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PAPER

m-Algebraic lattices in formal concept analysis

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Abstract

The notion of an m-algebraic lattice, where m stands for a cardinal number, includes numerous special cases, such as complete lattice, algebraic lattice, and prime algebraic lattice. In formal concept analysis, one fundamental result states that every concept lattice is complete, and conversely, each complete lattice is isomorphic to a concept lattice. In this paper, we introduce the notion of an m-approximable concept on each context. The m-approximable concept lattice derived from the notion is an m-algebraic lattice, and conversely, every m-algebraic lattice is isomorphic to an m-approximable concept lattice of some context. Morphisms on m-algebraic lattices and those on contexts are provided, called m-continuous functions and m-approximable morphisms, respectively. We establish a categorical equivalence between $\mathbf{LAT}_m$, the category of m-algebraic lattices and m-continuous functions, and $\mathbf{CXT}_m$, the category of contexts and m-approximable morphisms. We prove that $\mathbf{LAT}_m$ is cartesian closed whenever m is regular and $m > 2$. By the equivalence of $\mathbf{LAT}_m$ and $\mathbf{CXT}_m$, we obtain that $\mathbf{CXT}_m$ is also cartesian closed under same circumstances. The notions of a concept, an approximable concept, and a weak approximable concept are showed to be special cases of that of an m-approximable concept.

Keywords: Formal concept analysis; m-algebraic lattice; m-approximable concept, cartesian closed category; categorical equivalence

1. Introduction

Formal concept analysis (FCA) is a method for deriving a concept hierarchy from a collection of objects and their attributes, which has a significant potential for applications in fields including data mining, knowledge management, and machine learning (see, e.g., Davey and Priestley 2002; Ganter and Wille 1999; Kang and Miao 2016; Ren et al. 2017; Yu et al. 2018). The theory has a deep connection with the mathematical theory of lattices and ordered sets. One standard result in FCA states that the concept lattice of a context is a complete lattice, and, conversely, every complete lattice is isomorphic to a concept lattice of some context. To build connections between contexts and other mathematical structures, Zhang and Shen presented the notion of an approximable concept (Zhang and Shen 2006). The introduction of the notion was motivated by the idea in domain theory (Gierz et al. 2003) that pieces of information or results of a computation should either be finite or approximated by finite elements. Approximable concept lattices constituted by approximable concepts are exactly algebraic lattices, and every algebraic lattice is isomorphic to an approximable concept lattice of a context. Based on the work about approximable concepts, Hitzler and Zhang (2004) provided an appropriate notion of morphisms between contexts and showed that the resulting category is equivalent to the category of algebraic lattices and Scott continuous functions. As an abstract branch of mathematics, category theory has been widely applied in various areas (MacLane 1971). The theory provides a unified framework to different mathematical structures, and an equivalence of categories establishes a relation that these structures are essentially the same from the abstraction view. Hitzler et al. (2006) explored algebraicity in FCA from a category theoretical view. Following the work of Zhang and Shen (2006), Liu et al. (2011) presented the notion of a weak approximable concept, and they proved that the corresponding weak approximable concept lattices derived from the notion are exactly completely distributive algebraic lattices.

In mathematics, cardinal numbers are used to measure the size of sets (see, e.g., Dauben 1990; Deiser 2010). The concept of cardinality was formulated by Georg Cantor, who also developed a large portion of the theory of cardinal numbers. In this paper, we accept the axiom of choice. Then the sequence of natural numbers and aleph numbers indexed by ordinals:

$$0, 1, 2, ..., n, ...; \aleph_0, \aleph_1, \aleph_2, ..., \aleph_\alpha, ...$$

includes every cardinal number. A cardinal m is regular if and only if it cannot be expressed as the cardinal sum of a set whose cardinality is less than m, and the elements of which are cardinals less than m. Roughly speaking, regular cardinals are those which cannot be broken into a smaller collection of smaller parts. In this paper, we mainly deal with regular cardinals. m-Algebraic lattices were first appeared in Grätzer (1965), where m stands for a cardinal number. In the case that $m = \aleph_0$ the smallest infinite cardinal, m-algebraic lattices are exactly algebraic lattices. Thus m-algebraic lattices are also viewed as natural generalizations of algebraic lattices. In Lee (1988), Lee investigated countably algebraic lattices which are precisely $\aleph_1$-algebraic lattices ($\aleph_1$ is the smallest uncountable cardinal). Representation theory of m-algebraic lattices is an important research topic, where m is a regular cardinal (see Grätzer 1992; Grätzer et al. 1994). One question arises naturally: Is there a type of concepts on each context such that the resulting concept lattice is an m-algebraic lattice, and every m-algebraic lattice is isomorphic to the resulting concept lattice on some context? We deal with this problem by presenting the notion of an m-approximable concept. We also provide appropriate types of morphisms on contexts and functions on m-algebraic lattices such that the corresponding category of contexts and that of m-algebraic lattices are equivalent. Finally, we compare our work with other results and show some direct applications of the categorical equivalence.

2. Preliminaries

In this section, we recall some basic concepts in order theory, FCA, and category theory. We refer the reader to Abramsky and Jung (1994) and Gierz et al. (2003) for standard domain-theoretic notations and elementary facts about algebraic lattices, and to MacLane (1971) for more details about categorical results.

Let L be a complete lattice, $A \subseteq L$ and $x \in L$. We write $\bigvee A$ for the least upper bound of A in L, and

$$\downarrow x = \{y \in L : y \leq x\}, \uparrow x = \{y \in L : x \leq y\}, \downarrow A = \bigcup_{a \in A} \downarrow a.$$

For any set A, we denote by $card(A)$ the cardinality of A. A cardinal m is *regular* if for any family of sets $\{A_j : j \in J\}$, $card(J) < m$ and $card(A_j) < m$ for $j \in J$ imply $card(\bigcup_{j \in J} A_j) < m$. In this paper, m stands for a fixed cardinal.

Definition 2.1. *A context is a triple $K = (G, M, I)$ where G and M are sets of objects and attributes, respectively, and $I \subseteq G \times M$ is a relation.*

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For $A \subseteq G$ and $B \subseteq M$, define

$$A' = \{m \in M : aIm \text{ for all } a \in A\},$$
$$B' = \{g \in G : gIb \text{ for all } b \in B\}.$$

A *concept* is a pair (A, B), where $A \subseteq G$, $B \subseteq M$, $A' = B$, and $B' = A$. Then A is called the *extent* of the concept (A, B) and B is called the *intent*.

Lemma 2.2. (Davey and Priestley 2002) *Let (G, M, I) be a context, $A, A_j \subseteq G$ and $B, B_j \subseteq M$, for $j \in J$. Then*
(1) $A \subseteq A''$ and $B \subseteq B''$;
(2) $A' = A'''$ and $B' = B'''$;
(3) $(\bigcup_{j\in J} A_j)' = \bigcap_{j\in J} A'_j$ and $(\bigcup_{j\in J} B_j)' = \bigcap_{j\in J} B'_j$.

Definition 2.3. *A category $\mathcal{C}$ consists of the following:*
(1) a collection $\mathcal{C}_0$ of objects;
(2) for each pair A, B of objects, a collection $Hom(A, B)$ of morphisms from A to B;
(3) for each triple A, B, C of objects, a composition function

$$\circ : Hom(A, B) \times Hom(B, C) \to Hom(A, C)$$

satisfies the associative law: if $f \in Hom(A, B)$, $g \in Hom(B, C)$, and $h \in Hom(C, D)$, then $(h \circ g) \circ f = h \circ (g \circ f)$;
(4) for each object A, an identity morphism id_A on A satisfies the left and right unit laws: $f \in Hom(A, B)$ implies $f = f \circ id_A$, $g \in Hom(C, A)$ implies $g = id_A \circ g$.

Definition 2.4. *A category $\mathcal{C}$ is called cartesian closed if it satisfies the following three properties:*
(1) it has a terminal object T such that for any object A of $\mathcal{C}$, there is exactly one morphism from A to T;
(2) any two objects A and B of $\mathcal{C}$ have a product object $A \times B$ together with morphisms $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ such that for any object C and morphisms $f : C \to A$, $g : C \to B$, there is a unique morphism $f \times g : C \to A \times B$ with $\pi_1 \circ (f \times g) = f$ and $\pi_2 \circ (f \times g) = g$;
(3) any two objects A and B of $\mathcal{C}$ have an exponential object B^A in $\mathcal{C}$ together with a morphism $ev : B^A \times A \to B$ such that for each $f : C \times A \to B$, there exists a unique morphism $\Lambda_f : C \to B^A$ with $ev \circ (\Lambda_f \times id_A) = f$.

Category theory has been widely applied in many branches of mathematics, and even in areas of theoretical computer science. Cartesian closed categories are especially important in mathematical logic and the theory of functional programming. Semantic categories of higher-order functional programming languages are required to be cartesian closed, in which case higher order objects are just normal objects. For example, in domain theory, one main task is to find cartesian closed categories of domains to give rise to models of typed λ-calculi (see, e.g., Abramsky and Jung 1994; Gierz et al. 2003; Jia et al. 2015; Jung 1989; Zhang and Li 2017).

Definition 2.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$ is a map sending each object A of $\mathcal{C}$ to an object $\mathcal{F}(A)$ of $\mathcal{D}$ and each morphism $f : A \to B$ in $\mathcal{C}$ to a morphism $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$ in $\mathcal{D}$ such that*
(1) $\mathcal{F}$ preserves composition: $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$ for all compositions $f \circ g$;
(2) $\mathcal{F}$ preserves identity: $\mathcal{F}(id_X) = id_{\mathcal{F}(X)}$ for all objects X.

