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EXACT DISTRIBUTIONS IN A JUMP-DIFFUSION STORAGE MODEL

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We consider a reflected independent superposition of a Brownian motion and a compound Poisson process with positive and negative jumps, which can be interpreted as a model for the content process of a storage system with different types of customers under heavy traffic. The distributions of the duration of a busy cycle and the maximum content during a cycle are determined in closed form.

1. INTRODUCTION

We consider the the basic jump-diffusion process $X = (X(t))_{t\geq 0}$ defined by $X(t) = B(t) + Y(t)$, where (i) $B = (B(t))_{t\geq 0}$ is a Brownian motion (BM) with drift γ and variance σ^2 per unit time and (ii) $Y = (Y(t))_{t\geq 0}$ is a compound Poisson process with independent and identically distributed (i.i.d.) positive *and* negative jumps. The process of interest to us is the corresponding *storage process* $W = (W(t))_{t\geq 0}$, which is the reflection of X at zero; that is, $W(t) = X(t) + L(t)$, where $L(t) = -\inf_{0\leq s\leq t} X(s)$. This storage process has many interpretations. For example, in stochastic finance, W can represent a cash fund serving two types of customer. There are "small" customers, who very frequently withdraw and deposit small amounts, thereby contributing the Brownian component of W. The second type of transactions arises from "big" firms, which move large sums in and out of the fund; their contribution is modeled

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by the compound Poisson process Y. The reflection of X at 0 means that, for the cash fund, overdraft is not allowed. W can be also interpreted as the workload process of a queuing system that serves three types of customer: very frequent small service requirements, occasional big ones whose interarrival times are $\exp(\lambda)$ distributed and whose service times are nonnegligible, and occasional *negative* big ones which remove random amounts of work from the system and whose interarrival times are $\exp(\eta)$ distributed. For related models in queuing, we refer to Kella and Whitt [8,9], Bardhan [2], Perry and Stadje [13], and Perry [12]; for applications in stochastic finance, see for example, Schäl [17], Moeller [11], Bardhan and Chao [3], and Perry and Stadje [14]. One can think of several other applications to storage processes.

In this article, we study two important characteristics of W: the duration of a *busy cycle* and the *maximum value* attained during such a cycle. A busy cycle can be defined as follows: It starts when the system is (or becomes) empty, contains at least one (positive or negative) jump, and ends at the next time of emptiness. We will derive exact solutions for a wide class of phase-type distributions for the jump sizes. Under the assumption that all jumps are positive, the problem was addressed in Perry and Stadje [15] for general distributions; in the two-sided problem considered here, we have to restrict ourselves to the phase-type case to obtain explicit results.

Related first-exit problems are important in insurance mathematics and have therefore been studied from many angles in this context; see, for example, Picard and Lefèvre [16], De Vylder and Goovaerts [5,6], and Gerber and Shiu [7] and the references given in these articles. In the case when all jumps are positive, there are also general results for Lévy processes [4].

Let us now specify our model. Without loss of generality we assume that $\sigma^2 = 1$. The positive (negative) jumps have the common distribution function G (H) and arrive with intensity rate λ (η). Thus, the arrival rate of Y is $\lambda + \eta$ and the common Laplace transform (LT) of the jump sizes is

$$K^*(\alpha) = \frac{\lambda}{\lambda+\eta} G^*(\alpha) + \frac{\eta}{\lambda+\eta} H^*(\alpha), \tag{1.1}$$

where $G^*(\alpha)$ and $H^*(\alpha)$ are the LTs of G and H, respectively. An important function in our derivations is the *exponent* of X, defined by

$$\varphi(\alpha) = \log E(e^{-\alpha X(1)}) = \frac{\alpha^2}{2} - \gamma\alpha - (\lambda+\eta)[1 - K^*(\alpha)]. \tag{1.2}$$

