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ABSTRACT

We classify the horizontal $SL(2)$ s and \mathbb{R} -split polarized mixed Hodge structures on a Mumford–Tate domain.

1. Introduction

A variation of (pure, polarized) Hodge structure gives rise to a horizontal holomorphic mapping into a flag domain D ; here *horizontal* indicates that the image of the map satisfies a system of partial differential equations known as the *infinitesimal period relation* (or *Griffiths' transversality condition*). Such maps arise as (lifts of) period mappings associated with families of polarized algebraic manifolds. The celebrated nilpotent orbit and $SL(2)$ -orbit theorems of Schmid [Sch73] and Cattani *et al.* [CKS86], describe the asymptotic behavior of a horizontal mapping, and play a fundamental role in the analysis of singularities of the period mapping (equivalently, degenerations of Hodge structure); cf. the work of Kato and Usui [KU09]. Two of the more striking applications of the nilpotent and $SL(2)$ -orbit theorems are: (i) Cattani *et al.*'s [CDK95] proof of the algebraicity of Hodge loci, which provides some of the strongest evidence for the Hodge conjecture; and (ii) the proof of Deligne's conjectured isomorphism between the L^2 and intersection cohomologies of a polarized variation of Hodge structure with normal crossing singularities over a compact Kähler manifold (first proved by Zucker [Zuc79] in the case of a one-dimensional base, followed by Cattani and Kaplan's [CK85] proof in the case of dimension two and weight one, with the general case established independently by Cattani *et al.* [CK87] and Kashiwara and Kawai [KK85]), and the resulting corollary that the intersection cohomology carries a pure Hodge structure.

As a consequence it became an important problem to describe the $SL(2)$ s appearing in Schmid's theorem, and to that end partial results were obtained by Cattani and Kaplan [CK77, CK78] and Usui [Usu93]. The main result (Theorem 5.9) of this paper is a classification of those objects. It is a corollary of Theorem 5.5, which classifies the \mathbb{R} -split polarized mixed Hodge structures (PMHSs), and the familiar equivalence

$$\{\mathbb{R}\text{-split PMHS on } D\} \longleftrightarrow \{\text{horizontal } SL(2)\text{s on } D\}, \quad (1.1)$$

which follows from the work of Cattani *et al.* [CK82a, CK82b, CKS86, Del82]. These results are established for Mumford–Tate domains [GGK12]; the latter generalize period domains to include

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classifying spaces for Hodge structures with nongeneric Hodge tensors, that is, the Mumford–Tate group of a generic Hodge structure in the domain need not be the full automorphism group

$$G_{\mathbb{R}} = \text{Aut}(V_{\mathbb{R}}, Q).$$

A Mumford–Tate domain D is homogeneous with respect to a real, reductive Lie group

$$G_{\mathbb{R}} \subset \text{Aut}(V_{\mathbb{R}}, Q).$$

In particular, the classification describes (as \mathbb{R} -split PMHSs) the degenerations that may arise in a variation of Hodge structure subject to the constraint that (the connected identity component of) the Mumford–Tate group of the generic fiber lies in $G_{\mathbb{R}}$. There is a natural action of $G_{\mathbb{R}}$ on both the horizontal $SL(2)$ s and the \mathbb{R} -split PMHSs; the classification theorems enumerate these objects up to the action of $G_{\mathbb{R}}$, which we assume to be connected.

Cattani has pointed out that the problem of classifying horizontal $SL(2)$ -orbits in *period domains* is essentially solved by the possible Hodge diamonds. This is a consequence of: (i) the equivalence (1.1); (ii) the classification of subalgebras $\mathfrak{sl}_2\mathbb{R} \subset \mathfrak{g}_{\mathbb{R}}$ by signed Young diagrams when $\mathfrak{g}_{\mathbb{R}}$ is classical [CM93, ch. 9]; and (iii) the fact that the signed Young diagram is determined by the Hodge diamond (cf. [BPR15]). One subtlety to keep in mind here is that the Hodge diamonds suffice to classify the $SL(2)$ -orbits up to the action of the full automorphism group $G_{\mathbb{R}}$. However, in the case of even weight, the Hodge diamonds do *not* suffice to classify the orbits up to the action of the connected identity component $G_{\mathbb{R}}^{\circ}$. This is essentially due to the fact that the signed Young diagrams classify the $\mathfrak{sl}_2\mathbb{R} \subset \text{End}(V_{\mathbb{R}}, Q)$ up to the adjoint action of $G_{\mathbb{R}}$; and some of these $G_{\mathbb{R}}$ -conjugacy classes decompose into distinct $G_{\mathbb{R}}^{\circ}$ -conjugacy classes which the signed Young diagram/Hodge diamond fails to distinguish. (See § 5.5 for further discussion and examples.) I assume throughout that $G_{\mathbb{R}}$ is connected. A second point to keep in mind is that the classification of subalgebras $\mathfrak{sl}_2\mathbb{R} \subset \mathfrak{g}_{\mathbb{R}}$ by signed Young diagrams requires that $\mathfrak{g}_{\mathbb{R}}$ not only be classical, but also act by the ‘standard representation’. However, even when $\mathfrak{g}_{\mathbb{R}}$ is classical, it may not be possible to realize D as the Mumford–Tate domain for a Hodge representation¹ $(G_{\mathbb{R}}, V_{\mathbb{R}}, \varphi, Q)$ with $V_{\mathbb{R}}$ the standard representation. This means that in general we will not be able to classify the horizontal $SL(2)$ -orbits on D by Hodge diamonds when D cannot be realized as a period domain.

A second motivation behind Theorem 5.5 is the problem of identifying polarizable orbits. Recall that the flag domain D is an open $G_{\mathbb{R}}$ -orbit in the compact dual $\check{D} = G_{\mathbb{C}}/P$. In particular, the boundary $\text{bd}(D) \subset \check{D}$ is a union of $G_{\mathbb{R}}$ -orbits. We say that one of these boundary orbits is *polarizable* if it contains the limit of a nilpotent orbit; cf. [GGK13, KP14] and § 3.3.² We think of these as the ‘Hodge-theoretically accessible’ orbits. Then the natural partial order on the $G_{\mathbb{R}}$ -orbits in $\text{bd}(D)$ allows one to address, from a Hodge-theoretic perspective, the question ‘what is the most/least singular variety to which a smooth projective variety can degenerate?’ [GGR14]. Theorem 5.5(c) parameterizes the polarizable orbits (§ 5.4), and from that point of view generalizes [KR14, Theorem 6.38]. The parameterization is surjective by definition, and is shown to be injective in a forthcoming paper by Kerr and Robles. As a corollary to Theorem 5.5, and the fact that all codimension-one orbits $\mathcal{O} \subset \text{bd}(D)$ are polarized [KP14], we obtain a precise count of the number of codimension-one orbits in $\text{bd}(D)$ (Proposition 5.23); in the case that P is a maximal parabolic, this recovers [KR14, Proposition 6.56].

¹ Defined in § 3.1.

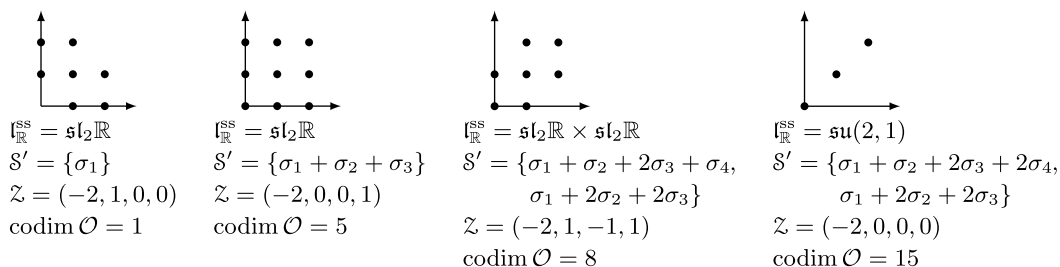
² This notion of a ‘polarized’ orbit is distinct from Wolf’s [Wol69, Definition 9.1]. In Wolf’s sense, the polarized orbits $\mathcal{O} = G_{\mathbb{R}} \cdot o$ in \check{D} are those that realize the minimal CR-structure on the homogeneous manifold $G_{\mathbb{R}}/\text{Stab}_{G_{\mathbb{R}}}(o)$; cf. [AMN10, Remark 5.5].

The key observation in the proof of Theorem 5.5 is that underlying every \mathbb{R} -split PMHS is a Hodge–Tate degeneration (Theorem 4.3), and from the latter we may recover the original \mathbb{R} -split PMHSs. Consequently, the *sine qua non* of the paper is the classification of the Hodge–Tate degenerations (Theorem 4.11). Theorem 4.3 may be viewed as describing the branching of a $\mathfrak{g}_{\mathbb{R}}$ -Hodge representation under a Levi algebra $\mathfrak{l}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$; cf. Remark 4.6. Let $L_{\mathbb{R}} \subset G_{\mathbb{R}}$ be the connected Lie subgroup with Lie algebra $\mathfrak{l}_{\mathbb{R}}$. As a corollary to Theorem 4.3, Mal’cev’s theorem and a result of Cattani, Kaplan and Schmid we find that the (open) nilpotent cone $\mathcal{C} \subset \mathfrak{g}_{\mathbb{R}}$ underlying a nilpotent orbit is contained in an $\text{Ad}(L_{\mathbb{R}}^Y)$ -orbit, where $L_{\mathbb{R}}^Y \subset L_{\mathbb{R}}$ is a connected, reductive Lie group (Corollary 4.9).

Both the statements of the classification theorems and their proofs are couched in representation theory; the necessary background material is reviewed in §2. Both Levi subalgebras, and their ‘distinguished’ parabolic subalgebras, play a key role in the classification theorems. This is not surprising as Bala and Carter’s classification [BC76a, BC76b] of the $\mathfrak{sl}_2\mathbb{C}$ s in a complex semisimple $\mathfrak{g}_{\mathbb{C}}$ is in terms of these pairs. Indeed, Theorem 5.9 could be viewed as the analog the Bala–Carter classification for horizontal $\mathfrak{sl}_2\mathbb{R}$ s, and from this perspective is related to both Vinberg’s classification [Vin75] of nilpotent elements of graded Lie algebras, and Noël’s classification [Noë98] of (not necessarily horizontal) $\mathfrak{sl}_2\mathbb{R} \subset \mathfrak{g}_{\mathbb{R}}$. The pertinent Hodge-theoretic material is reviewed in §3.

As I hope the examples presented here (most are concentrated in §§ 4.5 and 5.5) demonstrate, the classifications are computationally accessible: it is straightforward to describe the horizontal $\text{SL}(2)$ s and the Deligne splittings of the associated \mathbb{R} -split PMHSs. Here is one illustrative example.

Example 1.2. The exceptional simple Lie group F_4 of rank four admits a real form $G_{\mathbb{R}}$ with maximal compact subalgebra $\mathfrak{sp}(2) \oplus \mathfrak{su}(2)$.³ This real form admits a real Hodge representation $V_{\mathbb{R}}$ with Hodge numbers $(6, 14, 6)$;⁴ in particular, $G_{\mathbb{R}} \subset \text{SO}(14, 12)$. The horizontal distribution is a holomorphic contact distribution⁵ on the associated domain D . Theorem 5.9 identifies four horizontal $\text{SL}(2)$ s. The Hodge diamonds of the corresponding \mathbb{R} -split PMHSs are depicted below; see § 5.5 for further explanation of these diagrams and the notations $\mathfrak{l}_{\mathbb{R}}^{\text{ss}}$, S' , \mathcal{Z} and \mathcal{O} below.



Finally, I wish to mention that an inductive argument based on Theorem 5.9 yields a classification of the commuting $\text{SL}(2)$ s in Cattani, Kaplan and Schmid’s several-variables $\text{SL}(2)$ -orbit theorem, as will be demonstrated in a forthcoming paper with Matt Kerr in which we will also establish the injectivity of the parameterization of the polarized orbits by Theorem 5.5.

³ This real form is commonly denoted by FI or $F_4(4)$.

⁴ The highest weight of $V_{\mathbb{C}}$ the fourth fundamental weight.

⁵ See [KR14] for further discussion of the case that the horizontal distribution is contact.

Einstein summation convention. When an index appears as both a subscript and a superscript in a formula, it is meant to be summed over. For example, $z^i N_i$ denotes $\sum_i z^i N_i$.

2. Representation theory background

2.1 Parabolic subgroups and subalgebras

Let $G_{\mathbb{C}}$ be a connected, complex semisimple Lie group, and let $P \subset G_{\mathbb{C}}$ be a parabolic subgroup. Fix *Cartan* and *Borel subgroups* $H \subset B \subset P$. Let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{p} \subset \mathfrak{g}$ be the associated Lie algebras. The choice of Cartan determines a set of *roots* $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$. Given a root $\alpha \in \Delta$, let $\mathfrak{g}^\alpha \subset \mathfrak{g}$ denote the *root space*. Given a subspace $\mathfrak{s} \subset \mathfrak{g}$, let

$$\Delta(\mathfrak{s}) := \{\alpha \in \Delta \mid \mathfrak{g}^\alpha \subset \mathfrak{s}\}.$$

The choice of Borel determines *positive roots* $\Delta^+ = \Delta(\mathfrak{b}) = \{\alpha \in \Delta \mid \mathfrak{g}^\alpha \subset \mathfrak{b}\}$. Let $\mathcal{S} = \{\sigma_1, \dots, \sigma_r\}$ denote the *simple roots*, and set

$$I = I(\mathfrak{p}) := \{i \mid \mathfrak{g}^{-\sigma_i} \not\subset \mathfrak{p}\}. \tag{2.1}$$

Note that the parabolic \mathfrak{p} is maximal if and only if $I = \{i\}$; in this case we say that σ_i is the *simple root associated with the maximal parabolic* \mathfrak{p} . Likewise, $\mathfrak{p} = \mathfrak{b}$ if and only if $I = \{1, \dots, r\}$.

Every parabolic $P \subset G_{\mathbb{C}}$ is $G_{\mathbb{C}}$ -conjugate to one containing B . Thus, the conjugacy classes \mathcal{P}_I of parabolic subgroups are indexed by the subsets $I \subset \{1, \dots, r\}$. Let $\mathcal{B} = \mathcal{P}_{\{1, \dots, r\}}$ denote the conjugacy class of the Borel subgroups.

2.2 Grading elements and Levi subalgebras

Given a choice of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$, let $\Lambda_{\text{rt}} \subset \mathfrak{h}^*$ denote the root lattice. The set of *grading elements* is the lattice $\text{Hom}(\Lambda_{\text{rt}}, \mathbb{Z}) \subset \mathfrak{h}$ taking integral values on roots. As an element of the Cartan subalgebra, a grading element \mathbf{E} is necessarily semisimple. Therefore, any $\mathfrak{g}_{\mathbb{C}}$ module $V_{\mathbb{C}}$ decomposes into a direct sum of \mathbf{E} -eigenspaces

$$V_{\mathbb{C}} = \bigoplus_{\ell \in \mathbb{Q}} V^\ell \quad \text{where } V^\ell = \{v \in V_{\mathbb{C}} \mid \mathbf{E}(v) = \ell v\}.^6 \tag{2.2}$$

When specialized to $V_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$, (2.2) yields

$$\mathfrak{g} = \bigoplus_{\ell \in \mathbb{Z}} \mathfrak{g}^\ell \tag{2.3a}$$

where

$$\mathfrak{g}^\ell := \{\xi \in \mathfrak{g} \mid [\mathbf{E}, \xi] = \ell \xi\}. \tag{2.3b}$$

In terms of root spaces, we have

$$\begin{aligned} \mathfrak{g}^\ell &= \bigoplus_{\alpha(\mathbf{E})=\ell} \mathfrak{g}^\alpha, \quad \ell \neq 0, \\ \mathfrak{g}^0 &= \mathfrak{h} \oplus \bigoplus_{\alpha(\mathbf{E})=0} \mathfrak{g}^\alpha. \end{aligned} \tag{2.3c}$$

⁶ To see that the eigenvalues are necessarily rational, it suffices to observe that the eigenvalues are $\lambda(\mathbf{E})$, where $\lambda \in \Lambda_{\text{wt}} \subset \mathfrak{h}^*$ is a weight of $V_{\mathbb{C}}$, and to recall that the weights of $\mathfrak{g}_{\mathbb{C}}$ are rational linear combinations of the roots.

The E-eigenspace decomposition (2.3) is a *graded Lie algebra decomposition* in the sense that

$$[\mathfrak{g}^\ell, \mathfrak{g}^m] \subset \mathfrak{g}^{\ell+m}, \tag{2.4}$$

a straightforward consequence of the Jacobi identity. It follows that

$$\mathfrak{p}_E = \mathfrak{g}^0 \oplus \mathfrak{g}^+ \tag{2.5}$$

is a Lie subalgebra of $\mathfrak{g}_\mathbb{C}$; we call this the *parabolic subalgebra determined by the grading element E*.

From (2.4) we also see that \mathfrak{g}^0 is a Lie subalgebra of \mathfrak{g} (in fact, reductive), and each \mathfrak{g}^ℓ is a \mathfrak{g}^0 -module. In general, by *Levi subalgebra* we will mean any subalgebra $\mathfrak{l}_\mathbb{C} \subset \mathfrak{g}_\mathbb{C}$ that can be realized as the 0-eigenspace \mathfrak{g}^0 of a grading element.

Remark 2.6. By the second equation of (2.3c) every Levi subalgebra contains a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_\mathbb{C}$ and recall that the Weyl group $\mathcal{W} \subset \text{Aut}(\mathfrak{h}^*)$ is generated by the reflections in the hyperplanes orthogonal to the roots $\alpha \in \Delta$. Fix a choice of simple roots $\mathcal{S} \subset \Delta \subset \mathfrak{h}^*$. The Levi subalgebras containing \mathfrak{h} are in bijective correspondence with the subsets $\{w\mathcal{S}' \subset \Delta \mid w \in \mathcal{W}, \mathcal{S}' \subset \mathcal{S}\}$: $w\mathcal{S}'$ is a set of simple roots for the semisimple factor $\mathfrak{l}_\mathbb{C}^{\text{ss}} = [\mathfrak{l}_\mathbb{C}, \mathfrak{l}_\mathbb{C}]$ of the Levi subalgebra $\mathfrak{l}_\mathbb{C} \supset \mathfrak{h}$. In particular, there exist only finitely many Levi subalgebras containing \mathfrak{h} .

Remark 2.7. Recall that the simple reflections $(i) \in \mathcal{W}$ in the hyperplanes orthogonal to the simple roots $\sigma_i \in \mathcal{S}$ form a minimal set of generators for the Weyl group. Given a Levi subalgebra $\mathfrak{l}_\mathbb{C} \supset \mathfrak{h}$, by replacing \mathcal{S} with $w\mathcal{S}$ (the latter is also a set of simple roots for \mathfrak{h}), we may assume that the simple roots of $\mathfrak{l}_\mathbb{C}^{\text{ss}}$ are a subset \mathcal{S}' of the simple roots \mathcal{S} of $\mathfrak{g}_\mathbb{C}$. Then the Weyl group \mathcal{W}' of $\mathfrak{l}_\mathbb{C}$ is generated by the simple reflections $(i) \in \mathcal{W}$ with $\sigma_i \in \mathcal{S}'$.

Given a real form $\mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}_\mathbb{C}$, we will say that $\mathfrak{l}_\mathbb{R} \subset \mathfrak{g}_\mathbb{R}$ is a Levi subalgebra if the complexification $\mathfrak{l}_\mathbb{C} = \mathfrak{l}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ is a Levi subalgebra of $\mathfrak{g}_\mathbb{C}$; equivalently, a *Levi subalgebra of the real form $\mathfrak{g}_\mathbb{R}$* is the real form $\mathfrak{l}_\mathbb{R}$ of a conjugation-stable Levi subalgebra $\mathfrak{l}_\mathbb{C} \subset \mathfrak{g}_\mathbb{C}$.

Let $\{\mathcal{S}^1, \dots, \mathcal{S}^r\}$ be the basis of \mathfrak{h} dual to the simple roots $\{\sigma_1, \dots, \sigma_r\}$. Then any grading element $E = n_i \mathcal{S}^i$ is an integral linear combination of the $\{\mathcal{S}^i\}$; if \mathfrak{p}_E contains the Borel $\mathfrak{b} \supset \mathfrak{h}$ determining the simple roots, then $n_i \geq 0$ for all i . In this case, the index set (2.1) is

$$I(\mathfrak{p}_E) = \{i \mid n_i > 0\},$$

and the reductive Levi subalgebra $\mathfrak{g}^0 = \mathfrak{g}_{\text{ss}}^0 \oplus \mathfrak{z}$ has center $\mathfrak{z} = \text{span}_\mathbb{C}\{\mathcal{S}^i \mid i \in I(\mathfrak{p}_E)\}$ and semisimple subalgebra $\mathfrak{g}_{\text{ss}}^0 = [\mathfrak{g}^0, \mathfrak{g}^0]$. A set of simple roots for $\mathfrak{g}_{\text{ss}}^0$ is given by $\mathcal{S}(\mathfrak{g}^0) = \{\sigma_j \mid j \notin I(\mathfrak{p}_E)\}$. I emphasize that the sets $\mathcal{S}(\mathfrak{g}^0)$ and $I(\mathfrak{p}_E)$ encode the same information which describes the $G_\mathbb{C}$ -conjugacy class \mathcal{P}_E of the parabolic subgroup P_E .