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A functor $\mathcal{F}:\mathcal{C}\to\mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ induces a function

$$\mathcal{F}_{A,B}: Hom(A,B)\to Hom(\mathcal{F}(A),\mathcal{F}(B))$$

for every pair of A and B in $\mathcal{C}$. The functor $\mathcal{F}$ is called
(1) *full* if $\mathcal{F}_{A,B}$ is surjective for every pair A, B;
(2) *faithful* if $\mathcal{F}_{A,B}$ is injective for every pair A, B.

Definition 2.6. *The categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there is a functor $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$ such that $\mathcal{F}$ is full, faithful, and each object A in $\mathcal{D}$ is isomorphic to $\mathcal{F}(B)$ for some object B in $\mathcal{C}$.*

Category theory makes it possible to prove related results of different mathematical structures in a unified way. The concept of equivalence of categories is the category theoretic view of sameness of categories. Particularly, an equivalence of categories preserves cartesian closedness.

3. *m*-Algebraic Lattice

In this section, we first recall the notion of an m-algebraic lattice. We then introduce a type of continuous functions between m-algebraic lattices, with which a category of m-algebraic lattices yields. Cartesian closedness of the category is investigated.

Definition 3.1. (Grätzer 1965; Grätzer et al. 1994) *Let L be a complete lattice.*
(1) An element $x\in L$ is m-compact if for all subsets $A\subseteq L$, the relation $x\leq\bigvee A$ always implies the existence of $X\subseteq A$ with $card(X)<m$ and $x\leq\bigvee X$. Write $\kappa_m(L)$ for the set of m-compact elements of L.
(2) The lattice L is m-algebraic if each element x of L is a join of m-compact elements, i.e., $x=\bigvee(\downarrow x\bigcap\kappa_m(L))$.

The notion of an m-algebraic lattice is a natural generalization of that of an algebraic lattice. There is also an “m-version” of the concept of a directed set: A subset D of a complete lattice is called *m-directed* if for any $X\subseteq D$ with $card(X)<m$, there is an upper bound of X in D. By means of m-directed sets, we have the following definitions.

Definition 3.2. *Let L be a complete lattice.*
(1) We say that $x\in L$ is m'-compact if for all m-directed sets $D\subseteq L$, $x\leq\bigvee D$ implies $x\leq d$ for some $d\in D$. Let $\kappa'_m(L)$ denote the set of m'-compact elements of L.
(2) The lattice L is called m'-algebraic if for all $x\in L$, $\downarrow x\bigcap\kappa'_m(L)$ is m-directed and $x=\bigvee(\downarrow x\bigcap\kappa'_m(L))$.

Proposition 3.3. (1) *If an element x of a complete lattice L is m-compact, then x is m'-compact.*
(2) *If m is a regular cardinal, then m-compact elements coincide with m'-compact ones, and L is m-algebraic if and only if it is m'-algebraic.*

Proof. (1) Suppose that x is m-compact. For any m-directed set $D\subseteq L$ with $x\leq\bigvee D$, we have $x\leq\bigvee X$ for some $X\subseteq D$ with $card(X)<m$. Then, since D is m-directed, there is an upper bound d of X in D, and hence $x\leq d$. Thus x is m'-compact.

(2) Let $x\in L$ be an m'-compact element, and $A\subseteq L$ with $x\leq\bigvee A$. Define $D=\{\bigvee X: X\subseteq A$ and $card(X)<m\}$. We claim that D is m-directed. Indeed, for any $\{\bigvee X_j: j\in J\}\subseteq D$ such that $card(J)<m$, we have $card(\bigcup_{j\in J}X_j)<m$, and then $\bigvee\{\bigvee X_j: j\in J\}=\bigvee(\bigcup_{j\in J}X_j)\in D$, which proves the claim. Moreover, $x\leq\bigvee A=\bigvee D$. Then $x\leq\bigvee X_0$ for some $\bigvee X_0\in D$, where $card(X_0)<m$. Thus x is m-compact. By (1), we have m-compactness is equivalent to m'-compactness.

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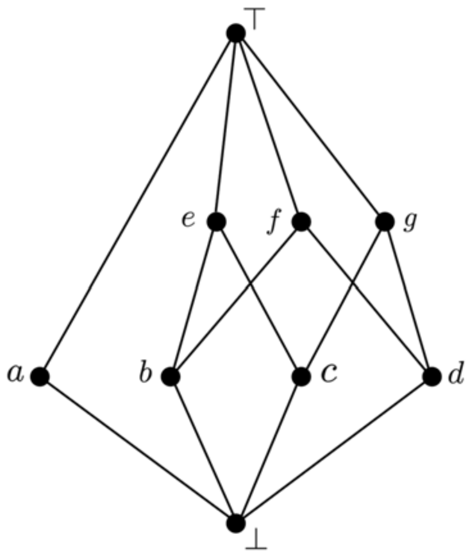

Figure 1. The Hasse diagram of the lattice L in Example 3.4.

To prove the equivalence between m-algebraicity and m'-algebraicity, it suffices to show that $\downarrow x \bigcap \kappa_m(L)$ is m-directed for all $x \in L$. Let $X \subseteq \downarrow x \bigcap \kappa_m(L)$ with $card(X) < m$. We shall show $\bigvee X$ is m-algebraic. Suppose $A \subseteq L$ such that $\bigvee X \leq \bigvee A$. Then, for each $y \in X$, we have $y \leq \bigvee X_y$ for some $X_y \subseteq A$, where $card(X_y) < m$. Hence $\bigvee X \leq \bigvee(\bigcup_{y \in X} X_y)$. Moreover, $card((\bigcup_{y \in X} X_y)) < m$ by the regularity of m. Thus $\bigvee X$ is m-algebraic, and then $\bigvee X \in \downarrow x \bigcap \kappa_m(L)$, which proves that $\downarrow x \bigcap \kappa_m(L)$ is m-directed. □

Example 3.4. We shall show that an m'-algebraic lattice is not necessarily m-algebraic. Consider the case of $m = 3$, which is not regular. A 3-directed set is exactly a directed set. And $3'$-algebraic lattices are algebraic lattices. Let $L = \{\bot, a, b, c, d, e, f, g, \top\}$, see Figure 1 for the order relation $\leq$ on L. It is easy to check that L is a complete lattice. Moreover, L is an algebraic lattice (in fact, all finite complete lattices are algebraic). Notice that $a \leq \bigvee\{b, c, d\} = \top$, but for any subset X of $\{b, c, d\}$ whose cardinality less than 3, we always have $a \nleq \bigvee X$. Then a is not 3-compact, and hence $a \neq \bigvee(\downarrow a \bigcap \kappa_m(L)) = \bot$. That is to say, L is not 3-algebraic.

Definition 3.5. *Let L be a complete lattice. A subset $A \subseteq L$ is called m-closed, if*

(1) $\downarrow A = A$;

(2) for any m-directed set $D \subseteq L$, $D \subseteq A$ implies $\bigvee D \in A$.

Write $\Gamma_m(L)$ for the set of m-closed subsets of L, and $\Gamma_m(L)$ is called the m-closure system on L.

It is clear that $\Gamma_m(L)$ is closed under arbitrary intersections, and thus $\Gamma_m(L)$ is indeed a closure system on L. Recall that a function $f : (X, \Gamma(X)) \to (Y, \Gamma(Y))$ between closure spaces is called *continuous* if $f^{-1}(A) \in \Gamma(X)$ for all $A \in \Gamma(Y)$. For a closure system $\Gamma(X)$ on X, if $\emptyset, X \in \Gamma(X)$ and $A \bigcup B \in \Gamma(X)$ for any $A, B \in \Gamma(X)$, then $\Gamma(X)$ is said to be a *topological closure system* (in which case the complements of elements in $\Gamma(X)$ form a topology on X). We call a function $f : L \to M$ between complete lattices *m-continuous* if f is continuous with respect to the m-closure systems.

Proposition 3.6. (1) *Unions of subfamilies $\{A_j : j \in J\}$ of an m-closure system $\Gamma_m(L)$, where $card(J) < m$ are still m-closed.*

(2) *If $m > 2$, then $\Gamma_m(L)$ is a topological closure system on the complete lattice L.*

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(3) *A function $f : L \to M$ between complete lattices is m-continuous iff $f(\bigvee D) = \bigvee f(D)$ for all m-directed subsets D of L.*

Proof. (1) If $card(J) < 2$, then the family $\{A_j : j \in J\}$ has at most one element, and hence $\bigcup_{j \in J} A_j$ is obviously m-closed. Now suppose that $card(J) \geq 2$. Let $D \subseteq \bigcup_{j \in J} A_j$ be m-directed. We shall show that $D \subseteq A_{i_0}$ for some $i_0 \in I$. Assume not, then, for each $j \in J$, there is $d_j \in D$ such that $d_j \notin A_j$. Then, since D is m-directed, there is an upper bound $d_0 \in D$ of $\{d_j : j \in J\}$. Moreover, there must be an $i_0 \in I$ such that $d_0 \in A_{i_0}$. Since A_{i_0} is a lower set, we have $\{d_j : j \in J\} \subseteq \downarrow d_0 \subseteq A_{i_0}$, which contradicts to $d_{i_0} \notin A_{i_0}$. Thus $D \subseteq A_{i_0}$ for some $i_0 \in I$, and hence $\bigvee D \in A_{i_0} \subseteq \bigcup_{j \in J} A_j$, which proves that $\bigcup_{j \in J} A_j$ is m-closed.

