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ADDITIVE COMPLETION OF THIN SETS

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Abstract

Two sets A, B of positive integers are called *exact additive complements* if $A+B$ contains all sufficiently large integers and $A(x)B(x)/x \to 1$. For $A = \{a_1 < a_2 < \cdots\}$, let $A(x)$ denote the counting function of A and let $a^*(x)$ denote the largest element in $A \bigcap [1, x]$. Following the work of Ruzsa ['Exact additive complements', *Quart. J. Math.* **68** (2017) 227–235] and Chen and Fang ['Additive complements with Narkiewicz's condition', *Combinatorica* **39** (2019), 813–823], we prove that, for exact additive complements A, B with $a_{n+1}/na_n \to \infty$,

$$A(x)B(x) - x \geqslant \frac{a^*(x)}{A(x)} + o\Big(\frac{a^*(x)}{A(x)^2}\Big) \quad \text{as } x \to +\infty.$$

We also construct exact additive complements A, B with $a_{n+1}/na_n \to \infty$ such that

$$A(x)B(x) - x \leqslant \frac{a^*(x)}{A(x)} + (1+o(1))\Big(\frac{a^*(x)}{A(x)^2}\Big)$$

for infinitely many positive integers x.

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1. Introduction

Two sets A, B of positive integers are called *additive complements*, if $A+B$ contains all sufficiently large integers. Let $A = \{a_1 < a_2 < \cdots\}$ be a set of positive integers. Denote by $A(x)$ the counting function of A and by $a^*(x)$ the largest element in $A \bigcap [1, x]$. If additive complements A, B satisfy

$$\frac{A(x)B(x)}{x} \to 1,$$

then we call such A, B *exact additive complements*. In 2001, Ruzsa [2] introduced the following notation which is powerful for the proof of additive complements.

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Let $m > a_1$ be an integer and let $k = A(m)$. Denote by $L(m)$ the smallest number l for which there are integers $b_1, \ldots, b_l$ such that the numbers $a_i + b_j$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$, contain every residue modulo m. Obviously, $L(m) \geqslant m/k$.

THEOREM 1.1 (Ruzsa [2]). *If*

$$\frac{a_{n+1}}{na_n} \to \infty, \tag{1.1}$$

then A has an exact complement.

THEOREM 1.2 (Ruzsa [2]). *Let A be a set satisfying $A(2x)/A(x) \to 1$. The following are equivalent.*

(a) *A has an exact complement.*
(b) $A(m)L(m)/m \to 1$.
(c) *There is a sequence $m_1 < m_2 < \cdots$ of positive integers such that $A(m_{i+1})/A(m_i) \to 1$ and $A(m_i)L(m_i)/m_i \to 1$.*

THEOREM 1.3 (Ruzsa [3]). *For exact additive complements A, B with $A(2x)/A(x) \to 1$,*

$$A(x)B(x) - x \geqslant (1 + o(1))\frac{a^*(x)}{A(x)} \quad \text{as } x \to +\infty.$$

In 2019, Chen and Fang [1] improved Theorem 1.3 by removing the *exact* condition. Chen and Fang also showed in [1] that Theorem 1.3 is the best possible.

THEOREM 1.4 (Chen and Fang [1]). *There exist exact additive complements A, B with $A(2x)/A(x) \to 1$ such that*

$$A(x)B(x) - x \leqslant (1 + o(1))\frac{a^*(x)}{A(x)}$$

for infinitely many positive integers x.

In this paper, under condition (1.1) from [2], we obtain the following result.

THEOREM 1.5. *For exact additive complements A, B with (1.1),*

$$A(x)B(x) - x \geqslant \frac{a^*(x)}{A(x)} + o\Big(\frac{a^*(x)}{A(x)^2}\Big) \quad \text{as } x \to +\infty. \tag{1.2}$$

Moreover, we also show that $a^*(x)/A(x)^2$ is the best possible.

