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PROBABILISTIC CELLULAR AUTOMATA WITH GENERAL ALPHABETS POSSESSING A MARKOV CHAIN AS AN INVARIANT DISTRIBUTION

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Abstract

This paper is devoted to probabilistic cellular automata (PCAs) on $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$, depending on two neighbors with a general alphabet E (finite or infinite, discrete or not). We study the following question: under which conditions does a PCA possess a Markov chain as an invariant distribution? Previous results in the literature give some conditions on the transition matrix (for positive rate PCAs) when the alphabet E is finite. Here we obtain conditions on the transition kernel of a PCA with a general alphabet E. In particular, we show that the existence of an invariant Markov chain is equivalent to the existence of a solution to a cubic integral equation. One of the difficulties in passing from a finite alphabet to a general alphabet comes from the problem of measurability, and a large part of this work is devoted to clarifying these issues.

Keywords: Probabilistic cellular automata; invariant measure; Markov chain

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1. Introduction

1.1. Cellular automata and probabilistic cellular automata with finite alphabet

Cellular automata (CAs), as described by Hedlund [11], are discrete local dynamical systems on a space $E^{\mathbb{L}}$, where $E = \{0, \dots, \kappa\}$ is a finite alphabet, the set of states of cells, and $\mathbb{L}$ is a discrete lattice. Formally, a cellular automaton A is a tuple $(\mathbb{L}, E, N, f)$, where

- $\mathbb{L}$ is a lattice, called a set of cells. In this paper, $\mathbb{L}$ is $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$.
- N is the neighborhood function for $i \in \mathbb{L}$, $N(i) = (i + l : l \in L)$, where $L \subset \mathbb{L}$ is finite. Each neighborhood has cardinality $|N| = |L|$. In this paper, $N(i) = (i, i + 1)$ when the lattice is $\mathbb{N}$ or $\mathbb{Z}$ and $N(i) = (i, i + 1 \pmod n)$ when the lattice is $\mathbb{Z}/n\mathbb{Z}$.
- f is the local rule. It is a function $f : E^{|N|} \to E$.

The CA $\boldsymbol{A} = (\mathbb{L}, E, N, f)$ defines a global function $F : E^{\mathbb{L}} \mapsto E^{\mathbb{L}}$ on the set of configurations $E^{\mathbb{L}}$. For any configuration $S_0 = (S_0(i) : i \in \mathbb{L})$, the image $S_1 = F(S_0)$ of S_0 by F is defined by, for any $j \in \mathbb{L}$, $S_1(j) = f((S_0(i) : i \in N(j)))$.

In words, the state of all cells are updated simultaneously and the state $S_1(j)$ of the cell j at time 1 depends only on the states $(S_0(i) : i \in N(j))$ of its neighborhood at time 0. Hence, the dynamics are as follows. Starting from an initial configuration $S_{t_0} \in E^{\mathbb{L}}$ at time t_0, the successive states of the system are $(S_t : t \ge t_0)$, where $S_{t+1} = F(S_t)$. The sequence of

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configurations $S = (S_t = (S_t(i) : i \in \mathbb{L}),\ t \geq t_0)$ is called the space–time diagram of $\boldsymbol{A}$. The state $S_t(i)$ of the cell i at time t will be denoted by $S(i, t)$.

Probabilistic cellular automata (PCAs) with finite alphabets are generalizations of CAs in which the states $(S(i, t) : i \in \mathbb{L},\ t \geq t_0)$ are random variables defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$, each of the random variables $S(i, t)$ taking almost surely its value in E. Seen as a random process, S is equipped with the σ-field generated by the cylinders. The definition of a PCA relies on a transition matrix T indexed by $E^{|N|} \times E$ (instead of a local rule f), which gives the distributions of the state of a cell at time $t + 1$ conditionally on those of its neighborhood at time t:

$$\mathbb{P}(S(j, t+1) = b \mid (S(i, t) = a_i : i \in N(j))) = T((a_i : i \in N(j)); b).$$

Conditionally on S_t, the states $(S(j, t+1) : j \in \mathbb{L})$ are independent (see (1)). The transition matrix T is then an array of nonnegative numbers satisfying, for any $(a_1, \dots, a_{|N|}) \in E^{|N|}$, $\sum_{b \in E} T((a_1, \dots, a_{|N|}); b) = 1$. Formally, a PCA $\boldsymbol{A}$ with a finite alphabet E is an operator $\mathcal{F} : \mathcal{M}(E^{\mathbb{L}}) \mapsto \mathcal{M}(E^{\mathbb{L}})$ on the set of probability distributions $\mathcal{M}(E^{\mathbb{L}})$ on the set of configurations. If S_0 has distribution μ_0 then S_1 has distribution $\mu_1 = \mathcal{F}(\mu_0)$. We can also define μ_1 directly from μ_0 and T by giving its finite-dimensional distribution (Kolmogorov extension theorem) as follows. For any finite subset $C \subset \mathbb{L}$ and for any $(b_j : j \in C) \in E^C$,

$$\mu_1((b_j : j \in C)) = \sum_{(a_i)_{i \in N(C)} \in E^{N(C)}} \mu_0((a_i : i \in N(C))) \prod_{j \in C} T((a_i : i \in N(j)); b_j), \qquad (1)$$

where $N(C) = \bigcup_{j \in C} N(j)$. A measure $\mu \in \mathcal{M}(E^{\mathbb{L}})$ is said to be *invariant* by $\boldsymbol{A}$ if $\mathcal{F}(\mu) = \mu$.

The simplest case of a PCA is the two colors case $E = \{0, 1\}$ on $\mathbb{Z}$ with neighborhood $N(i) = (i, i+1)$. This case has been studied in depth and there are many results in the literature; see Toom [19]. For example, Belyaev [2] characterized the set of a PCA possessing as an invariant distribution a Markov chain indexed by $\mathbb{Z}$. Nevertheless, there are still interesting open problems concerning them. For instance, it remains an open question as to whether all positive rate PCAs (i.e. for any $a, b, c \in \{0, 1\}$, $T(a, b; c) > 0$) are ergodic or not.

So far, it has been observed in different frameworks that explicit calculus of the invariant distribution of PCAs can be done only if the transition matrix satisfies some algebraic equations (that form a manifold in terms of the $(T(a, b; c) : a, b, c \in E)$). In Belyaev [2] this was shown for PCAs with a two-letter alphabet whose invariant distributions are Markov chains or product measures. In [13] and [21], this was shown for quasi-reversible PCAs on $\mathbb{Z}^d$ with finite alphabet whose invariant distributions are Markov chains or Gibbs measures. In [17], this was done for PCAs on $\mathbb{Z}^d$ with a two-letter alphabet and whose invariant distributions are Gibbs measures. In [6], the same phenomenon was observed for PCAs on $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ with a finite alphabet possessing a Markov chain as an invariant distribution. Hence, the literature has been focused on the characterization of PCAs having simple invariant measures: product measures and Markov chains for $|N| = 2$, and Gibbs measures for PCAs on $\mathbb{Z}^d$. In addition to [2], the study of PCAs on $\mathbb{Z}$ admitting an invariant product measure was carried out by Mairesse and Marcovici [15] (in a finite alphabet case). For PCAs possessing a Markov chain as an invariant distribution, in addition to [2] and [6], Bousquet-Mélou [4] characterized those on $\mathbb{Z}/n\mathbb{Z}$ with a two-letter alphabet and Toom [19] gave a sufficient condition for PCAs on $\mathbb{Z}$ with a finite alphabet.

The most general results are given in [6] where it is proved (in Theorem 2.6) that a positive PCA on $\mathbb{Z}$ with two neighbors and a finite alphabet $E = \{0, \dots, \kappa\}$ admits a horizontal zigzag

Markov chain (see Definition 3) as an invariant distribution if and only if the following two conditions are satisfied:

(i) for any $a, b, c \in E$,

$$T(a, b; c)T(a, 0; 0)T(0, b; 0)T(0, 0; c) = T(a, b; 0)T(a, 0; c)T(0, b; c)T(0, 0; 0);$$

(ii) $D^\gamma U^\gamma = U^\gamma D^\gamma$, where

$$D^\gamma(a; c) = \sum_{k\in E} \frac{\gamma(k)}{T(a, k; 0)} T(a, k; c) \bigg\{ \sum_{k\in E} \frac{\gamma(k)}{T(a, k; 0)} \bigg\}^{-1}$$

and

$$U^\gamma(c; b) = \frac{\gamma(b)}{T(0, b; 0)} T(0, b; c) \bigg\{ \sum_{k\in E} \frac{\gamma(k)}{T(0, k; 0)} T(0, k; c) \bigg\}^{-1} \quad \text{for any } a, b, c \in E,$$

where γ is an eigenvector of an explicit matrix that depends only on T.

This theorem is an extension of [2, Theorem 3] valid only for a 2-letter alphabet.

1.2. PCAs with general alphabets

Inspired by this recent work, in this paper we investigate the case where the alphabet E is general (finite or infinite, discrete or not). As we have to define probability distributions on E, as usual in probability theory, we will assume that E is a Polish space (a separable complete metrizable space) equipped with its Borel set $\mathcal{B}(E)$. It could be finite or infinite and discrete or not. In the following, when we write 'general alphabet', we are thinking about a Polish space alphabet.

CAs and PCAs with infinite alphabets appear in the literature under different forms. In [5], CAs with alphabet $E = [0, 1]$ were used to solve the classification problem with arbitrary precision: the classification problem consists of finding a CA such that, on any initial configuration of 1s and 0s on the line $\mathbb{Z}$, the CA configuration converges to the line colored 1 if the initial fraction r of 1s is greater than $\frac{1}{2}$ and to the line colored 0 if $r < \frac{1}{2}$. CAs with alphabet $E = \mathbb{R}$ are applicable to modeling the heat equation [18]. Theorems about surjectivity of CAs have been extended to CAs whose alphabets are (possibly infinite) objects in some concrete category and then guarantee that some CAs with infinite alphabets have a Garden of Eden configuration (a configuration that does not have a predecessor) [7]. Recently, complex PCAs with infinite and continuous alphabets have been proposed in [20] in order to model urban dynamics. In Section 3.3 we will see that the synchronous totally asymmetric simple exclusion process (TASEP) on $\mathbb{R}$ defined by Blank [3] (it is a discrete-time, synchronous, space-continuous version of the TASEP studied by Derrida *et al.* [8]) could be modeled by a PCA on $\mathbb{Z}$ with alphabet $E = \mathbb{R}$ and neighborhood $N(i) = (i, i + 1)$. Hence, PCAs with general alphabets are already present in the literature even if they are not generally studied as such.

We believe that the present approach of a PCA with general alphabets permits the connection between different domains and points of view. The structure of the set of a PCA having the distribution of a Markov chain as an invariant measure is shown to be characterized by some algebraic-integral equations. These equations are reminiscent of the standard algebraic relations (in the parameters space) appearing in

- statistical physics concerning the notion of integrable systems,
- combinatorics where it is often the case that exact computations can be performed only for simple structures for which generating functions solve 'simple' functional equations [10],

- probability theory where invariant distribution of Markov chain on $\mathbb{Z}$ can be computed in some rare cases (conditioned random walks, birth and death processes), this being again related to some algebraic questions.

Here, Theorem 2 and Proposition 1 have exactly this flavor and this is a case where everything is quite transparent. If a Markov chain is conserved by a PCA (see (2)) then an infinite system of algebraic-integral equations having the form 'a product equals a sum of products' possesses a solution. Underlying this paper is the following question: 'which PCA possesses a Markov chain?' (or anything else one may prefer) must be seen as an algebraic question in the discrete case, and as an algebraic-integral question in the continuous one, solved here.

Theorem 2 and Proposition 1 provide the form of the solutions, those which explain such 'miraculous' simplification in the infinite system (2).

First, let us define formally a PCA with general alphabets. In this case, transition matrices are replaced by transition kernels. Let F and G be two Polish spaces, $K = (K(x; Y) : x \in F,\ Y \in \mathcal{B}(G))$ is a *transition kernel* from F to G: if, for all $Y \in \mathcal{B}(G)$, $x \mapsto K(x; Y)$ is $\mathcal{B}(F)$-measurable and if, for all $x \in F$, $Y \mapsto K(x; Y)$ is a probability measure on $(G, \mathcal{B}(G))$.