(2) A direct consequence of (1).

(3) $\Rightarrow$: Suppose that f is m-continuous and $D \subseteq L$ is m-directed. For any $x, y \in L$ with $x \leq y$, we have $\downarrow f(y) \in \Gamma_m(M)$, then $y \in f^{-1}(\downarrow f(y)) \in \Gamma_m(L)$ which implies $x \in \downarrow y \subseteq f^{-1}(\downarrow f(y))$, and hence $f(x) \in f(f^{-1}(\downarrow f(y))) \subseteq \downarrow f(y)$, i.e., $f(x) \leq f(y)$. Thus f is monotone. Consequently, $\bigvee f(D) \leq f(\bigvee D)$. And $f(D) \subseteq \downarrow \bigvee f(D)$. Then $D \subseteq f^{-1}(\downarrow \bigvee f(D)) \in \Gamma_m(L)$, and hence $\bigvee D \in f^{-1}(\downarrow \bigvee f(D))$. Thus $f(\bigvee D) \in f(f^{-1}(\downarrow \bigvee f(D))) \subseteq \downarrow \bigvee f(D)$, that is to say, $f(\bigvee D) \leq \bigvee f(D)$. Therefore, $f(\bigvee D) = \bigvee f(D)$.

$\Leftarrow$: Conversely, suppose $A \in \Gamma_m(M)$ and $D \subseteq f^{-1}(A)$ is m-directed. Again, we first show that f is monotone. Let $x, y \in L$ with $x \leq y$. Notice that $x \in \downarrow y$ and $\downarrow y$ is m-directed. Then $f(x) \leq \bigvee f(\downarrow y) = f(\bigvee \downarrow y) = f(y)$, and hence f is monotone. We next prove that $f(D) \subseteq A$ is m-directed. Suppose that $X \subseteq f(D)$ with $card(X) < m$. For each $x \in X$, there is $d_x \in D$ such that $f(d_x) = x$. Let $X_0 = \{d_x : x \in X\}$. Then $card(X_0) = card(X)$ and $f(X_0) = X$. Let $d_0 \in D$ be an upper bound of X_0 in D. Then $f(d_0)$ is an upper bound of X in $f(D)$, i.e., $f(D)$ is m-directed. Then, since A is m-closed, $f(\bigvee D) = \bigvee f(D) \in A$. Hence $\bigvee D \in f^{-1}(A)$, and then $f^{-1}(A)$ is m-closed, which proves that f is m-continuous. □

Let L and M be m-algebraic lattices. For $d \in \kappa_m(L)$ and $e \in \kappa_m(M)$, define the *step function* $(d \searrow e) : L \to M$ by

$$(d \searrow e)(x) = \begin{cases} e & x \in \uparrow d, \\ \bot_M & \text{otherwise.} \end{cases}$$

Proposition 3.7. *The step functions $(d \searrow e) : L \to M$ are m-continuous.*

Proof. Suppose that $D \subseteq L$ is m-directed. If $\bigvee D \in \uparrow d$, then, by Proposition 3.3(1), there is $d_0 \in D$ such that $d \leq d_0$, hence $\bigvee (d \searrow e)(D) = e = (d \searrow e)(\bigvee D)$; else, $(d \searrow e)(\bigvee D) = \bot_M = \bigvee (d \searrow e)(D)$. Thus $(d \searrow e)$ is m-continuous. □

Consider the set $[L \to M]$ of all m-continuous functions between m-algebraic lattices L and M, under the pointwise ordering: $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in L$.

Lemma 3.8. *If m is a regular cardinal, then the function space $[L \to M]$ between m-algebraic lattices is still an m-algebraic lattice.*

Proof. We first show that $[L \to M]$ is a complete lattice. Let $\mathcal{F} \subseteq [L \to M]$. Define $h : L \to M$ by

$$h(x) = \bigvee_{f \in \mathcal{F}} f(x).$$

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Then h is the least upper bound of $\mathcal{F}$ as long as it is m-continuous. For any m-directed $D \subseteq L$, we have

$$h\left(\bigvee D\right) = \bigvee_{f\in\mathcal{F}} f\left(\bigvee D\right) = \bigvee_{f\in\mathcal{F}} \bigvee_{d\in D} f(d) = \bigvee_{d\in D} \bigvee_{f\in\mathcal{F}} f(d) = \bigvee_{d\in D} h(d),$$

which shows that h is indeed m-continuous. Consequently, $[L \to M]$ is a complete lattice.

It remains to show that $[L \to M]$ is m-algebraic. We claim that the step functions $(d \searrow e)$ are m-compact in $[L \to M]$. Indeed, for any $\mathcal{A} \subseteq [L \to M]$ with $(d \searrow e) \leq \bigvee \mathcal{A}$, we have $(d \searrow e)(d) = e \leq \bigvee \mathcal{A}(d)$, then, since e is m-compact, there is $\mathcal{X} \subseteq \mathcal{A}$ with $card(\mathcal{X}) < m$, such that $(d \searrow e)(d) \leq \bigvee \mathcal{X}(d)$. If $x \in \uparrow d$, then $(d \searrow e)(x) = e \leq \bigvee \mathcal{X}(d) \leq \bigvee \mathcal{X}(x)$ (we have proved that each m-continuous function is monotone in the proof of Proposition 3.6); if $x \in L \setminus \uparrow d$, then $(d \searrow e)(x) = \bot_M \leq \bigvee \mathcal{X}(x)$. Thus $(d \searrow e) \leq \bigvee \mathcal{X}$, which proves the claim. Suppose $f \in [L \to M]$. Define

$$\mathcal{F} = \{(d \searrow e) : d \in \kappa_m(L), e \in \kappa_m(M), (d \searrow e) \leq f\}.$$

Then $\mathcal{F} \subseteq \downarrow f \bigcap \kappa_m([L \to M])$. For every $x \in L$, define

$$\mathcal{G} = \{(d \searrow e) : d \in \downarrow x \bigcap \kappa_m(L), e \in \downarrow f(d) \bigcap \kappa_m(M)\}.$$

Then $\mathcal{G} \subseteq \mathcal{F}$. Moreover,

$$\begin{aligned}\left(\bigvee \mathcal{G}\right)(x) &= \bigvee_{d\in\downarrow x \bigcap \kappa_m(L)} \; \bigvee_{e\in\downarrow f(d)\bigcap \kappa_m(M)} (d \searrow e)(x) \\ &= \bigvee_{d\in\downarrow x \bigcap \kappa_m(L)} \; \bigvee_{e\in\downarrow f(d)\bigcap \kappa_m(M)} e \\ &= \bigvee_{d\in\downarrow x \bigcap \kappa_m(L)} f(d) \\ &= f\left(\bigvee(\downarrow x \bigcap \kappa_m(L))\right) \\ &= f(x),\end{aligned}$$

because in the case that m is regular, $\downarrow x \bigcap \kappa_m(L)$ is m-directed by Proposition 3.3(2). Then

$$f(x) = \left(\bigvee \mathcal{G}\right)(x) \leq \left(\bigvee \mathcal{F}\right)(x) \leq f(x),$$

and hence $f = \bigvee \mathcal{F}$, which proves that $[L \to M]$ is an m-algebraic lattice. □

We denote the category of all m-algebraic lattices and m-continuous functions by $\mathbf{LAT}_m$.

Theorem 3.9. *Let m be a regular cardinal with $m > 2$. Then the category $\mathbf{LAT}_m$ is cartesian closed.*

Proof. (1) A singleton m-algebraic lattice serves as a terminal object in the category $\mathbf{LAT}_m$.

(2) Suppose that L and M are m-algebraic lattices. Let $L \times M$ be the cartesian product together with the pointwise order, that is: $(x_1, y_1) \leq (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \leq y_2$. Define $\pi_1 : L \times M \to L$ by

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$$\pi_1(x, y) = x,$$

and $\pi_2 : L \times M \to M$ by

$$\pi_2(x, y) = y.$$

It is trivial to check that $L \times M$ is a complete lattice. We shall show that $L \times M$ is m-algebraic by proving $(\kappa_m(L) \times \{\perp_M\}) \bigcup (\{\perp_L\} \times \kappa_m(M)) \subseteq \kappa_m(L \times M)$, where $\perp_L, \perp_M$ are least elements of L and M, respectively. Suppose $(x, \perp_M) \in \kappa_m(L) \times \{\perp_M\}$. For any $A \subseteq L \times M$ with $(x, \perp_M) \leq \bigvee A$, we have $x \leq \bigvee \pi_1(A)$, then there is $X \subseteq \pi_1(A)$, where $card(X) < m$, such that $x \leq \bigvee X$. Notice that for each $a \in X$, there is $b_a \in M$ such that $(a, b_a) \in A$. Define $Y = \{(a, b_a) : a \in X\}$. Then $(x, \perp_M) \leq \bigvee Y$. Moreover, $card(Y) = card(X) < m$. Thus $(x, \perp_M) \in \kappa_m(L \times M)$. Dually, we also have $\{\perp_L\} \times \kappa_m(M) \subseteq \kappa_m(L \times M)$, which proves $(\kappa_m(L) \times \{\perp_M\}) \bigcup (\{\perp_L\} \times \kappa_m(M)) \subseteq \kappa_m(L \times M)$. For every $(x, y) \in L \times M$, we have

$$\left(\left(\downarrow x \bigcap \kappa_m(L)\right) \times \{\perp_M\}\right) \bigcup (\{\perp_L\} \times (\downarrow y \bigcap \kappa_m(M))) \subseteq \downarrow (x, y) \bigcap \kappa_m(L \times M).$$