We assume that the LTs G^* and H^* are of the form

$$G^*(\alpha) = \sum_{i=1}^{n} p_i \prod_{j=1}^{k_i} \frac{\mu_{ij}}{\mu_{ij}+\alpha}, \qquad H^*(\alpha) = \sum_{i=1}^{m} q_i \prod_{j=1}^{l_i} \frac{\nu_{ij}}{\nu_{ij}-\alpha}, \tag{1.3}$$

where $n, k_1, \ldots, k_n, m, l_1, \ldots, l_m \in \mathbb{N}$, $p_1, \ldots, p_n$ and $q_1, \ldots q_m$ are positive, $\sum_{i=1}^n p_i = \sum_{i=1}^m q_i = 1$, and $\mu_{ij} > 0$ and $\nu_{ij} > 0$. Thus, G and H are finite mixtures of finite convolutions of (not necessarily identical) exponential distributions. This class covers a wide range of phase-type distributions. For example, Coxian and hyperexponential distributions have LTs of this type (see Asmussen [1, p. 74]). G and H possess probability densities g and h which are easy to compute by convolution of exponential functions. For simplicity, we assume that no two μ_{ij}'s and no two ν_{ij}'s coincide, so that all poles of G^* and of H^* are simple (otherwise we obtain the desired functionals of W by taking a simple limit). We denote by $1/\mu$ and $-1/\nu$ the expected positive and negative jump sizes, respectively, and assume that the total drift $\gamma + (\lambda/\mu) - (\eta/\nu)$ is negative, so that W is a regenerative process. Let $\tau = \inf\{t \geq 0 | Y(t) > 0\}$ be the arrival time of the first big customer. Then, $T = \inf\{t \geq \tau | W(t) \leq 0\}$ is the length of the first busy cycle, and $M = \sup_{0\leq t\leq T} W(t)$ is the first cycle maximum. Note that $P(T = \tau) > 0$ because the first jump can be negative with absolute value greater than $W(\tau-)$. In the stochastic finance interpretation, T is the time until ruin and M is the maximal value achieved by the cash fund during its lifetime. We note here that T might be a point of continuity or discontinuity of the sample path $W(\cdot)$, depending on whether at time T level 0 is reached by the Brownian component of W or is crossed by a negative jump. In the latter case, $W(T-) > 0 > W(T)$; however, we always have $W(T+) = 0$, and $T+$ is the beginning of a new cycle.

We derive the distribution function of M in Section 2 and the LT of T is obtained in Section 3. Finally, we show how to extend this result to the case of arbitrarily distributed upward jumps.

2. CYCLE MAXIMUM, PHASE-TYPE JUMPS

The first cycle can be partitioned into the two parts $[0,\tau)$ and $[\tau, T]$, and, obviously, $M = \max(M_1, M_2)$ where $M_1 = \sup_{0\leq t\leq \tau} W(t)$ and $M_2 = \sup_{\tau\leq t\leq T} W(t)$. Regarding the first cycle, we need the following two lemmas which have been proved by Yor [18] and Perry and Stadje [15].

Lemma 1: *Let $\theta_+(\beta)$ and $\theta_-(\beta)$ be the positive and the negative root of the equation $\theta^2 - \gamma\theta - \beta = 0$; that is, $\theta_\pm(\beta) = [\gamma \pm (\gamma^2 + 4\beta^{1/2})]/2$. Let $\theta_\pm = \theta_\pm(\lambda + \eta)$. $W(\tau -)$ and $L(\tau -)$ are independent and*

$$W(\tau -) \backsim \exp(|\theta_-|), \qquad L(\tau -) \backsim \exp(|\theta_+|).$$

Lemma 2: *For all $0 \leq x \leq y$,*

$$P(M_1 > y, W(\tau -) \in dx) = \Gamma(\lambda + \eta, y) b_y(x)\, dx,$$

where

$$\Gamma(\beta, y) = \frac{\theta_+(\beta) - \theta_-(\beta)}{\theta_+(\beta)e^{-\theta_-(\beta)y} - \theta_-(\beta)e^{-\theta_+(\beta)y}}$$

and $b_y(\cdot)$ *is the density whose LT is given by*

$$B_y^*(\alpha) = \frac{\theta_+|\theta_-|(\alpha e^{-\theta_+ y} - \theta_+ e^{-\alpha y})}{(\alpha - \theta_+)(\alpha + |\theta_-|)}.$$