Two distinct grading elements may determine the same parabolic \mathfrak{p} . For example, any positive multiple $n\mathcal{S}^i$ will determine the same (maximal) parabolic as \mathcal{S}^i . However given a parabolic \mathfrak{p} , and a choice of Cartan and Borel subalgebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{p}$, there is a canonical choice of grading element $E = E_\mathfrak{p}$ with $\mathfrak{p}_E = \mathfrak{p}$ such that $\mathfrak{g}^{\pm 1}$ generates the nilpotent \mathfrak{g}^\pm as a subalgebra. The *grading element associated to $\mathfrak{p} \supset \mathfrak{b} \supset \mathfrak{h}$* is

$$E_\mathfrak{p} := \sum_{i \in I(\mathfrak{p})} \mathcal{S}^i. \tag{2.8}$$

For more detail on grading elements and parabolic subalgebras, see [Rob14, § 2.2] and the references therein.

2.3 Standard triples and three-dimensional semisimple subalgebras

Let \mathfrak{g} be a Lie algebra defined over a field $\mathbb{k} = \mathbb{R}, \mathbb{C}$. A *standard triple* in \mathfrak{g} is a set of three elements $\{N^+, Y, N\} \subset \mathfrak{g}$ such that

$$[Y, N^+] = 2N^+, \quad [N^+, N] = Y, \quad \text{and} \quad [Y, N] = -2N.$$

Note that $\{N^+, Y, N\}$ span a three-dimensional semisimple subalgebra (TDS) of \mathfrak{g} isomorphic to $\mathfrak{sl}_2\mathbb{k}$. We call Y the *neutral element*, N the *nilnegative element* and N^+ the *nilpositive element*, respectively, of the standard triple. The Jacobson–Morosov theorem asserts that every nilpotent $N \in \mathfrak{g}$ can be realized as the nilnegative of a standard triple.

Example 2.9. The matrices

$$\mathbf{n}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad \mathbf{n} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.10}$$

form a standard triple in $\mathfrak{sl}_2\mathbb{R}$; while the matrices

$$\bar{\mathbf{e}} = \frac{1}{2} \begin{pmatrix} -\mathbf{i} & 1 \\ 1 & \mathbf{i} \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{e} = \frac{1}{2} \begin{pmatrix} \mathbf{i} & 1 \\ 1 & -\mathbf{i} \end{pmatrix} \tag{2.11}$$

form a standard triple in $\mathfrak{su}(1, 1)$.

2.4 Jacobson–Morosov filtrations

Given a standard triple $\{N^+, Y, N\} \subset \mathfrak{g}$ and a representation $\mathfrak{g} \hookrightarrow \text{End}(V)$ of \mathfrak{g} , the theory of $\mathfrak{sl}_2\mathbb{k}$ -representations implies that the eigenvalues ℓ of Y are integers. In the case that $V = \mathfrak{g}$, this implies that

$$\text{the neutral element } Y \text{ is a grading element.} \tag{2.12}$$

The *Jacobson–Morosov filtration* (or *weight filtration*) of N is the unique filtration $W_\bullet(N, V)$ of V with the following properties:

- (i) the filtration is increasing, $W_\ell(N, V) \subset W_{\ell+1}(N, V)$;
- (ii) the nilpotent N maps $W_\ell(N, V)$ into $W_{\ell-2}(N, V)$;
- (iii) the induced map $N^\ell : \text{Gr}_\ell W(N, V) \rightarrow \text{Gr}_{-\ell} W(N, V)$ is an isomorphism for all $\ell \geq 0$, where

$$\text{Gr}_k W(N, V) := W_k(N, V) / W_{k-1}(N, V).$$

If $V = \bigoplus V_\ell$ is the Y -eigenspace decomposition, $V_\ell = \{v \in V \mid Y(v) = \ell v\}$, then

$$W_\ell(N, V) = \bigoplus_{m \leq \ell} V_m. \tag{2.13}$$

Note that

$$W_0(N^+, \mathfrak{g}) = \mathfrak{p}_Y.$$

Parabolic subalgebras of the form $W_0(N^+, \mathfrak{g})$ are *Jacobson–Morosov parabolics*.

Remark 2.14. (a) Some parabolic subalgebras cannot be realized as Jacobson–Morosov parabolics; cf. Example 2.17. Similarly, not every grading element can be realized as the neutral element of a standard triple.

(b) The neutral element Y may not be a grading element E_p canonically associated with $\mathfrak{p} = W_0(N^+, \mathfrak{g}) \supset \mathfrak{b} \supset \mathfrak{h}$ by (2.8). Moreover, it is possible that there exist nilpotents N_1 and N_2 that are *not* congruent under the action of $\text{Ad}(G)$ on \mathfrak{g} (equivalently, Y_1 and Y_2 are not congruent), but with $W_0(N_1, \mathfrak{g}) = W_0(N_2, \mathfrak{g})$. For an illustration of this, consider Example 2.16 where we have $W_0(N_{[3,1]}, \mathfrak{g}\mathbb{C}) = W_0(N_{[2,2]}, \mathfrak{g}\mathbb{C})$, but $Y_{[3,1]} = 2Y_{[2,2]} = 2(\mathbf{S}^1 + \mathbf{S}^3)$.

2.5 $\text{Ad}(G_{\mathbb{C}})$ -orbits in $\text{Nilp}(\mathfrak{g}_{\mathbb{C}})$

Given any Lie algebra \mathfrak{g} , let $\text{Nilp}(\mathfrak{g})$ denote the set of nilpotent elements. A *nilpotent orbit* is an $\text{Ad}(G)$ -orbit in $\text{Nilp}(\mathfrak{g})$.⁷ In this section we will review some properties of nilpotent orbits in a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$, including their classification by ‘characteristic vectors’ (a.k.a. ‘weighted Dynkin diagrams’);⁸ an excellent reference for the discussion that follows is [CM93].

Given a nilpotent $N \in \mathfrak{g}_{\mathbb{C}}$, fix a standard triple $\{N^+, Y, N\}$. We may choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ and a set of simple roots $\mathcal{S} = \{\sigma_1, \dots, \sigma_r\} \subset \mathfrak{h}^*$ such that $Y \in \mathfrak{h}$ and $\sigma_i(Y) \geq 0$ for all i . The (*complex*) *characteristic vector*

$$\sigma(Y) := (\sigma_1(Y), \dots, \sigma_r(Y))$$

is independent of our choices, and is an invariant of the nilpotent orbit $\mathcal{N} = \text{Ad}(G_{\mathbb{C}}) \cdot N \subset \mathfrak{g}_{\mathbb{C}}$ through N , so that

$$\sigma(\mathcal{N}) := \sigma(Y)$$

is well defined. For the *trivial orbit* $\mathcal{N}_{\text{triv}} = \{0\} \subset \text{Nilp}(\mathfrak{g}_{\mathbb{C}})$ we have $\sigma(\mathcal{N}_{\text{triv}}) = (0, \dots, 0)$. The nilpotent orbits are characterized by their characteristic vectors (the following result is [Dyn57, Theorem 8.3]; see also [Kos59, Lemma 5.1]).

THEOREM 2.15 (Dynkin). *The characteristic vector $\sigma(\mathcal{N})$ is a complete invariant of a nilpotent orbit; that is, $\sigma(\mathcal{N}) = \sigma(\mathcal{N}')$ if and only if $\mathcal{N} = \mathcal{N}'$. Moreover, $0 \leq \sigma_i(\mathcal{N}) \leq 2$.*

Example 2.16 (Nilpotent orbits in $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_n(\mathbb{C})$). The $\text{Ad}(G_{\mathbb{C}})$ -orbits in $\text{Nilp}(\mathfrak{g}_{\mathbb{C}})$ are indexed by partitions $\mathbf{d} = [d_i]$ of n [CM93, ch. 3]. Given a partition, the corresponding characteristic vector is obtained as follows. From a part d_i , we construct a set $(d_i) = \{d_i - 1, d_i - 3, \dots, 3 - d_i, 1 - d_i\}$. Take the union of these sets, reordering into a nonincreasing sequence $\bigcup_i (d_i) = \{h_1 \geq \dots \geq h_n\}$. Then the characteristic vector of the orbit $\mathcal{N}_{\mathbf{d}}$ indexed by \mathbf{d} is

$$\sigma(\mathcal{N}_{\mathbf{d}}) = (h_1 - h_2, h_2 - h_3, \dots, h_{n-1} - h_n).$$

For example, in the case that $n = 4$ there are five nilpotent orbits, indexed by

$$\begin{aligned} \sigma(\mathcal{N}_{[4]}) &= (2, 2, 2), & \sigma(\mathcal{N}_{[3,1]}) &= (2, 0, 2), \\ \sigma(\mathcal{N}_{[2^2]}) &= (0, 2, 0), & \sigma(\mathcal{N}_{[2,1^2]}) &= (1, 0, 1), \\ \sigma(\mathcal{N}_{[1^4]}) &= (0, 0, 0). \end{aligned}$$

The index set I (§ 2.1) corresponding to the conjugacy class of the Jacobson–Morosov parabolic $W_0(N, \mathfrak{g})$ is

$$I = \{i \mid \sigma_i(\mathcal{N}) \neq 0\}.$$

Equivalently, the simple roots of the reductive Levi factor are

$$\mathcal{S}(\mathfrak{g}_0) = \{\sigma_j \mid \sigma_j(\mathcal{N}) = 0\}.$$

⁷ Here we have a conflict in the nomenclature: the term ‘nilpotent orbit’ is used in both representation theory and Hodge theory to refer to two distinct, but related objects (see § 3.2 for the second). Which of the two meanings is intended should be clear from the context.

⁸ In the case that $\mathfrak{g}_{\mathbb{C}}$ is a classical Lie algebra, the nilpotent orbits may be classified by partitions (or Young diagrams); see Example 2.16 and [BPR15].

Example 2.17 (Jacobson–Morosov parabolics in $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_4\mathbb{C}$). The group $G_{\mathbb{C}}$ contains $2^3 - 1 = 7$ conjugacy classes \mathcal{P}_I of parabolic subgroups, indexed by nonempty $I \subset \{1, 2, 3\}$. From Example 2.16 we see that only three of the conjugacy classes are Jacobson–Morosov: the corresponding index sets are $I = \{2\}, \{1, 3\}, \{1, 2, 3\}$.

The neutral element Y is *even* if the Y -eigenvalues are all even. From Theorem 2.15 we see that the neutral element Y is even if and only if $\sigma_i(\mathcal{N}) \in \{0, 2\}$ for all i . Equivalently,

$$\begin{aligned} & \text{the neutral element } Y \text{ is even if and only if it is twice the grading element (2.8)} \\ & \text{canonically associated with a choice of Cartan and Borel } \mathfrak{h} \subset \mathfrak{b} \subset W_0(N^+, \mathfrak{g}_{\mathbb{C}}). \end{aligned} \tag{2.18}$$

When Y is even we say that $W_0(N^+, \mathfrak{g}_{\mathbb{C}})$ is an *even Jacobson–Morosov parabolic*.

There is a unique Zariski open orbit $\mathcal{N}_{\text{prin}} \subset \text{Nilp}(\mathfrak{g}_{\mathbb{C}})$ of dimension $\dim \mathfrak{g}_{\mathbb{C}} - \text{rank } \mathfrak{g}_{\mathbb{C}}$; this is the *principal* (or *regular*) *nilpotent orbit*. The orbit is represented by $N = \xi^1 + \dots + \xi^r$ with each simple root vector $\xi^i \in \mathfrak{g}^{\sigma^i}$ nonzero. In this case the characteristic vector is

$$\sigma(\mathcal{N}_{\text{prin}}) = (2, 2, \dots, 2).$$

In particular,

$$\text{the Borel } B \subset G_{\mathbb{C}} \text{ is an even Jacobson–Morosov parabolic.} \tag{2.19}$$

2.6 Compact roots

Let $G_{\mathbb{R}}$ be a real semisimple Lie algebra. Fix a Cartan decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{t}_{\mathbb{R}}^{\perp}$. There is a classification of nilpotent orbits in $\mathfrak{g}_{\mathbb{R}}$ that is analogous to that of Theorem 2.15 in the sense that the orbits are enumerated by characteristic vectors that are given by the roots of $\mathfrak{k}_{\mathbb{C}}$. This classification is reviewed in § 2.7; in anticipation of that discussion we briefly recall the relationship between the roots of $\mathfrak{g}_{\mathbb{C}}$ and the roots of $\mathfrak{k}_{\mathbb{C}}$.

Fix a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}_{\mathbb{R}}$. Let \mathfrak{h} be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing $\mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$. Given a choice of simple roots $\mathcal{S} = \{\sigma_1, \dots, \sigma_r\} \subset \mathfrak{h}^*$ of $\mathfrak{g}_{\mathbb{C}}$, let $\tilde{\alpha}$ denote the highest root, and set

$$\mathcal{S}_{\text{ext}} := \{\mathcal{S}\} \cup \{-\tilde{\alpha}\}.$$

For a suitable choice⁹ of \mathcal{S} there exists a subset $\mathcal{S}_{\mathfrak{k}} \subset \mathcal{S}_{\text{ext}}$ such that $\mathcal{S}_{\mathfrak{k}}|_{\mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}}$ is a set of simple roots of $\mathfrak{k}_{\mathbb{C}}$. We will assume throughout that $\text{rank } \mathfrak{k}_{\mathbb{C}} = \text{rank } \mathfrak{g}_{\mathbb{C}}$,¹⁰ so that $\mathfrak{h} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra of both $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$. There are two cases to consider.

(a) If $\mathfrak{g}_{\mathbb{R}}$ is of Hermitian symmetric type, then $\mathfrak{k}_{\mathbb{R}}$ is reductive with a one-dimensional center and we may take $\mathcal{S}_{\mathfrak{k}} \subset \mathcal{S}$. In this case, the center of $\mathfrak{k}_{\mathbb{C}}$ is spanned by the grading element dual to the simple noncompact root $\{\alpha'\} = \mathcal{S} \setminus \mathcal{S}_{\mathfrak{k}}$.

(b) If $\mathfrak{g}_{\mathbb{R}}$ is not of Hermitian symmetric type, then $\mathfrak{k}_{\mathbb{R}}$ is semisimple and $-\tilde{\alpha} \in \mathcal{S}_{\mathfrak{k}}$.

In both cases $\mathcal{S} \setminus \mathcal{S}_{\mathfrak{k}}$ consists of a single simple root α' , which we will refer to as *the noncompact simple root*.¹¹

Example 2.20. The algebra $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(p, q)$ is of Hermitian symmetric type. In this case we have $\alpha' = \sigma_p$ and $\mathcal{S}_{\mathfrak{k}} = \mathcal{S} \setminus \{\sigma_p\} \subset \mathcal{S}$.

⁹ This means we may need to replace \mathcal{S} with its image $w\mathcal{S}$ under an element $w \in \mathcal{W}$ of the Weyl group.

¹⁰ This is the case when $G_{\mathbb{R}}$ may be realized as a Mumford–Tate group [GGK12].

¹¹ The root α' corresponds to the painted node in the Vogan diagram of $\mathfrak{g}_{\mathbb{R}}$; cf. [Kna02, § VI.8].

Example 2.21. For the algebra $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(2p, 2q+1)$, we have $\alpha' = \sigma_p$. This real form is of Hermitian symmetric type if and only if $p = 1$.

Example 2.22. The algebra $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(r, \mathbb{R})$ is of Hermitian symmetric type; in this case $\alpha' = \sigma_r$. The real forms $\mathfrak{sp}(p, r-p)$, with $p \geq 1$, are not of Hermitian symmetric type; in this case $\alpha' = \sigma_p$.

2.7 $\text{Ad}(G_{\mathbb{R}})$ -orbits in $\text{Nilp}(\mathfrak{g}_{\mathbb{R}})$

This section is a terse review of the classification of the nilpotent orbits in a real semisimple Lie algebra $\mathfrak{g}_{\mathbb{R}}$ by the *Djoković–Kostant–Sekiguchi correspondence*¹²

$$\{\text{nilpotent Ad}(G_{\mathbb{R}})\text{-orbits in } \mathfrak{g}_{\mathbb{R}}\} \xleftrightarrow{\text{bij}} \{\text{nilpotent Ad}(K_{\mathbb{C}})\text{-orbits in } \mathfrak{k}_{\mathbb{C}}^{\perp}\}. \tag{2.23}$$

For details, consult [CM93, §9] and the references therein.

The correspondence is realized through refinements of the standard triples of §2.3. Let $G_{\mathbb{R}}$ be a real semisimple Lie algebra. Fix a Cartan decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$, and let θ be the associated Cartan involution. A *Cayley triple* is a standard triple $\{N^+, Y, N\}$ of $\mathfrak{g}_{\mathbb{R}}$ with the property that

$$\theta(N) = -N^+, \quad \theta(N^+) = -N, \quad \text{and} \quad \theta(Y) = -Y. \tag{2.24}$$

Remark 2.25. Every standard triple in $\mathfrak{g}_{\mathbb{R}}$ is $G_{\mathbb{R}}$ -conjugate to a Cayley triple [CM93, Theorem 9.4.1].

Example 2.26. Let $\{N^+, Y, N\}$ be a standard triple. Then $\text{span}_{\mathbb{R}}\{N^+, Y, N\}$ is isomorphic to $\mathfrak{sl}_2\mathbb{R}$. The standard triple is a Cayley triple with respect to the Cartan decomposition $\mathfrak{sl}_2\mathbb{R} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ given by $\mathfrak{k}_{\mathbb{R}} = \text{span}_{\mathbb{R}}\{N^+ - N\}$ and $\mathfrak{p}_{\mathbb{R}} = \text{span}_{\mathbb{R}}\{Y, N^+ + N\}$.

A *Djoković–Kostant–Sekiguchi triple* (DKS triple) is any standard triple in $\mathfrak{g}_{\mathbb{C}}$ of the form $\{\bar{\mathcal{E}}, \mathcal{Z}, \mathcal{E}\}$ with the property that $\mathcal{Z} \in \mathfrak{k}_{\mathbb{C}}$ and $\bar{\mathcal{E}}, \mathcal{E} \in \mathfrak{k}_{\mathbb{C}}^{\perp}$. The *Cayley transform* of a Cayley triple $\{N^+, Y, N\}$ is the DKS triple

$$\begin{aligned} \bar{\mathcal{E}} &= \frac{1}{2}(N^+ + N - \mathbf{i}Y), \\ \mathcal{Z} &= \mathbf{i}(N - N^+), \\ \mathcal{E} &= \frac{1}{2}(N^+ + N + \mathbf{i}Y). \end{aligned} \tag{2.27}$$

Note that

$$\{\bar{\mathcal{E}}, \mathcal{Z}, \mathcal{E}\} = \text{Ad}_{\varrho}\{N^+, Y, N\}, \tag{2.28}$$

where the element $\varrho \in G_{\mathbb{C}}$ is defined by

$$\varrho = \exp \mathbf{i}\frac{\pi}{4}(N^+ + N) \quad \left(= \exp \mathbf{i}\frac{\pi}{4}(\mathcal{E} + \bar{\mathcal{E}}) \right). \tag{2.29}$$

The Djoković–Kostant–Sekiguchi correspondence (2.23) identifies the $\text{Ad}(G_{\mathbb{R}})$ -orbit of N with the $\text{Ad}(K_{\mathbb{C}})$ -orbit of $\mathcal{E} = \text{Ad}_{\varrho}(N)$.

Example 2.30. Identify (2.10) as a Cayley triple with respect to the Cartan decomposition of Example 2.26. Then (2.11) is the Cayley transform of (2.10).

In summary, to distinguish the $\text{Ad}(G_{\mathbb{R}})$ -orbits in $\mathfrak{g}_{\mathbb{R}}$ it suffices to distinguish the $\text{Ad}(K_{\mathbb{C}})$ -orbits in $\mathfrak{k}_{\mathbb{C}}^{\perp}$. Let $\mathcal{S}_{\mathfrak{k}} = \{\gamma_1, \dots, \gamma_s\} \subset \mathfrak{h}^*$ denote the simple roots of $\mathfrak{k}_{\mathbb{C}}$ (§2.6). We may conjugate \mathcal{Z} by $\text{Ad}(K_{\mathbb{C}})$ so that $\mathcal{Z} \subset \mathfrak{h}$ and $\gamma_i(\mathcal{Z}) \geq 0$. The vector

$$\gamma(\mathcal{Z}) := (\gamma_1(\mathcal{Z}), \dots, \gamma_s(\mathcal{Z}))$$

¹² The correspondence was conjectured by Kostant, and proved independently by Djoković [Djo87] and Sekiguchi [Sek87].

is an invariant of the nilpotent orbit, so that

$$\gamma(\mathcal{N}) := \gamma(\mathcal{Z})$$

is well defined. However, in the case that $\mathfrak{g}_{\mathbb{R}}$ is of Hermitian symmetric type, it is not a complete invariant (two distinct orbits $\mathcal{N}' \neq \mathcal{N}$ may have $\gamma(\mathcal{N}) = \gamma(\mathcal{N}')$); we have lost information on the component of \mathcal{Z} lying in the center. Recall the noncompact simple root $\alpha' \in S \setminus S_{\mathfrak{k}}$ (§ 2.6). The integer $\alpha'(\mathcal{Z})$ is also an invariant of the nilpotent orbit, so that

$$\alpha'(\mathcal{N}) := \alpha'(\mathcal{Z})$$

is also well defined. The pair $(\gamma(\mathcal{Z}); \alpha'(\mathcal{Z}))$ is a complete invariant of the orbit, which we shall refer to as the (compact) characteristic vector of the orbit $\mathcal{N} = \text{Ad}(G_{\mathbb{R}}) \cdot N$ (or the orbit $\text{Ad}(K_{\mathbb{C}}) \cdot \mathcal{E}$). (In the case that $\mathfrak{g}_{\mathbb{R}}$ is not Hermitian symmetric, the simple roots $S_{\mathfrak{k}}$ span \mathfrak{h}^* so that $\alpha'(\mathcal{Z})$ is determined by $\gamma(\mathcal{N})$.) The following theorem may be found in [CM93, § 9.5].