Hence

$$\begin{aligned}(x, y) &= \bigvee \left(\left(\left(\downarrow x \bigcap \kappa_m(L)\right) \times \{\perp_M\}\right) \bigcup \left(\{\perp_L\} \times \left(\downarrow y \bigcap \kappa_m(M)\right)\right)\right) \\ &\leq \bigvee \left(\downarrow (x, y) \bigcap \kappa_m(L \times M)\right) \\ &\leq (x, y),\end{aligned}$$

and then $(x, y) = \bigvee \left(\downarrow (x, y) \bigcap \kappa_m(L \times M)\right)$, which proves that $L \times M$ is an object of the category $\mathbf{LAT}_m$. We next show that $L \times M$ is the product of L and M in $\mathbf{LAT}_m$. For any $D \subseteq L \times M$, we have

$$\begin{aligned}\pi_1\left(\bigvee D\right) &= \pi_1\left(\bigvee \pi_1(D), \bigvee \pi_2(D)\right) \\ &= \bigvee \pi_1(D).\end{aligned}$$

Thus π_1 is m-continuous by Proposition 3.6(3). In the same way, we conclude that π_2 is also m-continuous. Suppose that Y is an m-algebraic lattice, $f_1 : Y \to L$ and $f_2 : Y \to M$ are m-continuous functions. Define $f : Y \to L \times M$ by

$$f(x) = (f_1(x), f_2(x)).$$

For any m-directed $D \subseteq Y$, we have

$$\begin{aligned}f\left(\bigvee D\right) &= \left(f_1\left(\bigvee D\right), f_2\left(\bigvee D\right)\right) \\ &= \left(\bigvee f_1(D), \bigvee f_2(D)\right) \\ &= \bigvee f(D).\end{aligned}$$

Then f is m-continuous. Moreover, $\pi_1 \circ f = f_1$ and $\pi_2 \circ f = f_2$. Thus $L \times M$ is the product object of L and M.

(3) We shall show that the function space $[L \to M]$ is an exponential object for m-algebraic lattices L and M. By Lemma 3.8, we have that $[L \to M]$ is an object of $\mathbf{LAT}_m$. Consider the evaluation function $ev : [L \to M] \times L \to M$ defined by

$$ev(f, x) = f(x).$$

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Let $\mathcal{F}$ and D be an m-directed subsets of $[L \to M]$ and L, respectively. Then

$$\begin{aligned} ev\left(\bigvee \mathcal{F}, x\right) &= \left(\bigvee \mathcal{F}\right)(x) \\ &= \bigvee_{f \in \mathcal{F}} f(x) \\ &= \bigvee_{f \in \mathcal{F}} ev(f, x), \end{aligned}$$

and

$$\begin{aligned} ev\left(f, \bigvee D\right) &= f\left(\bigvee D\right) \\ &= \bigvee f(D) \\ &= \bigvee_{d \in D} ev(f, d). \end{aligned}$$

Assume now that $\mathcal{D} \subseteq [L \to M] \times L$ is m-directed. Then $\pi_1(\mathcal{D})$ and $\pi_2(\mathcal{D})$ are also m-directed. For each $f \in \pi_1(\mathcal{D})$ and $x \in \pi_2(\mathcal{D})$, since $m > 2$, there is $(g, y) \in \mathcal{D}$ such that $f \leq g$ and $x \leq y$. Thus

$$\begin{aligned} ev\left(\bigvee \mathcal{D}\right) &= ev\left(\bigvee \pi_1(\mathcal{D}), \bigvee \pi_2(\mathcal{D})\right) \\ &= \bigvee_{f \in \pi_1(\mathcal{D})} ev\left(f, \bigvee \pi_2(\mathcal{D})\right) \\ &= \bigvee_{f \in \pi_1(\mathcal{D})} \bigvee_{x \in \pi_2(\mathcal{D})} ev(f, x) \\ &= \bigvee_{(f,d) \in \mathcal{D}} ev(f, d), \end{aligned}$$

which proves ev is m-continuous. For each m-continuous function $f : P \times L \to M$, where P is an m-algebraic lattice, we define $\Lambda_f : P \to [L \to M]$ by

$$\Lambda_f(x) = f(x, \cdot).$$

The m-continuity follows easily from that of f. And $ev \circ (\Lambda_f \times id_L) = f$. Therefore $[L \to M]$ is the exponential object for L and M. □

4. *m*-Approximable Concept

In this section, we introduce the notion of an m-approximable concept and provide a representation theorem for m-algebraic lattices. By defining an appropriate type of morphisms on contexts, an equivalence of the resulting category of contexts and the category $\mathbf{LAT}_m$ of m-algebraic lattices is obtained. The cardinals m considered in this section are regular, in which case m-compactness and m-algebraicity coincide with m'-compactness and m'-algebraicity, respectively. We also require that $m \geq 2$.

Definition 4.1. *Let $K = (G, M, I)$ be a context. A subset $A \subseteq M$ is called an m-approximable (attribute) concept if for any $X \subseteq A$ with $card(X) < m$, we have $X'' \subseteq A$. We use the notation $\mathcal{A}(K)$ for the set of all m-approximable concepts in K, ordered by inclusion.*

Lemma 4.2. *Let $K = (G, M, I)$ be a context and $X \subseteq M$. Then*

(1) *the poset $\mathcal{A}(K)$ is a complete lattice;*

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(2) *X'' is an m-approximable concept;*

(3) *if $card(X) < m$, then X'' is an m-compact element in the set $\mathcal{A}(K)$;*

(4) *if X is an m-compact element in $\mathcal{A}(K)$, then there is $X_0 \subseteq X$ with $card(X_0) < m$, such that $X = X_0''$.*

Proof. (1) To prove that $\mathcal{A}(K)$ is a complete lattice, it suffices to show that $\mathcal{A}(K)$ is a closure system. It is equivalent to show that $\bigcap \mathcal{B} \in \mathcal{A}(K)$ for any $\mathcal{B} \subseteq \mathcal{A}(K)$. Suppose that $X \subseteq \bigcap \mathcal{B}$ with $card(X) < m$. Then, for each $B \in \mathcal{B}$, $X \subseteq B$, and we have $X'' \subseteq B$ because B is an m-approximable concept. Hence $X'' \subseteq \bigcap \mathcal{B}$, this proves $\bigcap \mathcal{B} \in \mathcal{A}(K)$. Consequently, $\mathcal{A}(K)$ is a complete lattice.

(2) By the fact that $F'' \subseteq (X'')'' = X''$ for any $F \subseteq X''$.

(3) Let $\mathcal{D} \subseteq \mathcal{A}(K)$ be an m-directed set such that $X'' \subseteq \bigvee \mathcal{D}$. We claim that $\bigcup \mathcal{D}$ is the supremum of $\mathcal{D}$ in $\mathcal{A}(K)$, which is equivalent to saying that $\bigcup \mathcal{D}$ is an m-approximable concept. Indeed, if $Y \subseteq \bigcup \mathcal{D}$ for any $card(Y) < m$, then, for all $y \in Y$, there is $D_y \in \mathcal{D}$ such that $y \in D_y$, and then, since $\mathcal{D}$ is m-directed, there is $D_Y \in \mathcal{D}$ such that $D_y \subseteq D_Y$ for all $y \in Y$. Hence $Y \subseteq D_Y$, and therefore $Y'' \subseteq D_Y \subseteq \bigcup \mathcal{D}$, which proves the claim.

Then $X'' \subseteq \bigcup \mathcal{D}$, and thus $X \subseteq \bigcup \mathcal{D}$. By an argument similar to that given above, there is $D_X \in \mathcal{D}$ such that $X \subseteq D_X$. Hence $X'' \subseteq D_X$, and thus X'' is m-compact.

(4) Since X is an m-approximable concept, we have $X = \bigvee \{Y'' : Y \subseteq X \text{ and } card(Y) < m\}$ (notice that $m \geq 2$). Moreover, $\{Y'' : Y \subseteq X \text{ and } card(Y) < m\}$ is m-directed by the regularity of m. Then, by m-compactness of X, $X = X_0''$ for some $X_0 \subseteq X$ with $card(X_0) < m$. □

Theorem 4.3. [Representation Theorem] *For any context $K = (G, M, I)$, the poset $\mathcal{A}(K)$ of m-approximable concepts is an m-algebraic lattice. Conversely, for every m-algebraic lattice L, there is a context K_L whose m-approximable concept lattice $\mathcal{A}(K_L)$ is order-isomorphic to L.*

Proof. We first show that $\mathcal{A}(K)$ is an m-algebraic lattice. Let $A \in \mathcal{A}(K)$. Then, $A = \bigcup \{X'' : X \subseteq A$ with $card(X) < m\}$, and by Lemma 4.2(3), we have $\mathcal{A}(K)$ is an m-algebraic lattice.