We remark that the probability measure corresponding to $b_y(\cdot)$ is given by

$$\exp(|\theta_-|) * \exp_-(\theta_+) * \epsilon_y - e^{-\theta_+ y} \exp(|\theta_-|) * \exp_-(\theta_+) - e^{-\theta_+ y} \exp(|\theta_-|),$$

where ϵ_y is the point mass at y, $*$ denotes convolution, and $\exp_-(\theta_+)$ is the distribution with density $\theta_+ \exp(\theta_+ x) 1_{(-\infty,0)}(x)$. It is thus easy to write down an explicit formula for $b_y(\cdot)$.

We now start with the analysis of M. Lemma 3 enables us to express its distribution in terms of those of M_1 and M_2, conditional on $W(\tau -)$ and $W(\tau)$, respectively.

LEMMA 3: *For all* $x > 0$,

$$\begin{aligned} P(M \le x) = \int_0^\infty & P(M_1 \le x | W(\tau -) = y) |\theta_-| e^{-|\theta_-| y} \\ & \times \left[\frac{\eta}{\lambda + \eta} (1 - G(y)) + \frac{\eta}{\lambda + \eta} \int_0^y P(M_2 \le x | W(\tau) = w) h(w - y)\, dw \right. \\ & \left. + \frac{\lambda}{\lambda + \eta} \int_y^\infty P(M_2 \le x | W(\tau) = w) g(y - w)\, dw \right] dy. \end{aligned}$$

PROOF: By the law of total probability and Lemma 1,

$$\begin{aligned} P(M \le x) &= \int_0^\infty \int_{-\infty}^\infty P(M_1 \le x, M_2 \le x | W(\tau -) = y, W(\tau) = w) \\ & \qquad \times P(W(\tau) \in dw | W(\tau -) = y) |\theta_-| e^{-|\theta_-| y}\, dy \\ &= \int_0^\infty \int_{-\infty}^\infty P(M_1 \le x | W(\tau -) = y) P(M_2 \le x | W(\tau) = w) \\ & \qquad \times P(W(\tau) \in dw | W(\tau -) = y) |\theta_-| e^{-|\theta_-| y} dy, \end{aligned} \tag{2.1}$$

where the second equality follows from the strong Markov property of W at τ, which implies that M_1 and M_2 are conditionally independent, given $W(\tau -)$ and $W(\tau)$. By the structure of the jump size distribution,

$$P(W(\tau) \in dw | W(\tau -) = y) = \begin{cases} \dfrac{\eta}{\eta + \lambda} g(y - w)\, dw, & y > w \\ \dfrac{\lambda}{\eta + \lambda} h(w - y)\, dw, & y < w. \end{cases}$$

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Moreover, $w < 0$ implies that $P(M_2 \le x | W(\tau) = w) = 1$ for all $x > 0$. The assertion now follows from (2.1). ■

By Lemmas 1 and 2, $P(M_1 \le y, W(\tau -) \in dx)$ has the density $|\theta_-|e^{-|\theta_-|x} - \Gamma(\lambda+\eta, y)b_y(x)$, so that

$$P(M_1 \le y | W(\tau -) = x) = 1 - |\theta_-|^{-1} e^{|\theta_-|x} \Gamma(\lambda+\eta, y) b_y(x).$$

By Lemma 3, it remains to compute $P(M_2 \le x | W(\tau) = w)$ for $w > 0$. The main tool in the derivation of this distribution is the process

$$Z(t) = (\varphi(\alpha) - \beta) \int_0^t e^{-\alpha X(s) - \beta s}\, ds + e^{-\alpha X(0)} - e^{-\alpha X(t) - \beta t}, \qquad t \ge 0,$$

where $\varphi(\alpha)$ is given by (1.2) (and (1.1) and (1.3)). We consider the stopping time

$$T_{w,x} = \inf\{t > 0 : X(t) \ge x - w \quad \text{or} \quad X(t) \le -w\}, \qquad x > w \ge 0.$$