THEOREM 2.31. *The compact characteristic vector $(\gamma(\mathcal{Z}); \alpha'(\mathcal{Z}))$ is a complete invariant of the orbit $\text{Ad}(G_{\mathbb{R}}) \cdot N \subset \text{Nilp}(\mathfrak{g}_{\mathbb{R}})$.*

3. Hodge theory background

3.1 Hodge representations and Mumford–Tate domains

Let $G_{\mathbb{R}}$ be a noncompact, reductive, real algebraic group with maximal compact subgroup $K_{\mathbb{R}}$ of equal rank

$$\text{rank } \mathfrak{g}_{\mathbb{C}} = \text{rank } \mathfrak{k}_{\mathbb{C}}.$$

A (real) Hodge representation (of weight n) of $G_{\mathbb{R}}$ is defined in [GGK12] and consists of:

- (i) a finite-dimensional vector space $V_{\mathbb{R}}$ defined over \mathbb{R} , a nondegenerate $(-1)^n$ -symmetric bilinear form $Q : V_{\mathbb{R}} \times V_{\mathbb{R}} \rightarrow \mathbb{R}$, and a homomorphism of real algebraic groups

$$\rho : G_{\mathbb{R}} \rightarrow \text{Aut}(V_{\mathbb{R}}, Q);$$

- (ii) a nonconstant homomorphism of real algebraic groups

$$\varphi : S^1 \rightarrow G_{\mathbb{R}}$$

such that $\rho \circ \varphi$ defines a Q -polarized (pure, real) Hodge structure of weight n on $V_{\mathbb{R}}$.

The latter condition means that

$$V^{p,q} = \{v \in V_{\mathbb{C}} \mid \rho \circ \varphi(z)v = z^{p-q}v \ \forall z \in S^1\} \tag{3.1}$$

defines a Hodge decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$ and $Q(\varphi(\mathbf{i})v, \bar{v}) > 0$ for all $0 \neq v \in V_{\mathbb{C}}$.

We always assume that the induced representation $d\rho : \mathfrak{g}_{\mathbb{R}} \rightarrow \text{End}(V_{\mathbb{R}}, Q)$ is faithful, and will often refer to φ as a ‘circle’. The Hodge representation is properly denoted $(V_{\mathbb{R}}, Q, \rho, \varphi)$, but will sometimes be indicated by $V_{\mathbb{R}}$ alone. Additionally, we will often suppress ρ , and view the circle φ as acting directly on $V_{\mathbb{C}}$; it is from this perspective that we will refer to φ as the Hodge structure on $V_{\mathbb{R}}$, and generally write $N \in \text{End}(V_{\mathbb{R}})$ in place of $d\rho(N) \in \text{End}(V_{\mathbb{R}})$.

Associated to the Hodge representation is the *Hodge flag*

$$F^p = \bigoplus_{r \geq p} V^{r, \bullet}. \tag{3.2}$$

The *Hodge numbers* are the dimensions $\mathbf{f} = (f^p = \dim_{\mathbb{C}} F^p)$. The Hodge flag is a point in the Q -isotropic flag variety $\text{Flag}_{\mathbf{f}}^Q(V_{\mathbb{C}})$. The $G_{\mathbb{R}}$ -orbit $D = G_{\mathbb{R}} \cdot F^{\bullet}$ is the *Mumford–Tate domain* of the Hodge representation; it is an open subset of the *compact dual* $\check{D} = G_{\mathbb{C}} \cdot F^{\bullet}$. When $G_{\mathbb{R}} = \text{Aut}(V_{\mathbb{R}}, Q)$, D is a *period domain*.

As homogeneous manifolds,

$$\check{D} = G_{\mathbb{C}}/P \quad \text{and} \quad D = G_{\mathbb{R}}/K_{\mathbb{R}}^0,$$

where $P = \text{Stab}_{G_{\mathbb{C}}} F^{\bullet}$ is a parabolic subgroup of $G_{\mathbb{C}}$ and $K_{\mathbb{R}}^0 = G_{\mathbb{R}} \cap P$ is compact. We say that *the Hodge representation* $(V_{\mathbb{R}}, Q, \rho, \varphi)$ *realizes the homogeneous manifold* $G_{\mathbb{R}}/K_{\mathbb{R}}^0$ *as a Mumford–Tate domain*. Such a realization is not unique. For example, given $(V_{\mathbb{R}}, Q, \rho, \varphi)$, there is an induced bilinear form $Q_{\mathfrak{g}}$ on $\mathfrak{g}_{\mathbb{R}} \subset \text{End}(V_{\mathbb{R}}, Q)$ that is nondegenerate and symmetric, and $(\mathfrak{g}_{\mathbb{R}}, Q_{\mathfrak{g}}, \text{Ad}, \varphi)$ is a weight zero Hodge representation that also realizes $G_{\mathbb{R}}/K_{\mathbb{R}}^0$ as a Mumford–Tate domain. (See §3.1.1 for further discussion of this induced representation.) These two realizations are isomorphic as Mumford–Tate domains. A key consequence of this is that

$$\begin{aligned} & \text{for the purposes of studying } G_{\mathbb{R}}/K_{\mathbb{R}}^0 \text{ as a Mumford–Tate domain } D, \\ & \text{we may work with either the Hodge representation } (V_{\mathbb{R}}, Q, \rho, \varphi) \text{ or} \\ & \text{the induced Hodge representation } (\mathfrak{g}_{\mathbb{R}}, Q_{\mathfrak{g}}, \text{Ad}, \varphi). \end{aligned} \tag{3.3}$$

What we have in mind is the case that $V_{\mathbb{R}}$ carries an effective Hodge structure of weight $n \geq 0$; for example, $V_{\mathbb{R}} = H^n(X, \mathbb{R})$, where X is a smooth projective variety. It is helpful to work with the induced, weight zero, Hodge representation on $\mathfrak{g}_{\mathbb{R}}$ because the latter is closely related to the geometry and representation theory associated with the flag domain $D \subset \check{D}$.

Remark 3.4 (A notational liberty). The Hodge flag F^{\bullet} and the circle φ are equivalent: given one, the second is determined; cf. [GGK12]. So we may identify φ with the point $F^{\bullet} \in D$. This will be especially convenient when we wish to downplay our choice of Hodge representation $(V_{\mathbb{R}}, \rho)$ that gives $D \simeq G_{\mathbb{R}}/K_{\mathbb{R}}^0$ the structure of a Mumford–Tate domain.

3.1.1 *Hodge structures and Cartan decompositions.* Given a Hodge representation $(V_{\mathbb{R}}, Q, \rho, \varphi)$, the induced Hodge structure on $\mathfrak{g}_{\mathbb{C}}$ is

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus \mathfrak{g}^{p,-p}, \tag{3.5a}$$

where

$$\begin{aligned} \mathfrak{g}^{p,-p} &= \{ \xi \in \mathfrak{g}_{\mathbb{C}} \mid \xi(V^{r,s}) \subset V^{r+p,s-p} \ \forall r, s \} \\ &= \{ \xi \in \mathfrak{g}_{\mathbb{C}} \mid \text{Ad}_{\varphi(z)} \xi = z^{2p} \xi \ \forall z \in S^1 \}. \end{aligned} \tag{3.5b}$$

The decomposition is a grading of the Lie algebra in the sense that

$$[\mathfrak{g}^{p,-p}, \mathfrak{g}^{q,-q}] \subset [\mathfrak{g}^{p+q,-p-q}].$$

This implies that

$$\mathfrak{k}_{\mathbb{C}} := \bigoplus_{p \text{ even}} \mathfrak{g}^{p,-p} \tag{3.6a}$$

is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$, and

$$\mathfrak{k}_{\mathbb{C}}^{\perp} := \bigoplus_{p \text{ odd}} \mathfrak{g}^{p,-p} \tag{3.6b}$$

is a $\mathfrak{k}_{\mathbb{C}}$ -submodule. Moreover, $\overline{\mathfrak{g}^{p,-p}} = \mathfrak{g}^{-p,p}$ implies that both $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{C}}^{\perp}$ are defined over \mathbb{R} , so that

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{k}_{\mathbb{R}}^{\perp} \tag{3.7}$$

where $\mathfrak{k}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{R}}^{\perp} = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{k}_{\mathbb{C}}^{\perp}$. The following lemma is well known; see, for example, [CK77, GGK12].

LEMMA 3.8. *The Weyl operator $\varphi(\mathbf{i})$ is a Cartan involution with Cartan decomposition (3.7).*

Remark 3.9. The projection $D = G_{\mathbb{R}}/K_{\mathbb{R}}^0 \rightarrow G_{\mathbb{R}}/K_{\mathbb{R}}$ may be viewed as the map taking the Hodge decomposition (3.5) to the Cartan decomposition (3.7).

Proof. In the case that $\mathfrak{g}_{\mathbb{C}}$ is simple, $Q_{\mathfrak{g}}$ is necessarily a negative multiple of the Killing form. This is because a simple complex Lie algebra admits a unique $\text{Ad}(G_{\mathbb{C}})$ -invariant symmetric bilinear form, the Killing form, up to scale. So the induced polarization is necessarily a constant multiple of the Killing form. The facts that $Q_{\mathfrak{g}}$ is positive definite on the subalgebra $\mathfrak{k}_{\mathbb{R}}$ and negative definite $\mathfrak{k}_{\mathbb{R}}^{\perp}$ imply that (3.7) is a Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$ and $Q_{\mathfrak{g}}$ is a negative multiple of the Killing form.

More generally, as a reductive algebra $\mathfrak{g}_{\mathbb{C}}$ decomposes as the direct sum $\mathfrak{z} \oplus \mathfrak{g}_{\mathbb{C}}^{\text{ss}}$ of its center and the semisimple factor $\mathfrak{g}_{\mathbb{C}}^{\text{ss}} = [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$. Note that $\mathfrak{z} \subset \mathfrak{g}^{0,0}$, so that the polarization $Q_{\mathfrak{g}}$ is positive definite on the real form $\mathfrak{z} \cap \mathfrak{g}_{\mathbb{R}} \subset \mathfrak{k}_{\mathbb{R}}$. As above, the restriction of $Q_{\mathfrak{g}}$ to any simple factor of $\mathfrak{g}_{\mathbb{C}}^{\text{ss}}$ will be a negative multiple of the Killing form (the multiple may vary from one simple factor to the next) and (3.7) is a Cartan decomposition. \square

Remark 3.10 (A reasonable assumption on $Q_{\mathfrak{g}}$). From the argument establishing Lemma 3.8 we see that there is no essential loss of generality in assuming that the induced polarization $Q_{\mathfrak{g}}$ on $\mathfrak{g}_{\mathbb{R}}$ is minus the Killing form.

Given a maximal compact Lie subgroup $K_{\mathbb{R}} \subset G_{\mathbb{R}}$, let $\theta : \mathfrak{g}_{\mathbb{R}} \rightarrow \mathfrak{g}_{\mathbb{R}}$ be the corresponding Cartan involution. A point $\varphi \in \check{D}$ is a *K-Matsuki point* if the Lie algebra \mathfrak{p} of the stabilizer $\text{Stab}_{G_{\mathbb{C}}}(\varphi)$ contains a conjugation and θ -stable Cartan subalgebra \mathfrak{h} of $\mathfrak{g}_{\mathbb{C}}$. As discussed in [FHW06, § 4.3],

$$\text{any two } K\text{-Matsuki points in } D \text{ are } K_{\mathbb{R}}\text{-conjugate.} \tag{3.11}$$

From Lemma 3.8 we obtain the following corollary.

COROLLARY 3.12. *The circle $\varphi \in D$ is a Matsuki point with respect to the maximal compact subgroup $K_{\mathbb{R}}$ determined by (3.7).*

3.1.2 *Hodge structures and grading elements.* As illustrated in [Rob14, § 2.3], grading elements (§ 2.2) are essentially infinitesimal Hodge structures. Briefly, given a circle $\varphi : S^1 \rightarrow G_{\mathbb{R}}$, we may assume that the image $\text{im } \varphi$ is contained in a compact maximal torus $T \subset G_{\mathbb{R}}$ and that the complexification $\mathfrak{h} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$ of the Lie algebra \mathfrak{t} of T is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Then the (rescaled) derivative

$$E_{\varphi} := \frac{1}{4\pi\mathbf{i}} \varphi'(1) \tag{3.13}$$

is a grading element. The relationship between the E_{φ} -eigenspace decomposition (2.2) and the Hodge decomposition (3.1) is

$$V^{(p-q)/2} = V^{p,q}.$$

In the case that $V_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$, we have

$$\mathfrak{g}^p = \mathfrak{g}^{p,-p}. \tag{3.14}$$

As a consequence, the Lie algebra \mathfrak{p}_{φ} of the stabilizer $P_{\varphi} = \text{Stab}_{G_{\mathbb{C}}}(\varphi)$ is the parabolic (2.5) associated with the grading element E_{φ} .

Observe that the holomorphic tangent space $T_{\varphi}D = \mathfrak{g}_{\mathbb{C}}/\mathfrak{p}_{\varphi}$ is naturally identified with $\bigoplus_{p>0} \mathfrak{g}^{-p,p}$. The horizontal subbundle $T^hD \subset TD$ is the $G_{\mathbb{R}}$ -homogeneous subbundle with fiber $T^h_{\varphi}D \simeq \mathfrak{g}^{-1,1}$. A holomorphic map $f : M \rightarrow D$ is horizontal if $f_*TM \subset T^hD$.

The horizontal subbundle is bracket-generating if and only if E_{φ} is the grading element $E_{\mathfrak{p}_{\varphi}}$ associated with \mathfrak{p}_{φ} by (2.8). One may always reduce to the case that the infinitesimal period relation is bracket-generating (cf. [Rob14, § 3.3]), and so we will

$$\begin{aligned} & \text{assume that the horizontal subbundle is} \\ & \text{bracket-generating; equivalently, } E_{\varphi} = E_{\mathfrak{p}_{\varphi}}. \end{aligned} \tag{3.15}$$

This assumption has the very significant consequence that

$$\text{the compact dual } \check{D} = G_{\mathbb{C}}/P \text{ determines the real form } G_{\mathbb{R}}. \tag{3.16}$$

This may be seen as follows. The choice of compact dual is equivalent to a choice of conjugacy class \mathcal{P} of parabolic subgroups $P \subset G_{\mathbb{C}}$. Modulo the action of $G_{\mathbb{C}}$, the conjugacy class determines the grading element E by (2.8). It then follows from (3.6) and (3.14) that the E -eigenspace decomposition (2.3) of $\mathfrak{g}_{\mathbb{C}}$ determines the complexified Cartan decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{k}_{\mathbb{C}}^{\perp}$. If $\mathfrak{g}_{\mathbb{C}}$ is simple, then $\mathfrak{k}_{\mathbb{C}}$ uniquely determines $\mathfrak{g}_{\mathbb{R}}$; cf. Appendix A. More generally, if $\mathfrak{g}_{\mathbb{C}}$ is semisimple then each simple ideal $\mathfrak{g}'_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ is a sub-Hodge structure; again the grading element/infinisimal Hodge structure determines a complexified Cartan decomposition, and the corresponding $\mathfrak{k}'_{\mathbb{C}}$ determines $\mathfrak{g}'_{\mathbb{R}}$. Finally, in the general case that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{z}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}^{\text{ss}}$ is reductive, the fact that the center $\mathfrak{z}_{\mathbb{C}}$ is contained in $\mathfrak{g}^{0,0} \subset \mathfrak{k}_{\mathbb{C}}$ forces $Z_{\mathbb{R}}$ to be a compact torus $S^1 \times \dots \times S^1$.

3.1.3 Levi subalgebras and sub-Hodge structures. A (real) sub-Hodge structure of a Hodge representation $(V_{\mathbb{R}}, Q, \rho, \varphi)$ is given by a real subspace $U_{\mathbb{R}} \subset V_{\mathbb{R}}$ that is preserved under the action of $\varphi(z)$ for all $z \in S^1$. In this case, we will say that the subspace $U_{\mathbb{R}}$ is φ -stable. The following lemma formalizes an observation made in the proof of [GGR14, Lemma V.23].

LEMMA 3.17. *Consider a Hodge representation $(\mathfrak{g}_{\mathbb{R}}, Q_{\mathfrak{g}}, \text{Ad}, \varphi)$ of $G_{\mathbb{R}}$ on the Lie algebra. A Levi subalgebra $\mathfrak{l}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ carries a sub-Hodge structure if and only if the image $\varphi(S^1)$ lies in the (connected) Lie subgroup $L_{\mathbb{R}} \subset G_{\mathbb{R}}$ with Lie algebra $\mathfrak{l}_{\mathbb{R}}$; equivalently, $E_{\varphi} \in \mathfrak{l}_{\mathbb{C}}$.*

Remark 3.18. A priori the condition that $\varphi(S^1) \subset L_{\mathbb{R}}$ is stronger than the condition that $\mathfrak{l}_{\mathbb{R}}$ carries a sub-Hodge structure: the former implies that $(\mathfrak{g}_{\mathbb{R}}, Q_{\mathfrak{g}}, \rho|_{L_{\mathbb{R}}}, \varphi)$ is a Hodge representation of $L_{\mathbb{R}}$.

Proof. (\Leftarrow) If the image of φ lies in $L_{\mathbb{R}}$, then it is clear that $\varphi(z)$ preserves $\mathfrak{l}_{\mathbb{R}}$ for all $z \in S^1$.

(\Rightarrow) Recall the (rescaled) derivative $E_{\varphi} = \varphi'(1)/4\pi i$ of (3.13). To show that the image of φ lies in $L_{\mathbb{R}}$, it suffices to show that $E_{\varphi} \in \mathfrak{l}_{\mathbb{C}}$. Let $\mathfrak{g}_{\mathbb{C}} = \bigoplus \mathfrak{g}^{p,q}$ be the Hodge decomposition. Then $\mathfrak{l}_{\mathbb{C}} = \bigoplus \mathfrak{l}^{p,q}$, where $\mathfrak{l}^{p,q} = \mathfrak{l}_{\mathbb{C}} \cap \mathfrak{g}^{p,q}$. As discussed in § 3.1.2, these Hodge decompositions may be viewed as E_{φ} -eigenspace decompositions for the grading element $E_{\varphi} \in \mathfrak{g}_{\mathbb{C}}$. In particular,

$$\mathfrak{l}_{\mathbb{C}} = \bigoplus \mathfrak{l}^{\alpha} \tag{3.19}$$

where $\mathfrak{l}^a = \mathfrak{l}_{\mathbb{C}} \cap \mathfrak{g}^a$, and $\mathfrak{g}_{\mathbb{C}} = \bigoplus \mathfrak{g}^a$ is given by (2.3). Moreover, (2.4) implies that (3.19) is a graded decomposition; that is, $[\mathfrak{l}^a, \mathfrak{l}^b] \subset \mathfrak{l}^{a+b}$.

As a reductive Lie algebra $\mathfrak{l}_{\mathbb{C}} = \mathfrak{z}_{\mathbb{C}} \oplus \mathfrak{l}_{\mathbb{C}}^{\text{ss}}$, where $\mathfrak{l}_{\mathbb{C}}^{\text{ss}} = [\mathfrak{l}_{\mathbb{C}}, \mathfrak{l}_{\mathbb{C}}]$ is the semisimple factor, and $\mathfrak{z}_{\mathbb{C}} \subset \mathfrak{l}^0$ is the center of $\mathfrak{l}_{\mathbb{C}}$. The graded decomposition of $\mathfrak{l}_{\mathbb{C}}$ induces a graded decomposition

$$\mathfrak{l}_{\mathbb{C}}^{\text{ss}} = \bigoplus \mathfrak{l}_a^{\text{ss}} \tag{3.20}$$

by $\mathfrak{l}_a^{\text{ss}} = \mathfrak{l}_{\mathbb{C}}^{\text{ss}} \cap \mathfrak{l}^a$. There exists a grading element $F \in \mathfrak{l}_{\mathbb{C}}^{\text{ss}}$ with the property that (3.20) is the F -eigenspace decomposition of $\mathfrak{l}_{\mathbb{C}}^{\text{ss}}$ [ČS09, Proposition 3.1.2]. Observe that $E_{\varphi} - F \in C_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{l}_{\mathbb{C}})$ lies in the centralizer of $\mathfrak{l}_{\mathbb{C}}$. Because $\mathfrak{l}_{\mathbb{C}}$ is a Levi subalgebra, this centralizer is equal to the center $\mathfrak{z}_{\mathbb{C}}$. Therefore, $E_{\varphi} - F \in \mathfrak{l}_{\mathbb{C}}$. Since $F \in \mathfrak{l}_{\mathbb{C}}$, this implies $E_{\varphi} \in \mathfrak{l}_{\mathbb{C}}$. \square

3.2 Polarized mixed Hodge structures

Let $(V_{\mathbb{R}}, Q)$ be a Hodge representation of $G_{\mathbb{R}}$ and let $D \subset \check{D}$ be the corresponding Mumford–Tate domain. An (m -variable) *nilpotent orbit* on D consists of a pair $(F^{\bullet}; N_1, \dots, N_m)$ such that $F^{\bullet} \in \check{D}$, the $N_i \in \mathfrak{g}_{\mathbb{R}}$ commute and $N_i F^p \subset F^{p-1}$, and the holomorphic map $\psi : \mathbb{C}^m \rightarrow \check{D}$ defined by

$$\psi(z^1, \dots, z^m) = \exp(z^i N_i) F^{\bullet} \tag{3.21}$$

has the property that $\psi(z) \in D$ for $\text{Im}(z^i) \gg 0$. The associated (open) *nilpotent cone* is

$$\mathcal{C} = \{t^i N_i \mid t^i > 0\}. \tag{3.22}$$

A *polarized mixed Hodge structure* on D is given by a pair (F^{\bullet}, N) such that $F^{\bullet} \in \check{D}$, $N \in \mathfrak{g}_{\mathbb{R}}$ and $N(F^p) \subset F^{p-1}$, $(F^{\bullet}, W_{\bullet}(N, V_{\mathbb{R}}))$ is a mixed Hodge structure, and the Hodge structure on

$$\text{Gr}_k(W_{\bullet}(N, V_{\mathbb{R}}))_{\text{prim}} := \ker\{N^k : \text{Gr}_k(W_{\bullet}(N, V_{\mathbb{R}})) \rightarrow \text{Gr}_{-k-2}(W_{\bullet}(N, V_{\mathbb{R}}))\}$$

is polarized by $Q(\cdot, N^k \cdot)$, for all $k \geq 0$. The notions of nilpotent orbit and PMHS are closely related. The following well-known results are due to Cattani, Kaplan, and Schmid [CK82a, CK89, CKS86, CK87, Sch73].

