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ON WHEN A TOPOLOGICALLY SIMPLE SEMIGROUP IS SIMPLE

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Abstract

The compact semigroups in which each topologically simple subsemigroup is simple are characterized as those in which no subgroup contains an element of infinite order. It is also shown that a locally compact topologically simple subsemigroup of a compact semigroup must be simple. The note closes with an open problem.

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This note is motivated by the recent note of Chow (1975) in which he conjectured that a topologically simple subsemigroup of a compact topological semigroup must be simple. A semigroup is called *simple* if it contains no proper ideal and is called *topologically simple* if each of its ideals is dense.

Chow's conjecture has rather immediate counterexamples. If a is an element of infinite order in a compact group, then the subsemigroup $\{a, a^2, a^3, \ldots\}$ is a topologically simple subsemigroup which is not simple. This is the prototype of all counterexamples, for, as is shown in Theorem A, any counterexample must contain one of this type.

We are able, however, to characterize the compact semigroups in which *each* topologically simple subsemigroup is simple as those in which no subgroup contains an element of infinite order. The question of *which* topologically simple subsemigroups of a compact semigroup must be simple is then addressed. It is shown that such holds for locally compact subsemigroups, but the general question remains open.

Our first lemma shows that those topologically simple semigroups which are embeddable in a compact semigroup are exactly the dense subsemigroups of compact completely simple semigroups.

LEMMA 1. *A subsemigroup T of a compact semigroup is topologically simple if, and only if, $\bar{T}$ is simple* (*and hence completely simple*).

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PROOF. We first suppose that T is topologically simple and show that $\bar{T}$ is simple. If J is a proper ideal of $\bar{T}$ and $a \in J$, then $\bar{T}a\bar{T}$ is a proper closed ideal of $\bar{T}$. Hence $(\bar{T}a\bar{T}) \cap T$ is a proper ideal of T which is closed in T, a fact contradicting the topological simplicity of T. The converse is immediate since the closure of an ideal of T must be an ideal of $\bar{T}$.

We can now characterize those compact semigroups in which each topologically simple subsemigroup is simple. Gratitude is expressed to Maria Klawe (University of Alberta) for sharing an unpublished result which motivated the next theorem.

THEOREM A. *In a compact semigroup S the following are equivalent:*

A. *Each topologically simple subsemigroup of S is simple.*

B. *No subgroup of S contains an element of infinite order.*

PROOF. Since the closure of any subgroup of S is again a subgroup of S (Paalman-de Miranda (1970, p. 19); Hofmann and Mostert (1966, p. 13)), it is immediate from the remark in our second paragraph that A implies B.

To show that B implies A, let T denote a topologically simple subsemigroup of S. We wish to show that T is simple. From Lemma 1 we know that $\bar{T}$ is completely simple and hence a union of groups. Thus from condition B we have that each element of $\bar{T}$, and hence T, is of finite order. T must then also be a union of groups. It follows from the known structure of completely simple semigroups that any subsemigroup of a completely simple semigroup which is a union of groups must itself be completely simple. Hence, T is simple.

We turn now to the question of which topologically simple subsemigroups of a compact semigroup are simple.

LEMMA 2. *A locally compact subsemigroup of a compact topological group is a compact subgroup of the group.*

PROOF. Let T be a locally compact subsemigroup of the group G. Since a closed subsemigroup of a compact group must be a group (Paalman-de Miranda (1970, p. 22); Hofmann and Mostert (1966, p. 15)), we may assume that $\bar{T} = G$. T is then open in G since it is locally compact and dense. We now show that $T = G$. For $g \in G$, we know that $T^{-1}g$ is open in G and so must intersect the dense subset T of G. That is, there exist elements x and y of T so that $s = y^{-1}g$. But then $g = yx \in T^2 \subset T$ and hence $G = T$.

THEOREM B. *A locally compact topologically simple subsemigroup of a compact semigroup is completely simple.*

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PROOF. Let T be a locally compact topologically simple subsemigroup of the compact semigroup S. In view of Lemma 1 we may assume that $\bar{T}=S$ with S completely simple. Then S is a union of groups, each maximal subgroup G of S is closed (Paalman-de Miranda (1970, p. 19); Hofmann and Mostert (1966, p. 13)), and $T\cap G$ is a locally compact subsemigroup of G so that, in view of Lemma 2, $T\cap G$ is a subgroup of G. It follows that T is a union of groups and hence, as in the proof of Theorem A, completely simple.

Clark, Mukherjea and Tserpes (1975) obtained independently a result analogous to Theorem B.

We close with a problem. A simple subsemigroup of a completely simple semigroup need not be completely simple, for as noted by Chow, $T=\{(x,y)\,|\,x>0,y>0\}$ is a simple subsemigroup of the group $G=\{(x,y)\,|\,x>0\}\subset R\times R$ with multiplication $(x.y)(u,v)=(xu,xv+y)$. The question arises whether, in a *compact* (completely) simple semigroup, simplicity and complete simplicity of subsemigroups are equivalent.

PROBLEM. *Must a simple subsemigroup T of a compact simple semigroup S be completely simple? In particular must a simple subsemigroup of a compact group be a subgroup?*

One may assume, in view of Lemma 1, that T is dense in S. We also note that since all idempotents of S are primitive, T will be completely simple exactly when it contains an idempotent.

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