THEOREM 1.6. *There exist exact additive complements A, B with (1.1) such that*

$$\liminf_{x\to\infty} \frac{A(x)B(x) - x - \frac{a^*(x)}{A(x)}}{\frac{a^*(x)}{A(x)^2}} \leqslant 1. \tag{1.3}$$

2. Proofs of the main results

Let

$$\sigma(x,n) = |\{(a,b) : a+b=n, a,b \leqslant x, a \in A, b \in B\}|$$

and

$$\delta(x,n) = |\{(a,b) : b-a=n, a,b \leqslant x, a \in A, b \in B\}|.$$

The ideas used in the proofs of the main results come from [1–3]. We use the following lemma of Ruzsa in the proof of Theorem 1.5.

LEMMA 2.1 [3, Lemma 2.1]. *Let U and V be finite sets of integers and let*

$$\sigma(n) = |\{(u,v) : u \in U, v \in V, u+v=n\}|$$

and

$$\delta(n) = |\{(u,v) : u \in U, v \in V, v-u=n\}|.$$

Then

$$\sum_{\sigma(n)>1} (\sigma(n)-1) \geqslant \frac{1}{|U|} \sum_{\delta(n)>1} (\delta(n)-1).$$

PROOF OF THEOREM 1.5. Assume the contrary. Suppose that (1.2) does not hold. Then there exist a positive number $\delta_0(<1)$ and a sequence $x_1 < x_2 < \cdots < x_k < \cdots$ such that

$$A(x_k)B(x_k) - x_k \leqslant \frac{a^*(x_k)}{A(x_k)} - \delta_0 \frac{a^*(x_k)}{A(x_k)^2}. \tag{2.1}$$

We know that

$$\begin{aligned} A(x_k)B(x_k) - x_k &= \sum_{\substack{a \leqslant x_k, b \leqslant x_k \\ a \in A, b \in B}} 1 - x_k = \sum_{n=1}^{2x_k} \sigma(x_k,n) - x_k \\ &= \sum_{\substack{n=1 \\ \sigma(x_k,n)\geqslant 1}}^{x_k} (\sigma(x_k,n)-1) + \sum_{\substack{n=x_k+1 \\ \sigma(x_k,n)\geqslant 1}}^{2x_k} \sigma(x_k,n) \\ &= \sum_{\substack{n=1 \\ \sigma(x_k,n)\geqslant 1}}^{2x_k} (\sigma(x_k,n)-1) + \sum_{\substack{n=x_k+1 \\ \sigma(x_k,n)\geqslant 1}}^{2x_k} 1 \\ &= \sum_{\substack{n=1 \\ \sigma(x_k,n)> 1}}^{2x_k} (\sigma(x_k,n)-1) + \sum_{\substack{n=x_k+1 \\ \sigma(x_k,n)\geqslant 1}}^{2x_k} 1. \end{aligned}$$

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Since $a^*(x_k) \in A$ and $a^*(x_k) + b > x_k$ for all $b \in B$ with $x_k - a^*(x_k) < b \leqslant x_k$,

$$\sum_{\substack{n=x_k+1 \\ \sigma(x_k,n)\geqslant 1}}^{2x_k} 1 \geqslant B(x_k) - B(x_k - a^*(x_k)).$$

Thus,

$$A(x_k)B(x_k) - x_k \geqslant \sum_{\substack{n \\ \sigma(x_k,n)>1}} (\sigma(x_k,n) - 1) + B(x_k) - B(x_k - a^*(x_k)).$$

From Ruzsa's Lemma 2.1,

$$A(x_k)B(x_k) - x_k \geqslant \frac{1}{A(x_k)} \sum_{\substack{n \\ \delta(x_k,n)>1}} (\delta(x_k,n) - 1) + B(x_k) - B(x_k - a^*(x_k)). \tag{2.2}$$

Let

$$D = \{(a,b) : a \in A, b \in B, a \leqslant b \leqslant x_k - a^*(x_k)\}.$$

Then

$$\sum_{\substack{n \\ \delta(x_k,n)>1}} (\delta(x_k,n) - 1) = \sum_{\substack{n \\ \delta(x_k,n)\geqslant 1}} (\delta(x_k,n) - 1) \geqslant |D| - (x_k - a^*(x_k) + 1). \tag{2.3}$$

Now we need a lower bound for $|D|$. We consider the following two cases.