THEOREM 3.23 (Cattani, Kaplan, and Schmid). *Let $D \subset \check{D}$ be a Mumford–Tate domain (and compact dual) for a Hodge representation $V_{\mathbb{R}}$ of $G_{\mathbb{R}}$.*

(a) *A pair $(F^{\bullet}; N)$ forms a one-variable nilpotent orbit if and only if it forms a polarized mixed Hodge structure.*

(b) *The weight filtration $W_{\bullet}(N, V_{\mathbb{R}})$ does not depend on the choice of $N \in \mathcal{C}$. Let $W_{\bullet}(\mathcal{C}, V_{\mathbb{R}})$ denote this common weight filtration.*

(c) *Fix $F^{\bullet} \in \check{D}$ and commuting nilpotent elements $\{N_1, \dots, N_m\} \subset \mathfrak{g}_{\mathbb{R}}$ with the properties that: (i) $N_i F^p \subset F^{p-1}$ for every i ; and (ii) the filtration $W_{\bullet}(N, V_{\mathbb{R}})$ does not depend on the choice of $N \in \mathcal{C}$, where the latter is given by (3.22). Then $(F^{\bullet}; N)$ is a polarized mixed Hodge structure for some $N \in \mathcal{C}$, if and only if $(F^{\bullet}; N_1, \dots, N_m)$ is an m -variable nilpotent orbit.*

In a mild abuse of nomenclature, given a nilpotent orbit $(F^{\bullet}; N_1, \dots, N_m)$, we will sometimes refer to $(F^{\bullet}, W_{\bullet}(\mathcal{C}, V_{\mathbb{R}}))$ as a PMHS (especially when we wish to emphasize the weight filtration $W_{\bullet}(\mathcal{C}, V_{\mathbb{R}})$ over the nilpotents $N \in \mathcal{C}$).

The *Deligne splitting* [CKS86, Del71]

$$V_{\mathbb{C}} = \bigoplus I^{p,q} \tag{3.24a}$$

of a mixed Hodge structure (F^\bullet, W_\bullet) on $V_{\mathbb{R}}$ is given by

$$I^{p,q} := F^p \cap W_{p+q} \cap \left(\overline{F^q} \cap W_{p+q} + \sum_{j \geq 1} \overline{F^{q-j}} \cap W_{p+q-j-1} \right). \tag{3.24b}$$

It is the unique bigrading of $V_{\mathbb{C}}$ with the properties that

$$F^p = \bigoplus_{r \geq p} I^{r,\bullet} \quad \text{and} \quad W_\ell = \bigoplus_{p+q \leq \ell} I^{p,q}, \tag{3.25}$$

and

$$\overline{I^{p,q}} = I^{q,p} \quad \text{mod} \quad \bigoplus_{r < q, s < p} I^{r,s}.$$

Any mixed Hodge structure (F^\bullet, W_\bullet) on V induces a mixed Hodge structure $(F_{\mathfrak{g}}^\bullet, W_{\mathfrak{g}}^\bullet)$ on \mathfrak{g} by

$$F_{\mathfrak{g}}^p = \{ \xi \in \mathfrak{g}_{\mathbb{C}} \mid \xi(F^r) \subset F^{p+r} \ \forall r \},$$

$$W_{\mathfrak{g}}^\ell = \{ \xi \in \mathfrak{g}_{\mathbb{R}} \mid \xi(W_m) \subset W_{m+\ell} \ \forall m \}.$$

The elements of $F_{\mathfrak{g}}^r \cap W_{2r}^{\mathfrak{g}} \cap \mathfrak{g}_{\mathbb{R}}$ are the (r, r) -morphisms of the mixed Hodge structure (F^\bullet, W_\bullet) . Alternatively, if $\mathfrak{g}_{\mathbb{C}} = \bigoplus I_{\mathfrak{g}}^{p,q}$ denotes the corresponding Deligne splitting

$$I_{\mathfrak{g}}^{p,q} = \{ \xi \in \mathfrak{g}_{\mathbb{C}} \mid \xi(I^{r,s}) \subset I^{p+r, q+s} \ \forall r, s \},$$

then the elements of $I_{\mathfrak{g}}^{r,r} \cap \mathfrak{g}_{\mathbb{R}}$ are the (r, r) -morphisms.

When $\overline{I^{p,q}} = I^{q,p}$ we say that the mixed Hodge structure is \mathbb{R} -split. When an \mathbb{R} -split mixed Hodge structure $(F^\bullet, W_\bullet(\mathcal{C}, V_{\mathbb{R}}))$ arises from a nilpotent orbit $(F^\bullet; N_1, \dots, N_m)$, we will say that the nilpotent orbit is \mathbb{R} -split.

Remark 3.26. If (F^\bullet, N) is \mathbb{R} -split, then so is the induced $(F_{\mathfrak{g}}^\bullet, N)$.

Observe that

$$L_{\mathfrak{g}}^{-1,-1} := \bigoplus_{p,q > 0} I_{\mathfrak{g}}^{-p,-q}$$

is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and is defined over \mathbb{R} . The following well-known results are due to Cattani, Deligne, Kaplan, and Schmid [CK82a, CKS86, Del71].

THEOREM 3.27 (Deligne, Cattani, Kaplan, and Schmid). *Let $D \subset \check{D}$ be a Mumford–Tate domain (and compact dual) for a weight n Hodge representation of $G_{\mathbb{R}}$ on $V_{\mathbb{R}}$.*

(a) *If $(F^\bullet; N)$ is an \mathbb{R} -split polarized mixed Hodge structure, then $\psi(z) = e^{zN} F^\bullet \in D$ for all $\text{Im}(z) > 0$ and ψ is a horizontal, $\text{SL}_2\mathbb{R}$ -equivariant embedding of the upper-half plane.*

(b) *Given a mixed Hodge structure (F^\bullet, W_\bullet) on $V_{\mathbb{R}}$, there exists a unique $\delta \in L_{\mathfrak{g}, \mathbb{R}}^{-1,-1}$ such that*

$$e^{-2i\delta} \cdot F^p = \bigoplus_{s \geq p} I^{\bullet,s}.$$

The element δ is real, commutes with all morphisms of (F^\bullet, W_\bullet) and, given

$$\tilde{F}^\bullet := e^{-i\delta} \cdot F^\bullet, \tag{3.28}$$

$(\tilde{F}^\bullet, W_\bullet)$ is an \mathbb{R} -split mixed Hodge structure. (From $L_{\mathfrak{g}}^{-1,-1} \subset W_{-2}^{\mathfrak{g}}$ we see that δ preserves the filtration W_\bullet and acts trivially on $\text{Gr}_\ell(W_\bullet)$.) It follows that both F^\bullet and \tilde{F}^\bullet determine the same filtrations on $\text{Gr}_\ell(W_\bullet)$. Moreover, every morphism of (F^\bullet, W_\bullet) commutes with δ , so that the morphisms of (F^\bullet, W_\bullet) are precisely those of $(\tilde{F}^\bullet, W_\bullet)$ that commute with δ .

(c) In the case that $W_\bullet = W_\bullet(N, V_{\mathbb{R}})[-n]$, the two nilpotent orbits $\psi(z) = e^{zN}F^\bullet$ and $\tilde{\psi}(z) = e^{zN}\tilde{F}^\bullet$ agree to first order at $z = \infty$, and that limit flag is

$$F_\infty^p := \lim_{\text{Im}(z) \rightarrow \infty} e^{zN}F^p = \bigoplus_{s \leq n-p} I^{\bullet,s}. \tag{3.29}$$

3.3 Reduced limit period mapping

Given commuting $N_1, \dots, N_m \in \text{Nilp}(\mathfrak{g}_{\mathbb{R}})$ defining a cone (3.22), the boundary component $B(\mathcal{C})$ is the set of nilpotent orbits $(F^\bullet; N_1, \dots, N_m)$ modulo reparametrization. That is, we say that two elements F_1^\bullet and F_2^\bullet of

$$\tilde{B}(\mathcal{C}) := \{F^\bullet \in \check{D} \mid (F^\bullet; N_1, \dots, N_m) \text{ is a nilpotent orbit}\}$$

are equivalent if $F_1^\bullet = \exp(z^i N_i)F_2^\bullet$ for some $z = (z^i) \in \mathbb{C}^m$; then

$$B(\mathcal{C}) := \tilde{B}(\mathcal{C}) / \sim.$$

In the case that $m = 1$, we write $B(\mathcal{C}) = B(N)$ and $\tilde{B}(\mathcal{C}) = \tilde{B}(N)$.

The reduced limit period mapping $\Phi_\infty : \tilde{B}(N) \rightarrow \text{cl}(D)$ defined by

$$\Phi_\infty(F^\bullet, N) := \lim_{\text{Im}(z) \rightarrow \infty} e^{zN} \cdot F^\bullet \tag{3.30}$$

descends to a well-defined map on $B(N)$; see [GGK13, Appendix to Lecture 10] and [KP14, § 5] for details.¹³ More generally, as observed in [KP14, Remark 5.6], the reduced limit period mapping is well defined on $B(\mathcal{C})$; that is, (3.30) does not depend on our choice of $N \in \mathcal{C}$. This may be seen as follows. First, by Theorem 3.23(b), the weight filtration $W_\bullet(\mathcal{C}, V_{\mathbb{R}})$ does not depend on our choice of $N \in \mathcal{C}$. Let $(\tilde{F}^\bullet, W_\bullet(\mathcal{C}, V_{\mathbb{R}}))$ be the \mathbb{R} -split mixed Hodge structure given by Theorem 3.27(b), and let $V_{\mathbb{C}} = \bigoplus \tilde{I}^{p,q}$ be the corresponding Deligne splitting (3.24). Then Theorem 3.27(c) and (3.30) assert that

$$\Phi_\infty(F^\bullet, N) = \Phi_\infty(\tilde{F}^\bullet, N) = \tilde{F}_\infty^\bullet \quad \text{and} \quad \tilde{F}_\infty^p = \bigoplus_{s \leq n-p} \tilde{I}^{\bullet,s}$$

is independent of $N \in \mathcal{C}$.¹⁴

4. Hodge–Tate degenerations

The main results of this section are that: (i) underlying every \mathbb{R} -split PMHS (F^\bullet, N) is a Hodge–Tate PMHS $(\mathcal{F}_1^\bullet, N)$ on a Levi subalgebra $\mathfrak{l}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ (Theorem 4.3); and (ii) the classification of the Hodge–Tate degenerations (Theorem 4.11). As corollaries to these results we will: (a) see that the nilpotent cone $\mathcal{C} \subset \mathfrak{g}_{\mathbb{R}}$ underlying a nilpotent orbit is contained in an $\text{Ad}(L_{\mathbb{R}}^Y)$ -orbit, where $L_{\mathbb{R}}^Y$ is a connected Lie subgroup of $G_{\mathbb{R}}$ with reductive Lie algebra $\mathfrak{l}_{\mathbb{R}}^Y \subset \mathfrak{l}_{\mathbb{R}}$ (Corollary 4.9); and (b) obtain the classification theorems of § 5.

¹³ In [KP14], Φ_∞ is called the *naïve limit map*.

¹⁴ See [BP13, HP15] for more general convergence results.

4.1 Definition

Let $(V_{\mathbb{R}}, Q, \rho, \varphi)$ be a Hodge representation of $G_{\mathbb{R}}$, and let $D \subset \check{D} = G_{\mathbb{C}}/P$ be the associated Mumford–Tate domain and compact dual. We say that D admits a Hodge–Tate degeneration if there exists a nilpotent orbit $(F^{\bullet}; N_1, \dots, N_m)$ with nilpotent cone \mathcal{C} such that the Deligne splitting (3.24) of $(F^{\bullet}, W_{\bullet}(\mathcal{C}, V_{\mathbb{R}}))$ satisfies

$$I^{p,q} = 0 \quad \text{for all } p \neq q.$$

In this case we say that the nilpotent orbit $(F^{\bullet}; N_1, \dots, N_m)$ is a Hodge–Tate degeneration.

We recall some properties of Hodge–Tate degenerations in the following proposition.

PROPOSITION 4.1. *Let $V_{\mathbb{R}}$ admit the structure of a Hodge representation of $G_{\mathbb{R}}$, and let $(F^{\bullet}; N_1, \dots, N_m)$ be a nilpotent orbit on the associated Mumford–Tate domain $D \subset \check{D}$.*

(a) *If $(F^{\bullet}; N_1, \dots, N_m)$ is Hodge–Tate, then so is the induced nilpotent orbit $(F_{\mathfrak{g}}^{\bullet}; N_1, \dots, N_m)$ on $\mathfrak{g}_{\mathbb{R}}$.*

(b) *Suppose that $G_{\mathbb{R}}$ is semisimple. Then $(F^{\bullet}; N_1, \dots, N_m)$ is Hodge–Tate if and only if $(F_{\mathfrak{g}}^{\bullet}; N_1, \dots, N_m)$ is Hodge–Tate.*

(c) *If $(F_{\mathfrak{g}}^{\bullet}; N_1, \dots, N_m)$ is Hodge–Tate, then the nilpotent orbit $(F; N_1, \dots, N_m)$ is a ‘maximal’ degeneration of Hodge structure in the sense that $\Phi_{\infty}(F^{\bullet}, N)$ lies in the unique closed $G_{\mathbb{R}}$ -orbit $\mathcal{O}_{\text{cl}} \subset \check{D}$, for any $N \in \mathcal{C}$.*

Proof. Part (b) is [GGR14, Proposition I.9], and (c) is [KP14, Corollary 4.3] or [GGR14, Proposition I.15]. In general, $G_{\mathbb{R}}$ is reductive and (a) follows from the arguments establishing (b). □

Remark 4.2. If $G_{\mathbb{R}}$ is not semisimple, then the converse to Proposition 4.1(a) need not hold: it is possible for a non-Hodge–Tate (F^{\bullet}, N) to induce a Hodge–Tate $(F_{\mathfrak{g}}^{\bullet}, N)$. Indeed, this is precisely the case in Theorem 4.3, where the nilpotent orbit $(F^{\bullet}; N_1, \dots, N_m)$ on the Hodge representation $(V_{\mathbb{R}}, Q, \rho, \varphi)$ of the reductive $L_{\mathbb{R}}$ will in general fail to be Hodge–Tate, while the induced $(\mathcal{F}_{\mathfrak{g}}^{\bullet}; N_1, \dots, N_m)$ is always Hodge–Tate; cf. Remark 4.6.

While the Hodge–Tate degenerations are ‘maximal’ in the sense of Proposition 4.1(b), the associated representation theory is relatively simple, as we will see in the classification of Theorem 4.11.

4.2 The underlying Hodge–Tate degeneration

In a suitably interpreted sense all degenerations are induced from a degeneration of Hodge–Tate type.¹⁵ The results of this section for $\dim_{\mathbb{R}} \mathcal{C} = 1$ first appeared in [GGR14]. Let

$$\mathcal{H}^m := \{z = (z^i) \in \mathbb{C}^m \mid \text{Im}(z^i) > 0\}.$$

THEOREM 4.3. *Let $(V_{\mathbb{R}}, Q, \rho, \varphi)$ be a Hodge representation of a semisimple Lie group $G_{\mathbb{R}}$, and let D be the associated Mumford–Tate domain. Suppose that $(F^{\bullet}; N_1, \dots, N_m)$ is a \mathbb{R} -split nilpotent orbit.*

¹⁵ Some care must be taken with this statement, as it is not necessarily the case that the underlying degeneration arises algebro-geometrically: this is a statement about the orbit structure and representation theory associated with the $\text{SL}(2)$ -orbit approximating an arbitrary degeneration, which may or may not arise algebro-geometrically.

(a) Let $\mathfrak{g}_{\mathbb{C}} = \bigoplus I_{\mathfrak{g}}^{p,q}$ be the associated Deligne splitting (cf. (3.3) and (3.24)), and set

$$\mathfrak{l}_{\mathbb{C}} := \bigoplus_p I_{\mathfrak{g}}^{p,p}. \tag{4.4}$$

Then $\mathfrak{l}_{\mathbb{C}}$ is a Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ defined over \mathbb{R} with real form $\mathfrak{l}_{\mathbb{R}} = \mathfrak{l}_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{R}}$ and $N_i \in \mathfrak{l}_{\mathbb{R}}$. Let $L_{\mathbb{R}} \subset G_{\mathbb{R}}$ be the connected Lie subgroup with Levi algebra $\mathfrak{l}_{\mathbb{R}}$.

(b) Given $z \in \mathcal{H}^m$, let $\varphi_z : S^1 \rightarrow G_{\mathbb{R}}$ denote the Hodge structure on $V_{\mathbb{R}}$ parameterized by $\exp(z^i N_i) \cdot F^{\bullet} \in D$. Then the circle φ_z is contained in $L_{\mathbb{R}}$ for all $z \in \mathcal{H}^m$; that is, $\text{im } \varphi_z \subset L_{\mathbb{R}}$. Equivalently, $(V_{\mathbb{R}}, Q, \rho|_{L_{\mathbb{R}}}, \varphi_z)$ is a Hodge representation of $L_{\mathbb{R}}$; let \mathcal{D} denote the associated Mumford–Tate domain.

(c) The induced nilpotent orbit $(\mathcal{F}_1^{\bullet}; N_1, \dots, N_m)$ on \mathcal{D} is a Hodge–Tate degeneration.

Remark 4.5. An immediate and important consequence of Theorem 4.3(b) is that any nilpotent orbit on \mathcal{D} induces a nilpotent orbit on D ; so we may think of the nilpotent orbit $(\mathcal{F}_1^{\bullet}; N_1, \dots, N_m)$ as ‘the Hodge–Tate degeneration underlying the nilpotent orbit $(F^{\bullet}; N_1, \dots, N_m)$ ’. From this perspective, Theorem 4.3 asserts that the essential structure/relationship is between the $\{N_1, \dots, N_m\}$ and the Levi subalgebra \mathfrak{l} ; the remaining structure on $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{l}^{\perp}$,¹⁶ that is, the Hodge structure on \mathfrak{l}^{\perp} , is induced from the \mathfrak{l} -module structure on \mathfrak{l}^{\perp} .¹⁷

Remark 4.6. Each

$$\mathcal{V}_{\ell} = \bigoplus_{p-q=\ell} I_{\mathfrak{g}}^{p,q}$$

is an $\mathfrak{l}_{\mathbb{C}}$ -module, and $\mathcal{V}_{\ell} + \mathcal{V}_{-\ell}$ naturally has the structure of a Hodge representation of $L_{\mathbb{R}}$. In particular, $V = \bigoplus_{\ell \geq 0} \mathcal{V}_{\ell}$ is a coarse branching of V as an $L_{\mathbb{R}}$ -Hodge representation. (‘Coarse’ because the \mathcal{V}_{ℓ} need not be irreducible.)

Proof. The fact that the nilpotent orbit is \mathbb{R} -split implies $\mathfrak{l}_{\mathbb{C}}$ is a conjugation-stable subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and

$$N_i \in I_{\mathfrak{g}, \mathbb{R}}^{-1,-1} \subset \mathfrak{l}_{\mathbb{R}}.$$

As the zero eigenspace for the grading element $E - \bar{E}$, the subalgebra $\mathfrak{l}_{\mathbb{C}}$ is necessarily a Levi subalgebra. This establishes Theorem 4.3(a).