Definition 1. (*PCAs with a general alphabet.*) Let E be a Polish space, $\mathbb{L}$ a lattice, N a neighborhood function, and T a transition kernel from $E^{|N|}$ to E. A PCA A is a tuple $(\mathbb{L}, E, N, T)$ that defines an operator $\mathcal{F} : \mathcal{M}(E^{\mathbb{L}}) \mapsto \mathcal{M}(E^{\mathbb{L}})$, where, for any $\mu_0 \in \mathcal{M}(E^{\mathbb{L}})$, $\mu_1 = \mathcal{F}(\mu_0)$ is such that for any finite subset $C \subset \mathbb{L}$, for any $(B_j : j \in C) \in \mathcal{B}(E)^C$,

$$\mu_1((B_j : j \in C)) = \int_{E^{N(C)}} \left(\prod_{j \in C} T((a_i : i \in N(j)); B_j) \right) \mathrm{d}\mu_0((a_i : i \in N(C))).$$

As usual, the measure μ_1 is defined by its finite-dimensional distributions. If E is finite, this definition is similar to the classical definition of a PCA.

Example 1. (*Gaussian PCAs.*) For any $m, \sigma > 0$, we define a PCA $(\boldsymbol{G}_{m,\sigma})$ on $\mathbb{N}$ with alphabet $\mathbb{R}$ and neighborhood $N(i) = (i, i+1)$ as follows. The transition kernel of $\boldsymbol{G}_{m,\sigma}$ is as follows. For all $a, b \in \mathbb{R}$ and Borel set $C \in \mathcal{B}(\mathbb{R})$,

$$T(a, b; C) = \mathbb{P}\left(\mathcal{N}\left(\frac{a+b}{m}, \sigma^2 \right) \in C \right),$$

where $\mathcal{N}(c, \sigma^2)$ is a Gaussian random variable with mean c and variance σ^2. In Section 3.2.1 we prove that an invariant measure of this PCA is related to autoregressive processes of order 1.

The aim of this paper is to shed some light on the structure of the set of a PCA with a general alphabet (finite or infinite, discrete or not) having a Markovian invariant distribution on lattices $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$. In this case, some important complications arise (compared with the finite case).

The first problem is that in the case of a finite alphabet, it is known that each PCA admits at least one invariant probability distribution [19, p. 25]. This property fails when the alphabet size is infinite.

Example 2. Consider the following (infinite) transition matrix T defined, for any $a, b, c \in \mathbb{N}$, by

$$T(a, b; c) = \tfrac{1}{2}(\mathbf{1}_{\{\max(a,b)+1\}}(c) + \mathbf{1}_{\{a+b+1\}}(c)).$$

The PCA indexed by $\mathbb{N}$ having transition matrix T does not admit any invariant probability measure since, for any (t, i), $S(i, t+1) \geq S(i, t) + 1$ and so $S(i, t) \to \infty$ as $t \to \infty$.

The second problem is due to measurability issues. In continuous probability, two distributions having a density are equal if these densities differ at most on a Lebesgue negligible set. This fact holds in a more general context. If ν_1 and ν_2 are two probability measures absolutely continuous with respect to a σ-finite measure μ, then $\nu_1 = \nu_2$ if and only if their Radon–Nikodym derivatives with respect to μ are equal μ-almost everywhere (a.e.).

Now, assume that M and M' are two Markov kernels such that $M(x; \cdot) = M'(x; \cdot)$ except possibly for some x in a μ-negligible set. Under this condition, some distribution ρ may exist such that the two Markov chain with initial distribution ρ and respective Markov kernels M and M' do not coincide in distribution.

For a PCA with any general alphabet, the same complications arise: a unique PCA can have some 'plural behaviors'. Hence, in this paper, each time a PCA $\boldsymbol{A}$ is studied, a σ-finite measure μ is specified and, formally, it is on the pair $(\boldsymbol{A}, \mu)$ that the conditions and/or results hold.

Example 3. (*Gaussian PCAs except on the diagonal.*) Let $m, \sigma > 0$. We define the PCA $(\tilde{\boldsymbol{G}}_{m,\sigma})$ on $\mathbb{N}$ with alphabet $\mathbb{R}$. Its transition kernel $\tilde{T}$ is the same as that of $\boldsymbol{G}_{m,\sigma}$ (see Example 1) except when $a = b$; in this case, for any $C \in \mathcal{B}(\mathbb{R})$, $\tilde{T}(a, a; C) = \delta_a(C)$, where δ_a is the Dirac measure in a.

The PCA $\tilde{\boldsymbol{G}}_{m,\sigma}$ has the same behavior as $\boldsymbol{G}_{m,\sigma}$ if the initial state S_{t_0} does not contain two consecutive cells in the same state, i.e. for any i, $S(i, t_0) \neq S(i+1, t_0)$. But, if, for example, its initial state is $0^{\mathbb{N}}$ then it will stay in this configuration until the end.

1.3. μ-supported and μ-positive transition kernels

Before stating our main results we recall some facts concerning the Radon–Nikodym theorem. Recall that if μ and ν are two measures on E such that μ is *absolutely continuous* with respect to ν ($\mu \ll \nu$), there exists a unique (up to a ν-null set) ν-measurable function $f \colon E \longrightarrow \mathbb{R}^+$ such that for all $A \in \mathcal{B}(E)$, $\mu(A) = \int_A f \,\mathrm{d}\nu$. The function f is denoted by $\mathrm{d}\mu/\mathrm{d}\nu$ and called the Radon–Nikodym derivative of μ with respect to ν (or ν-density). We say that ν and μ are *positive equivalent* if $\nu \ll \mu$ and $\mu \ll \nu$. In that case, $\mathrm{d}\mu/\mathrm{d}\nu > 0$ and $\mathrm{d}\nu/\mathrm{d}\mu > 0$, μ-a.e.

If μ is a measure on E and $d \in \mathbb{N}$ then μ^d will stand for the product measure on E^d.

Now, we define the two crucial notions used throughout this paper: μ-supported and μ-positive transition kernels.

Definition 2. Let E be a Polish space, μ a σ-finite measure on E, and $d \in \mathbb{N}$. Let K be a transition kernel from E^d to E; K is said to be μ*-supported* if for μ^d-a.e. $(x_1, \ldots, x_d)$, $K(x_1, \ldots, x_d; \cdot) \ll \mu$. If, moreover, for μ^d-a.e. $(x_1, \ldots, x_d)$, $\mu \ll K(x_1, \ldots, x_d; \cdot)$, then K is said to be μ*-positive*.

For K, a μ-supported transition kernel from E^d to E, the μ*-density* of K is the μ^{d+1}-measurable function k such that

$$k \colon E^{d+1} \to \mathbb{R}, \qquad k(x_1, \ldots, x_d; y) \longmapsto \frac{\mathrm{d}K(x_1, \ldots, x_d; \cdot)}{\mathrm{d}\mu}(y).$$

If, moreover, K is μ-positive then for μ^{d+1}-a.e. $(x_1, \ldots, x_d, y)$, $k(x_1, \ldots, x_d; y) > 0$.

In the following, we will work with μ-supported or μ-positive kernels for $d = 1$ (transition kernels of a Markov chain) or $d = |N| = 2$ (transition kernels of a PCA). We will see that such transition kernels permit us to work with densities instead of measures. In the following, the Radon–Nikodym derivative of any measure with respect to μ will be also shorter in μ-density.

An example of a Lebesgue-supported transition kernel is the transition kernel T of a Gaussian PCAs (defined in Example 1). This transition kernel is even Lebesgue-positive. In the following, we call a μ-supported (respectively μ-positive) PCA a PCA whose transition kernel is μ-supported (respectively μ-positive).

Remark 1. (i) There exists some transition kernel that is not μ-supported by any σ-finite measure μ. For example, the transition kernel T from $\mathbb{R}^2$ to $\mathbb{R}$ defined by, for any $a, b \in \mathbb{R}$, $C \in \mathcal{B}(\mathbb{R})$,

$$T(a, b; C) = \begin{cases} \delta_a(C) & \text{if } a \neq b, \\ \displaystyle\int_C \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-(c-a)^2/2}\,\mathrm{d}c & \text{if } a = b \end{cases}$$

is not μ-supported. Indeed, any measure μ that could support this PCA has necessarily an atom at each x in $\mathbb{R}$. Then μ is not a σ-finite measure.

(ii) At the opposite, there exists some transition kernel that is supported by several singular measures. The PCA $\tilde{\boldsymbol{G}}_{m,\sigma}$ of Example 3 is Lebesgue-positive and also δ_a-positive for any $a \in \mathbb{R}$.

(iii) Nevertheless, if a PCA $\boldsymbol{A}$ is μ and ν-positive then μ and ν are positive equivalent or singular. Indeed, if there exist $(a, b) \in E^2$ such that the measure $T(a, b; \cdot)$ is both μ and ν-positive, then they are positive equivalent by transitivity. Otherwise, $P_\mu = \{(a, b) \colon T(a, b; \cdot) \text{ is } \mu\text{-positive}\}$ and $P_\nu = \{(a, b) \colon T(a, b; \cdot) \text{ is } \nu\text{-positive}\}$ are measurable and disjoint, and so taking $N = P_\nu \subset P_\mu^c$, $\mu(N) = 0$ and $\nu(N^c) = 0$, i.e. $\mu \perp \nu$.

We will make apparent below (in particular in Section 3.1 and Section 3.2.1) that to describe the invariant distribution of a PCA, at least in the case where it admits a Markov chain as an invariant distribution, we have to work under a reference measure μ which, depending on the case, can be the Lebesgue measure, a discrete measure, or any σ-finite measure. The idea is that the PCA can be seen to be trapped on some subsets of $E^{\mathbb{Z}}$ of the type $A^{\mathbb{Z}}$, where A is the support of a measure μ. When such a trap exists, the criterion for it to be an invariant distribution will depend on μ only (and its support). An example of this is the PCAs $\tilde{\boldsymbol{G}}_{m,\sigma}$ of Example 3 for which we will find different invariant distributions according to whether the reference measure is the Lebesgue measure or δ_a.

The PCAs studied in this paper correspond to a μ-supported PCA and its subset of a μ-positive PCA for μ, a σ-finite measure. For both sets, we characterize PCAs that have an invariant horizontal zigzag Markov chain, as defined now.

Let us define the horizontal zigzag Markov chains (HZMCs) on $\mathbb{N}$. First, the geometrical structure of a *horizontal zigzag* is such that the tth horizontal zigzag (HZ) on a space–time diagram is given by

$$\mathrm{HZ}_{\mathbb{N}}(t) = \left\{ \left(\left\lfloor \frac{i}{2} \right\rfloor, t + \frac{1 + (-1)^{i+1}}{2} \right), i \in \mathbb{N} \right\}$$

as illustrated in Figure 1.

Since $\mathrm{HZ}_{\mathbb{N}}(t)$ is made by two lines corresponding to two successive times, a PCA $\boldsymbol{A}$ on $\mathbb{N}$ can be seen as acting on the configurations of $\mathrm{HZ}_{\mathbb{N}}$. The image of a configuration $(S(i, t), S(i, t + 1) \colon i \in \mathbb{N})$ on $\mathrm{HZ}_{\mathbb{N}}(t)$ by the PCA $\boldsymbol{A}$ is $(S(i, t + 1), S(i, t + 2) \colon i \in \mathbb{N})$ on $\mathrm{HZ}_{\mathbb{N}}(t + 1)$, where the configuration of the second line of $\mathrm{HZ}_{\mathbb{N}}(t)$ becomes the configuration of the first line of $\mathrm{HZ}_{\mathbb{N}}(t + 1)$, and the configuration of the second line of $\mathrm{HZ}_{\mathbb{N}}(t + 1)$ is the image by $\boldsymbol{A}$ of the second line of $\mathrm{HZ}_{\mathbb{N}}(t)$.