Case 1: $a^*(x_k) > \frac{1}{2}x_k$ for infinitely many k. By (1.1),

$$A\Big(\frac{\delta_0}{5}\frac{a^*(x_k)}{A(x_k)}\Big) = A(x_k) - 1 \quad \text{for all sufficiently large integers } k.$$

Thus, in this case, by Theorem 1.3 and $A(x)B(x)/x \to 1$,

$$\begin{aligned}
|D| &\geqslant \sum_{\substack{\frac{\delta_0}{5}\frac{a^*(x_k)}{A(x_k)} \leqslant b \leqslant x_k - a^*(x_k) \\ b\in B}} A(b) \geqslant A\Big(\frac{\delta_0}{5}\frac{a^*(x_k)}{A(x_k)}\Big)\Big(B(x_k - a^*(x_k)) - B\Big(\frac{\delta_0}{5}\frac{a^*(x_k)}{A(x_k)}\Big)\Big) \\
&= (A(x_k) - 1)B(x_k - a^*(x_k)) - A\Big(\frac{\delta_0}{5}\frac{a^*(x_k)}{A(x_k)}\Big)B\Big(\frac{\delta_0}{5}\frac{a^*(x_k)}{A(x_k)}\Big) \\
&= A(x_k)B(x_k) + A(x_k)(B(x_k - a^*(x_k)) - B(x_k)) - B(x_k - a^*(x_k)) \\
&\quad - A\Big(\frac{\delta_0}{5}\frac{a^*(x_k)}{A(x_k)}\Big)B\Big(\frac{\delta_0}{5}\frac{a^*(x_k)}{A(x_k)}\Big) \\
&\geqslant x_k + \Big(1 - \frac{\delta_0}{4}\Big)\frac{a^*(x_k)}{A(x_k)} + A(x_k)(B(x_k - a^*(x_k)) - B(x_k)) - B(a^*(x_k)) - \frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)} \\
&\geqslant x_k + \Big(1 - \frac{\delta_0}{4}\Big)\frac{a^*(x_k)}{A(x_k)} + A(x_k)(B(x_k - a^*(x_k)) - B(x_k))
\end{aligned}$$

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$$-\Big(1+\frac{\delta_0}{4}\Big)\frac{a^*(x_k)}{A(x_k)}-\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}$$
$$= x_k-\frac{3\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}+A(x_k)(B(x_k-a^*(x_k))-B(x_k))$$

for sufficiently large k. It follows from (2.2) and (2.3) that

$$\begin{aligned}A(x_k)B(x_k)-x_k &\geqslant \frac{x_k}{A(x_k)}-\frac{3\delta_0}{4}\frac{a^*(x_k)}{A(x_k)^2}+B(x_k-a^*(x_k))-B(x_k)-\frac{x_k-a^*(x_k)+1}{A(x_k)}\\ &\quad +B(x_k)-B(x_k-a^*(x_k))\\ &> \frac{a^*(x_k)}{A(x_k)}-\delta_0\frac{a^*(x_k)}{A(x_k)^2}\end{aligned}$$

for sufficiently large k, which is in contradiction with (2.1).

Case 2: $a^*(x_k)\leqslant \frac{1}{2}x_k$ for infinitely many k. By (1.1),

$$A\Big(\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)=A(x_k)-1 \quad \text{for all sufficiently large integers } k.$$