Let \mathcal{C} be the nilpotent cone (3.22) underlying the nilpotent orbit. Observe that the PMHS $(F_{\mathfrak{g}}^{\bullet}, W_{\bullet}(\mathcal{C}, \mathfrak{g}_{\mathbb{R}}))$ on $\mathfrak{g}_{\mathbb{R}}$ induces a polarized mixed Hodge substructure $(\mathcal{F}_1^{\bullet}, \mathcal{W}_{\bullet}(\mathcal{C}, \mathfrak{l}_{\mathbb{R}}))$ on $\mathfrak{l}_{\mathbb{R}}$ by

$$\mathcal{F}_1^p := F^p \cap \mathfrak{l}_{\mathbb{C}} = \bigoplus_{q \geq p} I_{\mathfrak{g}}^{q,q} \quad \text{and} \quad \mathcal{W}_{\ell}(\mathcal{C}, \mathfrak{l}_{\mathbb{R}}) := W_{\ell}(\mathcal{C}, \mathfrak{g}_{\mathbb{R}}) \cap \mathfrak{l}_{\mathbb{R}} = \bigoplus_{2q \leq \ell} I_{\mathfrak{g}, \mathbb{R}}^{q,q}. \tag{4.7}$$

Theorem 3.23(c) implies that the Hodge flag $\exp(z^i N_i) \cdot \mathcal{F}_1^{\bullet}$ defines a Hodge structure on $\mathfrak{l}_{\mathbb{R}}$; equivalently, $\mathfrak{l}_{\mathbb{R}}$ is a sub-Hodge structure of $(\mathfrak{g}_{\mathbb{R}}, \varphi_z)$. Theorem 4.3(b) now follows from Lemma 3.17.

Finally, (4.4) and (4.7) yield Theorem 4.3(c). □

¹⁶ This \mathfrak{l} -module decomposition of \mathfrak{g} exists because \mathfrak{l} is reductive.

¹⁷ This sort of idea goes back to Bala and Carter’s classification [BC76a, BC76b] of nilpotent orbits $\mathcal{N} \subset \mathfrak{g}_{\mathbb{C}}$, where the idea is to look at minimal Levi subalgebras \mathfrak{l} containing a fixed $N \in \mathcal{N}$, and to classify the pairs (N, \mathfrak{l}) . (In fact, the idea goes back farther to Dynkin [Dyn57], who looked at minimal reductive subalgebras containing N , but this approach does not seem to work as well.)

Remark 4.8 (Mumford–Tate domain for the Hodge structures $\varphi_z|_{\mathfrak{l}_{\mathbb{R}}}$). The Mumford–Tate domain \mathcal{D} for the Hodge structures φ_z on $\mathfrak{l}_{\mathbb{R}}$ may be viewed as a subset of D , the Mumford–Tate domain for the Hodge structure φ on $\mathfrak{g}_{\mathbb{R}}$ (or $V_{\mathbb{R}}$). Let $L_{\mathbb{C}} \subset G_{\mathbb{C}}$ be the connected Lie subgroup with Lie algebra $\mathfrak{l}_{\mathbb{C}}$, and set $\tilde{\mathcal{D}} = L_{\mathbb{C}} \cdot F^{\bullet}$. Then $\mathcal{D} \simeq \tilde{\mathcal{D}} \cap D$.

COROLLARY 4.9. *Given an \mathbb{R} -split nilpotent orbit (3.21) on a Mumford–Tate domain $D = G_{\mathbb{R}}/K_{\mathbb{R}}^0$ with nilpotent cone \mathcal{C} as in (3.22), let $\mathfrak{l}_{\mathbb{R}}$ be the Levi subalgebra (4.4). Let $Y \in \mathfrak{l}_{\mathbb{R}}$ be the grading element defined by*

$$Y|_{I_{\mathfrak{g}}^{p,p}} = 2p,$$

and let $L_{\mathbb{R}}^Y$ denote the connected subgroup of $L_{\mathbb{R}}$ stabilizing Y under the adjoint action. Then the Lie algebra $\mathfrak{l}_{\mathbb{R}}^Y = \{\xi \in \mathfrak{l}_{\mathbb{R}} \mid [\xi, Y] = 0\}$ is Levi and $\mathcal{C} \subset \text{Nilp}(\mathfrak{l}_{\mathbb{R}})$ is contained in an $\text{Ad}(L_{\mathbb{R}}^Y)$ -orbit.

Proof. Recall that Y is a grading element, cf. (2.12); it then follows from the definition (§ 2.2) that

$$\mathfrak{l}_{\mathbb{R}}^Y = I_{\mathfrak{g},\mathbb{R}}^{0,0}$$

is Levi.

Cattani and Kaplan [CK82a, (3.3)] proved that the Jacobson–Morosov filtration $W_{\bullet}(N')$ is independent of our choice of $N' \in \mathcal{C}$; we denote this weight filtration by $W_{\bullet}(\mathcal{C})$. Let

$$\mathcal{W}_{\mathcal{C}} = \{N' \in I_{\mathfrak{g},\mathbb{R}}^{-1,-1} \mid W(N') = W(\mathcal{C})\}.$$

Of course, $\mathcal{C} \subset \mathcal{W}_{\mathcal{C}}$. It suffices to show that $\mathcal{W}_{\mathcal{C}}$ is a disjoint union of open $L_{\mathbb{R}}^Y$ -orbits in $I_{\mathfrak{g},\mathbb{R}}^{-1,-1}$; for it then follows from the connectedness of \mathcal{C} that the cone is contained in an $\text{Ad}(L_{\mathbb{R}}^Y)$ -orbit.

First observe that property (iii) in § 2.4 implies that, for each $N' \in \mathcal{W}_{\mathcal{C}}$, there exists a unique $N' +_{\in} I_{\mathfrak{g},\mathbb{R}}^{1,1}$ such that $\{N'_+, Y, N'\}$ is a standard triple. Define

$$P_{N'}^{\ell} := \{\xi \in \mathfrak{l}_{\mathbb{R}} \mid [Y, \xi] = \ell\xi, [N'_+, \xi] = 0\} \quad \text{for all } \ell \geq 0.$$

This is the vector space of ‘highest weight vectors’ in the ‘isotypic component of weight ℓ ’ for the action of $\mathfrak{sl}_{\mathbb{R}}2 = \text{span}_{\mathbb{R}}\{N'_+, Y, N'\}$ on $\mathfrak{l}_{\mathbb{R}}$. It is a basic result of $\mathfrak{sl}(2)$ -representation theory that

$$\mathfrak{l}_{\mathbb{R}} = \bigoplus_{\substack{\ell \geq 0 \\ 0 \leq a \leq \ell}} (N')^a P_{N'}^{\ell}.$$

From this, and $(N')^{\ell+1}(P_{N'}^{\ell}) = 0$, we may deduce that the $\text{Ad}(L_{\mathbb{R}}^Y)$ -orbit of N' is open in $I_{\mathfrak{g},\mathbb{R}}^{-1,-1}$. □

4.3 Classification of Hodge–Tate degenerations

In [GGR14, Lemma V.7, Theorem V.15] it is shown that a *period domain* parameterizing weight n polarized Hodge structures admits a Hodge–Tate degeneration if and only if the Hodge numbers satisfy

$$h^{n,0} \leq h^{n-1,1} \leq \dots \leq h^{n-m,m}, \tag{4.10}$$

with m defined by $n \in \{2m, 2m + 1\}$. In the more general setting of Mumford–Tate domains, (4.10) is a necessary, but not sufficient, condition for the existence of a Hodge–Tate degeneration [GGR14, Lemma V.7 and Remark V.16]. Here we extend the classification to arbitrary Mumford–Tate domains D with the property that the infinitesimal period relation is bracket-generating (we can always reduce to this case [Rob14, § 3.3]).

THEOREM 4.11. *Suppose that $V_{\mathbb{R}}$ is a Hodge representation of a real semisimple algebraic group $G_{\mathbb{R}}$. Let $D \subset G_{\mathbb{C}}/P$ be the associated Mumford–Tate domain and compact dual, and assume that the infinitesimal period relation is bracket-generating. Then D admits a Hodge–Tate degeneration (F^{\bullet}, N) if and only if there exists a standard triple $\{N^+, Y, N\} \subset \mathfrak{g}_{\mathbb{R}}$ such that the following two conditions hold:*

(a) *The neutral element Y is even, and $\mathfrak{p} = W_0(N^+, \mathfrak{g}_{\mathbb{C}})$. In this case, $\frac{1}{2}Y$ is the grading element (2.8) associated with \mathfrak{p} and $F_{\mathfrak{g}}^p = W_{-2p}(N^+, \mathfrak{g}_{\mathbb{C}})$.*

(b) *The compact characteristic vector (§ 2.7) of the nilpotent orbit $\mathcal{N} = \text{Ad}(G_{\mathbb{R}}) \cdot N$ satisfies the following conditions: $\gamma_i(\mathcal{N}) \equiv 0 \pmod{4}$, for all i ; and for the noncompact simple root, $\alpha'(\mathcal{N})$ is even and $\alpha'(\mathcal{N})/2$ is odd.*

If it exists, then the orbit \mathcal{N} is unique. That is, given a second Hodge–Tate nilpotent orbit $(\tilde{F}^{\bullet}, \tilde{N})$, it is the case that $\tilde{N} \in \mathcal{N}$.

The necessity of Theorem 4.11(a) was observed in [GGR14]. It implies that the Lie algebra $\mathfrak{p} = F_{\mathfrak{g}}^0$ of the stabilizer $P = \text{Stab}_{G_{\mathbb{C}}}(F^{\bullet})$ is an even Jacobson–Morosov parabolic. As illustrated by the examples at the end of this section, this constrains the (conjugacy classes of the) parabolics P , and therefore the compact duals, that may arise.

Proof. (\implies) Suppose that there exists a Hodge–Tate nilpotent orbit (F^{\bullet}, N) . Then the induced nilpotent orbit $(F_{\mathfrak{g}}^{\bullet}, N)$ is also Hodge–Tate (Proposition 4.1). Thus the Lie algebra of the parabolic subgroup $P \subset G_{\mathbb{C}}$ stabilizing the Hodge flag F^{\bullet} is

$$\mathfrak{p} = F_{\mathfrak{g}}^0 = \bigoplus_{p \geq 0} I_{\mathfrak{g}}^{p, \bullet} = \bigoplus_{p \geq 0} I_{\mathfrak{g}}^{p, p} = \bigoplus_{p+q \geq 0} I_{\mathfrak{g}}^{p, q}; \tag{4.12}$$

here, the second equality is due to (3.25), and the last two follow from the hypothesis that $(F_{\mathfrak{g}}^{\bullet}, N)$ is Hodge–Tate. Without loss of generality, the PMHS $(F^{\bullet}, W_{\bullet}(N, V_{\mathbb{R}}))$ is \mathbb{R} -split; then the induced PMHS $(F_{\mathfrak{g}}^{\bullet}, W_{\bullet}(N, \mathfrak{g}_{\mathbb{R}}))$ is also \mathbb{R} -split. Therefore, we may complete N to a standard triple (§ 2.3) with

$$N \in I_{\mathfrak{g}}^{-1, -1}, \quad Y \in I_{\mathfrak{g}}^{0, 0}, \quad \text{and} \quad N^+ \in I_{\mathfrak{g}}^{1, 1}. \tag{4.13}$$

It follows that

$$\mathfrak{p} = W_0(N^+, \mathfrak{g}_{\mathbb{C}})$$

is a Jacobson–Morosov parabolic subalgebra. Moreover, the neutral element

$$Y \text{ acts on } I_{\mathfrak{g}}^{p, p} \text{ by the scalar } 2p, \tag{4.14}$$

establishing the necessity of (a).

Since the infinitesimal period relation is bracket-generating, the grading element (2.8) associated with \mathfrak{p} necessarily acts on $I_{\mathfrak{g}}^{p, q}$ by the eigenvalue p . Given this, from (4.12) and (4.14) we see that

$$\frac{1}{2}Y \text{ is the grading element equation (2.8) associated with } \mathfrak{p}. \tag{4.15}$$

Let $\mathfrak{sl}_2\mathbb{R} \subset \mathfrak{g}_{\mathbb{R}}$ be the TDS spanned by the standard triple (4.13), and let $SL_2\mathbb{R} \subset G_{\mathbb{R}}$ be the corresponding subgroup. By Theorem 3.27(a), the map $z \mapsto \exp(zN) \cdot F^{\bullet}$ is a holomorphic, $SL_2\mathbb{R}$ -equivariant, horizontal embedding of the upper-half plane into D . Let $\mathcal{H} \subset D$ denote the image.

Recall the element ϱ of (2.29) and the triple $\{\bar{\mathcal{E}}, \mathcal{Z}, \mathcal{E}\}$ of (2.28). Note that ϱ lies in the image of $\mathrm{SL}_2\mathbb{C}$ and

$$\varphi := \varrho(F^\bullet) \in \mathcal{H} \subset D. \tag{4.16}$$

Taken with (4.15), this implies that

$$\begin{aligned} &\text{the grading element equation (2.8) associated with the} \\ &\text{stabilizer } \mathrm{Ad}_\varrho(\mathfrak{p}) \text{ of } \varphi \text{ is } \frac{1}{2}\mathrm{Ad}_\varrho(Y) = \frac{1}{2}\mathcal{Z}. \end{aligned} \tag{4.17}$$

Then the hypothesis that the infinitesimal period relation is bracket-generating implies that

$$\frac{1}{2}\mathcal{Z} = \mathbf{E}_\varphi, \tag{4.18}$$

where the latter is the grading element (3.13) associated with φ ; cf. §3.1.2. Therefore, by (3.6), Lemma 3.8, and (3.14), the $\frac{1}{2}\mathcal{Z}$ -graded decomposition (2.3) of $\mathfrak{g}_\mathbb{C}$ must satisfy

$$\mathfrak{g}^{\mathrm{even}} = \mathfrak{k}_\mathbb{C} \quad \text{and} \quad \mathfrak{g}^{\mathrm{odd}} = \mathfrak{k}_\mathbb{C}^\perp \tag{4.19}$$

where $\mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} \oplus \mathfrak{k}_\mathbb{R}^\perp$ is the Cartan decomposition given by the Cartan involution $\varphi(\mathbf{i})$. Observe that

$$\bar{\mathcal{E}} \in \mathfrak{g}^1 \subset \mathfrak{k}_\mathbb{C}^\perp, \quad \mathcal{Z} \in \mathfrak{g}^0 \subset \mathfrak{k}_\mathbb{C}, \quad \text{and} \quad \mathcal{E} \in \mathfrak{g}^{-1} \subset \mathfrak{k}_\mathbb{C}^\perp.$$

Since the Cartan involution acts on $\mathfrak{g}^1 \oplus \mathfrak{g}^{-1} = \mathfrak{g}^{1,-1} \oplus \mathfrak{g}^{-1,1}$ by the scalar -1 , and on $\mathfrak{g}^0 = \mathfrak{g}^{0,0}$ by the scalar 1 , we see that $\{N^+, Y, N\}$ is a Cayley triple (with respect to \mathfrak{k}); equivalently, $\{\bar{\mathcal{E}}, \mathcal{Z}, \mathcal{E}\}$ is a DKS triple. Equation (4.19) implies that the compact characteristic vector $(\gamma(\mathcal{Z}); \alpha'(\mathcal{Z}))$ of the orbit \mathcal{N} satisfies (b), establishing necessity.

(Uniqueness) At this point we may observe that if the $\mathrm{Ad}(G_\mathbb{R})$ -orbit \mathcal{N} exists, then it is unique: the compact characteristic vector $(\gamma(\mathcal{N}); \alpha'(\mathcal{N}))$ is uniquely determined by (4.18) and (4.19). Uniqueness of the orbit \mathcal{N} then follows from Theorem 2.31.

(\Leftarrow) Assume that conditions (a) and (b) hold. Fix a Cartan decomposition $\mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} \oplus \mathfrak{k}_\mathbb{R}^\perp$ and a Cayley triple $\{N^+, Y, N\}$ (§2.7). Set $F_\mathfrak{g}^p = W_{-2p}(N^+, \mathfrak{g}_\mathbb{C})$. Expression (2.13) implies that $Y, N^+ \in W_0(N^+, \mathfrak{g}_\mathbb{R})$. Therefore, Y and N^+ stabilize $F_\mathfrak{g}^\bullet$. Given hypothesis (a), this implies that the $\mathrm{SL}_2\mathbb{C}$ -orbit of $F_\mathfrak{g}^\bullet$ is a holomorphic, equivariant, horizontal embedding $\mathbb{P}^1 \hookrightarrow G_\mathbb{C}/P$. Arguing as above, the conditions of (b) imply (4.19); equivalently, $\varphi = \varrho(F_\mathfrak{g}^\bullet) \in D \cap \mathbb{P}^1$. This implies that $D \cap \mathbb{P}^1 = \mathcal{H}$ and $z \mapsto \exp(zN)F^\bullet$ is a nilpotent orbit. Then Theorem 3.23(a) ensures that (F^\bullet, N) is a PMHS. Finally, from $F_\mathfrak{g}^p = W_{-2p}(N^+, \mathfrak{g}_\mathbb{C})$, and the fact that Y is even, we see that $\mathbf{E} = \frac{1}{2}Y$ splits $F_\mathfrak{g}^\bullet$, while Y splits $W_\bullet(N, \mathfrak{g})$; it follows that (F^\bullet, N) is Hodge–Tate. \square

It will be helpful later for us to observe that

$$\Phi_\infty(F_\mathfrak{g}^\bullet, N) = \varrho(\varphi) = \varrho^2(F_\mathfrak{g}^\bullet). \tag{4.20}$$

The second equality is (4.16). To see why the first equality holds, set $F_{\mathfrak{g},\infty}^\bullet = \Phi_\infty(F_\mathfrak{g}^\bullet, N)$ and observe that (3.29) implies

$$F_{\mathfrak{g},\infty}^p = \bigoplus_{s \geq p} I^{\bullet, -s} = \bigoplus_{s \geq p} I^{-s, -s}.$$

At the same time

$$F_\mathfrak{g}^p = \bigoplus_{s \geq p} I^{s, s}.$$

The assertion now follows from (4.15) and the easily verified

$$\mathrm{Ad}_\varrho^2 Y = -Y.$$

Remark 4.21 (Cayley triples and Matsuki points). As we observed in the proof of (\implies) above, the standard triple $\{N^+, Y, N\}$ of Theorem 4.11 is a Cayley triple with respect to the Cartan involution $\varphi(\mathbf{i})$ defined by (4.16); cf. Lemma 3.8. This implies that (4.16) is a Matsuki point (with respect to the Cartan involution $\varphi(\mathbf{i})$).

4.4 Distinguished grading elements

It may be the case that a Hodge–Tate degeneration $(F_{\mathfrak{g}}^\bullet, N)$ on D is itself induced from a Hodge–Tate degeneration $(\mathcal{F}_l^\bullet, N)$ on a Mumford–Tate subdomain $\mathcal{D} \subset D$. More precisely, let φ be the circle (4.16) and suppose that $\mathfrak{l}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ is a φ -stable Levi subalgebra containing N . Then $\varphi(S^1) \subset L_{\mathbb{R}}$ by Lemma 3.17. In this case, setting $\mathcal{F}_l^\bullet = F_{\mathfrak{g}}^\bullet \cap \mathfrak{l}_{\mathbb{C}}$ defines a Hodge–Tate degeneration $(\mathcal{F}_l^\bullet, N)$ on $\mathcal{D} = L_{\mathbb{R}} \cdot \varphi$. (Here $\mathcal{D} \subset D$ is the Mumford–Tate domain for the Hodge representation $(L_{\mathbb{R}}, Q_l, \text{Ad}, \varphi)$ of $L_{\mathbb{R}}$; cf. Remark 4.8.) A simple test of the neutral element Y will determine whether or not $\mathfrak{g}_{\mathbb{R}}$ is the minimal such Levi subalgebra (i.e., whether or not there exists $\mathfrak{l}_{\mathbb{R}} \subsetneq \mathfrak{g}_{\mathbb{R}}$); cf. Lemma 4.23.

A grading element $Y \in \mathfrak{g}_{\mathbb{C}}$ is *distinguished* if $\frac{1}{2}Y$ is the grading element (2.8) associated with the parabolic \mathfrak{p}_Y and the Y -eigenspace decomposition $\mathfrak{g}_{\mathbb{C}} = \bigoplus \mathfrak{g}_\ell$ satisfies $\dim \mathfrak{g}_0 = \dim \mathfrak{g}_2$.

THEOREM 4.22 (Bala and Carter [BC76a]). *A grading element $Y \in \mathfrak{g}_{\mathbb{C}}$ is distinguished if and only if it can be realized as the neutral element of a standard triple $\{N^+, Y, N\}$ with the property that no proper Levi subalgebra $\mathfrak{l}_{\mathbb{C}} \subsetneq \mathfrak{g}_{\mathbb{C}}$ contains the standard triple. (Equivalently, no proper Levi subalgebra contains N .)*

LEMMA 4.23. *Given a Hodge–Tate degeneration $(F_{\mathfrak{g}}^\bullet, N)$ on D , let $\{N^+, Y, N\}$ be the standard triple of Theorem 4.11, and let φ be given by (4.16). The neutral element Y is distinguished if and only if $\mathfrak{g}_{\mathbb{R}}$ is the only φ -stable Levi subalgebra of $\mathfrak{g}_{\mathbb{R}}$ containing N^+ (equivalently, N).*

Remark 4.24. The hypothesis that (F^\bullet, N) is a Hodge–Tate degeneration on a Mumford–Tate domain is essential: there exist nilpotent $N \in \mathfrak{g}_{\mathbb{R}}$ with the property that $\mathfrak{g}_{\mathbb{R}}$ is the minimal φ -stable Levi subalgebra of $\mathfrak{g}_{\mathbb{R}}$ containing N , but for which Y is not even, let alone distinguished. Such nilpotents are *noticed* [Noë98].