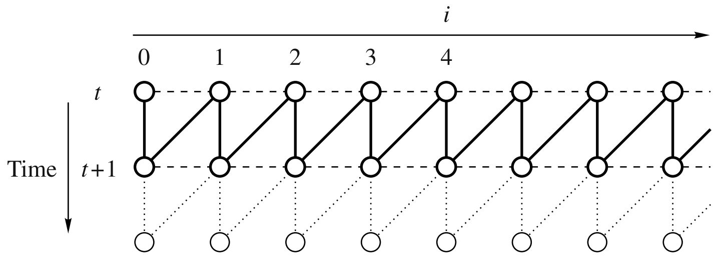

FIGURE 1: Shown in bold is $\mathrm{HZ}_{\mathbb{N}}(t)$, the tth horizontal zigzag on $\mathbb{N}$ on a space–time diagram.

Definition 3. A HZMC on $\mathrm{HZ}_{\mathbb{N}}(t)$ with general alphabet E is a Markov chain with two transition kernels D (for down) and U (for up) from E to E and an initial probability distribution ρ_0 on E such that

(i) the distribution of state $S(0,t)$ is ρ_0,

(ii) the distribution of state $S(i,t+1)$ knowing $S(i,t)=x_i$ is $D(x_i;\cdot)$, and

(iii) the distribution of state $S(i+1,t)$ knowing $S(i,t+1)=y_i$ is $U(y_i;\cdot)$.

In the following, we study the conditions under which a PCA admits a HZMC as an invariant distribution. For μ-supported PCAs, the HZMC itself will be μ-supported: a (ρ_0, D, U)-HZMC is μ-supported if $\rho_0 \ll \mu$ and D and U are μ-supported. In this case, we denote r_0, d, and u their respective μ-densities. Hence, a μ-supported (ρ_0, D, U)-HZMC is invariant by a μ-supported PCA with transition kernel T, if, for any $k \geq 0$, for μ-a.e. $b_0, b_1, \dots, b_{k+1}, c_0, \dots, c_k \in E$,

$$r_0(b_0)\bigg(\prod_{i=0}^{k} d(b_i;c_i)u(c_i;b_{i+1})\bigg) = \int_{E^{k+3}} r_0(a_0)\bigg(\prod_{i=0}^{k+1} d(a_i;b_i)u(b_i;a_{i+1})\bigg)\bigg(\prod_{i=0}^{k} t(b_i,b_{i+1};c_i)\bigg)\,\mathrm{d}\mu^{k+3}(a_0,\dots,a_{k+2}). \tag{2}$$

The *support* $\tilde{E}_{(\rho_0,D,U)}$ of a (ρ_0, D, U)-HZMC on $\mathrm{HZ}_{\mathbb{N}}(t)$ is the union of the support of the marginals of the first line of the HZMC, i.e. $\tilde{E}_{(\rho_0,D,U)} = \bigcup_{i\in\mathbb{N}} \mathrm{supp}(\rho_i)$, where ρ_i is the distribution of $S(i,t)$. When the (ρ_0, D, U)-HZMC is μ-supported then, for μ-a.e. $x \in \tilde{E}_{(\rho_0,D,U)}$, there exists $i \in \mathbb{N}$ such that $r_i(x) > 0$ (that holds because E is a Polish space). In the case of a μ-positive (ρ_0, D, U)-HZMC, $\tilde{E}_{(\rho_0,D,U)} = \mathrm{supp}(\mu)$. When the context is clear, $\tilde{E}_{(\rho_0,D,U)}$ will be denoted $\tilde{E}$.

Remark 2. Take two μ-supported PCAs $\boldsymbol{A}$ and $\boldsymbol{A}'$ with transition kernel T and T' with support $\tilde{E}$ such that T and T' coincide except on a μ^2-negligible set ($\mu^2(\{a,b\colon T(a,b;\cdot) \neq T'(a,b;\cdot)\}) = 0$). Such PCAs are said to be μ-*equivalent*. They have the same set of invariant μ-supported HZMCs. To see this, change t by t' in (2).

Let μ be a measure on E and $d\colon (a,c) \mapsto d(a;c)$ and $u\colon (c,b) \mapsto u(c;b)$ be two μ^2-measurable functions from E^2 to $\mathbb{R}$, then the μ^2-measurable function $\overline{du}$ from E^2 to $\mathbb{R}$ is

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defined by $\overline{du}(a; b) = \int_E d(a; c)u(c; b)\,\mathrm{d}\mu(c)$. For a μ-supported HZMC, $\overline{du}(a; b)$ is the μ-density of the transition kernel (DU) of the Markov chain (induced by the HZMC) on the first line $S_t = (S(i, t) : i \in \mathbb{N})$ of $\mathrm{HZ}_{\mathbb{N}}(t)$.

1.4. Content

In the next section we present out main results; Theorems 1 and 2 and Proposition 1.

Section 3 is dedicated to some examples of PCAs. In Section 3.1 we show applications of Theorems 1 and 2 and Proposition 1 to PCAs with finite alphabets. In Section 3.2.1 we use Theorem 2 and Proposition 1 to show that the law of an autoregressive process of order 1 (AR(1) process) is invariant by both Gaussian PCAs $\boldsymbol{G}_{m,\sigma}$ and $\tilde{\boldsymbol{G}}_{m,\sigma}$ (defined in Examples 1 and 3). In Section 3.2.2 we present a Lebesgue-supported PCA called a beta PCA. In Section 3.3 we present first a PCA with alphabet $\mathbb{R}$ that simulates a synchronous TASEP on $\mathbb{R}$ as defined by Blank [3] and then a PCA with alphabet $\mathbb{R}$ that simulates the first-passage percolation as presented by Kesten [12] on a particular graph $\mathcal{G}$. Unfortunately, Theorems 1 and 2 do not apply to these two PCAs.

In Section 4, Theorems 1 and 2 and Proposition 1, the main contributions of this paper, are proved.

Section 5 is devoted to extensions of Theorems 1 and 2 for PCAs on $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$. First, we extend in both cases the notion of HZMC: $\mathrm{HZMC}_{\mathbb{Z}}$ on $\mathbb{Z}$ and cyclic-HZMC (CHZMC) on $\mathbb{Z}/n\mathbb{Z}$ (if E is finite, a CHZMC is a HZMC conditioned to be periodic and, in the general case, it is a Gibbs measure). Then we characterize PCAs allowing the $\mathrm{HZMC}_{\mathbb{Z}}$ to be invariant, and also PCAs allowing the CHZMC to be invariant.

2. Main results

We start with a generalization to Polish space alphabets of [6, Lemma 2.3].

Theorem 1. *Let μ be a σ-finite measure on a general alphabet E. Let $\boldsymbol{A} := (\mathbb{N}, E, N, T)$ be a μ-supported PCA and (ρ_0, D, U) a μ-supported HZMC with support $\tilde{E}$. The (ρ_0, D, U)-HZMC is invariant by $\boldsymbol{A}$ if and only if the following three conditions are satisfied.*

(C.1) *For μ^3-a.e. $(a, b, c) \in \tilde{E}^3$, $t(a, b; c)\overline{du}(a; b) = d(a; c)u(c; b)$.*

(C.2) *For μ^2-a.e. $(a, b) \in \tilde{E}^2$, $\overline{du}(a; b) = \overline{ud}(a; b)$.*

(C.3) *The Markov chain with transition kernel D possesses ρ_0 as an invariant distribution, i.e. for μ-a.e. c, $r_0(c) = \int_E r_0(a)d(a; c)\,\mathrm{d}\mu(a)$.*

We arrive at our main result, Theorem 2. When a PCA with transition kernel T is μ-positive, we can go further and reduce the existence of an invariant HZMC for the PCA to the existence of a function η that is a solution to a cubic integral equation on T. In case of existence, we can express the kernels of the invariant HZMC using η and T. Let us first provide an introduction.

Let $\boldsymbol{A}$ be a PCA with transition kernel T whose μ-density is t. Define, for any positive measurable function $\phi \in L^1(\mu)$ (i.e. for μ-a.e. $x \in E$, $\phi(x) > 0$, and $\int_E \phi(x)\,\mathrm{d}\mu(x) < \infty$), the two μ^2-measurable functions $d^\phi : E^2 \mapsto \mathbb{R}$ and $u^\phi : E^2 \mapsto \mathbb{R}$ by

$$d^\phi(a; c) = \int_E \frac{\phi(x)}{t(a, x; c_0)} t(a, x; c)\,\mathrm{d}\mu(x) \biggl\{ \int_E \frac{\phi(x)}{t(a, x; c_0)}\,\mathrm{d}\mu(x) \biggr\}^{-1} \tag{3}$$

and

$$u^\phi(c; b) = \frac{\phi(b)}{t(a_0, b; c_0)} t(a_0, b; c) \biggl\{ \int_E \frac{\phi(x)}{t(a_0, x; c_0)} t(a_0, x; c)\,\mathrm{d}\mu(x) \biggr\}^{-1}. \tag{4}$$

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Theorem 2. *Let μ be a σ-finite measure on a general alphabet E. Let $\boldsymbol{A} := (\mathbb{N}, E, N, T)$ be a μ-positive PCA. It holds that $\boldsymbol{A}$ admits a μ-positive invariant HZMC if and only if the following three conditions are satisfied.*

(C.4) *There exists a triplet $(a_0, b_0, c_0) \in E^3$ such that $T(a_0, b_0; \cdot)$ and μ are positive equivalent and, for μ^3-a.e. (a, b, c),*

$$\begin{aligned} &t(a, b; c)t(a_0, b_0; c)t(a_0, b; c_0)t(a, b_0; c_0) \\ &\quad = t(a_0, b_0; c_0)t(a, b; c_0)t(a, b_0; c)t(a_0, b; c). \end{aligned} \tag{5}$$

(C.5) *There exists a positive function $\eta \in L^1(\mu)$ that is a solution to, for μ^2-a.e. (a, b) and for the (a_0, c_0) of (C.4),*

$$\begin{aligned} &\frac{\eta(b)}{t(a, b; c_0)}\bigg\{\int_E \frac{\eta(x)}{t(a, x; c_0)}\,\mathrm{d}\mu(x)\bigg\}^{-1} \\ &\quad = \int_E \bigg[\frac{\eta(c)}{t(a_0, c; c_0)}t(a_0, c; a)\bigg\{\int_E \frac{\eta(x)}{t(c, x; c_0)}\,\mathrm{d}\mu(x)\bigg\}^{-1} \\ &\qquad\quad \times \int_E \frac{\eta(x)}{t(c, x; c_0)}t(c, x; b)\,\mathrm{d}\mu(x) \\ &\qquad\quad \times \bigg\{\int_E \frac{\eta(x)}{t(a_0, x; c_0)}t(a_0, x; a)\,\mathrm{d}\mu(x)\bigg\}^{-1}\bigg]\,\mathrm{d}\mu(c). \end{aligned} \tag{6}$$

(C.6) *The Markov chain with transition kernel D^η, whose μ-density is d^η given by (3) and (4), possesses a (unique) invariant probability distribution ρ_0 such that ρ_0 and μ are positive equivalent.*

In this case, the (ρ_0, D^η, U^η)-HZMC, where D^η and U^η are transition kernel of μ-densities given by (3) and (4) is invariant by $\boldsymbol{A}$.

Remark 3. The uniqueness of ρ_0 comes from Lemma 1 presented below. It implies that the μ-positive (ρ_0, D^η, U^η)-HZMC is necessarily taken under its invariant probability distribution, i.e. for any $i \in \mathbb{L}$, $\rho_i = \rho_0$.

If (C.4) and (C.5) hold and if E is finite, the Markov chain with transition kernel D^η is irreducible and aperiodic (because, for any $a, c \in E$, $D^\eta(a, c) > 0$) and, so it possesses a unique invariant distribution, i.e. (C.6) always holds. If E is not finite, we refer the reader to the book of Meyn and Tweedie [16] for the conditions on D^η for which the Markov chain with transition kernel D^η possesses an invariant distribution.