It is easy to check that $(Z(t))_{t \ge 0}$ is a martingale for every α satisfying $-\mu < \operatorname{Re} \alpha < \nu$ and every $\beta \ge 0$ (see, e.g., Perry and Stadje [14]). Applying the optional sampling theorem yields

$$(\varphi(\alpha) - \beta) E\left(\int_0^{T_{w,x}} e^{-\alpha X(s) - \beta s}\, ds \right) = -1 + E(e^{-\alpha X(T_{w,x}) - \beta T_{w,x}}). \qquad \textbf{(2.2)}$$

By the structure of $G^*(\cdot)$ and $H^*(\cdot)$, any jump of W can be thought of as being generated by first choosing between G and H with probabilities $\lambda/(\lambda+\eta)$ and $\eta/(\lambda+\eta)$, respectively, and depending on this choice selecting either an index i from $\{1,\dots,n\}$ or from $\{1,\dots,m\}$ according to the probability distribution $(p_1,\dots,p_n)$ or $(q_1,\dots,q_m)$ and then carrying out k_i or l_i successive phases which are independent and exponentially distributed with means $1/\mu_{i1},\dots,1/\mu_{ik_i}$ or $1/\nu_{i1},\dots,1/\nu_{il_i}$. Let $C_{G,i,j}$ ($C_{H,i,j}$) be the event that at time $T_{w,x}$, the level $x - w$ ($-w$) is crossed by an upward (downward) phase with distribution $\exp(\mu_{ij})$ ($\exp(\nu_{ij})$). At time $T_{w,x}$ there is, of course, also the possibility of hitting $x - w$ or $-w$ exactly, due to the Brownian component: Let C_0 and C_1 be the events $\{X(T_{w,x}) = x - w\}$ and $\{X(T_{w,x}) = -w\}$, respectively. Let $h_k = h_k(\beta | w, x) = E(e^{-\beta T_{w,x}} 1_{C_k})$, $k = 0,1$, and

$$h_{G,i,j} = h_{G,i,j}(\beta | w, x) = E(e^{-\beta T_{w,x}} 1_{C_{G,i,j}}), \qquad i = 1,\dots,n,\ j = 1,\dots,k_i$$

$$h_{H,i,j} = h_{H,i,j}(\beta | w, x) = E(e^{-\beta T_{w,x}} 1_{C_{H,i,j}}), \qquad i = 1,\dots,m,\ j = 1,\dots,l_i.$$

Given $C_{G,i,j}$, the overshoot $X(T_{w,x}) - (x - w)$ is $\exp(\mu_{ij})$ distributed and independent of $T_{w,x}$. Similarly, conditional on $C_{H,i,j}$, the undershoot $X(T_w) + w$ is $\exp(\nu_{ij})$

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distributed and independent of $T_{w,x}$. Using the formula of total probability, we can thus rewrite the expected value $E(e^{-\alpha X(T_{w,x})-\beta T_{w,x}})$ on the right-hand side of (2.2) as

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{k_i} P(C_{G,i,j})e^{-\alpha(x-w)}\,\frac{\mu_{ij}}{\mu_{ij}+\alpha}\,E(e^{-\beta T_{w,x}}\,|\,C_{G,i,j})\\
&\quad + P(C_0)e^{-\alpha(x-w)}E(e^{-\beta T_{w,x}}\,|\,C_0) + P(C_1)e^{\alpha w}(e^{-\beta T_{w,x}}\,|\,C_1)\\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{l_i} P(C_{H,i,j})e^{\alpha w}\,\frac{\nu_{ij}}{\nu_{ij}-\alpha}\,E(e^{-\beta T_{w,x}}\,|\,C_{H,i,j})\\
&= \sum_{i=1}^{n}\sum_{j=1}^{k_i} e^{-\alpha(x-w)}\,\frac{\mu_{ij}}{\mu_{ij}+\alpha}\,h_{G,i,j} + e^{-\alpha(x-w)}h_0 + e^{\alpha w}h_1\\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{l_i} e^{\alpha w}\,\frac{\nu_{ij}}{\nu_{ij}-\alpha}\,h_{H,i,j}.
\end{aligned}\qquad \textbf{(2.3)}$$