Remark 4.25. Let $\{\bar{\mathcal{E}}, \mathcal{Z}, \mathcal{E}\} = \text{Ad}_\rho\{N^+, Y, N\}$ be the DKS triple in the proof of Theorem 4.11. Note that Y is distinguished if and only if \mathcal{Z} is. Moreover, $\mathfrak{g}_{\mathbb{R}}$ is the minimal φ -stable Levi subalgebra of $\mathfrak{g}_{\mathbb{R}}$ containing N^+ if and only if $\mathfrak{g}_{\mathbb{C}}$ is the minimal conjugation and φ -stable Levi subalgebra containing $\bar{\mathcal{E}}$.

Proof. (\implies) If Y is distinguished, then $\mathfrak{g}_{\mathbb{C}}$ is the smallest Levi subalgebra containing N^+ by Theorem 4.22.

(\impliedby) By Lemma 3.17 and (4.18) a Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ is φ -stable if and only if it contains \mathcal{Z} . Suppose that $\mathfrak{g}_{\mathbb{C}}$ is the only Levi subalgebra of \mathfrak{g} that: (i) contains the DKS triple $\{\bar{\mathcal{E}}, \mathcal{Z}, \mathcal{E}\}$; and (ii) can be expressed as the centralizer of an element in $\mathfrak{it}_{\mathbb{R}}$. Any such Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ is both conjugation-stable and $\varphi(\mathbf{i})$ -stable. Then $\bar{\mathcal{E}}$ is a *noticed* nilpotent, in the terminology of [Noë98]. Whence [Noë98, Lemma 2.1.1] yields $\dim \mathfrak{g}_0 \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{g}_2 \cap \mathfrak{k}_{\mathbb{C}}^\perp$, where $\mathfrak{g}_{\mathbb{C}} = \bigoplus \mathfrak{g}_\ell$ is the \mathcal{Z} -eigenspace decomposition. From (4.18) and (4.19), we see that $\dim \mathfrak{g}_0 = \dim \mathfrak{g}_2$, and \mathcal{Z} is distinguished by definition. The lemma now follows from Remark 4.25. \square

4.5 Examples

In the following examples, given $G_{\mathbb{C}}$, we apply Theorem 4.11 to identify the compact duals $\check{D} = G_{\mathbb{C}}/P$ with an open $G_{\mathbb{R}}$ -orbit admitting the structure of a Mumford–Tate domain with a Hodge–Tate degeneration. Keep in mind that, since we are assuming that the infinitesimal period relation is bracket-generating, the compact dual determines the real form; cf. (3.16).

Example 4.26 (The symplectic group $\mathrm{Sp}_8\mathbb{C}$). Of the $2^4 - 1 = 15$ conjugacy classes of parabolic subgroups in $G_{\mathbb{C}}$, only six are even Jacobson–Morosov; the indexing sets (§ 2.1) are $I = \{1, 2, 3, 4\}, \{1, 2, 4\}, \{2, 4\}, \{1, 4\}, \{2\}, \{4\}$ (cf. [CM93] or [BPR15]). Therefore, the pairs of compact duals \check{D} with an open $G_{\mathbb{R}}$ -orbit D admitting the structure of a Mumford–Tate domain with a Hodge–Tate degeneration are as listed in the table below.

\check{D}	$\mathrm{Flag}_{1,2,3,4}^Q(\mathbb{C}^8)$	$\mathrm{Flag}_{1,2,4}^Q(\mathbb{C}^8)$	$\mathrm{Flag}_{2,4}^Q(\mathbb{C}^8)$	$\mathrm{Flag}_{1,4}^Q(\mathbb{C}^8)$	$\mathrm{Gr}^Q(2, \mathbb{C}^8)$	$\mathrm{Gr}^Q(4, \mathbb{C}^8)$
$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{sp}(4, \mathbb{R})$	$\mathfrak{sp}(4, \mathbb{R})$	$\mathfrak{sp}(4, \mathbb{R})$	$\mathfrak{sp}(4, \mathbb{R})$	$\mathfrak{sp}(2, 2)$	$\mathfrak{sp}(4, \mathbb{R})$
$V_{\mathbb{R}}$	\mathbb{R}^8	\mathbb{R}^8	\mathbb{R}^8	\mathbb{R}^8	$\bigwedge^2 \mathbb{R}^8$	\mathbb{R}^8
\mathfrak{h}	$(1, \dots, 1)$	$(1, 1, 2, 2, 1, 1)$	$(2, 2, 2, 2)$	$(1, 3, 3, 1)$	$(1, 8, 9, 8, 1)$	$(4, 4)$

The table also lists a Hodge representation $V_{\mathbb{R}}$ realizing D as a Mumford–Tate domain, and the corresponding Hodge numbers. In all but one of these cases we have $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(4, \mathbb{C})$ and $V_{\mathbb{R}} = \mathbb{R}^8$; this realizes the Mumford–Tate domain as a period domain. In the case that $\check{D} = \mathrm{Gr}^Q(2, \mathbb{C}^8)$, the standard representation \mathbb{R}^8 does not admit the structure of a Hodge representation (because \mathbb{C}^8 is quaternionic, rather than real, with respect to the real form $\mathfrak{g}_{\mathbb{R}}$). However, the second exterior power $\bigwedge^2 \mathbb{R}^8$ does admit the structure of Hodge representation that realizes $\mathrm{Gr}^Q(2, \mathbb{C}^8)$ as the compact dual of a Mumford–Tate domain.

Example 4.27 (The orthogonal group $\mathrm{SO}_9\mathbb{C}$). Of the $2^4 - 1 = 15$ conjugacy classes of parabolic subgroups in $G_{\mathbb{C}}$, only five are even Jacobson–Morosov; the indexing sets (§ 2.1) are $I = \{1, 2, 3, 4\}, \{1, 2, 3\}, \{1, 3\}, \{3\}, \{1\}$. It follows that the pairs of compact duals \check{D} with an open $G_{\mathbb{R}}$ -orbit D admitting the structure of a Mumford–Tate domain with a Hodge–Tate degeneration are

\check{D}	$\mathrm{Flag}_{1,2,3,4}^Q(\mathbb{C}^8)$	$\mathrm{Flag}_{1,2,3}^Q(\mathbb{C}^8)$	$\mathrm{Flag}_{1,3}^Q(\mathbb{C}^8)$	$\mathrm{Gr}^Q(3, \mathbb{C}^8)$	$\mathrm{Gr}^Q(1, \mathbb{C}^8) = \mathbb{Q}^6$
$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{so}(4, 5)$	$\mathfrak{so}(4, 5)$	$\mathfrak{so}(4, 5)$	$\mathfrak{so}(6, 3)$	$\mathfrak{so}(2, 7)$

Here we may take $V_{\mathbb{R}} = \mathbb{R}^9$ in each case, and the Mumford–Tate domains are all period domains.

Example 4.28 (The exceptional Lie group $G_2(\mathbb{C})$). The complex Lie group $G_{\mathbb{C}} = G_2(\mathbb{C})$ contains three conjugacy classes \mathcal{P}_I of parabolic subgroups; as discussed in § 2.1, they are indexed by the nonempty subsets $I \subset \{1, 2\}$. Parabolics in two of the three may be realized as even Jacobson–Morosov parabolics: the Borel subgroups $\mathcal{B} = \mathcal{P}_{\{1,2\}}$ and the maximal parabolics \mathcal{P}_2 (cf. [CM93, § 8.4]). (The parabolics in the third class \mathcal{P}_1 may also be realized as Jacobson–Morosov parabolics, but not as *even* Jacobson–Morosov parabolics.) The complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ admits a single noncompact real form $\mathfrak{g}_{\mathbb{R}}$. The maximal compact subalgebra is $\mathfrak{k}_{\mathbb{R}} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. In both cases we may take $V_{\mathbb{R}}$ to be the standard representation \mathbb{R}^7 .

(a) In the case of the Borel conjugacy class \mathcal{B} , as discussed in § 2.5, we have $\sigma(\mathcal{N}) = (2, 2)$. From the tables of [CM93, § 9.6] we see that $\mathcal{N} \cap \mathfrak{g}_{\mathbb{R}}$ consists of a single $\mathrm{Ad}(G_{\mathbb{R}})$ -orbit \mathcal{N} and $\gamma(\mathcal{N}) = (4, 8)$ and $\alpha'(\mathcal{N}) = \sigma_2(\mathcal{N}) = -10$. It follows from Theorem 4.11 that $D \subset G_{\mathbb{C}}/B$ admits a Hodge–Tate degeneration. The Hodge numbers are $\mathfrak{h} = (1, 1, 1, 1, 1, 1, 1)$.

(b) For $P \in \mathcal{P}_2$ we have $\sigma(\mathcal{N}) = (0, 2)$. From the tables of [CM93, § 9.6] we see that $\mathcal{N} \cap \mathfrak{g}_{\mathbb{R}}$ consists of two $\text{Ad}(G_{\mathbb{R}})$ -orbits. One of these has characteristic vector $\gamma(\mathcal{N}) = (0, 4)$ and $\alpha'(\mathcal{N}) = \sigma_2(\mathcal{N}) = -2$. Theorem 4.11 implies that $D \subset G_{\mathbb{C}}/P$ has a Hodge–Tate degeneration. The Hodge numbers are $\mathbf{h} = (2, 3, 2)$.

More generally, the polarized $G_{\mathbb{R}}$ -orbits in a $G_2(\mathbb{C})$ -homogeneous compact dual have been determined by Kerr and Pearlstein in [KP14, § 6.1.3].

4.6 Constraints on the existence of Hodge–Tate degenerations

In the case that the compact dual is the full flag variety $\check{D} = G_{\mathbb{C}}/B$, that is, $P = B$ is a Borel subgroup, we may be explicit about the real forms $G_{\mathbb{R}}$ that yield a $G_{\mathbb{R}}$ -orbit $D \subset \check{D}$ admitting the structure of a Mumford–Tate domain with a Hodge–Tate degeneration.

PROPOSITION 4.29. *Let $G_{\mathbb{C}}$ be a simple complex Lie group and consider the full flag variety $\check{D} = G_{\mathbb{C}}/B$. Given a real form $G_{\mathbb{R}}$ of $G_{\mathbb{C}}$ there exists a $G_{\mathbb{R}}$ -flag domain $D \subset \check{D}$ admitting the structure of a Mumford–Tate domain (with bracket-generating infinitesimal period relation) with a Hodge–Tate degeneration if and only if $\mathfrak{g}_{\mathbb{R}}$ is one of the following:*

$$\begin{aligned} & \mathfrak{su}(p, p), \quad \mathfrak{su}(p, p \pm 1), \quad \mathfrak{sp}(n, \mathbb{R}), \\ & \mathfrak{so}(2p \pm 1, 2p), \quad \mathfrak{so}(2p, 2p), \quad \mathfrak{so}(2p + 2, 2p), \\ & \text{E II}, \quad \text{E V}, \quad \text{E VIII}, \quad \text{F I}, \quad \text{G}. \end{aligned}$$

Proof. Hodge–Tate degenerations in full flag varieties are discussed in [GGR14, Remark V.12]. There it was observed that, if $G_{\mathbb{C}}$ is classical (special linear, symplectic or orthogonal), then $\mathfrak{g}_{\mathbb{R}}$ is necessarily one of the algebras listed above. Additionally, for each of the symplectic and orthogonal algebras, a Mumford–Tate domain and Hodge–Tate degeneration are exhibited.

Now consider the special linear algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_n \mathbb{C}$. If the Mumford–Tate domain admits a Hodge–Tate degeneration, then the complex characteristic vector $\sigma(\mathcal{N})$ is necessarily of the form $(2, \dots, 2)$. Moreover, (4.15) implies $(1, \dots, 1) = (\sigma_1(\mathbf{E}), \dots, \sigma_r(\mathbf{E}))$, where $r = n - 1$ and $\mathbf{E} = \mathbf{E}_{\varphi}$ is the grading element (2.8) associated with the Borel. Therefore the simple roots σ_i are all noncompact. Whence the collection $S' = \{\sigma_1 + \sigma_2, \sigma_2 + \sigma_3, \sigma_3 + \sigma_4, \dots, \sigma_{r-1} + \sigma_r\}$ forms a set of simple roots for $\mathfrak{k}_{\mathbb{C}}$. Attaching the noncompact $-\sigma_1$ completes S' to a set of simple roots for $\mathfrak{g}_{\mathbb{C}}$. From this choice of simple roots we see that Theorem 4.11(b) holds; whence D admits a Hodge–Tate degeneration. To see that the real form is either $\mathfrak{su}(p, p)$ or $\mathfrak{su}(p \pm 1, p)$, observe that $-\sigma_1$ is the unique noncompact simple root in the system $S' \cup \{-\sigma_1\}$. In the Vogan diagram classification of real forms [Kna02, § VI.10], this corresponds to painting either the $(p \pm 1)$ th or p th node in the Dynkin diagram.

In the case that $G_{\mathbb{C}}$ is exceptional, the proposition follows from Theorem 4.11 and the tables in [CM93, § 9.6]. □

5. Classification theorems

In this section we prove the two main results of the paper: the classifications of the \mathbb{R} -split PMHSs (Theorem 5.5), and of the horizontal $SL(2)$ s (Theorem 5.9).

5.1 \mathbb{R} -split polarized mixed Hodge structures

Let (F^{\bullet}, N) be an \mathbb{R} -split PMHS on a Mumford–Tate domain D . Given any $g \in G_{\mathbb{R}}$,

$$g \cdot (F^{\bullet}, N) := (g \cdot F^{\bullet}, \text{Ad}_g N)$$

is also an \mathbb{R} -split PMHS on D ; let

$$[F^\bullet, N] := \{g \cdot (F^\bullet, N) \mid g \in G_{\mathbb{R}}\}$$

denote the corresponding $G_{\mathbb{R}}$ -conjugacy class, and let

$$\Psi_D := \{[F^\bullet, N] \mid (F^\bullet, N) \text{ is an } \mathbb{R}\text{-split PMHS on } D\} \tag{5.1}$$

denote the set of all such conjugacy classes.

Fix a point $\varphi \in D$. Recall the grading element E_φ of (3.13) and let $\mathfrak{t} \ni \mathbf{i}E_\varphi$ be a compact Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$. Given a Levi subalgebra $\mathfrak{l}_{\mathbb{C}} \supset \mathfrak{h}$, recall from § 2.2 that $\mathfrak{l}_{\mathbb{C}} = \mathfrak{l}_{\mathbb{C}}^{\text{ss}} \oplus \mathfrak{z}$, where \mathfrak{z} is the center of $\mathfrak{l}_{\mathbb{C}}$ and $\mathfrak{l}_{\mathbb{C}}^{\text{ss}} = [\mathfrak{l}_{\mathbb{C}}, \mathfrak{l}_{\mathbb{C}}]$ is semisimple; let $\pi_{\mathfrak{l}_{\mathbb{C}}}^{\text{ss}} : \mathfrak{l}_{\mathbb{C}} \rightarrow \mathfrak{l}_{\mathbb{C}}^{\text{ss}}$ denote the projection. Set

$$\mathcal{L}_{\varphi, \mathfrak{t}} := \left\{ \begin{array}{l} \varphi\text{-stable Levi subalgebras } \mathfrak{l}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}} \text{ such that } \mathfrak{t} \subset \mathfrak{l}_{\mathbb{R}} \text{ and} \\ 2\pi_{\mathfrak{l}_{\mathbb{C}}}^{\text{ss}}(E_\varphi) \text{ is a distinguished semisimple element of } \mathfrak{l}_{\mathbb{C}}^{\text{ss}} \end{array} \right\}.$$

(The condition, in the definition of $\mathcal{L}_{\varphi, \mathfrak{t}}$, that $\mathfrak{l}_{\mathbb{R}}$ be φ -stable is added for emphasis/clarity; it follows from $\mathbf{i}E_\varphi \in \mathfrak{t} \subset \mathfrak{l}_{\mathbb{R}}$, which implies that the image of the circle is contained in $L_{\mathbb{R}}$.) In computations it is helpful to note that $2\pi_{\mathfrak{l}_{\mathbb{C}}}^{\text{ss}}(E_\varphi)$ is a distinguished semisimple element of $\mathfrak{l}_{\mathbb{C}}$ if and only if

$$\text{rank } \mathfrak{l}_{\mathbb{C}}^{\text{ss}} + \#\{\alpha \in \Delta(\mathfrak{l}) \mid \alpha(E_\varphi) = 0\} = \#\{\alpha \in \Delta(\mathfrak{l}) \mid \alpha(E_\varphi) = 1\}. \tag{5.2}$$

LEMMA 5.3. *Given $\mathfrak{l}_{\mathbb{R}} \in \mathcal{L}_{\varphi, \mathfrak{t}}$, there exists a DKS triple $\{\bar{\mathcal{E}}, \mathcal{Z}, \mathcal{E}\} \subset \mathfrak{l}_{\mathbb{C}}^{\text{ss}}$ with neutral element $\mathcal{Z} = 2\pi_{\mathfrak{l}_{\mathbb{C}}}^{\text{ss}}(E_\varphi)$.*

The lemma is proved in § 5.3.

Let $\mathfrak{g}_{\mathbb{C}} = \bigoplus \mathfrak{g}^p$ be the E_φ -eigenspace decomposition (2.3). Recall from (3.14) that the Hodge filtration $F_{\varphi, \mathfrak{g}}^\bullet$ of $\mathfrak{g}_{\mathbb{C}}$ induced by φ is given by $F_{\varphi, \mathfrak{g}}^p = \bigoplus_{q \geq p} \mathfrak{g}^q$. The parabolic $\mathfrak{p}_\varphi = \mathfrak{g}^0 \oplus \mathfrak{g}^+$ is the Lie algebra of the stabilizer $P_\varphi \subset G_{\mathbb{C}}$ of φ , and the 0-eigenspace \mathfrak{g}^0 is a Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ (§ 2.1) containing the Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$. Let $\mathcal{W}^0 \subset \mathcal{W} \subset \text{Aut}(\mathfrak{h})$ denote the Weyl group of \mathfrak{g}^0 (Remark 2.7). Then \mathcal{W}^0 acts on $\mathcal{L}_{\varphi, \mathfrak{t}}$. Given $\mathfrak{l}_{\mathbb{R}} \in \mathcal{L}_{\varphi, \mathfrak{t}}$, let $[\mathfrak{l}_{\mathbb{R}}]$ denote the \mathcal{W}^0 -conjugacy class, and let

$$\Lambda_{\varphi, \mathfrak{t}} := \{[\mathfrak{l}_{\mathbb{R}}] \mid \mathfrak{l}_{\mathbb{R}} \in \mathcal{L}_{\varphi, \mathfrak{t}}\}$$

be the corresponding set of \mathcal{W}^0 -conjugacy classes.¹⁸

Finally, we note that (3.6) and (3.14) imply \mathfrak{g}^0 has compact real form

$$\mathfrak{k}_{\mathbb{R}}^0 := \mathfrak{g}^0 \cap \mathfrak{g}_{\mathbb{R}} = \mathfrak{p}_\varphi \cap \mathfrak{k}_{\mathbb{R}};$$

let $K_{\mathbb{R}}^0 = P_\varphi \cap K_{\mathbb{R}}$ denote the corresponding Lie subgroup. (Note that $K_{\mathbb{R}}^0$ is the stabilizer of $\varphi \in D$ in $G_{\mathbb{R}}$.) Then

$$\text{elements of } \mathcal{W}^0 \text{ admit representatives in } K_{\mathbb{R}}^0. \tag{5.4}$$

THEOREM 5.5. *Let $V_{\mathbb{R}}$ be a Hodge representation of $G_{\mathbb{R}}$, and assume that the infinitesimal period relation on the associated Mumford–Tate domain $D = G_{\mathbb{R}}/K_{\mathbb{R}}^0$ is bracket-generating. With the notation above, we have:*

¹⁸ The role of \mathcal{W}^0 here is anticipated by Cattani and Kaplan’s [CK78, Proposition 3.29].

(a) There is a bijection $\Psi_D \leftrightarrow \Lambda_{\varphi,t}$. That is, up to the action of $G_{\mathbb{R}}$, the \mathbb{R} -split polarized mixed Hodge structures on D are indexed by the W^0 -conjugacy classes of $\mathcal{L}_{\varphi,t}$.

(b) Given $\mathfrak{l}_{\mathbb{R}} \in \mathcal{L}_{\varphi,t}$, let $\{\bar{\mathcal{E}}, \mathcal{Z}, \mathcal{E}\} \subset \mathfrak{l}_{\mathbb{C}}^{\text{ss}}$ be a DKS triple with neutral element $\mathcal{Z} = 2\pi_1^{\text{ss}}(\mathbf{E}_{\varphi})$; cf. Lemma 5.3. The element $[F^{\bullet}, N] \in \Psi_D$ corresponding to $[\mathfrak{l}_{\mathbb{R}}] \in \Lambda_{\varphi,t}$ is represented by $(F^{\bullet}, N) = \varrho^{-1} \cdot (\varphi, \mathcal{E})$, where

$$\varrho := \exp \mathbf{i} \frac{\pi}{4} (\mathcal{E} + \bar{\mathcal{E}}) \in L_{\mathbb{C}}.$$

(c) The image of the reduced limit period mapping is $\Phi_{\infty}(F^{\bullet}, N) = \varrho(\varphi) = \varrho^2(F^{\bullet})$.