When the alphabet is finite, we can go further and show that η satisfying (6) is, in fact, an eigenvector of a computable matrix [6]. That allows us to simplify (C.5). For a PCA with a general alphabet, this cannot be performed because we are not permitted to take $a = b$ in (6) in general. Nevertheless, under stronger conditions on t, we can characterize a set of functions that contains the set of functions η that are solutions to (6).

Proposition 1. *Let μ be a σ-finite measure on a general alphabet E. Let $\boldsymbol{A} := (\mathbb{Z}, E, N, T)$ be a μ-positive PCA. Suppose that (C.4) and the following two conditions are satisfied.*

(C.7) *For the same triplet (a_0, b_0, c_0) of (C.4), for μ^2-a.e. (a, c),*

$$\begin{aligned} &t(a, a; c)t(a_0, b_0; c)t(a_0, a; c_0)t(a, b_0; c_0) \\ &\quad = t(a_0, b_0; c_0)t(a, a; c_0)t(a, b_0; c)t(a_0, a; c). \end{aligned}$$

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(C.8) *There exists a positive function $\eta \in L^1(\mu)$ that is a solution to, for μ-a.e. a and for the (a_0, c_0) of (C.4),*

$$\frac{\eta(a)}{t(a,a;c_0)}\left\{\int_E \frac{\eta(k)}{t(a,x;c_0)}\,\mathrm{d}\mu(x)\right\}^{-1}$$
$$= \int_E\Bigg[\frac{\eta(c)}{t(a_0,c;c_0)}t(a_0,c;a)\left\{\int_E \frac{\eta(x)}{t(c,x;c_0)}\,\mathrm{d}\mu(x)\right\}^{-1} \times \int_E \frac{\eta(x)}{t(c,x;c_0)}t(c,x;a)\,\mathrm{d}\mu(x) \times \left\{\int_E \frac{\eta(x)}{t(a_0,x;c_0)}t(a_0,x;a)\,\mathrm{d}\mu(x)\right\}^{-1}\Bigg]\mathrm{d}\mu(c).$$

Then η is a positive eigenfunction of

$$\mathcal{A}_2\colon f \mapsto \left(\mathcal{A}_2(f)\colon a \mapsto \int_E f(k)\frac{t(a,a;c_0)}{t(a,x;c_0)}\nu(a)\,\mathrm{d}\mu(x)\right),$$

where ν is a positive eigenfunction (unique up to a multiplicative constant) in $L^1(\mu)$ of

$$\mathcal{A}_1\colon f \mapsto \left(\mathcal{A}_1(f)\colon a \mapsto \int f(c)t(c,c;a)\,\mathrm{d}\mu(c)\right).$$

Remark 4. Any positive PCA with finite alphabet E (i.e. for all a, b, c, $T(a,b;c) > 0$) is a μ_E-positive PCA, where μ_E is the counting measure on E. Hence, (C.7) and (C.8) are necessarily implied by (C.4) and (C.5) in the case of finite alphabets. Moreover, in this case, $\mathcal{A}_1$ and $\mathcal{A}_2$ have their own unique eigenfunction (due to the Perron–Frobenius theorem) and (C.6) holds necessarily. So, applying Theorem 2 and Proposition 1 to a positive PCA gives [6, Theorem 2.6].

Let $E = \mathbb{R}$ and μ be the Lebesgue measure. In the case where t is *continuous at any point* of E^3, then (C.4) and (C.5) imply (C.7) and (C.8), respectively, by continuity. So a solution η to (6) is a function η given by Proposition 1.

If for a PCA $\boldsymbol{A}$ the conditions of Proposition 1 do not hold, it is in general a complex task to find a function η that is a solution to (6). But it may happen that a μ-equivalent PCA $\boldsymbol{A}'$ to $\boldsymbol{A}$ (see Remark 2) satisfies the conditions of Proposition 1. Hence, in the best-case scenario, we can characterize a (ρ_0, D^η, U^η)-HZMC invariant by $\boldsymbol{A}'$ using Proposition 1. This HZMC is also invariant by $\boldsymbol{A}$. The μ-equivalence gives us some 'degrees of freedom' to solve the 'rigid' integral cubic equation (6). An application of this method is shown in Section 3.2.1, where it is proved that the AR(1) process is an invariant distribution of $\tilde{\boldsymbol{G}}_{m,\sigma}$ (defined in Example 3).

The uniqueness (up to a multiplicative constant) of the eigenfunction ν (in Proposition 1) is a consequence of the following lemma.

Lemma 1. (Durrett [9, Theorem 6.8.7].) *Let $\mathcal{A}\colon f \mapsto (\mathcal{A}(f) : y \mapsto \int_E f(x)m(x;y)\mu(\mathrm{d}x))$ be an integral operator of kernel m. If m is the μ-density of a μ-positive transition kernel M from E to E, then $\mathcal{A}$ possesses at most one positive eigenfunction in $L^1(\mu)$ (up to a multiplicative constant).*

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3. Examples

Our first examples are PCAs with finite alphabets. Then we introduce two new models: Gaussian PCAs and beta PCAs to illustrate our theorems. Finally, we present PCAs with infinite alphabets that model existing problems in the literature. One PCA models a synchronous TASEP on $\mathbb{R}$ as defined by Blank [3] and another one a variant of directed first-passage percolation.

All PCAs presented in this section are PCAs on $\mathbb{N}$ (except the PCA modeling the TASEP that is on $\mathbb{Z}$) and neighborhood $N(i) = (i, i+1)$.

3.1. PCAs with finite alphabets

For positive PCAs, see the first paragraph of Remark 4. For any finite set E denote by $\mu_E = \sum_{x\in E}\delta_x$ the counting measure on E. In the following example, we focus on the PCAs that are not positive and we take a PCA not μ_E-positive, but μ_F-positive for some F subsets of E.

Let $\boldsymbol{A}$ be the PCA with alphabet $E = \{0, 1, 2\}$ and transition matrix:

- $T(0, i; i) = T(i, 0; i) = 1$ for all $i \in \{0, 1, 2\}$;
- $T(1, 1; 1) = T(1, 1; 2) = T(2, 2; 1) = T(2, 2; 2) = \frac{1}{2}$;
- $T(1, 2; 1) = T(2, 1; 2) = \frac{4}{5}$;
- $T(1, 2; 2) = T(2, 1; 1) = \frac{1}{5}$.

This PCA is not positive ($T(0, 1; 0) = 0$), nevertheless it is $\mu_{\{0\}}$-positive ($T(0, 0; \cdot) = \mu_{\{0\}}(\cdot)$) and also $\mu_{\{1,2\}}$-positive. These two measures are singular as 'predicted' by Remark 1(iii).

An application of Theorem 2 and Proposition 1 to $\boldsymbol{A}$ seen as a $\mu_{\{0\}}$-positive (respectively $\mu_{\{1,2\}}$-positive) PCA allows us to compute an invariant $\mu_{\{0\}}$-positive (respectively $\mu_{\{1,2\}}$-positive) HZMC. They are the only possible invariant HZMCs for $\boldsymbol{A}$. In fact, these invariant HZMCs could also be obtained using [2, Theorem 3] and [6, Theorem 2.6] to $\boldsymbol{A}$ restricted to having its value in alphabet $\{0\}$ or $\{1, 2\}$.

A μ-supported PCA. Let $\boldsymbol{A}$ be the PCA with alphabet $E = \mathbb{Z}/\kappa\mathbb{Z}$ with transition kernel T such that $T(a, b; \cdot)$ is the uniform distribution on E if $b = a$ or $b = a + 1 \pmod{\kappa}$, and on the circular interval $\{a + 1, \dots, b - 1\}$ otherwise. This PCA is a μ_E-supported PCA, but not μ-positive for any measure μ on E. It has an invariant (ρ_0, D, U)-HZMC with $D(a; a + 1 \pmod{\kappa}) = U(a; a+1 \pmod{\kappa}) = 1$ for all $a \in \mathbb{Z}/\kappa\mathbb{Z}$ and for any $a \in \mathbb{Z}/\kappa\mathbb{Z}$, $\rho_0(\kappa) = 1/\kappa$. This invariant HZMC seems to be the unique invariant HZMC for $\boldsymbol{A}$ (proved for $\kappa = 3, 4, 5$ by Theorem 1 and MAPLE® to solve the conditions of this theorem). But we do not know if there exists some other invariant distribution(s) (which would not be a HZMC) for $\boldsymbol{A}$.

3.2. Two new models of PCAs with infinite alphabet

3.2.1. *Gaussian PCAs.* Denote by $g[m, \sigma]$ the density of the Gaussian distribution of mean m and variance σ^2.

Gaussian PCA $\boldsymbol{G}_{m,\sigma}$. For $\boldsymbol{G}_{m,\sigma}$, it can be checked that (C.4) holds for any triplet (a_0, b_0, c_0) in $\mathbb{R}^3$, so let us choose $(a_0, b_0, c_0) = (0, 0, 0)$. We use Proposition 1 to obtain a function η. The first step consists of studying the eigenfunctions of

$$\mathcal{A}_1 : L^1 \longrightarrow L^1, \qquad f \longmapsto \mathcal{A}_1(f) : c \mapsto \int_{\mathbb{R}} f(a) g\left[\frac{2a}{m}, \sigma\right](c)\,\mathrm{d}a.$$

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The function $\nu(x) = \exp(-(c_m/2\sigma^2)x^2)$ is a positive eigenfunction of $\mathcal{A}_1$ for $c_m = 1 - 4/m^2$. Moreover, we need ν to be in L^1, hence c_m must be positive and so we need $|m| > 2$. Without this condition, for any i, the function $t \to \operatorname{var}(S(i,t))$ increases and goes to ∞ with t. When $|m| > 2$, we can go further with Proposition 1 and study the eigenfunctions of

$$\mathcal{A}_2 \colon L^1 \longrightarrow L^1, \qquad f \longmapsto \mathcal{A}_2(f) \colon a \mapsto \int_{\mathbb{R}} f(b) \frac{t(a,a;0)\nu(a)}{t(a,b;0)} \, \mathrm{d}b$$

with

$$\frac{t(a,a;0)\nu(a)}{t(a,b;0)} = \exp\biggl(-\frac{b^2}{2\sigma^2}\biggr) \exp\biggl(\frac{((a+b)/m)^2}{2\sigma^2}\biggr).$$

One can check that $\eta(x) = \exp(-(1+\sqrt{c_m})x^2/4\sigma^2)$ is a positive eigenfunction of $\mathcal{A}_2$ associated to the eigenvalue $\sqrt{\pi\sigma^2}/(1+\sqrt{c_m})^2$. Moreover, η satisfies (6) (this is an example where Proposition 1 allows us to compute a solution η to (6)). We obtain

$$d^\eta(a;c) = g\biggl[\frac{2}{ml}a, \sqrt{\frac{2}{l}}\sigma\biggr](c), \qquad u^\eta(c;b) = g\biggl[\frac{2}{ml}c, \sqrt{\frac{2}{l}}\sigma\biggr](b) \quad \text{for } l = 1 + \sqrt{c_m}.$$

To end, we need to find an invariant probability distribution ρ_0 for the Markov chain of transition kernel D^η (with density d^η). The measure ρ_0 with density r_0 is satisfactory: $r_0(x) = g[0, c_m^{-1/4}\sigma](x)$.

This allows us to conclude that the (ρ_0, D^η, U^η)-HZMC is an invariant measure for the Gaussian PCA. In fact, this invariant HZMC is an AR(1) process (see [22]) that is a process (X_i) such that $X_i = \theta + \phi X_{i-1} + \varepsilon_i$, where θ and ϕ are two real numbers and (ε_i) are independent and identically distributed (i.i.d.), of Gaussian law $\mathcal{N}(0, \sigma'^2)$. In our case, the invariant HZMC is an AR(1) process on $\mathrm{HZ}_{\mathbb{N}}$ with $\theta = 0$, $\phi = 2/ml$, and $\sigma'^2 = 2\sigma^2/l$.

Gaussian PCAs except on diagonal $\tilde{\boldsymbol{G}}_{m,\sigma}$. As already seen in Remark 1(ii), this PCA is Lebesgue-positive and also $\mu_{\{a\}}$-positive for any $a \in \mathbb{R}$.