For the converse part, suppose that $(L, \leq)$ is an m-algebraic lattice. Define the context $K_L = (L, \kappa_m(L), \geq)$, where $\geq$ is the dual of $\leq$. We shall show that L is order-isomorphic to $\mathcal{A}(K_L)$. We claim that $A \subseteq \kappa_m(L)$ is an m-approximable concept if and only if $A = \downarrow a \bigcap \kappa_m(L)$ for some $a \in L$. Since L is an m-algebraic lattice, we have $a = \bigvee (\downarrow a \bigcap \kappa_m(L))$ for every $a \in L$. Then the function $i : L \to \mathcal{A}(K_L)$ defined by

$$i(a) = \downarrow a \bigcap \kappa_m(L)$$

is clearly an isomorphism. To prove the claim, first observe that for any $X \subseteq \kappa_m(L)$,

$$\begin{aligned} X' &= \{a \in L : \forall x \in X, a \geq x\} \\ &= \uparrow \left(\bigvee X\right), \end{aligned}$$

and

$$\begin{aligned} X'' &= \{b \in \kappa_m(L) : \forall y \in X', y \geq b\} \\ &= \left\{b \in \kappa_m(L) : \forall y \in \uparrow \left(\bigvee X\right), y \geq b\right\} \\ &= \left\{b \in \kappa_m(L) : \bigvee X \geq b\right\} \\ &= \downarrow \left(\bigvee X\right) \bigcap \kappa_m(L). \end{aligned}$$

If $A = \downarrow a \bigcap \kappa_m(L)$ for some $a \in L$, then for any $X \subseteq \downarrow a \bigcap \kappa_m(L)$, we have $\bigvee X \leq a$, and then $X'' = \downarrow (\bigvee X) \bigcap \kappa_m(L) \subseteq \downarrow a \bigcap \kappa_m(L)$, hence A is an m-approximable concept. On the other hand,

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assume that A is an m-approximable concept. We will complete the proof by showing $A = {\downarrow}(\bigvee A) \bigcap \kappa_m(L)$. Clearly, $A \subseteq {\downarrow}(\bigvee A) \bigcap \kappa_m(L)$. Let

$$D = \left\{ \bigvee X : X \subseteq A \text{ and } card(X) < m \right\}.$$

We have D which is an m-directed set and $\bigvee A = \bigvee D$. Suppose $a \in {\downarrow}(\bigvee A) \bigcap \kappa_m(L)$. We have a which is m-compact and $a \leq \bigvee A = \bigvee D$. Then there is some $\bigvee X_0 \in D$, where $X_0 \subseteq A$ with $card(X_0) < m$, such that $a \leq \bigvee X_0$. Thus $a \in {\downarrow}(\bigvee X_0) \bigcap \kappa_m(L) = X_0'' \subseteq A$. That is to say, ${\downarrow}(\bigvee A) \bigcap \kappa_m(L) \subseteq A$, which completes the proof. □

The following definition carries the notion of an m-approximable concept from attributes to objects.

Definition 4.4. *Let $K = (G, M, I)$ be a context. A subset $A \subseteq G$ is called an m-approximable object concept if for any $X \subseteq A$ with $card(X) < m$, we have $X'' \subseteq A$. Let $\mathcal{O}(K)$ denote the set of all m-approximable object concepts of K, together with the inclusion order.*

Lemma 4.5. *Let $K = (G, M, I)$ be a context and $X \subseteq G$. Then*

(1) *the poset $\mathcal{O}(K)$ is a complete lattice;*

(2) *X'' is an m-approximable object concept;*

(3) *if $card(X) < m$, then X'' is an m-compact element in the set $\mathcal{O}(K)$ under inclusion;*

(4) *if X is an m-compact element in $\mathcal{O}(K)$, then there is $X_0 \subseteq X$ with $card(X_0) < m$, such that $X = X_0''$.*

Proof. Similar to that of Lemma 4.2. □

Theorem 4.6. *The poset $\mathcal{O}(K)$ of m-approximable object concepts of a context K is an m-algebraic lattice. Conversely, every m-algebraic lattice L is order-isomorphic to an m-approximable concept lattice $\mathcal{O}(K_L)$ for some context K_L.*

Proof. By a process analogous to that employed in the proof of Theorem 4.3. Notice that for the converse part, K_L is defined to be $(\kappa_m(L), L, \leq)$. □

For any set A, let $\mathcal{P}_m(A)$ denote the set of all subsets of A whose cardinality is less than m.

Definition 4.7. *Given the contexts $K_1 = (G_1, M_1, I_1)$ and $K_2 = (G_2, M_2, I_2)$, an m-approximable morphism $\rightsquigarrow$ from K_1 to K_2 is a subset of $\mathcal{P}_m(M_1) \times \mathcal{P}_m(M_2)$ such that for all $X, X_1, X_2 \in \mathcal{P}_m(M_1)$ and $Y_1, Y_2, Y_j \in \mathcal{P}_m(M_2)$, where $j \in J$ and $card(J) < m$, the following axioms are satisfied:*

(a1) $\emptyset \rightsquigarrow \emptyset$;

(a2) $X \rightsquigarrow Y_j$ for all $j \in J$ implies $X \rightsquigarrow \bigcup_{j \in J} Y_j$;

(a3) $X_1 \subseteq X_2''$, $X_1 \rightsquigarrow Y_1$ and $Y_2 \subseteq Y_1''$ imply $X_2 \rightsquigarrow Y_2$.

Proposition 4.8. *Let $\rightsquigarrow$ be an m-approximable morphism from $K_1 = (G_1, M_1, I_1)$ to $K_2 = (G_2, M_2, I_2)$, $X, X_1, X_2, X_j \in \mathcal{P}_m(M_1)$ and $Y, Y_1, Y_2, Y_j \in \mathcal{P}_m(M_2)$, for all $j \in J$ and $card(J) < m$. We have the following conclusions.*

(1) *If $X_1 \subseteq X_2$ and $X_1 \rightsquigarrow Y$, then $X_2 \rightsquigarrow Y$.*

(2) *If $X \rightsquigarrow Y_1$ and $Y_2 \subseteq Y_1$, then $X \rightsquigarrow Y_2$.*

(3) *If $X_j \rightsquigarrow Y_j$ for all $j \in J$, then $\bigcup_{j \in J} X_j \rightsquigarrow \bigcup_{j \in J} Y_j$.*

Proof. (1) Since $X_2 \subseteq X_2''$, we have $X_1 \subseteq X_2''$. Moreover, $Y \subseteq Y''$. Then, by Axiom (a3), we have $X_2 \rightsquigarrow Y$.

(2) By a method similar to the one used in (1).

(3) By (1), we have $\bigcup_{j\in J} X_j \rightsquigarrow Y_{j_0}$ for all $j_0 \in J$. Then, by Axiom (a2), we have $\bigcup_{j\in J} X_j \rightsquigarrow \bigcup_{j\in J} Y_j$. □

We write $\mathbf{CXT}_m$ for the collection of all contexts together with m-approximable morphisms.

Theorem 4.9. *The collection* $\mathbf{CXT}_m$ *is a category.*

Proof. The composition of m-approximable morphisms is taken to be the composition of the relations. We first show that the composition of m-approximable morphisms is still an m-approximable morphism. Let $K_1 = (G_1, M_1, I_1)$, $K_2 = (G_2, M_2, I_2)$, and $K_3 = (G_3, M_3, I_3)$ be contexts, and let $\rightsquigarrow_1 : K_1 \to K_2$ and $\rightsquigarrow_2 : K_2 \to K_3$ be m-approximable morphisms. We shall show that $\rightsquigarrow_2 \circ \rightsquigarrow_1$ is an m-approximable morphism from K_1 to K_3:

(1) It is clear that $\emptyset(\rightsquigarrow_2 \circ \rightsquigarrow_1)\emptyset$.

(2) Suppose that $X \in \mathcal{P}_m(M_1)$, $Z_j \in \mathcal{P}_m(M_3)$ and $X(\rightsquigarrow_2 \circ \rightsquigarrow_1)Z_j$ for all $j \in J$, where $card(J) < m$. Then for each $j \in J$, there exists $Y_j \in \mathcal{P}_m(M_2)$ such that $X \rightsquigarrow_1 Y_j$ and $Y_j \rightsquigarrow_2 Z_j$. By Axiom (a2), we have $X \rightsquigarrow_1 \bigcup_{j\in J} Y_j$. And by Proposition 4.8(3), we have $\bigcup_{j\in J} Y_j \rightsquigarrow_2 \bigcup_{j\in J} Z_j$. Hence $X(\rightsquigarrow_2 \circ \rightsquigarrow_1)\bigcup_{j\in J} Z_j$.

(3) Let $X_1, X_2 \in \mathcal{P}_m(M_1)$, $Z_1, Z_2 \in \mathcal{P}_m(M_3)$ such that $X_1 \subseteq X_2''$, $X_1(\rightsquigarrow_2 \circ \rightsquigarrow_1)Z_1$ and $Z_2 \subseteq Z_1''$. Then there is $Y \in \mathcal{P}_m(M_2)$ such that $X_1 \rightsquigarrow_1 Y$ and $Y \rightsquigarrow_2 Z_1$. Notice that $Y \subseteq Y''$. By Axiom (a3), we have $X_2 \rightsquigarrow_1 Y$ and $Y \rightsquigarrow_2 Z_2$. Thus $X_2(\rightsquigarrow_2 \circ \rightsquigarrow_1)Z_2$.