(d) If $V_{\mathbb{C}} = \bigoplus V^{\mu}$ is the weight space decomposition (with respect to \mathfrak{h}), then the Deligne splitting $V_{\mathbb{C}} = \bigoplus I^{p,q}$ induced by (F^{\bullet}, N) is given by

$$\varrho(I^{p,q}) = \bigoplus_{\substack{\mu(\mathbf{E}_{\varphi})=p \\ \mu(\mathcal{Z})=p+q}} V^{\mu}. \tag{5.6}$$

(e) With respect to the Deligne splitting $\mathfrak{g}_{\mathbb{C}} = \bigoplus I_{\mathfrak{g}}^{p,q}$ we have $\mathfrak{l}_{\mathbb{C}} \subset \bigoplus I_{\mathfrak{g}}^{p,p}$.

Theorem 5.5 is proved in § 5.3, and a number of examples are worked out in § 5.5. As will be discussed in § 5.4, Theorem 5.5(c) yields a parameterization of the polarized orbits in $\text{bd}(D) \subset \check{D}$.

5.2 Horizontal $SL(2)$ s

In this section we will show that Theorem 5.5 yields a classification of the horizontal $SL(2)$ s on D , up to the action of $G_{\mathbb{R}}$.

Let

$$\begin{aligned} \mathfrak{sl}_2 \mathbb{R} &= \text{span}_{\mathbb{R}} \{ \mathbf{n}^+, \mathbf{y}, \mathbf{n} \}, \\ \mathfrak{sl}_2 \mathbb{C} &= \text{span}_{\mathbb{C}} \{ \mathbf{n}^+, \mathbf{y}, \mathbf{n} \} = \text{span}_{\mathbb{C}} \{ \bar{\mathbf{e}}, \mathbf{z}, \mathbf{e} \} \end{aligned}$$

be the algebras defined by (2.10) and (2.11). Given $\varphi \in D$, recall the grading element \mathbf{E}_{φ} given by (3.13), and let $\mathfrak{g}_{\mathbb{C}} = \bigoplus \mathfrak{g}^p$ be the corresponding eigenspace decomposition given by (2.3). The latter is also the Hodge decomposition by (3.14). A horizontal $SL(2)$ at φ is given by a representation $v : SL(2, \mathbb{C}) \rightarrow G_{\mathbb{C}}$ such that

$$v(SL(2, \mathbb{R})) \subset G_{\mathbb{R}} \tag{5.7a}$$

and

$$v_* \bar{\mathbf{e}} \in \mathfrak{g}^1, \quad v_* \mathbf{z} \in \mathfrak{g}^0, \quad v_* \mathbf{e} \in \mathfrak{g}^{-1}. \tag{5.7b}$$

We will say that v is a horizontal $SL(2)$ if it is horizontal at some $\varphi \in D$.

Remark 5.8. Observe that (5.7) implies that $v_* \{ \bar{\mathbf{e}}, \mathbf{z}, \mathbf{e} \}$ is a DKS triple with respect to the maximal compact subgroup $K_{\mathbb{R}} \subset G_{\mathbb{R}}$ determined by the Cartan involution $\varphi(\mathbf{i})$; likewise $v_* \{ \mathbf{n}^+, \mathbf{y}, \mathbf{n} \}$ is a Cayley triple.

Note that $g \in G_{\mathbb{R}}$ acts on the set of horizontal $SL(2)$ s by $v \mapsto g \cdot v$. Let

$$\Upsilon_D := \{ [v] \mid v \text{ is a horizontal } SL(2) \}$$

be the set of $G_{\mathbb{R}}$ -equivalence classes.

THEOREM 5.9. *With the notation and assumptions of Theorem 5.5, we have:*

(i) There is a bijection $\Upsilon_D \leftrightarrow \Lambda_{\varphi,t}$. That is, up to the action of $G_{\mathbb{R}}$, the horizontal $SL(2)$ s on D are parameterized by the W^0 -conjugacy classes of $\mathcal{L}_{\varphi,t}$.

(ii) Given $\mathfrak{l}_{\mathbb{R}} \in \mathcal{L}_{\varphi, \mathfrak{t}}$, let $\{\bar{\mathcal{E}}, \mathcal{Z}, \mathcal{E}\} \subset \mathfrak{t}_{\mathbb{C}}^{\text{ss}}$ be a DKS triple with neutral element $\mathcal{Z} = 2\pi_1^{\text{ss}}(\mathbf{E}_{\varphi})$; cf. Lemma 5.3. The equivalence class $[v] \in \Upsilon_D$ corresponding to $[\mathfrak{l}_{\mathbb{R}}] \in \Lambda_{\varphi, \mathfrak{t}}$ is represented by the $v : \text{SL}_2\mathbb{C} \rightarrow L_{\mathbb{C}}^{\text{ss}}$ given by

$$v_*\bar{\mathbf{e}} = \bar{\mathcal{E}}, \quad v_*\mathbf{z} = \mathcal{Z}, \quad v_*\mathbf{e} = \mathcal{E}. \tag{5.10}$$

Proof. The result will follow from Theorem 5.5 and the $G_{\mathbb{R}}$ -equivariant bijection (1.1). This bijection is well known (cf. [CK82a, CKS86, CK77, Sch73, Usu93]); the following proof is given for the sake of completeness.

Given (F^{\bullet}, N) , the Deligne splitting $\mathfrak{g}_{\mathbb{C}} = \bigoplus I_{\mathfrak{g}}^{p,q}$ defines a semisimple $Y \in \mathfrak{g}_{\mathbb{R}}$ by $Y|_{I_{\mathfrak{g}}^{p,q}} = (p+q)\mathbf{1}$. There exists a unique $N^+ \in \mathfrak{g}_{\mathbb{R}}$ completing the pair $\{Y, N\}$ to a standard triple [CKS86, pp. 477]. As discussed in Remark 4.21, this standard triple is a Cayley triple with respect to the Cartan involution $\varphi(\mathbf{i})$ defined by (4.16). The corresponding Cayley transform (2.27) defines a horizontal $\text{SL}(2)$ v at $\varphi = \varrho(F^{\bullet})$ by (5.10). This defines the map from \mathbb{R} -split PMHSs to horizontal $\text{SL}(2)$ s.

Conversely, suppose that v is a horizontal $\text{SL}(2)$ at $\varphi \in D$. By Remark 5.8, (5.10) defines a DKS triple $\{\bar{\mathcal{E}}, \mathcal{Z}, \mathcal{E}\}$. Let $\{N^+, Y, N\} = v_*\{\mathbf{n}^+, \mathbf{y}, \mathbf{n}\}$ be the corresponding Cayley triple, which is defined by (2.28) and (2.29). Recalling that ϱ is given by (2.29), define $F^{\bullet} = \varrho^{-1}(\varphi)$. Then (F^{\bullet}, N) is a nilpotent orbit. Moreover, the Deligne splitting $V_{\mathbb{C}} = \bigoplus I^{p,q}$ of the corresponding PMHS is as given by (5.16) in Remark 5.15, and is manifestly \mathbb{R} -split. This defines the map from horizontal TDS to \mathbb{R} -split PMHSs. Moreover, this map is easily seen to be the inverse of the map defined in the previous paragraph. \square

5.3 Proof of Theorem 5.5

We begin with the promised proof of Lemma 5.3.

Proof of Lemma 5.3. As a distinguished semisimple element, \mathcal{Z} is the neutral element of a standard triple $\{\mathcal{E}^+, \mathcal{Z}, \mathcal{E}\}$.

Let $\mathfrak{l}_{\mathbb{C}} = \bigoplus \mathfrak{l}^p$ be the \mathbf{E}_{φ} -eigenspace decomposition. We have $\mathfrak{l}^p = \mathfrak{l}_{\mathbb{C}} \cap \mathfrak{g}^p$, $\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}} = \mathfrak{l}^{\text{even}}$, and $\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}^{\perp} = \mathfrak{l}^{\text{odd}}$. Since \mathcal{Z} differs from $2\mathbf{E}_{\varphi}$ by an element of the center of $\mathfrak{l}_{\mathbb{C}}$, we see that \mathcal{Z} acts on \mathfrak{l}^p by the eigenvalue $2p$. From $[\mathcal{Z}, \mathcal{E}^+] = 2\mathcal{E}^+$ and $[\mathcal{Z}, \mathcal{E}] = -2\mathcal{E}$, we see that $\mathcal{E}^+ \in \mathfrak{l}^1 \subset \mathfrak{k}_{\mathbb{C}}^{\perp}$ and $\mathcal{E} \in \mathfrak{l}^{-1} \subset \mathfrak{k}_{\mathbb{C}}^{\perp}$. It remains to show that \mathcal{E}^+ and \mathcal{E} may be chosen so that $\mathcal{E}^+ = \bar{\mathcal{E}}$.

Let $K' = L \cap K$ and $\mathfrak{k}' = \mathfrak{l} \cap \mathfrak{k}$. It is a consequence of the Djoković–Kostant–Sekiguchi correspondence and Remark 2.25 that $\{\mathcal{E}^+, \mathcal{Z}, \mathcal{E}\}$ is $K'_{\mathbb{C}}$ -conjugate to a DKS triple $\{\bar{\mathcal{E}}', \mathcal{Z}', \mathcal{E}'\}$ in $\mathfrak{t}_{\mathbb{C}}^{\text{ss}}$; cf. [Sek87]. By construction $\mathcal{Z} \in \mathfrak{it}$, and \mathfrak{t} is a Cartan subalgebra of $\mathfrak{k}'_{\mathbb{R}}$. Therefore $\mathcal{Z}' \in \mathfrak{it}'_{\mathbb{R}}$ is $K'_{\mathbb{R}}$ -conjugate to an element of \mathfrak{it} . So, without loss of generality, $\mathcal{Z}' \in \mathfrak{it}$.

The claim will follow once we show that \mathcal{Z} and \mathcal{Z}' are conjugate under the Weyl group $\mathcal{W}_{K'} \subset \text{Aut}(\mathfrak{t})$ of $\mathfrak{k}'_{\mathbb{R}}$. First, observe that \mathcal{Z} and \mathcal{Z}' (i) lie in the same Cartan \mathfrak{h} , and (ii) are (twice) the grading elements associated with parabolic subalgebras $\mathfrak{p}_{\mathcal{Z}}$ and $\mathfrak{p}_{\mathcal{Z}'}$ that are $K'_{\mathbb{C}}$ -conjugate; it follows that \mathcal{Z} and \mathcal{Z}' are conjugate under an element w of the Weyl group of $\mathfrak{l}_{\mathbb{C}}$. Because $\mathfrak{p}_{\mathcal{Z}}$ and $\mathfrak{p}_{\mathcal{Z}'}$ are conjugate under $K'_{\mathbb{C}}$, the element w must preserve the set of compact roots $\Delta(\mathfrak{k}_{\mathbb{C}}) \subset \Delta$, and is therefore an element of $\mathcal{W}_{K'}$. \square

We now turn to the proof of Theorem 5.5. To establish the bijection $\Lambda_{\varphi, \mathfrak{t}} \leftrightarrow \Psi_D$, first suppose that we are given a Levi subalgebra $\mathfrak{l}_{\mathbb{C}} \in \mathcal{L}_{\varphi, \mathfrak{t}}$; the corresponding $[F^{\bullet}, N] \in \Psi_D$ is obtained as follows. Given the DKS triple of Lemma 5.3, let $\{N^+, Y, N\} = \text{Ad}_{\varrho}^{-1}\{\bar{\mathcal{E}}, \mathcal{Z}, \mathcal{E}\} \subset \mathfrak{t}_{\mathbb{R}}^{\text{ss}}$ be the corresponding Cayley triple.

From Lemma 3.17 we see that φ induces a sub-Hodge structure on the real form $\mathfrak{l}_{\mathbb{R}}$. Let $D = L_{\mathbb{R}} \cdot \varphi \subset D$ denote the corresponding Mumford–Tate domain.

CLAIM 5.11. Define $\mathcal{F}_1^\bullet \in \check{\mathcal{D}}$ by $\mathcal{F}_1^{-p} = W_{2p}(N^+, \mathfrak{l}_\mathbb{C})$. Then the pair $(\mathcal{F}_1^\bullet, N)$ defines a Hodge–Tate degeneration on \mathcal{D} .

Proof. The fact that $\mathfrak{l}_\mathbb{C}$ is $\varphi(\mathbf{i})$ -stable implies that

$$\mathfrak{l}_\mathbb{C} = (\mathfrak{l}_\mathbb{C} \cap \mathfrak{k}_\mathbb{C}) \oplus (\mathfrak{l}_\mathbb{C} \cap \mathfrak{k}_\mathbb{C}^\perp) \quad \text{and} \quad \mathfrak{l}_\mathbb{C}^{\text{ss}} = (\mathfrak{l}_\mathbb{C}^{\text{ss}} \cap \mathfrak{k}_\mathbb{C}) \oplus (\mathfrak{l}_\mathbb{C}^{\text{ss}} \cap \mathfrak{k}_\mathbb{C}^\perp)$$

are both Cartan decompositions. Since $\mathfrak{l}_\mathbb{C} \supset \mathfrak{h}$, we may identify the roots of $\mathfrak{l}_\mathbb{C}$ with a subset of the roots of $\mathfrak{g}_\mathbb{C}$, and under this identification the (non)compact roots of $\mathfrak{l}_\mathbb{C}$ are (non)compact roots of $\mathfrak{g}_\mathbb{C}$. It follows from (3.6) and (3.14) that: (i) $\alpha(\mathcal{Z}) \equiv 0 \pmod{4}$ for all compact roots of $\mathfrak{l}_\mathbb{C}^{\text{ss}}$; (ii) $\beta(\mathcal{Z})$ is even and $\frac{1}{2}\beta(\mathcal{Z})$ is odd for all noncompact roots. The claim now follows from Theorem 4.11. \square

Remark 4.5 and Claim 5.11 imply that $(\mathcal{F}_1^\bullet, N)$ induces a nilpotent orbit (F_g^\bullet, N) on D . At this point Theorem 5.5(b,c) follows from (2.28), (2.29), (4.16) and (4.20).

The nilpotent orbit (F^\bullet, N) depends on both the Levi subalgebra $\mathfrak{l}_\mathbb{R}$ and our choice of DKS triple $\{\bar{\mathcal{E}}, \mathcal{Z}, \mathcal{E}\}$. Suppose that $\{\bar{\mathcal{E}}', \mathcal{Z}, \mathcal{E}'\}$ is a second DKS triple, also containing \mathcal{Z} as the neutral element. Then a theorem due to Rao [CM93, Theorem 9.4.6] implies that the triples are conjugate under $G^0 \cap L_\mathbb{R}^{\text{ss}}$. It is then straightforward to confirm that the nilpotent orbit (F^\bullet, N') associated with the second DKS triple is $G^0 \cap L_\mathbb{R}^{\text{ss}}$ -congruent to (F^\bullet, N) . Whence the two nilpotent orbits determine the same conjugacy class $[F^\bullet, N] \in \Psi_D$, and we have a well-defined map $\mathcal{L}_{\varphi,t} \rightarrow \Psi_D$. Finally, (5.4) and Theorem 5.5(b) imply that the map descends to $\Lambda_{\varphi,t} \rightarrow \Psi_D$.

To address the second half of the correspondence asserted in Theorem 5.5(a), suppose that $[F^\bullet, N] \in \Psi_D$. We normalize our choice of representative (F^\bullet, N) as follows. Let $\mathfrak{g}_\mathbb{C} = \bigoplus I_g^{p,q}$ be the associated Deligne splitting, and let $\tilde{\mathfrak{l}}_\mathbb{C} = \bigoplus I_g^{p,p}$ be the Levi subalgebra of Theorem 4.3. Since (F^\bullet, N) is \mathbb{R} -split, $\mathfrak{l}_\mathbb{C}$ is necessarily stable under conjugation. Moreover, we may complete N to the standard triple $\{N^+, Y, N\} \subset \tilde{\mathfrak{l}}_\mathbb{R}^{\text{ss}}$ so that (4.13) holds. Conjugating (F^\bullet, N) by an element $g \in G_\mathbb{R}$ if necessary, we may assume that this is a Cayley triple. Then $\tilde{\mathfrak{l}}_\mathbb{C}$ is $\varphi(\mathbf{i})$ -stable. Let $\{\bar{\mathcal{E}}, \mathcal{Z}, \mathcal{E}\} \subset \tilde{\mathfrak{l}}_\mathbb{C}^{\text{ss}}$ be the Cayley transform (2.27) of the Cayley triple, and let $\tilde{\varphi}$ be as given by (4.16). Then $\tilde{\varphi}$ is a K -Matsuki point of D (Remark 4.21), and therefore $K_\mathbb{R}$ -conjugate to φ by (3.11). So, conjugating (F^\bullet, N) by an element $g \in K_\mathbb{R}$, if necessary, we may assume that $\tilde{\varphi} = \varphi$.

Let $\mathfrak{l}_\mathbb{C} \subset \tilde{\mathfrak{l}}_\mathbb{C}$ be a minimal conjugation and φ -stable Levi subalgebra containing the DKS triple. (Such a Levi is not unique. However, any two such are conjugate under the reductive centralizer $Z(\bar{\mathcal{E}}, \mathcal{E})$ of $\bar{\mathcal{E}}$ and \mathcal{E} in $K_\mathbb{R}^0 \cap L_\mathbb{R}$; see the proof of [Noë98, Proposition 1.1.3].) Then \mathcal{Z} is a distinguished semisimple element of the semisimple factor $\mathfrak{l}_\mathbb{C}^{\text{ss}}$ by Lemma 4.23 and Remark 4.25. By construction, $L_\mathbb{R}$ admits the Hodge representation $(\mathfrak{l}_\mathbb{R}, Q_t, \text{Ad}, \varphi)$. Therefore, $\mathfrak{l}_\mathbb{R}$ has a compact Cartan subalgebra $\tilde{\mathfrak{t}} \ni \mathbf{i}E_\varphi$. Since both Cartans \mathfrak{t} and $\tilde{\mathfrak{t}}$ contain $\mathbf{i}E_\varphi$, they are necessarily Cartan subalgebras of the compact $\mathfrak{k}_\mathbb{R}^0$. Therefore, up to conjugation by $g \in K_\mathbb{R}^0$, we may assume that $\mathfrak{t} = \tilde{\mathfrak{t}}$. Thus $\mathfrak{l}_\mathbb{R} \in \mathcal{L}_{\varphi,t}$. At this point, the ambiguity in our choice of minimal $\mathfrak{l}_\mathbb{C}$ (see the parenthetical remark above) is up to the action of the Weyl group of $Z(\bar{\mathcal{E}}, \mathcal{E})$. Since the latter is a subgroup of W^0 , we have a well-defined map $\Psi_D \rightarrow \Lambda_{\varphi,t}$. This completes the proof of Theorem 5.5(a).

It remains to establish Theorem 5.5(d). The induced Deligne splitting $V_\mathbb{C} = \bigoplus I^{p,q}$ may be obtained as follows. Let $V_\mathbb{C} = \bigoplus V^\lambda$ be the weight space decomposition of $V_\mathbb{C}$. That is, $\lambda \in \mathfrak{h}^*$ and $v \in V^\lambda$ if and only if $\xi(v) = \lambda(\xi)v$ for all $\xi \in \mathfrak{h}$. It is immediate from (4.20) that $I^{p,\bullet}$ is the

$$E' := \text{Ad}_\rho^{-1}(E_\varphi)$$

eigenspace for the eigenvalue p . That is,

$$I^{p,\bullet} = \bigoplus_{\mu(E')=p} V^\mu;$$

here $V_{\mathbb{C}} = \bigoplus' V^{\mu}$ is the weight space decomposition with respect to the Cartan subalgebra $\mathfrak{h}' = \text{Ad}_{\varrho}^{-1}(\mathfrak{h})$. On the other hand, by (3.25),

$$\bigoplus_{p+q=\ell} I^{p,q} = \bigoplus_{\mu(Y)=\ell} 'V^{\mu}$$

is the ℓ -eigenspace for $Y = \varrho^{-1}(\mathcal{Z})$. Thus

$$I^{p,q} = \bigoplus_{\substack{\mu(E')=p \\ \mu(Y)=p+q}} 'V^{\mu}. \tag{5.12}$$

Applying Ad_{ϱ} to (5.12) yields (5.6), and completes the proof of Theorem 5.5. □

Remark 5.13 (Computing \mathcal{Z}). If we wish to compute the Deligne splitting (5.6) it is necessary to determine \mathcal{Z} . As a reductive algebra, $\mathfrak{l}_{\mathbb{C}}$ decomposes into a direct sum of its center and a semisimple factor

$$\mathfrak{l}_{\mathbb{C}} = \mathfrak{z}_{\mathbb{C}} \oplus \mathfrak{l}_{\mathbb{C}}^{\text{ss}};$$

the key is to recall (Theorem 5.5(b)) that

$$\mathcal{Z} \text{ is the image of } 2E_{\varphi} \text{ under the projection } \mathfrak{l}_{\mathbb{C}} \rightarrow \mathfrak{l}_{\mathbb{C}}^{\text{ss}}. \tag{5.14}$$

Let $S' \subset S \subset \mathfrak{h}^*$ be a choice of simple roots for $\mathfrak{l}_{\mathbb{C}}^{\text{ss}} \subset \mathfrak{g}_{\mathbb{C}}$. We have

$$\mathfrak{z}_{\mathbb{C}} = \text{span}_{\mathbb{C}}\{S^j \mid \sigma_j \notin S'\}.$$

Likewise, the Cartan subalgebra of the semisimple factor is

$$\mathfrak{h} \cap \mathfrak{l}_{\mathbb{C}}^{\text{ss}} = \text{span}_{\mathbb{C}}\{H^i \mid \sigma_i \in S'\},$$

where $H^i \in [\mathfrak{g}^{\sigma_i}, \mathfrak{g}^{-\sigma_i}] \subset \mathfrak{h}$ is defined by $\sigma_i(H^i) = 2$ (there is no sum over i). The sets $\{S^i\}_{i=1}^r$ and $\{H^i\}_{i=1}^r$ are the bases \mathfrak{h} dual to the simple roots and fundamental weights, respectively. In particular, if $C = (C_j^i)$ is the Cartan matrix, so that $\sigma_i = C_j^i \omega_j$, then $H^j = C_i^j S^i$. Moreover, $\{S^j \mid \sigma_j \notin S'\} \cup \{H^i \mid \sigma_i \in S'\}$ is a basis of \mathfrak{h} . Therefore, we may write

$$E_{\varphi} = \sum_{\sigma_j \notin S'} n_j S^j + \sum_{\sigma_i \in S'} m_i H^i,$$

and (5.14) yields

$$\mathcal{Z} = 2 \sum_{\sigma_i \in S'} m_i H^i \in \mathfrak{h} \cap \mathfrak{l}_{\mathbb{C}}^{\text{ss}}.$$

Remark 5.15 (The Deligne splitting). By (5.14) we have $E_{\varphi} = \frac{1}{2}\mathcal{Z} + \zeta$ with $\zeta \in \mathfrak{z}_{\mathbb{C}}$. Indeed, the discussion of Remark 5.13 yields

$$\zeta = \sum_{\sigma_j \notin S'} n_j S^j \in \mathfrak{h} \cap \mathfrak{z}_{\mathbb{C}}.$$

Since both E_{φ} and \mathcal{Z} are imaginary (i.e., they lie in $\mathfrak{ig}_{\mathbb{R}}$), ζ is as well. Observe that

$$E' = \frac{1}{2}Y + \zeta,$$

and from this we may conclude that

$$\bar{E}' = \frac{1}{2}Y - \zeta.$$

Since $\mathcal{Z}, \zeta \in \mathfrak{h}$, we have $Y, \zeta \in \mathfrak{h}'$ so that $\bar{E}' \in \mathfrak{h}'$. It follows that the Deligne splitting (5.12) is alternatively given by

$$I^{p,q} = \bigoplus_{\substack{\mu(\bar{E}')=p \\ \mu(\bar{E})=q}} 'V^\mu \tag{5.16}$$

where $V_{\mathbb{C}} = \bigoplus 'V^\mu$ is the weight space decomposition with respect to $\mathfrak{h}' = \text{Ad}_g^{-1}(\mathfrak{h})$.