When we consider $\tilde{\boldsymbol{G}}_{m,\sigma}$ as a Lebesgue-positive PCA, Proposition 1 cannot be used to find a solution η to (6). Hopefully, $\tilde{\boldsymbol{G}}_{m,\sigma}$ is Lebesgue-equivalent to $\boldsymbol{G}_{m,\sigma}$. Hence, by Remark 2, the invariant Lebesgue-positive (ρ_0, D^η, U^η)-HZMC that corresponds to an AR(1) process obtained for $\boldsymbol{G}_{m,\sigma}$ is also invariant for $\tilde{\boldsymbol{G}}_{m,\sigma}$. Besides, for any $a \in \mathbb{R}$, the constant process equal to a everywhere is also an invariant measure to $\tilde{\boldsymbol{G}}_{m,\sigma}$.

3.2.2. *Beta PCAs.* We define a class of PCAs with alphabet $\mathbb{R}$ depending on three positive real parameters α, β, and m. The transition kernel is as follows. For all $a, b \in \mathbb{R}$ and $C \in \mathcal{B}(\mathbb{R})$,

$$T(a,b;C) = \mathbb{P}((b-a)X + a - m \in C),$$

where X is a beta(α,β) random variable. In words, the PCA takes a random (following a beta law) number between the two values of its two neighbors and subtracts m from it.

This PCA is Lebesgue-supported but not Lebesgue-positive.

Now we try to search for an invariant (ρ_0, D, U)-HZMC to this PCA. Let θ be a positive real number. Let $D_1(a;C) = \mathbb{P}(X_1 + a - m \in C)$ and $U_1(c;B) = \mathbb{P}(X_2 + c + m \in B)$, where X_1 (respectively X_2) is a gamma(α,θ) (respectively gamma(β,θ)) random variable. For $D = D_1$ and $U = U_1$, (C.1) and (C.2) hold; unfortunately, there does not exist a probability distribution ρ_0 that satisfies (C.3). Hence, this PCA does not possess a Lebesgue-supported HZMC as an invariant distribution. Nevertheless, the image of a Lebesgue-supported (ρ, D_1, U_1)-HZMC by this PCA is the $(\rho D_1, D_1, U_1)$-HZMC, meaning that one can describe simply the distribution of the successive image of a (ρ, D_1, U_1)-HZMC by this PCA.

3.3. PCAs with infinite alphabet in the literature

3.3.1. *PCAs modeling TASEP.* We model the synchronous TASEP on $\mathbb{R}$ introduced by Blank [3] by a PCA on $\mathbb{Z}$ with alphabet $\mathbb{R}$. In the following, when we say the TASEP, we refer to this variant of the TASEP. The TASEP models the behavior of an infinite number of particles of the same radius $r \geq 0$ on the real line, that move to the right that do not bypass, that do not overlap, and, at each step of time, each particle moves with probability p $(0 < p \leq 1)$, independently of each others. When a particle moves, it travels a distance $v \geq 0$ to the right, except if it can create a collision with the next particle; in that case, it moves to the rightmost allowed position. In this model, the state of a cell i at time t is the position x_i^t of the ith particle at time t. Formally, the evolution of (x_i^t) is defined as

$$x_i^{t+1} = \begin{cases} \min(x_i^t + v, x_{i+1}^t - 2r) & \text{with probability } p, \\ x_i^t & \text{with probability } 1-p. \end{cases}$$

Here we propose to model this TASEP by a PCA $\boldsymbol{A}$ on $\mathbb{Z}$ with alphabet $\mathbb{R}$. The transition kernel of the PCA is as follows. For any $a, b \in \mathbb{R}$ such that $a+r \leq b-r$ and for any $C \in \mathscr{B}(\mathbb{R})$,

$$T(a, b; C) = \begin{cases} (1-p)\delta_a(C) + p\delta_{a+v}(C) & \text{if } a+v \leq b-2r, \\ (1-p)\delta_a(C) + p\delta_{b-2r}(C) & \text{if } a+v > b-2r. \end{cases}$$

The transition kernel for other pairs (a, b) is not specified since they concern forbidden configurations. Hence, if we start with an admissible configuration at time 0 for the PCA (i.e. for any $i \in \mathbb{Z}$, $S(i, 0) + r \leq S(i+1, 0) - r$), then the PCA models the TASEP.

We remark that if $v = 2r$ and, for any $i \in \mathbb{Z}$, $x_i(t) \in 2r\mathbb{Z}$, then $x_i(t+1) \in 2r\mathbb{Z}$ for any i. In terms of the PCA, this says that the PCA $\boldsymbol{A}$ is μ-supported for $\mu = \sum_{i\in\mathbb{Z}} \delta_{2ri}$. Rescaling this alphabet by $1/(2r)$, this PCA models a synchronous TASEP on $\mathbb{Z}$ [14, Section 2.3]. It is known that this TASEP possesses a family of invariant Markov chain distributions indexed by a parameter $q \in (0, p)$ [14, Section 4.3]. As a consequence, the corresponding PCA owns some 'quasi' invariant distributions (see below to understand the exact meaning of 'quasi'). It appears that these 'quasi' invariant distributions are HZMC$_{\mathbb{Z}}$s (see Section 5.1). The transition kernels of the HZMC$_{\mathbb{Z}}$ are D and U defined by, for any $a \in \mathbb{Z}$, $C \in \mathscr{B}(\mathbb{Z})$,

$$D(a; C) = \frac{1-p}{1-q}\delta_a(C) + \frac{p-q}{1-q}\delta_{a+1}(C), \qquad U(a; C) = \sum_{m=1}^{\infty} q\,\frac{(p-q)^{m-1}}{p^m}\delta_{a+m}(C).$$

One can verify that these two transition kernel satisfy (C.1) and (C.2) in the context of Theorem 3 (see Section 5.1). In Theorem 3, (C.10) holds only in the degenerated case where $q = p$. In that case, all the (deterministic) HZMC$_{\mathbb{Z}}$s which satisfy $S(i+1, t) = S(i+1, t+1) = S(i, t) + 1$ for any i and t are invariant by $\boldsymbol{A}$. They correspond to the infinite traffic jam where nobody can move. Otherwise, for $q \in (0, p)$, D does not possess an invariant probability measure and so (C.10) cannot hold. Nevertheless, the image of a (R, D, U)-HZMC$_{\mathbb{Z}}$ is the (R', D, U)-HZMC$_{\mathbb{Z}}$ with $\rho_i' = \rho_i D$, meaning that we can describe simply the distribution of the successive images of a (R, D, U)-HZMC$_{\mathbb{Z}}$ by $\boldsymbol{A}$ (that is the sense of the 'quasi'). In addition, with this view of the TASEP, the mean speed of particles is simple to obtain: it is $D(a; a+1) = (p-q)/(1-q)$.

3.3.2. *PCAs modeling a variant of first-passage percolation.* We propose a model of directed first-passage percolation on a directed graph which can be seen as a PCA with alphabet $[0, \infty)$. We use the same notation as [12] to present the classical model of first-passage percolation.

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The set of nodes of $\mathcal{G}$ is $\mathbb{N}^2$ and the set of directed edges is $\mathcal{E} = \{((i, j), (i, j+1)) : i, j \in \mathbb{N}\} \cup \{((i+1, j), (i, j+1)) : i, j \in \mathbb{N}\}$. We let $L_0 = \{(i, 0) : i \in \mathbb{N}\}$ the set of the nodes of the first line. Now associate with the edges some i.i.d. weights $(t(e), e \in \mathcal{E})$ with common distribution F, where $t(e)$ is the time needed to pass through edge e. The passage time of a directed path $r = (e_1, \dots, e_n)$ on $\mathcal{G}$ is $T(r) = \sum_{i=1}^{n} t(e_i)$. The travel time from a node u to a node v is defined as $T(u, v) = \inf\{T(r) : r \text{ is a directed path from } u \text{ to } v\}$. If there is no directed path from u to v then $T(u, v) = \infty$. We define the travel time from $U \subset \mathbb{N}^2$ to a node v by $T(U, v) = \inf\{T(u, v) : u \in U\}$. Finally, we define $\mathcal{V}(t) = \{v \in \mathcal{N} : T(L_0, v) \leq t\}$ as the set of nodes visited at time t. The object of study in the first-passage percolation is this set $\mathcal{V}(t)$.

The first-passage percolation on $\mathcal{G}$ can be seen as a PCA $\boldsymbol{A}$ on $\mathbb{N}$ with alphabet $[0, \infty)$ as follows. Let $S(i, j)$ represent the travel time $T(L_0, (i, j))$ from L_0 to the node (i, j). The transition kernel of the PCA is as follows. For any $a, b \in [0, \infty)$, for any $C \in \mathcal{B}([0, \infty))$, $T(a, b; C) = L_{a,b}(C)$, where $L_{a,b}$ is the distribution of the random variable $X = \min\{(a + T_1), (b + T_2)\}$, where T_1 and T_2 are i.i.d. with common law F.

If F is nontrivial, $\boldsymbol{A}$ cannot have an invariant distribution because $E[S(i, j)] \longrightarrow \infty$ as $j \to \infty$ for all i. Nevertheless, (for some F) two transition kernel D and U could exist such that if at time 0, the initial distribution is a (ρ_0, D, U)-HZMC, then at time 1 it is the $(\rho_0 D, D, U)$-HZMC. Such a property should allow us to describe the evolution of $\boldsymbol{A}$ as in Section 3.2.2.

4. Proofs of the main results

Proof of Theorem 1. First, let (ρ_0, D, U) be a μ-supported HZMC invariant by $\boldsymbol{A}$ with transition kernel T, a μ-supported PCA. For all $A, B, C \in \mathcal{B}(E)$, and for all $i \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}(S(i, t) \in A, S(i+1, t) \in B, S(i, t+1) \in C) \\ &= \int_{A\times B\times C} r_i(a) d(a; c) u(c; b) \, \mathrm{d}\mu^3(a, b, c) \\ &= \int_{A\times B\times C} r_i(a) \overline{du}(a; b) t(a, b; c) \, \mathrm{d}\mu^3(a, b, c),\end{aligned}$$

where ρ_i is the law of cell i of μ-density r_i. Taking the difference, we obtain, for all $A, B, C \in \mathcal{B}(E)$,

$$\int_{A\times B\times C} (r_i(a) d(a; c) u(c; b) - r_i(a) \overline{du}(a; b) t(a, b; c)) \, \mathrm{d}\mu^3(a, b, c) = 0.$$

Hence, since this holds for any Borel set $A \times B \times C$,

$$r_i(a) d(a; c) u(c; b) = r_i(a) \overline{du}(a; b) t(a, b; c) \quad \text{for } \mu^3\text{-a.e. } (a, b, c) \in E^3.$$

If $a \in \tilde{E}$, there exists i such that $r_i(a) > 0$ almost surely and then (C.1) holds.

We also have, for all $A, B \in \mathcal{B}(E)$, on the one hand,

$$\begin{aligned}\mathbb{P}(S(i, t+1) \in A, S(i+1, t+1) \in B) \\ &= \mathbb{P}(S(i, t+1) \in A, S(i+1, t+1) \in B, S(i+1, t) \in E) \\ &= \int_{A\times B} r_i(a) \overline{ud}(a; b) \, \mathrm{d}\mu^2(a, b)\end{aligned}$$

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because $(S(0,t), S(0,t+1), S(1,t), \dots)$ is a (ρ_0, D, U)-HZMC and, on the other hand,

$$\begin{aligned}\mathbb{P}(S(i,t+1)\in A, S(i+1,t+1)\in B)\\ &= \mathbb{P}(S(i,t+1)\in A, S(i+1,t+1)\in B, S(i,t+2)\in E)\\ &= \int_{A\times B} r_i(a)\overline{du}(a;b)\,\mathrm{d}\mu^2(a,b)\end{aligned}$$

because $(S(0,t+1), S(0,t+2), S(1,t+1), \dots)$ is also a (ρ_0, D, U)-HZMC due to its invariance by $\boldsymbol{A}$. Then, as before, $r_i(a)\overline{ud}(a;b) = r_i(a)\overline{du}(a;b)$ for μ^2-a.e. $(a,b)\in E^2$ and so (C.2) holds.