We next show that for each context K_1, there is an identity m-approximable morphism. Let $\rightsquigarrow_{id} : K_1 \to K_1$ be defined by

$$X_1 \rightsquigarrow_{id} X_2 \text{ iff } X_2 \subseteq X_1''.$$

It is easy to check that $\rightsquigarrow_{id}$ is an m-approximable morphism. Moreover, for every $X \in \mathcal{P}_m(M_1)$, we have $X \rightsquigarrow_{id} X$. Then, for any m-approximable morphism $\rightsquigarrow : K_1 \to K_2$, $X \rightsquigarrow Y$ implies $X(\rightsquigarrow \circ \rightsquigarrow_{id})\,Y$. Conversely, if $X(\rightsquigarrow \circ \rightsquigarrow_{id})Y$, then there is $X_0 \in \mathcal{P}_m(M_1)$ such that $X_0 \subseteq X''$ and $X_0 \rightsquigarrow Y$, and hence $X \rightsquigarrow Y$ by Axiom (a3). Thus $\rightsquigarrow \circ \rightsquigarrow_{id} = \rightsquigarrow$. The left unit law is proved similarly.

The associativity of m-approximable morphisms follows from that of composition of relations. Therefore, $\mathbf{CXT}_m$ is a category. □

Lemma 4.10. *Let $f, g : L \to M$ be m-continuous functions between m-algebraic lattices. Then $f = g$ iff $f(c) = g(c)$ for all $c \in \kappa_m(L)$.*

Proof. Suppose $f(c) = g(c)$ for all $c \in \kappa_m(L)$. Then for any $x \in L$,

$$\begin{aligned} f(x) &= f\left(\bigvee\left(\downarrow x \bigcap \kappa_m(L)\right)\right) \\ &= \bigvee f\left(\downarrow x \bigcap \kappa_m(L)\right) \\ &= \bigvee g\left(\downarrow x \bigcap \kappa_m(L)\right) \\ &= g\left(\bigvee\left(\downarrow x \bigcap \kappa_m(L)\right)\right) \\ &= g(x), \end{aligned}$$

and thus $f = g$. The converse is clear true. □

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Theorem 4.11. *The category* $\mathbf{CXT}_m$ *is equivalent to* $\mathbf{LAT}_m$.

Proof. We first construct a functor $\mathcal{F}$ from $\mathbf{LAT}_m$ to $\mathbf{CXT}_m$. For every m-algebraic lattice L, define $\mathcal{F}(L) = (L, \kappa_m(L), \geq)$. And for each m-continuous function $f : L \to M$ between m-algebraic lattices, let $\mathcal{F}(f) = \rightsquigarrow_f : (L, \kappa_m(L), \geq) \to (M, \kappa_m(M), \geq)$ be defined by

$$X \rightsquigarrow_f Y \text{ iff } Y \subseteq \downarrow f\left(\bigvee X\right) \bigcap \kappa_m(M).$$

It is trivial to check that $\rightsquigarrow_f$ satisfies Axioms (a1) and (a2). Assume that $X_1 \subseteq X_2''$, $X_1 \rightsquigarrow_f Y_1$, and $Y_2 \subseteq Y_1''$, where $X_1, X_2 \in \mathcal{P}_m(\kappa_m(L))$ and $Y_1, Y_2 \in \mathcal{P}_m(\kappa_m(M))$. Then

$$\begin{aligned} Y_2 &\subseteq \downarrow\left(\bigvee Y_1\right) \bigcap \kappa_m(M) \ \left(\text{because } Y_1'' = \downarrow\left(\bigvee Y_1\right) \bigcap \kappa_m(M)\right) \\ &\subseteq \downarrow f\left(\bigvee X_1\right) \bigcap \kappa_m(M) \ \left(\text{because } X_1 \rightsquigarrow_f Y_1 \Rightarrow Y_1 \subseteq \downarrow f\left(\bigvee X_1\right) \bigcap \kappa_m(M)\right) \\ &\subseteq \downarrow f\left(\bigvee X_2\right) \bigcap \kappa_m(M) \ \left(\text{because } X_1 \subseteq X_2'' \Rightarrow \bigvee X_1 \leq \bigvee X_2\right), \end{aligned}$$

and hence $X_2 \rightsquigarrow_f Y_2$. Thus $\rightsquigarrow_f$ is an m-approximable morphism. Let 1_L be the identity function on L. We prove $\mathcal{F}(1_L) = \rightsquigarrow_{1_L}$ is exactly the identity morphism $\rightsquigarrow_{id}$ on $(L, \kappa_m(L), \geq)$:

$$\begin{aligned} X \rightsquigarrow_{1_L} Y &\Leftrightarrow Y \subseteq \downarrow 1_L\left(\bigvee X\right) \bigcap \kappa_m(L) = \downarrow\left(\bigvee X\right) \bigcap \kappa_m(L) = X'' \\ &\Leftrightarrow X \rightsquigarrow_{id} Y. \end{aligned}$$

Suppose $g : M \to N$ also is an m-continuous function between m-algebraic lattices. Then $\mathcal{F}$ is a functor once we prove $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$. Assume that $X \rightsquigarrow_{g \circ f} Z$. Then $Z \subseteq \downarrow g \circ f(\bigvee X) \bigcap \kappa_m(M)$. Since $\downarrow g \circ f(\bigvee X) \bigcap \kappa_m(M)$ is m-directed, there is $z \in \downarrow g \circ f(\bigvee X) \bigcap \kappa_m(M)$ such that $\bigvee Z \leq z$. Moreover, $z \leq g(f(\bigvee X)) = \bigvee g(\downarrow f(\bigvee X) \bigcap \kappa_m)$. Notice that $g(\downarrow f(\bigvee X) \bigcap \kappa_m)$ is m-directed because g is monotone. Then there is $y \in \downarrow f(\bigvee X) \bigcap \kappa_m$ such that $z \leq g(y)$. Thus $X \rightsquigarrow_f \{y\}$ and $\{y\} \rightsquigarrow_g Z$, and hence $X(\rightsquigarrow_g \circ \rightsquigarrow_f)Z$. Conversely,

$$\begin{aligned} X(\rightsquigarrow_g \circ \rightsquigarrow_f)Z &\Rightarrow X \rightsquigarrow_f Y \text{ and } Y \rightsquigarrow_g Z \text{ for some } Y \in \mathcal{P}_m(\kappa_m(M)) \\ &\Rightarrow Y \subseteq \downarrow f\left(\bigvee X\right) \bigcap \kappa_m(M) \text{ and } Z \subseteq \downarrow g\left(\bigvee Y\right) \bigcap \kappa_m(M) \\ &\Rightarrow Z \subseteq \downarrow g(f\left(\bigvee X\right)) \bigcap \kappa_m(M) \\ &\Rightarrow X \rightsquigarrow_{g \circ f} Z. \end{aligned}$$

Then $\rightsquigarrow_{g \circ f} = \rightsquigarrow_g \circ \rightsquigarrow_f$, and hence $\mathcal{F}$ is a functor. The following three parts together imply that $\mathbf{CXT}_m$ is equivalent to $\mathbf{LAT}_m$:

(1) The functor $\mathcal{F}$ is faithful. Suppose $f, g \in [L \to M]$ with $f \neq g$. By Lemma 4.10, there is $c \in \kappa_m(L)$ such that $f(c) \neq g(c)$. Then $\downarrow f(c) \bigcap \kappa_m(M) \neq \downarrow g(c) \bigcap \kappa_m(M)$. Without loss of generality, there exists $d \in \downarrow f(c) \bigcap \kappa_m(M)$ but $d \notin \downarrow g(c) \bigcap \kappa_m(M)$. Then

$$\{c\} \rightsquigarrow_f \{d\} \text{ but } \{c\} \not\rightsquigarrow_g \{d\},$$

we conclude that $\rightsquigarrow_f \neq \rightsquigarrow_g$.

(2) The functor $\mathcal{F}$ is full. Let $\rightsquigarrow$ be an m-approximable morphism from $(L, \kappa_m(L), \geq)$ to $(M, \kappa_m(M), \geq)$. Define $f_\rightsquigarrow : L \to M$ by

$$f_\rightsquigarrow(x) = \bigvee \bigcup \{Y \in \mathcal{P}_m(\kappa_m(M)) : \exists X \in \mathcal{P}_m(\kappa_m(L)) \text{ such that } X \subseteq \downarrow x \text{ and } X \rightsquigarrow Y\}.$$

We check first that $f_\rightsquigarrow$ is m-continuous. Suppose that $D \subseteq L$ is m-directed. For any $X \in \mathcal{P}_m(\kappa_m(L))$ with $\bigvee X \leq \bigvee D$, we have $\bigvee X \leq d$ for some $d \in D$ by the m-compactness and m-directedness.