The left-hand side of (2.2) becomes zero for values of α satisfying the equation $\varphi(\alpha)-\beta=0$, which can be written as

$$\Gamma(\alpha)\equiv\lambda\sum_{i=1}^{n}p_i\prod_{j=1}^{k_i}\frac{\mu_{ij}}{\mu_{ij}+\alpha}+\eta\sum_{i=1}^{m}q_i\prod_{j=1}^{l_i}\frac{\nu_{ij}}{\nu_{ij}-\alpha}=\gamma\alpha-\frac{\alpha^2}{2}+\lambda+\eta+\beta. \qquad \textbf{(2.4)}$$

Let $N=k_1+\cdots+k_n$ and $M=l_1+\cdots+l_m$ and let $c_1<\cdots<c_N<0$ and $0<d_1<\cdots<d_M$ be the values of $-\mu_{ij}$ and ν_{ij}, respectively, in ascending order. Since (2.4) is equivalent to a polynomial equation of degree $N+M+2$ for α, there are exactly $N+M+2$ (possibly complex) solutions, each counted with its multiplicity. Let us show that, for any $\beta>0$, (2.4) has a real solution in each of the intervals $(0,d_1),(d_1,d_2),\ldots,(d_{M-1},d_M),(d_M,\infty)$. The function $\Gamma(\alpha)$ is continuous on each of the intervals $(0,d_1),(d_1,d_2),\ldots,(d_{M-1},d_M),(d_M,\infty)$ and satisfies $\lim_{\alpha\nearrow d_i}\Gamma(\alpha)=\infty$ and $\lim_{\alpha\searrow d_i}\Gamma(\alpha)=-\infty$ for $i=1,\ldots,M$. For any $\beta>0$, the parabola on the right-hand side of (2.4) takes a larger value than $\Gamma(\alpha)$ at $\alpha=0$ and tends to $-\infty$ as $\alpha\to\infty$. Thus, there is a solution of (2.4) in each of the intervals $(0,d_1),(d_1,d_2),\ldots,(d_{M-1},d_M),(d_M,\infty)$. Similarly, it is seen that there is a real solution in each interval $(-\infty,c_1),(c_1,c_2),\ \ldots,(c_{N-1},c_N),(c_N,0)$. Hence, all $N+M+2$ solutions of (2.4) are distinct and real; we denote them by $\alpha_1(\beta)<\cdots<\alpha_{N+M+2}(\beta)$.

By (2.3), the right-hand side of (2.2) can be analytically extended to $\mathbb{C}$ (as a function of α). The expected value on the left-hand side of (2.2) is also analytic in every $\alpha\in\mathbb{C}$; note that the integrand $e^{-\alpha X(s)-\beta s}$ is bounded by $\exp(|\alpha|\max[x-w,w])$ for $s\in[0,T_{w,x})$. Therefore, after inserting (2.3) in (2.2) we obtain an identity that is valid for all $\alpha\in\mathbb{C}\setminus\{c_1,\ldots,c_N,d_1,\ldots,d_M\}$. In particular, let $\alpha=\alpha_\ell(\beta)$, $\ell=1,\ldots,N+M+2$. Then the left-hand side of (2.2) vanishes and we get the $N+M+2$ equations