Moreover, the law of $S(0,t)$ and $S(0,t+1)$ must be the same because (ρ_0, D, U) is invariant by the PCA. Hence, the law of $S(0,t+1)$ of μ-density $\int_E r_0(a)d(a;c)\,\mathrm{d}\mu(a)$ must be equal to ρ_0 of μ-density $r_0(c)$, i.e. (C.3) holds.

Conversely, suppose that (C.1), (C.2), and (C.3) are satisfied. Suppose that the horizontal zigzag $\mathrm{HZ}_{\mathbb{N}}(t)$ is distributed as a (ρ_0, D, U)-HZMC. Now, compute the push forward measure of this HZMC by $\boldsymbol{A}$. For any $n\ge 0$, and for any

$$F_{2n+1} = B_0\times\cdots\times B_{n+1}\times C_0\times\cdots\times C_n \in \mathcal{B}(E)^{2n+1},$$

we have

$$\begin{aligned}&\mathbb{P}(S(0,t+1)\in B_0, S(0,t+2)\in C_0, \dots, S(n+1,t+1)\in B_{n+1})\\ &\quad= \int_{E^{n+2}\times F_{2n+1}} r_0(a_0)\prod_{i=0}^{n+1} d(a_i;b_i)u(b_i;a_{i+1})t(b_i,b_{i+1};c_i)\\ &\qquad\qquad \times \mathrm{d}\mu^{3n+6}(a_0,\dots,a_{n+2},b_0,\dots,b_{n+1},c_0,\dots,c_n)\\ &\quad= \int_{F_{2n+1}}\bigg(\int_E r_0(a_0)d(a_0;b_0)\,\mathrm{d}\mu(a_0)\bigg)\prod_{i=0}^{n}\bigg(\int_E u(b_i;a_{i+1})d(a_{i+1};b_{i+1})\,\mathrm{d}\mu(a_{i+1})\bigg)\\ &\qquad \times\bigg(\int_E u(b_{n+1};a_{n+2})\,\mathrm{d}\mu(a_{n+2})\bigg)\prod_{i=0}^{n} t(b_i,b_{i+1};c_i)\,\mathrm{d}\mu^{2n+3}(b_0,\dots,b_{n+1},c_0,\dots,c_n)\\ &\quad= \int_{F_{2n+1}} r_0(b_0)\prod_{i=0}^{n}\overline{ud}(b_i;b_{i+1})t(b_i,b_{i+1};c_i)\,\mathrm{d}\mu^{2n+3}(b_0,\dots,b_{n+1},c_0,\dots,c_n)\\ &\quad= \int_{F_{2n+1}} r_0(b_0)\prod_{i=0}^{n} d(b_i;c_i)u(c_i;b_{i+1})\,\mathrm{d}\mu^{2n+3}(b_0,\dots,b_{n+1},c_0,\dots,c_n).\end{aligned}$$

This shows that the push forward measure of a (ρ_0, D, U)-HZMC is a (ρ_0, D, U)-HZMC. Hence, the (ρ_0, D, U)-HZMC is an invariant measure of $\boldsymbol{A}$. □

In the case of a μ-positive HZMC, taking $\tilde{E}$ or E does not make any difference in Theorem 1. Indeed, by the basic properties of measurability for any property P, $P(x)$ holds for μ-a.e. $x\in E$ if and only if $P(x)$ holds for μ-a.e. $x\in\operatorname{supp}(\mu)\cap E$ (set equal to $\tilde{E}$ here). In addition, for a μ-positive (ρ_0, D, U)-HZMC for μ^2-a.e. $(a,b)\in E^2$, $\overline{du}(a,b)>0$.

To prove Theorem 2, we first prove Lemmas 2 and 3.

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Lemma 2. *Let A be a μ-positive PCA with transition kernel T. The three conditions (C.1), (C.4), and*

(C.9) *for μ^6-a.e. (a, a', b, b', c, c'),*

$$t(a, b; c)t(a, b'; c')t(a', b; c')t(a', b'; c) = t(a', b'; c')t(a', b; c)t(a, b'; c)t(a, b; c') \tag{7}$$

are equivalent.

Proof. From (C.1) to (C.9), replace in (C.9) the expressions of t by the ones given in (C.1).

From (C.9) to (C.4), we prove its contrapositive. Suppose that, for all (a_0, b_0, c_0), (C.4) does not hold. Hence, for all $(a_0, b_0, c_0) \in E^3$, either $T(a_0, b_0; \cdot)$ and μ are not positive equivalent, or

$$\mu^3(\{(a, b, c) \text{ such that (5) does not hold}\}) > 0. \tag{8}$$

But, by the definition of μ-positivity, the set of (a_0, b_0) such that $T(a_0, b_0; \cdot)$ and μ are not positive equivalent is μ^2-negligible. Hence, for μ^3-a.e. (a_0, b_0, c_0), (8) holds. But, by Fubini's theorem,

$$\begin{aligned} &\mu^6(\{(a, b, c, a', b', c') \text{ such that (7) does not hold}\}) \\ &\quad = \int_{E^3} \mu^3(\{(a, b, c) \text{ such that (5) does not hold}\})\, \mathrm{d}\mu(a_0, b_0, c_0) \\ &\quad > 0 \end{aligned}$$

and, on the other hand, (C.9) is equivalent to

$$\mu^6(\{(a, b, c, a', b', c') \text{ such that (7) does not hold}\}) = 0.$$

From (C.4) to (C.1), set

$$d(a; c) = K_a \frac{t(a, b_0; c)}{t(a_0, b_0; c)} \int_E t(a_0, b; c)\, \mathrm{d}\mu(b) \left\{ \int_E \frac{t(a, b_0; x)}{t(a_0, b_0; x)}\, \mathrm{d}\mu(x) \right\}^{-1}$$

and

$$u(c; b) = t(a_0, b; c) \left\{ \int_E t(a_0, x; c)\, \mathrm{d}\mu(x) \right\}^{-1},$$

where K_a is a normalization constant such that $\int_E d(a; c)\, \mathrm{d}\mu(c) = 1$. Then

$$\overline{du}(a; b) = K_a \int_E \frac{t(a, b_0; c)t(a_0, b; c)}{t(a_0, b_0; c)} \left\{ \int_E \frac{t(a, b_0; x)}{t(a_0, b_0; x)}\, \mathrm{d}\mu(x) \right\}^{-1} \mathrm{d}\mu(c)$$

and

$$\frac{d(a; c)u(c; b)}{\overline{du}(a; b)} = \frac{t(a, b_0; c)t(a_0, b; c)}{t(a_0, b_0; c)} \left\{ \int_E \frac{t(a, b_0; x)t(a_0, b; x)}{t(a_0, b_0; x)}\, \mathrm{d}\mu(x) \right\}^{-1} \tag{9}$$

$$= \frac{t(a, b_0; c)t(a_0, b; c)}{t(a_0, b_0; c)} \left\{ \int_E t(a, b; x) \frac{t(a, b_0; c_0)t(a_0, b; c_0)}{t(a_0, b_0; c_0)t(a, b; c_0)}\, \mathrm{d}\mu(x) \right\}^{-1} \tag{10}$$

$$= \frac{t(a, b_0; c)t(a_0, b; c)t(a_0, b_0; c_0)t(a, b; c_0)}{t(a_0, b_0; c)t(a, b_0; c_0)t(a_0, b; c_0)} \left\{ \int_E t(a, b; x)\, \mathrm{d}\mu(x) \right\}^{-1} \tag{11}$$

$$= t(a, b; c). \tag{12}$$

In this computation, we pass from (9) to (10) and from (11) to (12) by using (C.4). □

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Lemma 2 says that (C.1) is equivalent to (C.4) for μ-positive PCA. In the next lemma we give some necessary conditions for a (ρ_0, D, U)-HZMC to be invariant by a μ-positive PCA.

Lemma 3. *Let $\boldsymbol{A}$ be a μ-positive PCA. If $\boldsymbol{A}$ satisfies the conditions of Lemma 2 then there exists H, a μ-positive probability distribution on $(E, \mathcal{B}(E))$ of μ-density η such that the respective μ-densities of D and U are, for μ^3-a.e. (a, b, c), d^η, and u^η as defined in (3) and (4).*

Proof. Suppose that, for μ^3-a.e. (a, b, c),

$$\overline{du}(a; b) = \frac{d(a; c)u(c; b)}{t(a, b; c)} = \frac{d(a; c_0)u(c_0; b)}{t(a, b; c_0)}.$$

Then $d(a; c)u(c; b) = d(a; c_0)(u(c_0; b)/t(a, b; c_0))t(a, b; c)$. Integrating with respect to b, we have

$$d(a; c) = d(a; c_0) \int_E \frac{u(c_0; b)}{t(a, b; c_0)} t(a, b; c)\, \mathrm{d}\mu(b)$$

and then

$$u(c; b) = \frac{u(c_0; b)}{t(a, b; c_0)} t(a, b; c) \bigg\{ \int_E \frac{u(c_0; x)}{t(a, x; c_0)} t(a, x; c)\, \mathrm{d}\mu(x) \bigg\}^{-1}. \tag{13}$$

Then (C.4) and (C.9) allow us to replace a by a_0 on the right-hand side of (13). Then taking $\eta(b) = u(c_0; b)$ completes the proof. □

Now we end with the proof of Theorem 2.

Proof of Theorem 2. Let $\boldsymbol{A}$ be a μ-positive PCA. If (ρ_0, D, U) is an invariant HZMC for $\boldsymbol{A}$ then there exists $\eta \in L^1(\mu)$ such that (3) and (4) hold by (C.1), Lemma 2, and Lemma 3.

Moreover, u and d satisfy (C.2). Hence, writing $\overline{du}$ and $\overline{ud}$ in terms of η, we obtain

$$\begin{aligned}
\overline{du}(a; b) &= \int_E \bigg[\int_E \frac{\eta(x)}{t(a, x; c_0)} t(a, x; c)\, \mathrm{d}\mu(x) \bigg\{ \int_E \frac{\eta(x)}{t(a, x; c_0)}\, \mathrm{d}\mu(x) \bigg\}^{-1} \\
&\qquad \times \frac{\eta(b)}{t(a_0, b; c_0)} t(a_0, b; c) \bigg\{ \int_E \frac{\eta(x)}{t(a_0, x; c_0)} t(a_0, x; c)\, \mathrm{d}\mu(x) \bigg\}^{-1} \bigg] \mathrm{d}\mu(c) \qquad (14) \\
&= \int_E \bigg[\int_E \frac{\eta(x)}{t(a, x; c_0)} t(a, x; c)\, \mathrm{d}\mu(x) \bigg\{ \int_E \frac{\eta(x)}{t(a, x; c_0)}\, \mathrm{d}\mu(x) \bigg\}^{-1} \\
&\qquad \times \frac{\eta(b)}{t(a, b; c_0)} t(a, b; c) \bigg\{ \int_E \frac{\eta(x)}{t(a, x; c_0)} t(a, x; c)\, \mathrm{d}\mu(x) \bigg\}^{-1} \bigg] \mathrm{d}\mu(c) \qquad (15) \\
&= \frac{\eta(b)}{t(a, b; c_0)} \bigg\{ \int_E \frac{\eta(x)}{t(a, x; c_0)}\, \mathrm{d}\mu(x) \bigg\}^{-1}.
\end{aligned}$$

We pass from (14) to (15), replacing $t(a_0, b; c)t(a_0, x; c_0)/t(a_0, b; c_0)t(a_0, x; c)$ by $t(a, b; c)t(a, x; c_0)/t(a, b; c_0)t(a, x; c)$ using (C.4) and (C.9); and

$$\begin{aligned}
\overline{ud}(a; b) &= \int_E \bigg[\frac{\eta(c)}{t(a_0, c; c_0)} t(a_0, c; a) \bigg\{ \int_E \frac{\eta(x)}{t(a_0, x; c_0)} t(a_0, x; a)\, \mathrm{d}\mu(x) \bigg\}^{-1} \\
&\qquad \times \int_E \frac{\eta(x)}{t(c, x; c_0)} t(c, x; b)\, \mathrm{d}\mu(x) \bigg\{ \int_E \frac{\eta(x)}{t(c, x; c_0)}\, \mathrm{d}\mu(x) \bigg\}^{-1} \bigg] \mathrm{d}\mu(c).
\end{aligned}$$

Hence, η is a solution of (6) which implies (C.5).