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Then

$$f_{\rightsquigarrow}\left(\bigvee D\right) = \bigvee\bigcup\left\{Y \in \mathcal{P}_m(M) : \exists X \in \mathcal{P}_m(L) \text{ such that } X \subseteq \downarrow\left(\bigvee D\right) \text{ and } X \rightsquigarrow Y\right\}$$
$$= \bigvee\bigcup\{Y \in \mathcal{P}_m(M) : \exists X \in \mathcal{P}_m(L) \text{ such that } X \subseteq \downarrow d \text{ for some } d \in D \text{ and } X \rightsquigarrow Y\}$$
$$= \bigvee f_{\rightsquigarrow}(D),$$

which implies that $f_{\rightsquigarrow}$ is m-continuous. We next show that $\rightsquigarrow$ is the image $\mathcal{F}(f_{\rightsquigarrow}) = \rightsquigarrow_{f_{\rightsquigarrow}}$. Let $X \in \mathcal{P}_m(\kappa_m(L))$ and $Y \in \mathcal{P}_m(\kappa_m(M))$. Then $X \rightsquigarrow_{f_{\rightsquigarrow}} Y$ iff

$$Y \subseteq \downarrow f_{\rightsquigarrow}\left(\bigvee X\right) \bigcap \kappa_m(M)$$
$$= \left(\downarrow\bigvee\bigcup\left\{Y_0 \in \mathcal{P}_m(M) : \exists X_0 \in \mathcal{P}_m(M) \text{ such that } X_0 \subseteq \downarrow\bigvee X \text{ and } X_0 \rightsquigarrow Y_0\right\}\right) \bigcap \kappa_m(M).$$

The set $\{Y_0 \in \mathcal{P}_m(M) : \exists X_0 \in \mathcal{P}_m(L) \text{ such that } X_0 \subseteq \downarrow\bigvee X \text{ and } X_0 \rightsquigarrow Y_0\}$ is m-directed by Proposition 4.8(3). Thus

$$X \rightsquigarrow_{f_{\rightsquigarrow}} Y$$
$$\Leftrightarrow \exists Y_0 \in \mathcal{P}_m(M), \exists X_0 \in \mathcal{P}_m(L) \text{ such that } Y \subseteq \downarrow\bigvee Y_0 \bigcap \kappa_m(M) = Y_0'', X_0 \subseteq X'' \text{ and } X_0 \rightsquigarrow Y_0$$
$$\Rightarrow X \rightsquigarrow Y \text{ (by Axiom (a3))}.$$

Conversely, if $X \rightsquigarrow Y$, then $X \rightsquigarrow_{f_{\rightsquigarrow}} Y$ by setting $X_0 = X$ and $Y_0 = Y$. Thus $\rightsquigarrow = \rightsquigarrow_{f_{\rightsquigarrow}}$, and hence $\mathcal{F}$ is full.

(3) For each context $K = (G, M, I)$, we shall show $K_0 = \mathcal{F}(\mathcal{A}(K)) = (\mathcal{A}(K), \kappa_m(\mathcal{A}(K)), \supseteq)$ is isomorphic to K in $\mathbf{CXT}_m$. Define $\rightsquigarrow_1 \subseteq \mathcal{P}_m(M) \times \mathcal{P}_m(\kappa_m(\mathcal{A}(K)))$ by

$$X \rightsquigarrow_1 \mathcal{X} \text{ iff } \bigvee \mathcal{X} \subseteq X''.$$

It is obvious that $\rightsquigarrow_1$ satisfies Axioms (a1) and (a2). Suppose $X_1, X_2 \in \mathcal{P}_m(M)$ and $\mathcal{X}_1, \mathcal{X}_2 \in \mathcal{P}_m(\kappa_m(\mathcal{A}(K)))$ such that $X_1 \subseteq X_2''$, $X_1 \rightsquigarrow_1 \mathcal{X}_1$ and $\mathcal{X}_2 \subseteq \mathcal{X}_1''$. By Lemma 4.2(4), we have $\mathcal{X}_1$ is of the form $\{A_j'' : j \in J\}$, where $card(J) < m$ and $card(A_j) < m$ for all $j \in J$. Then

$$\mathcal{X}_2 \subseteq \mathcal{X}_1''$$
$$= \left(\downarrow\bigvee \mathcal{X}_1\right) \bigcap \kappa_m(\mathcal{A}(K))$$
$$= \left(\downarrow\left(\bigcup\{A_j : j \in J\}\right)''\right) \bigcap \kappa_m(\mathcal{A}(K))$$

implies $B \subseteq (\bigcup\{A_j : j \in J\})''$ for all $B \in \mathcal{X}_2$. And

$$X_1 \rightsquigarrow_1 \mathcal{X}_1 \Rightarrow A_j'' \subseteq X_1'' \text{ for all } j \in J$$
$$\Rightarrow \left(\bigcup\{A_j : j \in J\}\right)'' \subseteq X_1'' \subseteq X_2''.$$

Then $B \subseteq X_2''$ for all $B \in \mathcal{X}_2$, i.e., $X_2 \rightsquigarrow_1 \mathcal{X}_2$. Thus $\rightsquigarrow_1$ is an m-approximable morphism from K to K_0. Define $\rightsquigarrow_2 \subseteq \mathcal{P}_m(\kappa_m(\mathcal{A}(K))) \times \mathcal{P}_m(M)$ by

$$\mathcal{X} \rightsquigarrow_2 X \text{ iff } X \subseteq \bigvee \mathcal{X}.$$

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Again, it is trivial to check that $\rightsquigarrow_2$ satisfies Axioms (a1) and (a2). With the notions defined above, suppose $\mathcal{X}_1 \subseteq \mathcal{X}_2''$, $\mathcal{X}_1 \rightsquigarrow_2 X_1$, and $X_2 \subseteq X_1''$. Then

$$\begin{aligned} X_2 &\subseteq X_1'' \\ &\subseteq \bigvee \mathcal{X}_1 \left(\text{because } X_1 \subseteq \bigvee \mathcal{X}_1 \text{ and } \bigvee \mathcal{X}_1 \text{ is an } m\text{-approximable concept}\right) \\ &\subseteq \bigvee \mathcal{X}_2 \left(\text{because } \bigvee \mathcal{X}_1 \subseteq \bigvee \mathcal{X}_2'' = \left(\downarrow \bigvee \mathcal{X}_2\right) \bigcap \kappa_m(\mathcal{A}(K))\right), \end{aligned}$$

and thus $\mathcal{X}_2 \rightsquigarrow_2 X_2$. Hence $\rightsquigarrow_2$ is an m-approximable morphism from K_0 to K. We claim that $\rightsquigarrow_2 \circ \rightsquigarrow_1$ is the identity morphism $\rightsquigarrow_{id(K)}$ on K. Indeed,

$$\begin{aligned} X_1 \rightsquigarrow_{id(K)} X_2 &\Rightarrow X_2 \subseteq X_1'' \\ &\Rightarrow \{X_1''\} \rightsquigarrow_2 X_2 \\ &\Rightarrow X_1(\rightsquigarrow_2 \circ \rightsquigarrow_1)X_2 \text{ (because } X_1 \rightsquigarrow_1 \{X_1\}), \end{aligned}$$

and

$$\begin{aligned} X_1(\rightsquigarrow_2 \circ \rightsquigarrow_1)X_2 &\Rightarrow X_1 \rightsquigarrow_1 \mathcal{X} \text{ and } \mathcal{X} \rightsquigarrow_2 X_2 \text{ for some } \mathcal{X} \in \mathcal{P}_m(\kappa_m(\mathcal{A}(K))) \\ &\Rightarrow \bigvee \mathcal{X} \subseteq X_1'' \text{ and } X_2 \subseteq \bigvee \mathcal{X} \\ &\Rightarrow X_2 \subseteq X_1'' \\ &\Rightarrow X_1 \rightsquigarrow_{id(K)} X_2. \end{aligned}$$

It remains to show that $\rightsquigarrow_1 \circ \rightsquigarrow_2$ is the identity morphism $\rightsquigarrow_{id(K_0)}$ on K_0:

$$\begin{aligned} \mathcal{X}_1 \rightsquigarrow_{id(K_0)} \mathcal{X}_2 &\Rightarrow \mathcal{X}_2 \subseteq \mathcal{X}_1'' = \left(\downarrow \left(\bigcup_{j\in J} A_j\right)''\right) \bigcap \kappa_m(\mathcal{A}(K)) \text{ (where } A_j \text{ are those defined above)} \\ &\Rightarrow \mathcal{X}_1 \rightsquigarrow_2 \bigcup_{j\in J} A_j \text{ and } \bigcup_{j\in J} A_j \rightsquigarrow_1 \mathcal{X}_2 \\ &\Rightarrow \mathcal{X}_1(\rightsquigarrow_1 \circ \rightsquigarrow_2)\mathcal{X}_2, \end{aligned}$$

and conversely,

$$\begin{aligned} \mathcal{X}_1(\rightsquigarrow_1 \circ \rightsquigarrow_2)\mathcal{X}_2 &\Rightarrow \mathcal{X}_1 \rightsquigarrow_2 X \text{ and } X \rightsquigarrow_1 \mathcal{X}_2 \text{ (for some } X \in \mathcal{P}_m(M)) \\ &\Rightarrow X'' \subseteq \bigvee \mathcal{X}_1 \text{ and } \bigvee \mathcal{X}_2 \subseteq X'' \\ &\Rightarrow \bigvee \mathcal{X}_2 \subseteq \bigvee \mathcal{X}_1 \\ &\Rightarrow \mathcal{X}_2 \subseteq \left(\downarrow \bigvee \mathcal{X}_1\right) \bigcap \kappa_m(\mathcal{A}(K)) = \mathcal{X}_1'' \\ &\Rightarrow \mathcal{X}_1 \rightsquigarrow_{id(K_0)} \mathcal{X}_2. \end{aligned}$$

Thus K is isomorphic to K_0, which completes the proof. □

Since an equivalence of categories preserves cartesian closedness, we obtain the following corollary:

Corollary 4.12. If m is a regular cardinal with $m > 2$, then the category $\mathbf{CXT}_m$ is cartesian closed.