Thus, in this case, by Theorem 1.3 and $A(x)B(x)/x\to 1$,

$$\begin{aligned}|D| &\geqslant \sum_{\substack{\frac{\delta_0}{2}\frac{a^*(x_k)}{A(x_k)}<b\leqslant x_k-a^*(x_k)\\ b\in B}} A\Big(b-\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)\\
&= \sum_{\substack{\frac{\delta_0}{2}\frac{a^*(x_k)}{A(x_k)}<b<a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\\ b\in B}} A\Big(b-\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)+\sum_{\substack{a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\leqslant b\leqslant x_k-a^*(x_k)\\ b\in B}} A\Big(b-\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)\\
&= (A(x_k)-1)\Big(B\Big(a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)-B\Big(\frac{\delta_0}{2}\frac{a^*(x_k)}{A(x_k)}\Big)\Big)\\
&\quad +A(x_k)\Big(B(x_k-a^*(x_k))-B\Big(a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)\Big)\\
&= A\Big(a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)B\Big(a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)-B\Big(a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)\\
&\quad -A\Big(\frac{\delta_0}{2}\frac{a^*(x_k)}{A(x_k)}\Big)B\Big(\frac{\delta_0}{2}\frac{a^*(x_k)}{A(x_k)}\Big)+A(x_k)B(x_k)+A(x_k)(B(x_k-a^*(x_k))-B(x_k))\\
&\quad -A\Big(a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)B\Big(a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)\\
&= A(x_k)B(x_k)+A(x_k)(B(x_k-a^*(x_k))-B(x_k))-B\Big(a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)\\
&\quad -A\Big(\frac{\delta_0}{2}\frac{a^*(x_k)}{A(x_k)}\Big)B\Big(\frac{\delta_0}{2}\frac{a^*(x_k)}{A(x_k)}\Big)\end{aligned}$$

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$$\begin{aligned}
&\geqslant x_k + \Big(1 - \frac{\delta_0}{10}\Big)\frac{a^*(x_k)}{A(x_k)} + A(x_k)(B(x_k - a^*(x_k)) - B(x_k))\\
&\quad - \Big(1 + \frac{\delta_0}{10}\Big)\frac{a^*(x_k) + \frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}}{A(x_k)} - \frac{3\delta_0}{5}\frac{a^*(x_k)}{A(x_k)}\\
&\geqslant x_k - \frac{9\delta_0}{10}\frac{a^*(x_k)}{A(x_k)} + A(x_k)(B(x_k - a^*(x_k)) - B(x_k)),
\end{aligned}$$

for sufficiently large k. It follows from (2.2) and (2.3) that

$$\begin{aligned}
A(x_k)B(x_k) - x_k &\geqslant \frac{x_k}{A(x_k)} - \frac{9\delta_0}{10}\frac{a^*(x_k)}{A(x_k)^2} + B(x_k - a^*(x_k)) - B(x_k) - \frac{x_k - a^*(x_k) + 1}{A(x_k)}\\
&\quad + B(x_k) - B(x_k - a^*(x_k))\\
&> \frac{a^*(x_k)}{A(x_k)} - \delta_0\frac{a^*(x_k)}{A(x_k)^2}
\end{aligned}$$

for sufficiently large k, which is in contradiction with (2.1).

This completes the proof of Theorem 1.5. □

PROOF OF THEOREM 1.6. Let $a_1 = 1$ and $a_2 = 4$. We construct the sequence $a_3, a_4, \dots$ with

$$a_{n+1} \gg n^4 a_n \tag{2.4}$$

and a sequence $n_1, n_2, \dots$ such that $a_1, a_2, \dots, a_{n_k}$ form a complete residue system modulo n_k and $n_k \mid a_{n_k}$. We get such a sequence by a greedy algorithm: let $n_1 = 2$, and if $n_1, n_2, \dots, n_k$ are already defined, then let $n_{k+1} = a_{n_k}$. Since $a_1, \dots, a_{n_k}$ are distinct residues modulo a_{n_k}, we can choose $a_{n_k+1}, \dots, a_{n_{k+1}}$ such that $a_{m+1} \gg m^4 a_m$ for $m = n_k, \dots, n_{k+1} - 1$, $a_{n_k} \mid a_{a_{n_k}}$ and $a_1, \dots, a_{n_{k+1}}$ form a complete residue system modulo n_{k+1}.