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$$1 = \sum_{i=1}^{n}\sum_{j=1}^{k_i} e^{-\alpha_\ell(\beta)(x-w)} \frac{\mu_{ij}}{\mu_{ij}+\alpha_\ell(\beta)} h_{G,i,j} + e^{-\alpha_\ell(\beta)(x-w)} h_0 + e^{\alpha_\ell(\beta)w} h_1 + \sum_{i=1}^{m}\sum_{j=1}^{l_i} e^{\alpha_\ell(\beta)w} \frac{\nu_{ij}}{\nu_{ij}-\alpha_\ell(\beta)} h_{H,i,j}, \qquad \ell = 1,\dots,N+M+2. \tag{2.5}$$

The restricted Fourier transforms $h_0(\beta)$, $h_1(\beta)$, $h_{G,i,j}(\beta|w,x)$, and $h_{H,i,j}(\beta|w,x)$ can now be computed from (2.5). However, this also yields the distribution of M_2 by the following lemma.

LEMMA 4:

$$P(M_2 \le x|W(\tau)=w) = h_1(0|w,x) + \sum_{i=1}^{m}\sum_{j=1}^{l_i} h_{H,i,j}(0|w,x).$$

Summarizing, we have proved the following result.

THEOREM 1: *The distribution of M is given by*

$$\begin{aligned} P(M \le x) = \int_0^\infty &[|\theta_-|e^{-|\theta_-|y} - \Gamma(\lambda+\eta,y)b_x(y)] \\ &\times \Bigg(\frac{1}{\lambda+\eta}(\eta(1-G(y))) \\ &+ \eta\int_0^y \Bigg[h_1(0|w,x) + \sum_{i=1}^{m}\sum_{j=1}^{l_i} h_{H,i,j}(0|w,x)\Bigg] h(w-y)\,dw \\ &+ \lambda\int_y^\infty \Bigg[h_1(0|w,x) + \sum_{i=1}^{m}\sum_{j=1}^{l_i} h_{H,i,j}(0|w,x)\Bigg] h(w-y)\,dw\Bigg)\,dy. \end{aligned}$$

3. THE DURATION OF A BUSY CYCLE

Using the results of Section 2 we can also derive an explicit formula for the LT of T. Let $h_1(\beta|w) = \lim_{x\to\infty} h_1(\beta|w,x)$ and $h_{H,i,j}(\beta|w) = \lim_{x\to\infty} h_{H,i,j}(\beta|w,x)$.

THEOREM 2:

$$\begin{aligned} E(e^{-\beta T}) = &\frac{\lambda}{\lambda+\alpha}\frac{\eta}{\lambda+\eta}\int_0^\infty H(-x)|\theta_-|e^{-|\theta_-|x}\,dx \\ &+ \frac{\lambda}{\lambda+\alpha}\frac{\eta}{\lambda+\eta}\int_0^\infty \Bigg(h_1(\beta|w) + \sum_{i=1}^{m}\sum_{j=1}^{l_i} h_{H,i,j}(\beta|w)\Bigg)|\theta_-|e^{-|\theta_-|w}dw \\ &+ \frac{\lambda}{\lambda+\alpha}\frac{\lambda}{\lambda+\eta}\int_0^\infty \Bigg(h_1(\beta|w) + \sum_{i=1}^{m}\sum_{j=1}^{l_i} h_{H,i,j}(\beta|w)\Bigg) \\ &\times |\theta_-|e^{-|\theta_-|w}\Bigg(\int_0^w e^{|\theta_-|s} g(s)\,ds\Bigg)\,dw. \end{aligned} \tag{3.1}$$

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PROOF: The busy cycle $[0, T)$ can be decomposed in $[0, \tau)$ and $[\tau, T)$. Therefore, all three terms on the right-hand side of (3.1) include the LT $\lambda/(\lambda + \alpha)$. According to the jump size at τ, three cases can be distinguished:

(i) The first jump is negative and its absolute value is greater than $W(\tau-)$. By Lemma 1, $W(\tau-) \backsim \exp(|\theta_-|)$. The probability that a negative jump arrives before a positive jump is $\eta/(\lambda + \eta)$. The absolute value of the negative jump at τ is greater than $W(\tau-)$ with probability $P(|V| > U) = \int_0^\infty P(V < -x)|\theta_-|e^{-|\theta_-|x}\,dx$, where U and V are independent; $U \backsim \exp(|\theta_-|)$ and V has the distribution function H. Multiplying together, we get the first term on the right-hand side of (3.1).