Finally, we need a distribution ρ_0 to satisfy (C.3) with $D = D^\eta$. This is possible only if (C.6) holds.

Conversely, if we suppose that (C.4), (C.5), and (C.6) hold, then all the previous computations hold and then we obtain (C.1), (C.2), and (C.3) for $D = D^\eta$, $U = U^\eta$, and ρ_0. Then we conclude the proof by using Theorem 1. □

Proof of Proposition 1. Let $\boldsymbol{A}$ be a PCA and suppose that (C.4), (C.7), and (C.8) hold. Then we can replace in (C.5) the a_0 by c using (C.4) and (C.7). Then η must verify the following equation. For μ-a.e. a, and for the c_0 of (C.4),

$$\frac{\eta(a)}{t(a,a;c_0)}\bigg\{\int_E \frac{\eta(x)}{t(a,x;c_0)}\,\mathrm{d}\mu(x)\bigg\}^{-1} = \int_E\bigg[\frac{\eta(c)}{t(c,c;c_0)}\bigg\{\int_E \frac{\eta(x)}{t(c,x;c_0)}\,\mathrm{d}\mu(x)\bigg\}^{-1}\bigg]t(c,c;a)\,\mathrm{d}\mu(c).$$

So, we see that

$$\bigg(a \mapsto \frac{\eta(a)}{t(a,a;c_0)}\bigg\{\int_E \frac{\eta(x)}{t(a,x;c_0)}\,\mathrm{d}\mu(x)\bigg\}^{-1}\bigg)$$

is an eigenfunction of the operator $\mathcal{A}_1\colon f \mapsto (\mathcal{A}_1(f)\colon a \mapsto \int_E f(c)t(c,c;a)\,\mathrm{d}\mu(c))$. Hence, by Lemma 1, if there exists a positive eigenfunction ν in $L^1(\mu)$ for $\mathcal{A}_1$, it is unique up to a multiplicative constant. Hence, there exists $\lambda > 0$ such that, for μ-a.e. a,

$$\frac{\eta(a)}{t(a,a;c_0)}\bigg\{\int_E \frac{\eta(x)}{t(a,x;c_0)}\,\mathrm{d}\mu(x)\bigg\}^{-1} = \lambda\nu(a),$$

which is equivalent to

$$\eta(a) = \lambda\int_E \eta(x)\frac{t(a,a;c_0)}{t(a,x;c_0)}\nu(a)\,\mathrm{d}\mu(x).$$

Hence, η is an eigenfunction of

$$\mathcal{A}_2\colon f \mapsto \bigg(\mathcal{A}_2(f)\colon a \mapsto \int_E f(x)\frac{t(a,a;c_0)}{t(a,x;c_0)}\nu(a)\,\mathrm{d}\mu(x)\bigg).$$ □

5. Extension to $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$

5.1. PCAs on $\mathbb{Z}$

In this section we extend Theorems 1 and 2 to $\mathbb{Z}$. The main change is that ρ_0, the initial probability distribution for a HZMC on $\mathbb{N}$, is replaced on $\mathbb{Z}$ by a sequence of probability distributions $R = (\rho_i)_{i\in\mathbb{Z}}$ indexed by $\mathbb{Z}$.

Let us define a $\mathrm{HZMC}_\mathbb{Z}$ on $\mathbb{Z}$. The geometrical structure is now

$$\mathrm{HZ}_\mathbb{Z}(t) = \bigg\{\bigg(\bigg\lfloor\frac{i}{2}\bigg\rfloor, t + \frac{1+(-1)^{i+1}}{2}\bigg), i \in \mathbb{Z}\bigg\};$$

see Figure 2 for a graphical representation. On this structure, a (R, D, U)-$\mathrm{HZMC}_\mathbb{Z}$ is a Markov chain with two transition kernels D and U and a family of probability distributions $R = (\rho_i)_{i\in\mathbb{Z}}$ such that

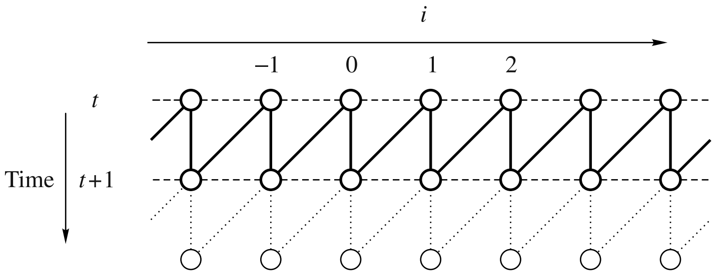

FIGURE 2: Shown in bold is $\mathrm{HZ}_{\mathbb{Z}}(t)$, the tth horizontal zigzag on $\mathbb{Z}$ on a space–time diagram.

- for all $i \in \mathbb{Z}$, the distribution of state $S(i,t)$ is ρ_i,
- the distribution of $S(i,t+1)$ knowing $S(i,t)$ is given by D, and that of $S(i+1,t)$ knowing $S(i,t+1)$ is given by U.

Hence, for any $i \in \mathbb{Z}$, the distributions ρ_i, ρ_{i+1}, D, and U are constrained such that $\rho_i DU = \rho_{i+1}$. In the case of a μ-supported (R,D,U)-$\mathrm{HZMC}_{\mathbb{Z}}$ (i.e. for all $i \in \mathbb{Z}$, $\rho_i \ll \mu$, D and U are μ-supported transition kernel), we have

$$r_{i+1}(x_{i+1}) = \int_E r_i(x_i)\overline{du}(x_i; x_{i+1})\,\mathrm{d}\mu(x_i). \tag{16}$$

A family of probability distributions R with this property is said to be compatible with (D,U).

As before, we define the support $\tilde{E} = \bigcup_{i\in\mathbb{Z}} \mathrm{supp}(\rho_i)$ of a $\mathrm{HZMC}_{\mathbb{Z}}$. If the $\mathrm{HZMC}_{\mathbb{Z}}$ is μ-supported then, for μ-a.e. $a \in E$, there exists i such that $r_i(a) > 0$ and, in the case of a μ-positive $\mathrm{HZMC}_{\mathbb{Z}}$ (i.e. for all $i \in \mathbb{Z}$, ρ_i and μ are positive equivalent and D and U are μ-positive transition kernels), $\tilde{E} = \mathrm{supp}(\mu)$.

The following theorem is an extension of Theorem 1 for the PCA on $\mathbb{Z}$.

Theorem 3. *Let μ be a σ-finite measure on E. Let $\boldsymbol{A} := (\mathbb{Z}, E, N, T)$ be a μ-supported PCA and (R,D,U) a μ-supported $\mathrm{HZMC}_{\mathbb{Z}}$. The (R,D,U)-$\mathrm{HZMC}_{\mathbb{Z}}$ is invariant by $\boldsymbol{A}$ if and only if (C.1), (C.2), and the following condition are satisfied.*

(C.10) *The distribution ρ is invariant by transition kernel D, i.e. for all $i \in \mathbb{Z}$, $\rho_i D = \rho_i$, i.e. for all $i \in \mathbb{Z}$, $r_i(c) = \int_E r_i(a)d(a;c)\,\mathrm{d}\mu(a)$.*

Proof. This is an immediate consequence of Theorem 1 because we just need, for all $i \in \mathbb{Z}$, the (ρ_i, D, U)-HZMC to be invariant by $\boldsymbol{A}$. □

As in Theorem 2 where we go further for μ-positive PCA on $\mathbb{N}$, we obtain a necessary and sufficient condition on μ-positive PCA to have an invariant $\mathrm{HZMC}_{\mathbb{Z}}$.

Theorem 4. *Let μ be a σ-finite measure on E. Let $\boldsymbol{A} := (\mathbb{Z}, E, N, T)$ be a μ-positive PCA. It holds that $\boldsymbol{A}$ admits a μ-positive invariant $\mathrm{HZMC}_{\mathbb{Z}}$ if and only if (C.4), (C.5), and (C.6) hold.*

In this case, the (R,D,U)-$\mathrm{HZMC}_{\mathbb{Z}}$ has for μ-densities d^η and u^η as defined in (3) and (4) and, for any $i \in \mathbb{Z}$, $\rho_i = \rho_0$.

Proof. It is an immediate consequence of Theorem 2. The only new thing to verify is that $R = \rho_0^{\mathbb{Z}}$ is (D,U) compatible, i.e. r_0 satisfies (16) in order to check that (R,D,U) defines a

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$\text{HZMC}_{\mathbb{Z}}$. By (C.6), for μ-a.e. y_i,

$$\int_E r_0(x_i)d(x_i; y_i)\,\mathrm{d}\mu(x_i) = r_0(y_i). \tag{17}$$

But, satisfying (16) and (17) is equivalent to satisfying, for μ-a.e. x_{i+1},

$$\int_E r_0(y_i)u(y_i; x_{i+1})\,\mathrm{d}\mu(y_i) = r_0(x_{i+1}).$$

Now, from (17), for μ-a.e. x_{i+1},

$$\iint_{E^2} r_0(x_i)d(x_i; y_i)u(y_i; x_{i+1})\,\mathrm{d}\mu(x_i)\,\mathrm{d}\mu(y_i) = \int_E r_0(y_i)u(y_i; x_{i+1})\,\mathrm{d}\mu(y_i).$$

But as $\overline{du} = \overline{ud}$,

$$\int_E \biggl(\int_E r_0(x_i)u(x_i; y_i)\,\mathrm{d}\mu(x_i)\biggr) d(y_i; x_{i+1})\,\mathrm{d}\mu(y_i) = \int_E r_0(y_i)u(y_i; x_{i+1})\,\mathrm{d}\mu(y_i).$$

So $f\colon y \to \int_E r_0(x)u(x; y)\,\mathrm{d}\mu(x)$ is a positive eigenfunction of the integral operator $\mathcal{A}$ of kernel d. By Lemma 1, this eigenfunction is unique (up to a multiplicative constant) equal to r_0, so $\int_E r_0(x)u(x; y)\,\mathrm{d}\mu(x) = \lambda r_0(y)$ and $\lambda = 1$ because they both integrate (with respect to μ) to 1. This completes the proof. □

Due to the uniqueness of ρ_0 in (C.6) (deduced from Lemma 1), the (R, D, U)-$\text{HZMC}_{\mathbb{Z}}$ is, in fact, necessarily taken under its invariant probability distribution.

In that case, Proposition 1 still holds and Remark 2 also holds if the (ρ_0, D, U)-HZMC is replaced by the (R, D, U)-$\text{HZMC}_{\mathbb{Z}}$.

5.2. PCAs on $\mathbb{Z}/n\mathbb{Z}$

In this section we have results, similar to Theorems 1 and 2, on the lattice $\mathbb{Z}/n\mathbb{Z}$. The main change is that we characterize the PCA whose invariant distribution is a cyclic-HZMC (CHZMC).

Consider, as represented in Figure 3,

$$\text{CHZ}(t) = \biggl\{\biggl(\biggl\lfloor \frac{i}{2} \biggr\rfloor, t + \frac{1 + (-1)^{i+1}}{2}\biggr), i \in \frac{\mathbb{Z}}{(2n)\mathbb{Z}}\biggr\}.$$

Let (D, U) be two μ-supported transition kernel from E to E such that

$$Z(D, U) = \int_{E^{2n}} u(y_{n-1}; x_0)d(x_0; y_0)\ldots d(x_{n-1}; y_{n-1})\,\mathrm{d}\mu^{2n}(x_0, y_0, x_1, \ldots, y_{n-1}) \notin \{0, +\infty\}.$$

We define the measure M on the cyclic horizontal zigzag (CHZ) called a (μ-supported) (D, U)-CHZMC by its μ^{2n}-density m, that is, for μ-a.e. $x_0, y_0, \ldots, y_{n-1} \in E$,

$$m(x_0, y_0, \ldots, y_{n-1}) = \frac{u(y_{n-1}; x_0)d(x_0; y_0)\ldots d(x_{n-1}; y_{n-1})}{Z(D, U)}.$$

For simplicity, we define, formally, only μ-supported (D, U)-CHZMCs (D and U are μ-supported transition kernels from E to E).