For every positive integer k, we further take

$$b_1 = n_k, \quad b_2 = 2n_k, \dots, b_{a_{n_k}/n_k} = \frac{a_{n_k}}{n_k} \cdot n_k.$$

Then $a_i + b_j$ for $1 \leqslant i \leqslant p, 1 \leqslant j \leqslant a_{n_k}/n_k$, form a complete residue system modulo a_{n_k}. From the definition of $L(a_{n_k})$,

$$L(a_{n_k}) = \frac{a_{n_k}}{n_k}. \tag{2.5}$$

For the set $A = \{a_k\}_{k=1}^{\infty}$ and every positive integer k, define q_k by

$$q_k = \Big\lfloor \frac{a_{k+1}}{k^4 a_k} \Big\rfloor, \quad \text{that is,} \quad q_k \cdot k^4 a_k < a_{k+1} \leqslant (q_k + 1) \cdot k^4 a_k. \tag{2.6}$$

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Define the same sets A, B as in [2, Theorem 3] (replacing m_k by a_k) and write $A_k = A \bigcap [1, a_k]$. Take $U_k \subseteq [1, a_k]$ so that $|U_k| = L(a_k)$ and $A_k + U_k$ contains every residue module a_k. Let

$$V_k = U_k + \left\{(q_k - 1)a_k, q_k a_k, (q_k + 1)a_k, \ldots, \left\lfloor \frac{q_{k+1}a_{k+1}}{a_k} \right\rfloor a_k \right\} \quad \text{and} \quad B = \bigcup_{k=1}^{\infty} V_k.$$

Let $q_k a_k \leqslant x \leqslant q_{k+1}a_{k+1}$. The sequence $\{q_k\}_{k=1}^{\infty}$ defined in (2.6) is increasing to infinity by (2.4) and $A(q_k a_k) \sim A(a_k)$. (In fact, $A(q_k a_k) = k = A(a_k)$ by (2.6).) By the same proof as in [2, Theorem 3], A, B are additive complements and $A(x)B(x) \sim x$. Thus, the set A with (2.4) has an exact complement B. Obviously, such an A with (2.4) satisfies (1.1).

Finally, we prove that (1.3) holds for infinitely many x_k. For x with $q_k a_k \leqslant x < (q_{k+1} - 1)a_{k+1}$, we have $k \leqslant A(x) \leqslant k + 1$ and

$$B(x) \leqslant \left(\left\lfloor \frac{x}{a_k} \right\rfloor - q_k + 2\right)L(a_k) + \sum_{i=2}^{k} \left(\left\lfloor \frac{q_i a_i}{a_{i-1}} \right\rfloor - q_{i-1} + 2\right)L(a_{i-1}). \tag{2.7}$$

By Theorem 1.2(b), $L(a_{k-1}) \leqslant 2a_{k-1}/(k-1)$ for large k. From (2.6),

$$\sum_{i=2}^{k} \left(\left\lfloor \frac{q_i a_i}{a_{i-1}} \right\rfloor - q_{i-1} + 2\right)L(a_{i-1}) \leqslant (k-1)\frac{2q_k a_k}{k-1} = O(q_k a_k) = O\left(\frac{a_{k+1}}{(k+1)^4}\right).$$

It is easy to see that, for large k,

$$(q_k - 2)L(a_k) \leqslant 2\frac{q_k a_k}{k} = O\left(\frac{a_{k+1}}{(k+1)^5}\right).$$

It follows from (2.7) that

$$B(x) \leqslant \frac{x}{a_k}L(a_k) + O\left(\frac{a_{k+1}}{(k+1)^4}\right). \tag{2.8}$$

Choose $x_k = a_{n_k+1}$, where n_k is the index satisfying (2.5). Then by (2.8),

$$\begin{aligned} A(x_k)B(x_k) - x_k - \frac{a^*(x_k)}{A(x_k)} &\leqslant (n_k + 1)\frac{x_k}{n_k} - x_k - \frac{x_k}{n_k + 1} + O\left(\frac{x_k}{(n_k+1)^3}\right) \\ &= \frac{x_k}{A(x_k)^2} + O\left(\frac{x_k}{A(x_k)^3}\right). \end{aligned}$$

This completes the proof of Theorem 1.6. □

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