(ii) The first jump is negative and its absolute value is smaller than $W(\tau-)$. The conditional distribution of $W(\tau)$ given the event $\{W(\tau-) > W(\tau) > 0\}$ is equal to $P(U - |V| \in dw \mid U - |V| > 0)$ with U and V as in (i). But this distribution is $\exp(\theta_-)$. For any $w > 0$, the conditional LT of $T - \tau$, given $\{W(\tau) = w\}$, is equal to $h_1(\beta|w) + \sum_{i=1}^m \sum_{j=1}^{l_i} h_{H,i,j}(\beta|w)$. This case contributes the second summand in (3.1).

(iii) The first jump is positive. The probability for this to happen is $\lambda/(\lambda + \eta)$, and the distribution of $W(\tau)$ after a positive jump at τ is the convolution $\exp(|\theta_-|)$ and G. This explains the third term in (3.1). ■

If there are no upward jumps and the downward jumps have a phase-type distribution function H with LT H^* as given in (1.3), the following alternative derivation of the LT of T and some related functionals is possible. We use the Kella–Whitt [10] martingale for Lévy processes with reflection at zero and optional sampling for T to obtain the identity

$$(\varphi(\alpha) - \beta)E\left(\int_0^T e^{-\alpha W(s) - \beta s}\,ds\right) = -1 + E(e^{-\alpha W(T) - \beta T}) + \alpha E\left(\int_0^T e^{-\beta s}\,dL(s)\right), \tag{3.2}$$

where $L(t) = -\inf_{0 \le s \le t} X(s)$ is the associated local time process. The crossing of zero at time T is either due to the Brownian motion part of W or to one of the $\exp(\nu_{ij})$ distributed phases. Proceeding similarly as in Section 2, we define

$$\Gamma_0(\beta) = E(e^{-\beta T} 1_{\{W(T)=0\}}),$$
$$\Gamma_{ij}(\beta) = E(e^{-\beta T} 1_{C_{ij}}),$$

where C_{ij} is the event that zero is first crossed by the jth phase of the ith mixture component of H. It is now clear that $E(e^{-\alpha W(T) - \beta T})$ can be decomposed as follows:

$$E(e^{-\alpha W(T) - \beta T}) = \Gamma_0(\beta) + \sum_{i=1}^m \sum_{j=1}^{l_i} \frac{\nu_{ij}}{\nu_{ij} - \alpha} - \alpha \Gamma_{ij}(\beta). \tag{3.3}$$

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The equation $\varphi(\alpha) - \beta = 0$ can be written as

$$\eta H^*(\alpha) = \beta + \eta + \gamma\alpha - \frac{\alpha}{2}. \quad \textbf{(3.4)}$$

As in Section 2, it is shown that (3.4) has a root in each of the intervals $(-\infty,0),(0,d_1),(d_1,d_2),\ldots,(d_{M-1},d_M),(d_M,\infty)$ and, thus, has $M+2$ real roots $\alpha_1^*(\beta) < 0 < \alpha_2^*(\beta) < \cdots < \alpha_{M+2}^*(\beta)$. Inserting (3.3) in (3.2) and setting $\alpha = \alpha_i^*(\beta)$, $i = 1,\ldots,M+2$, yields $M+2$ linear equations for the $M+2$ unknowns $\Gamma_0(\beta)$ and $\Gamma_{ij}(\beta)$ $(i = 1,\ldots,m, j = 1,\ldots,l_i)$ and $E(\int_0^T e^{-\beta s}\, dL(s))$. Thus, we obtain explicit expressions for the functionals $E(e^{-\alpha W(T)-\beta T})$, $E(\int_0^T e^{-\beta s}\, dL(s))$, and $E(\int_0^T e^{-\alpha W(s)-\beta s}\, ds)$.

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