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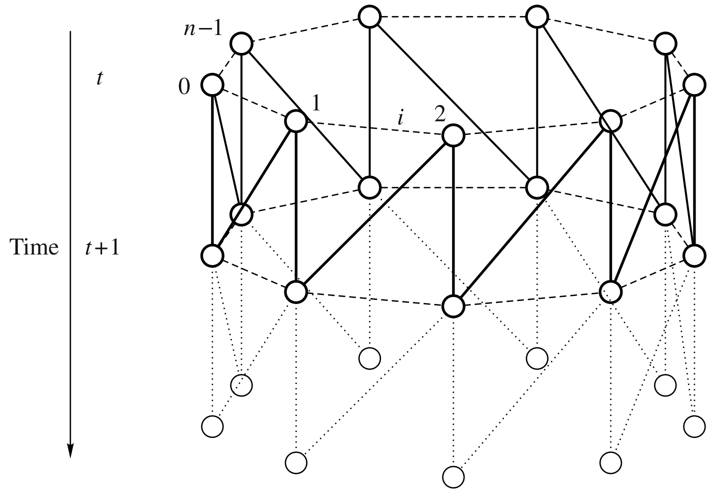

FIGURE 3: Shown in bold is CHZ(t), the tth cyclic horizontal zigzag on a space–time diagram.

When E is finite, a CHZMC is a HZMC conditioned to be periodic. In general, a CHZMC is a Gibbs measure in the cyclic horizontal zigzag the CHZ.

The denomination 'cyclic Markov chain' were first introduced by Albenque [1] to define a periodic Markov chain on $\mathbb{Z}/n\mathbb{Z}$. This notion is the same as Markovian fields in [4].

The distribution of the line S_t (respectively S_{t+1}) is denoted $M^{(1)}$ (respectively $M^{(2)}$) and its μ^n-density is obtained by integration of m with respect to the n variables $y_0, \ldots, y_{n-1}$ (respectively to the n variables $x_0, \ldots, x_{n-1}$). The distribution of the state $S(i,t)$ is denoted $M_i^{(1)}$ and its μ-density is obtained by integration of m with respect to the $2n-1$ variables $x_0, y_0, \ldots, x_{i-1}, y_{i-1}, y_i, x_{i+1}, \ldots, x_{n-1}$.

For any $j \in \mathbb{N}$, for μ-a.e. a, b, we let

$$\overline{(du)^j}(a;b) = \int_{E^{2j-1}} d(a;y_0)u(y_0;x_1)\cdots u(y_{j-1};b)\,\mathrm{d}\mu^{2j-1}(y_0,x_1,\ldots,y_{j-1}).$$

First, we obtain a theorem about μ-supported PCAs having μ-supported CHZMC.

Theorem 5. *Let μ be a σ-finite measure on E. Let $\mathbf{A} := (\mathbb{Z}/n\mathbb{Z}, E, N, T)$ be a μ-supported PCA and (D, U) a μ-supported CHZMC. The (D,U)-CHZMC is invariant by $\mathbf{A}$ if and only if the two following conditions are satisfied.*

(C.11) *For μ-a.e. $a, b, c \in E$,*

$$\overline{du}(a;b)t(a,b;c) = d(a;c)u(c;b) \text{ or } \overline{(du)^{n-1}}(b;a) = 0.$$

(C.12) *For μ-a.e. $x_0, x_1, \ldots, x_{n-1} \in \tilde{E}$,*

$$\overline{du}(x_0;x_1)\overline{du}(x_1;x_2)\cdots\overline{du}(x_{n-1};x_0) = \overline{ud}(x_0;x_1)\overline{ud}(x_1;x_2)\cdots\overline{ud}(x_{n-1};x_0).$$

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Proof. Let a (D, U)-CHZMC be invariant by $\boldsymbol{A}$. For all $A, B, C \in \mathcal{B}(E)$, and for all $i \in \mathbb{Z}/n\mathbb{Z}$,

$$\begin{aligned}
&\mathbb{P}(S(i,t) \in A, S(i+1,t) \in B, S(i,t+1) \in C) \\
&\quad = \frac{1}{Z(D,U)} \int_{A\times C\times B} d(x_i; y_i)u(y_i; x_{i+1})\overline{(du)^{n-1}}(x_{i+1}; x_i)\,\mathrm{d}\mu^3(x_i, y_i, x_{i+1}) \\
&\quad = \frac{1}{Z(D,U)} \int_{A\times C\times B} \overline{du}(x_i; x_{i+1})t(x_i, x_{i+1}; y_i)\overline{(du)^{n-1}}(x_{i+1}; x_i)\,\mathrm{d}\mu^3(x_i, y_i, x_{i+1}).
\end{aligned}$$

Hence, for μ-a.e. $x_i, y_i, x_{i+1} \in E$,

$$\overline{du}(x_i; x_{i+1})t(x_i, x_{i+1}; y_i)\overline{(du)^{n-1}}(x_{i+1}; x_i) = d(x_i; y_i)u(y_i; x_{i+1})\overline{(du)^{n-1}}(x_{i+1}; x_i),$$

i.e. (C.11).

To prove (C.12), we use the fact that the second line of the (D, U)-CHZMC at time t is the first line at time $t+1$ and, since the CHZMC is invariant, the law of the CHZMC at time t and at time $t+1$ is the same M. But $M^{(1)}$ is the law of the first line and $M^{(2)}$ of the second, so $M^{(1)} = M^{(2)}$. In terms of μ^n-densities, $m^{(1)} = m^{(2)}$. But

$$m^{(1)}(x_0, \dots, x_{n-1}) = \frac{1}{Z(D,U)}\overline{du}(x_0; x_1)\cdots\overline{du}(x_{n-1}; x_0)$$

and

$$m^{(2)}(x_0, \dots, x_{n-1}) = \frac{1}{Z(D,U)}\overline{ud}(x_0; x_1)\cdots\overline{ud}(x_{n-1}; x_0),$$

and we obtain (C.12).

Conversely, we suppose that (C.11) and (C.12) are satisfied. Then the push forward measure of the (D, U)-CHZMC by $\boldsymbol{A}$ is also the (D, U)-CHZMC (the computation is an adaptation of that performed in the proof of Theorem 1 to compute the push forward measure of a HZMC). This completes the proof. □

For μ-positive PCAs, (C.11) could be exploited a little more.

Theorem 6. *Let μ be a σ-finite measure on E. Let $\boldsymbol{A} := (\mathbb{Z}/n\mathbb{Z}, E, N, T)$ be a μ-positive PCA. It holds that $\boldsymbol{A}$ admits a μ-positive invariant CHZMC if and only if (C.4) and the following condition are satisfied.*

(C.13) *There exists a positive function $\eta \in L^1(\mu)$ that is a solution of*

$$\begin{aligned}
&\overline{d^\eta u^\eta}(x_0; x_1)\overline{d^\eta u^\eta}(x_1; x_2)\cdots\overline{d^\eta u^\eta}(x_{n-1}; x_0) \\
&\quad = \overline{u^\eta d^\eta}(x_0; x_1)\overline{u^\eta d^\eta}(x_1; x_2)\cdots\overline{u^\eta d^\eta}(x_{n-1}; x_0)
\end{aligned}$$

for μ-a.e. $x_0, \dots, x_{n-1} \in E$ with d^η and u^η as defined in (3) and (4). In this case, the (D, U)-CHZMC holds for μ-densities d^η and u^η as defined in (3) and (4).

Proof. First, when a PCA is μ-positive, (C.11) can be written, for μ-a.e. a, b, c, as

$$t(a, b; c) = d(a; c)u(c; b)\overline{du}(a; b)$$

because both $\overline{du}(a; b)$ and $\overline{(du)^{n-1}(b; a)}$ are positive. Hence, we use Lemma 2 to prove that (C.11) is equivalent to (C.4). Moreover, Lemma 3 still applies and the state space of possible solutions for (D, U) is parametrized by η, a function in $L^1(\mu)$. With (C.12) applied on d^η and u^η, we obtain (C.13). □

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References

[1] Albenque, M. (2009). A note on the enumeration of directed animals via gas considerations. *Ann. Appl. Prob.* **19,** 1860–1879.

[2] Belyaev, Y. K., Gromak, Y. I. and Malyshev, V. A. (1969). Invariant random Boolean fields. *Math. Notes Acad. Sci. USSR* **6,** 792–799.

[3] Blank, M. (2012). Stochastic stability of traffic maps. *Nonlinearity* **25,** 3389–3408.

[4] Bousquet-Mélou, M. (1998). New enumerative results on two-dimensional directed animals. *Discrete Mathematics* **180,** 73–106.

[5] Briceño, R., Moisset de Espanés, P., Osses, A. and Rapaport, I. (2013). Solving the density classification problem with a large diffusion and small amplification cellular automaton. *Physica D* **261,** 70–80.

[6] Casse, J. and Marckert, J.-F. (2015). Markovianity of the invariant distribution of probabilistic cellular auomata on the line. *Stoch. Process. Appl.* **125,** 3458–3483.

[7] Ceccherini-Silberstein, T. and Coornaert, M. (2013). Surjunctivity and reversibility of cellular automata over concrete categories. In *Trends in Harmonic Analysis*, Springer, Milan, pp. 91–133.

[8] Derrida, B., Domany, E. and Mukamel, D. (1992). An exact solution of a one-dimensional asymmetric exclusion model with open boundaries. *J. Statist. Phys.* **69,** 667–687.

[9] Durrett, R. (2010). *Probability: Theory and Examples*, 4th edn. Cambridge University Press.

[10] Flajolet, P. and Sedgewick, R. (2009). *Analytic Combinatorics*. Cambridge University Press.

[11] Hedlund, G. A. (1969). Endomorphisms and automorphisms of the shift dynamical system. *Math. Systems Theory* **3,** 320–375.

[12] Kesten, H. (1987). Percolation theory and first-passage percolation. *Ann. Prob.* **15,** 1231–1271.

[13] Kozlov, O. and Vasilyev, N. (1980). Reversible Markov chains with local interaction. In *Multicomponent Random Systems* (Adv. Prob. Relat. Topics **6**), Dekker, New York, pp. 451–469.

[14] Mairesse, J. and Marcovici, I. (2014). Around probabilistic cellular automata. *Theoret. Comput. Sci.* **559,** 42–72.

[15] Mairesse, J. and Marcovici, I. (2014). Probabilistic cellular automata and random fields with i.i.d. directions. *Ann. Inst. H. Poincaré Prob. Statist.* **50,** 455–475.

[16] Meyn, S. and Tweedie, R. L. (2009). *Markov Chains and Stochastic Stability*, 2nd edn. Cambridge University Press.

[17] Pra, P. D., Louis, P.-Y. and Roelly, S. (2002). Stationary measures and phase transition for a class of probabilistic cellular automata. *ESAIM Prob. Statist.* **6,** 89–104.

[18] Schiff, J. L. (2008). *Cellular Automata: A Discrete View of the World*. John Wiley, Hoboken, NJ.

[19] Toom, A. L. *et al*. (1990). Part I: Discrete local Markov systems. In *Stochastic Cellular Systems: Ergodicity, Memory, Morphogenesis*, eds R. L. Dobrushin *et al.*, Manchester University Press, pp. 1–182.

[20] Vancheri, A., Giordano, P., Andrey, D. and Albeverio, S. (2005). Continuous valued cellular automata and decision process of agents for urban dynamics. In *Computers in Urban Planning and Urban Management (CUPUM 2005)*. Paper 293.

[21] Vasilyev, N. B. (1978). Bernoulli and Markov stationary measures in discrete local interactions. In *Development in Statistics*, Vol. 1, Academic Press, New York, pp. 99–112.

[22] West, M. and Harrison, J. (1997). *Bayesian Forecasting and Dynamic Models*, 2nd edn. Springer, New York.