{1}{2}n}. \tag{4·28}$$

From (4·26) and (4·28) we see that

$$L_i^{(n)} \sim \delta^{-n}\{2n(1+\delta)\}^{-i}(1-\delta)^{i-1}. \tag{4·29}$$

Hence, by (4·23),

$$V_{2n} \sim \{n\pi(1+\delta)\,\delta^n\}^{-1}. \tag{4·30}$$

It follows from (4·19), (4·30) and (4·6) that

$$R_{2n,2}(\delta) \sim R_{2n,2}(0)\left(\frac{1-\delta}{1+\delta}\right). \tag{4·31}$$

One interesting consequence of (3·20), (4·5) and (4·31) is that, for large n,

$$R_{2n,2}(\delta) \sim \{R_{2n,1}(\delta)\}^2, \tag{4·32}$$

an (asymptotic) extension of the first equation of (4·5).

5. *Space of even dimension $d \geqslant 4$.* The relation (4·15) remains valid except that we now define

$$u = \frac{1}{d}\sum_{i=1}^{d}\xi_i \tag{5·1}$$

and $v = \frac{1}{2}(\delta^{\frac{1}{2}} + \delta^{-\frac{1}{2}})u$ as before. Equation (4·19) still holds but, in this case, the V's are obtained from the corresponding U's by replacing the terms v^{2j} by

$$w^{2j}R_{2j,d}(0) \tag{5·2}$$

respectively.

It follows from (4·6) that

$$R_{2n,d}(0) \sim 2\left(\frac{d}{2\pi}\right)^{\frac{1}{2}d}\left\{\frac{1}{(2n+1)\dots(2n+\frac{1}{2}d)}+\frac{a}{(2n+1)\dots(2n+\frac{1}{2}d+1)}+\dots\right\}, \quad (5\cdot3)$$

and so, in this case,

$$V_{2n} \sim 2\left(\frac{d}{2\pi}\right)^{\frac{1}{2}d}\{L^{(n)}_{\frac{1}{2}d}+aL^{(n)}_{\frac{1}{2}d+1}+\dots\} \quad \text{[cf. (4·21) and (4·23)]}$$

$$\sim 2\left(\frac{d}{4\pi n}\right)^{\frac{1}{2}d}\delta^{-n}(1-\delta)^{\frac{1}{2}d-1}(1+\delta)^{-\frac{1}{2}d}, \quad \text{by (4·29)},$$

$$\sim R_{2n,d}(0)\,\delta^{-n}(1-\delta)^{\frac{1}{2}d-1}(1+\delta)^{-\frac{1}{2}d}. \quad (5\cdot4)$$

Hence
$$R_{2n,d}(\delta) = \frac{\delta^n}{1+\delta}(V_{2n}-\delta V_{2n-2}) \sim R_{2n,d}(0)\left(\frac{1-\delta}{1+\delta}\right)^{\frac{1}{2}d}. \quad (5\cdot5)$$

The maximum possible positive value for δ is given by $\delta = p = r = 1/(2d-1)$, $q = 0$. In that case we have, for even d,

$$\frac{R_{2n,d}(\delta)}{R_{2n,d}(0)} \sim \left(1-\frac{1}{d}\right)^{\frac{1}{2}d} \to e^{-\frac{1}{2}} \quad \text{for large } d. \quad (5\cdot6)$$

This gives an estimate of the extent to which the value of $R_{2n,d}$ may be influenced by a positive correlation of this sort. There is, of course, no bound in the opposite direction, and the effect of a negative value of δ is seen clearly from (5·5), including the degeneracy which arises when δ takes the limiting value -1.

6. We raise here the possibility of extending to the general case ($d > 1$, unrestricted p, q, r) the results obtained in §3(*a*). Consider the generating function, as defined in (2·32),

$$A(\mathbf{x}, z) = \sum_{m_1, m_2, \dots = -\infty}^{\infty} \sum_{n=0}^{\infty} P_n(m)\, x_1^{m_1} x_2^{m_2} \dots x_d^{m_d} z^n. \quad (6\cdot1)$$

Now let $\mu = \epsilon_1 m_1 + \epsilon_2 m_2 + \dots + \epsilon_d m_d$, where each ϵ_i is fixed and is equal to ± 1. Then it is clear from the definitions that the value of μ varies in accordance with a one-dimensional walk. Analytically we can see it by writing $x = x_1^{\epsilon_1} = x_2^{\epsilon_2} = \dots = x_d^{\epsilon_d}$. Then $\xi_1 = \xi_2 = \dots = \xi_d = \xi$ (say). The expression (2·32) for $A(x, z)$ reduces to that in (3·1), while the right-hand side of (6·1) becomes

$$\sum_{\mu=-\infty}^{\infty} \sum_{n=0}^{\infty} T_n(\mu)\, x^{\mu} z^n,$$

where $T_n(\mu)$ is the probability that, at the nth step, $m_1\epsilon_1 + m_2\epsilon_2 + \dots + m_d\epsilon_d = \mu$. Thus the generating function for $T_n(\mu)$ is precisely the one-dimensional function

$$\frac{1-\delta\xi z}{1-(1+\delta)\xi z+\delta z^2}. \quad (6\cdot2)$$

It is interesting to observe that this function depends only on $p-q$ and not at all on r.

If we denote by $S_n(\epsilon_1, \epsilon_2, \ldots, \epsilon_d; \mu; p, q, r)$ the probability that the nth step will end on the hyperplane $\epsilon_1 m_1 + \epsilon_2 m_2 + \ldots + \epsilon_d m_d = \mu_d$ we deduce that, for small non-zero values of $|p-q|$,

$$S_n(\epsilon_1, \epsilon_2, \ldots, \epsilon_d;\ \mu;\ p, q, r) - S_n(\epsilon_1, \epsilon_2, \ldots, \epsilon_d;\ \mu;\ p', p', r') \tag{6·3}$$

has the same sign as $(\mu^2 - n)(p-q)$ without any further restriction on p, q, r, p', r'. This argument, of course, proves nothing about the probability of ending a walk at any specific point.

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