Proof. By Theorems 3.9 and 4.11. □

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5. Special Cases of *m*-Algebraic Lattices and *m*-Approximable Concepts

5.1 The case of $m = 2$

The cardinal 2 is regular. Let L be a complete lattice. It is easy to check that any nonempty subset D of L is 2-directed. Then we have the following:

Proposition 5.1. (1) *An element x of a complete lattice L is 2-compact iff for any nonempty subset $A \subseteq L$, $x \leq \bigvee A$ implies that there is $a \in A$ such that $x \leq a$.*

(2) *A function $f : L \to M$ between complete lattices is 2-continuous iff f preserves arbitrary joins, i.e., $f(\bigvee A) = \bigvee f(A)$ for all $A \subseteq L$.*

Definition 5.2. *(1) An element x of a complete lattice L is called a complete prime* (Nielsen et al. 1981) *(also called a completely join-irreducible element* Raney 1952, *a super-compact element* Zhao and Zhou (2006), *etc.) if for any $A \subseteq L$, $x \leq \bigvee A$ implies that there is $a \in A$ such that $x \leq a$.*

(2) A complete lattice L is prime algebraic (Nielsen et al. 1981) *(or superalgebraic lattice* Zhao and Zhou 2006, *completely algebraic lattice* Liu et al. 2011, *etc.) if each element of L is the join of complete primes.*

The notion of a 2-compact and that of a complete prime differ only on the least element $\bot$: $\bot$ is a 2-compact element, but not a complete prime (because $\bot \leq \bigvee \emptyset$). However, a 2-algebraic lattice is exactly a prime algebraic one since $\bot = \bigvee\{\bot\} = \bigvee \emptyset$.

For a context $K = (G, M, I)$, a subset $A \subseteq M$ is an 2-*approximable concept* iff $\emptyset'' \subseteq A$ and $\{a\}'' \subseteq A$ for all $a \in A$. The Axiom (a2) plays no role in the definition of a 2-*approximable morphism* between contexts. By Theorem 4.3, we have the following.

Theorem 5.3. *For any context $K = (G, M, I)$, the poset $\mathcal{A}(K)$ of 2-approximable concepts is a prime algebraic lattice. Conversely, for any prime algebraic lattice L, there is a context K_L whose 2-approximable concept lattice $\mathcal{A}(K_L)$ is order-isomorphic to L.*

Liu et al. (2011) introduced a type of concepts as follows:

Definition 5.4. (Liu et al. 2011) *Let $K = (G, M, I)$ be a concept. A subset $A \subseteq M$ is called a weak approximable attribute concept if $\{a\}'' \subseteq A$ for all $a \in A$.*

The notion of a weak approximable attribute concept is slightly different from that of a 2-approximable concept: $\emptyset$ is always a weak approximable attribute concept; however, it is a 2-approximable concept only in the case that $\emptyset'' = \emptyset$. Liu et al. also obtained the above representation by means of weak approximable attribute concepts (see Theorems 6 and 7 in Liu et al. 2011). By Theorem 4.11, a categorical equivalence is obtained:

Theorem 5.5. *The category* $\mathbf{LAT}_2$ *of prime algebraic lattices and 2-continuous functions is equivalent to the category* $\mathbf{CXT}_2$ *of contexts and 2-approximable morphisms.*

5.2 The case of $m = \aleph_0$

The smallest infinite cardinal $\aleph_0$ is regular. A subset D of a complete lattice L is $\aleph_0$-*directed* iff every finite subset of D has an upper bound in D iff D is directed. An $\aleph_0$-*algebraic lattice* is exactly an algebraic one. A function $f : L \to M$ between complete lattice is $\aleph_0$-*continuous* iff $f(\bigvee D) = \bigvee f(D)$ for all directed $D \subseteq L$ iff f is Scott continuous (see Gierz et al. 2003).

The notion of an *approximable concept*, introduced by Zhang and Shen (2006), coincides with that of an $\aleph_0$-approximable concept, and a *context morphism*, proposed by Hitzler and Zhang (2004), is precisely an $\aleph_0$-approximable morphism:

Definition 5.6. (Zhang and Shen 2006; Hitzler and Zhang 2004) *(1) Let $K = (G, M, I))$ be a context. A subset $A \subseteq M$ is an approximable concept iff $F'' \subseteq A$ for any finite subset F of A.*

(2) A context morphism $\rightsquigarrow: (G_1, M_1, I_1) \to (G_2, M_2, I_2)$ between contexts is a subset of $Fin(M_1) \times Fin(M_2)$, where $Fin(M_1)$ and $Fin(M_2)$ are all finite subsets of M_1 and M_2, respectively, such that for all $X, X_1, X_2 \in Fin(M_1)$ and $Y_1, Y_2 \in Fin(M_2)$:

(i) $\emptyset \rightsquigarrow \emptyset$;

(ii) $X \rightsquigarrow Y_1$ and $X \rightsquigarrow Y_2$ imply $X \rightsquigarrow Y_1 \bigcup Y_2$;

(ii) $X_1 \subseteq X_2''$, $X_1 \rightsquigarrow Y_1$ and $Y_2 \subseteq Y_1''$ imply $X_2 \rightsquigarrow Y_2$.

The following "$\aleph_0$-versions" of Theorems 4.3 and 4.11 have also been obtained in Zhang and Shen (2006); Hitzler and Zhang (2004):

Theorem 5.7. (1) *For any context $K = (G, M, I)$, the set $\mathcal{A}(K)$ of approximable concepts ordered by inclusion is an algebraic lattice. Conversely, for any algebraic lattice L, there is a context K_L whose approximable concept lattice $\mathcal{A}(K_L)$ is order-isomorphic to L.*

(2) *The category* $\mathbf{CXT}_{\aleph_0}$ *of contexts and context morphisms is equivalent to the category* $\mathbf{LAT}_{\aleph_0}$ *of algebraic lattices and Scott continuous functions.*

5.3 The case of $m = \aleph_1$

The cardinal $\aleph_1$ is the smallest one, which is bigger than $\aleph_0$. A set whose cardinality is less than $\aleph_1$ is a countable set. The cardinal $\aleph_1$ is regular because unions of countably many countable sets are still countable. Lee (1988) investigated $\aleph_1$-algebraic lattices under the name *countably algebraic lattices.*

Definition 5.8. (Lee 1988) *Let L be a complete lattice.*

(1) A subset $D \subseteq L$ is countably directed if every countable subset of D has an upper bound in D.

(2) An element $x \in L$ is said to be a Lindelöf element if for any countably directed $D \subseteq L$ with $x \leq \bigvee D$, there is $d \in D$ such that $x \leq d$.

(3) The lattice L is called countably algebraic if every $x \in L$ is a join of Lindelöf elements.

By Theorem 4.3, we have a representation theorem for countably algebraic lattices:

Theorem 5.9. *For any countably algebraic lattice L, there is a context K_L whose $\aleph_1$-approximable concept lattice $\mathcal{A}(K_L)$ is order-isomorphic to L. Conversely, for any context $K = (G, M, I)$, the poset $\mathcal{A}(K)$ of $\aleph_1$-approximable concepts is a countably algebraic lattice.*

A function $f : L \to M$ between complete lattices is $\aleph_1$-continuous if f preserves countably directed joins, i.e., $f(\bigvee D) = \bigvee f(D)$ for all countably directed subsets D of L. By Theorem 3.9, we have the following:

Theorem 5.10. *The category* $\mathbf{LAT}_{\aleph_1}$ *of countably algebraic lattices and functions which preserves countably directed joins is cartesian closed.*

Concerning the execution of a nondeterministic or parallel program, one natural assumption is that of fairness, which states that no process is forever denied its turn for execution (see, e.g., Apt and Olderog 1983; Park 1980). Apt and Plotkin (1986) came up with four semantics for a programming language admitting unbounded (but countable) nondeterminism, providing frameworks for investigating fairness via translation into a language for countable nondeterminism. Involving countable nondeterminism leads to a lack of continuity (preserving directed sups) of various semantic functions. The notion of a countably directed set is also called an ω-directed one, and that of $\aleph_1$-continuity (which is weaker than the notion of continuity) is called ω_1-continuity in

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Apt and Plotkin (1986). By the categorical equivalence and the representation theorem provided in this paper, $\aleph_1$-approximable concepts and $\aleph_1$-approximable morphisms might be considerable theoretical interest in programming semantics which resort to the use of countable ordinals.

5.4 The case of $m = \infty$

We set ∞ to be the limit of all cardinals, i.e., every cardinal is less than ∞. A subset D of a complete lattice L is an ∞-*directed set* iff $\bigvee D \in D$. Then every element of L is ∞-*compact*, and hence every ∞-*algebraic lattice* is just a complete lattice.

For a context $K = (G, M, I)$, a subset $A \subseteq M$ is an ∞-*approximable concept* iff $X'' \subseteq A$ for every $X \subseteq A$ iff $A'' = A$ iff (A', A) is a concept (in which case, A is called the intent of (A', A)). Similarly, a subset $B \subseteq G$ is an ∞-*approximable object concept* iff (B, B') is a concept iff B is the *extent* of (B, B'). Then Theorem 4.6 can be viewed as a generalization of the following well-known result in FCA:

Theorem 5.11. (Davey and Priestley 2002; Ganter and Wille 1999) *The poset $\mathcal{O}(K)$ of extents of a context K is a complete lattice. Conversely, every complete lattice L is order-isomorphic to $\mathcal{O}(K_L)$ for some context K_L.*

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