5.4 Polarized orbits

Let $D \subset \check{D}$ be a Mumford–Tate domain. We say that a $G_{\mathbb{R}}$ -orbit $\mathcal{O} \subset \text{cl}(D)$ is *polarized (relative to D)* if it contains the image $\Phi_\infty(F^\bullet, N)$ of a point $F^\bullet \in \check{B}(N)$ under the reduced limit period mapping (3.30). We think of the polarized orbits as the ‘Hodge-theoretically accessible’ orbits.

Let (\tilde{F}^\bullet, N) be the \mathbb{R} -split PMHS (3.28) associated with (F^\bullet, N) . From Theorem 3.27(c) and (3.30), we see that $\Phi_\infty(F^\bullet, N) = \Phi_\infty(\tilde{F}^\bullet, N)$. So, *for the purpose of studying polarized orbits, it suffices to consider \mathbb{R} -split PMHSs*. From Theorem 3.27(a, c) one may also deduce that \tilde{F}^\bullet and \tilde{F}_∞^\bullet lie in the same $G_{\mathbb{R}}$ -orbit in \check{D} . (See the proof of [CKS86, Lemma 3.12].)

Since $\Phi_\infty(g \cdot F^\bullet, \text{Ad}_g N) = g \cdot \Phi_\infty(F^\bullet, N)$, we see that any two $G_{\mathbb{R}}$ -congruent \mathbb{R} -split PMHSs parameterize the same $G_{\mathbb{R}}$ -orbit $\mathcal{O} \subset \text{bd}(D)$. We say that \mathcal{O} is *the orbit polarized by $[F^\bullet, N] \in \Psi_D$* . Theorem 5.5(c) describes the image of the surjection

$$\Psi_D \rightarrow \{\text{polarized } \mathcal{O} \subset \text{cl}(D)\}. \tag{5.17}$$

Remark 5.18. The parameterization (5.17) of the polarized orbits generalizes a construction of [KR14] which obtains polarized $G_{\mathbb{R}}$ -orbits $\mathcal{O} \subset \text{bd}(D)$ from sets of strongly orthogonal noncompact roots. In the case that D is Hermitian symmetric, all the boundary orbits $\mathcal{O} \subset \text{bd}(D)$ are polarizable; they are all parameterized by the [KR14] construction [FHW06, Theorem 3.2.1]; and the parameterization is essentially that given by the Harish-Chandra compactification of D .

Polarized orbits have received much attention recently; cf. [GGK13, GGR14, KP14, KR14]. One basic result is the following.

THEOREM 5.19 [KP14]. *The complexified normal space to $\mathcal{O} \subset \check{D}$ at the point $\varrho(\varphi) = \Phi_\infty(F^\bullet, N)$ is*

$$N_{\varrho(\varphi)}\mathcal{O} \otimes \mathbb{C} = \bigoplus_{p,q>0} I_{\mathfrak{g}}^{-p,-q}. \tag{5.20}$$

In particular, the (real) codimension of the (polarized) $G_{\mathbb{R}}$ -orbit \mathcal{O} is

$$\text{codim}_{\check{D}}\mathcal{O} = \dim_{\mathbb{C}} \bigoplus_{p,q>0} I_{\mathfrak{g}}^{p,q}. \tag{5.21}$$

Moreover, the boundary $\text{bd}(D) \subset \check{D}$ contains codimension-one $G_{\mathbb{R}}$ -orbits and they are all polarized. In this case the normal space

$$N_{\varrho(\varphi)}\mathcal{O} = \mathfrak{g}_{\mathbb{R}}^{-\alpha} \tag{5.22}$$

is naturally identified with a real root space.

Recall the set $I(\mathfrak{p}) = \{i \mid \mathfrak{g}^{-\sigma_i} \not\subset \mathfrak{p}\} = \{i \mid \sigma_i(\mathbf{E}) = 1\}$ of (2.1). We will see that Theorems 5.5 and 5.19 yield the following proposition.

PROPOSITION 5.23. *The boundary $\text{bd}(D) \subset \check{D}$ contains exactly $|I(\mathfrak{p})|$ codimension-one $G_{\mathbb{R}}$ -orbits.*

Remark 5.24. In the case that P is maximal (equivalently, $|I(\mathfrak{p})| = 1$), Proposition 5.23 was proven in [KR14].

Proof. From (4.13), (5.20), and (5.22) we see that N spans $I_{\mathfrak{g}}^{-1,-1} = \mathfrak{g}^{-\alpha}$ when $\varrho(\varphi)$ lies in a codimension-one $G_{\mathbb{R}}$ -orbit. It follows that the Levi subalgebra $\mathfrak{l}_{\mathbb{C}}$ of Theorem 5.5 has rank one, and the semisimple factor is the $\mathfrak{sl}_2\mathbb{C} \subset \mathfrak{g}_{\mathbb{C}}$ with simple root $\{\alpha\}$. In particular, $\alpha = \sigma_i$ for some $i \in I(\mathfrak{p})$. Whence $\text{bd}(D)$ contains at most $|I(\mathfrak{p})|$ codimension-one orbits.

Since no two σ_i , with $i \in I(\mathfrak{p})$, are congruent under the Weyl group \mathcal{W}^0 of \mathfrak{g}^0 , in order to see that equality holds we must show that every $i \in I(\mathfrak{p})$ yields a codimension-one orbit. Let $\mathfrak{l}_{\mathbb{C}}$ be the rank-one Levi subalgebra with simple root σ_i . Then $\mathfrak{l}_{\mathbb{R}} = \mathfrak{l}_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{R}} \in \mathcal{L}_{\varphi, \mathfrak{t}}$. Moreover, in this case $\mathcal{Z} = \mathbb{H}^i$, where $\{\mathbb{H}^j\}_{j=1}^r$ is the basis of \mathfrak{h} dual to the fundamental weights. The fact that the $G_{\mathbb{R}}$ -orbit through $\varrho(\varphi)$ has codimension one is [KR14, Lemma 6.52]. \square

5.5 Examples

Suppose that D is a Mumford–Tate domain for a Hodge representation $(V_{\mathbb{R}}, Q, \varphi)$ of $G_{\mathbb{R}}$. In the examples that follow we use Theorem 5.9 to enumerate the set Υ_D of horizontal $\text{SL}(2)$ s on D (modulo the action of $G_{\mathbb{R}}$). More precisely, given $[v] \in \Upsilon_D$, let $[\mathfrak{l}_{\mathbb{R}}] \in \Lambda_{\mathfrak{t}, \varphi}$ be the corresponding conjugacy class under Theorem 5.9, and let $[F^{\bullet}, N] \in \Psi_D$ be the corresponding (conjugacy class of) nilpotent orbit under Theorem 5.5. We will do the following.

(1) Identify a representative of $[\mathfrak{l}_{\mathbb{R}}]$ by describing the simple roots S' of $\mathfrak{l}_{\mathbb{C}}$ as a subset of the roots Δ of $\mathfrak{g}_{\mathbb{C}}$. The Levis $\mathfrak{l}_{\mathbb{R}}$ of $\mathcal{L}_{\varphi, \mathfrak{t}}$ are identified as follows. As discussed in Remark 2.6, the Levi subalgebras $\mathfrak{l}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ that contain $\mathfrak{h} = \mathfrak{t} \otimes \mathbb{C}$ are in bijection with the subsets $\{w\mathcal{S}_0 \mid w \in \mathcal{W}, \mathcal{S}_0 \subset \mathcal{S}\}$. This is a finite collection of subsets. For each subset we consider the corresponding Levi \mathfrak{l} and compute $\mathcal{Z} = \pi_1^{\text{ss}}(\mathbf{E}_{\varphi})$; cf. Remark 5.13. We then compute the \mathcal{Z} -eigenspace decomposition of $\mathfrak{l}_{\mathbb{C}}^{\text{ss}}$ to determine whether or not \mathcal{Z} is a distinguished element of $\mathfrak{l}_{\mathbb{C}}^{\text{ss}}$, cf. (5.2).

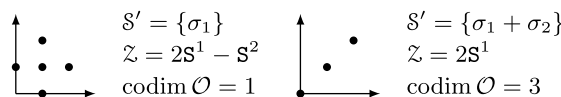
(2) Compute the codimension (5.21) of the $G_{\mathbb{R}}$ -orbit \mathcal{O} polarized by $[F^{\bullet}, N]$.

(3) Determine the Deligne splitting (5.6) of the \mathbb{R} -split PMHS (F^{\bullet}, N) . The splittings will be depicted by pictures in the pq -plane that place a \bullet at the point (p, q) if $I^{p,q} \neq 0$. When considering those pictures, keep in mind that $N \in I_{\mathfrak{g}}^{-1,-1}$, so that $N : I^{p,q} \rightarrow I^{p-1, q-1}$.

The Hodge diamond may fail to distinguish two distinct conjugacy classes in Υ_D ; see Remarks 5.37 and 5.29(a).

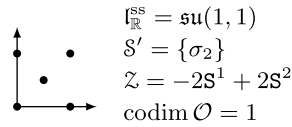
Throughout $(i) \in \mathcal{W}$ will denote the simple reflection in the hyperplane $\sigma_i^{\perp} \subset \mathfrak{h}^*$.

Example 5.25 (Period domain for $\mathfrak{h} = (1, 3, 1)$). We have $G_{\mathbb{R}} = \text{SO}(3, 2)^{\circ}$ and $\mathbf{E}_{\varphi} = \mathbb{S}^1$ so that $\mathcal{W}^0 = \{1, (2)\}$. In this case D is Hermitian symmetric and all the $G_{\mathbb{R}}$ -orbits of $\text{bd}(D)$ are polarized. Applying Theorem 5.9, we find that Υ_D consists of two elements:



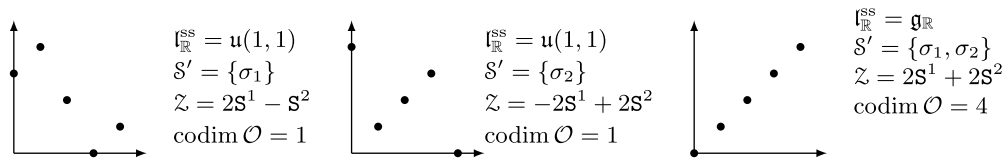
In both cases $\mathfrak{l}_{\mathbb{R}}^{\text{ss}} = \mathfrak{su}(1, 1)$.

Example 5.26 (Period domain for $\mathbf{h} = (2, 1, 2)$). We have $G_{\mathbb{R}} = SO(1, 4)^{\circ}$ and $E_{\varphi} = \mathbf{S}^2$, so that $\mathcal{W}^0 = \{\mathbf{1}, (1)\}$. Applying Theorem 5.9, we find that Υ_D consists of a single element:

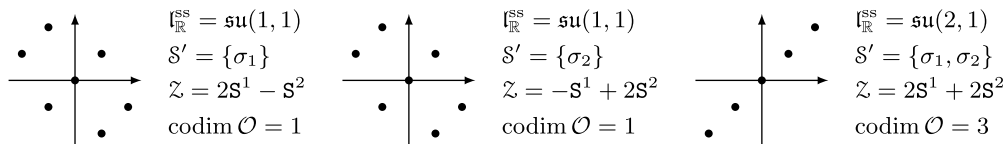


Moreover, from [GGR14, Lemma III.20] we may deduce that the codimension-one polarized orbit \mathcal{O} is closed, giving an example of a closed orbit that is polarized, but not by a Hodge–Tate degeneration.

Example 5.27 (Period domain for $\mathbf{h} = (1, 1, 1, 1, 1)$). We have $G_{\mathbb{R}} = SO(3, 2)$ and $E_{\varphi} = \mathbf{S}^1 + \mathbf{S}^2$, so that $\mathcal{W}^0 = \{\mathbf{1}\}$. Applying Theorem 5.9, we find that Υ_D consists of three elements:



Example 5.28 ($G_{\mathbb{R}} = SU(2, 1)$ and $\check{D} = \text{Flag}_{1,2}\mathbb{C}^3$). We have $E_{\varphi} = \mathbf{S}^1 + \mathbf{S}^2$, and consider the Mumford–Tate domain $D \subset \check{D}$ for the Hodge representation $(\mathfrak{g}_{\mathbb{R}}, Q_{\mathfrak{g}}, \text{Ad}, \varphi)$. This domain is well studied; indeed, it is known that $\text{bd}(D)$ contains three $G_{\mathbb{R}}$ -orbits, all of which are polarized; cf. [GGK13, KP14]. We have $\mathcal{W}^0 = \{\mathbf{1}\}$. Applying Theorem 5.5, we find that Υ_D consists of three elements; the corresponding data are:



Remark 5.29. (a) Observe that the first two Hodge diamonds in Example 5.28 are identical; in particular, they fail to distinguish the two distinct $G_{\mathbb{R}}$ -conjugacy classes of horizontal $SL(2)$ s.

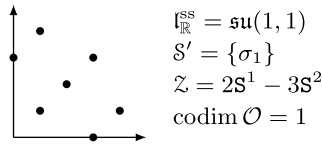
(b) Moreover, while the two nilpotent elements $N \in \text{Nilp}(\mathfrak{g}_{\mathbb{R}})$ of these examples lie in the same $\text{Ad}(G_{\mathbb{C}})$ -orbit (the minimal orbit \mathcal{N}_{\min}), they lie in distinct $\text{Ad}(G_{\mathbb{R}})$ -orbits. This may be seen by computing the invariants $(\gamma(\mathcal{Z}); \alpha'(\mathcal{Z}))$ of § 2.7, and observing that they differ. For this, we work with the simple roots $\tilde{\mathbf{S}} = (1)\mathbf{S} = \{-\sigma_1, \sigma_1 + \sigma_2\}$. Then $\tilde{\mathbf{S}}_{\mathfrak{k}} = \{\sigma_1 + \sigma_2\}$ is a set of simple roots for $\mathfrak{k}_{\mathbb{C}} = \mathfrak{gl}_2\mathbb{C}$, and the noncompact root is $\alpha' = -\sigma_1$. In both cases the compact characteristic vector satisfies

$$\gamma(\mathcal{Z}) = ((\sigma_1 + \sigma_2)(\mathcal{Z})) = (1);$$

however, in the first example we have $\alpha'(\mathcal{Z}) = -2$, while in the second we have $\alpha'(\mathcal{Z}) = 1$.

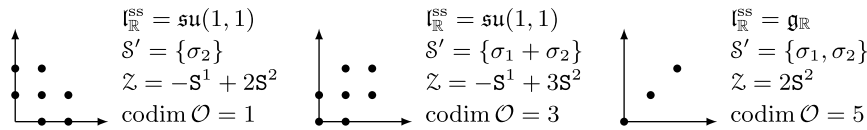
Example 5.30 ($G_{\mathbb{R}} = G_2 \subset SO(3, 4)$ and $\mathbf{h} = (1, 2, 1, 2, 1)$). We have $E_{\varphi} = \mathbf{S}^1$ and $G_{\mathbb{R}} \subset SO(3, 4)$, and consider the Mumford–Tate domain $D \subset \check{D}$ for the Hodge representation on $V_{\mathbb{R}} = \mathbb{R}^7$ with Hodge numbers $\mathbf{h} = (1, 2, 1, 2, 1)$. Kerr and Pearlstein have shown that $\text{bd}(D)$ contains three $G_{\mathbb{R}}$ -orbits, only one of which is polarized [KP14, § 6.1.3].

Here $\mathcal{W}^0 = \{\mathbf{1}, (2)\}$. Applying Theorem 5.9, we find that Υ_D consists of a single element:



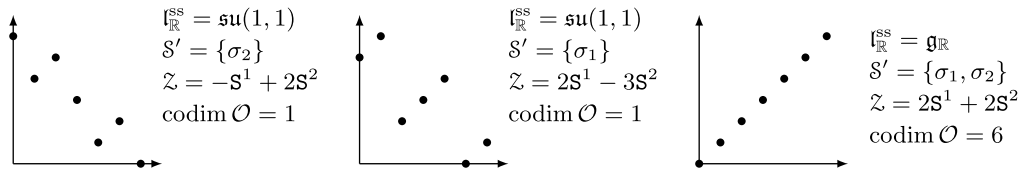
Example 5.31 ($G_{\mathbb{R}} = G_2 \subset \text{SO}(3, 4)$ and $\mathbf{h} = (2, 3, 2)$). We have $E_{\varphi} = \mathbf{S}^2$ and $G_{\mathbb{R}} \subset \text{SO}(3, 4)$, and consider the Mumford–Tate domain $D \subset \check{D}$ for the Hodge representation on $V_{\mathbb{R}} = \mathbb{R}^7$ with Hodge numbers $\mathbf{h} = (2, 3, 2)$. Kerr and Pearlstein have shown that $\text{bd}(D)$ contains three $G_{\mathbb{R}}$ -orbits, all of which are polarized [KP14, § 6.1.3].

Here $\mathcal{W}^0 = \{\mathbb{1}, (1)\}$. Applying Theorem 5.9, we find that Υ_D consists of three elements:

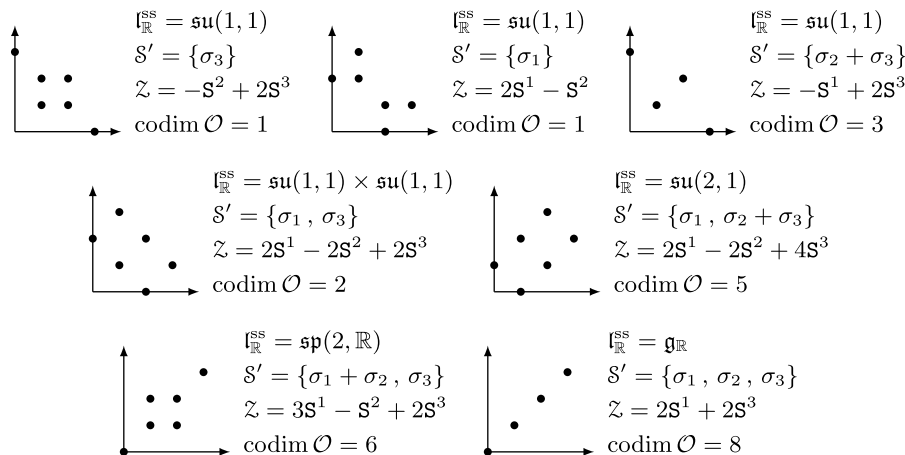


Example 5.32 ($G_{\mathbb{R}} = G_2 \subset \text{SO}(3, 4)$ and $\mathbf{h} = (1, 1, 1, 1, 1, 1, 1)$). We have $E_{\varphi} = \mathbf{S}^1 + \mathbf{S}^2$ and $G_{\mathbb{R}} \subset \text{SO}(3, 4)$, and consider the Mumford–Tate domain $D \subset \check{D}$ for the Hodge representation on $V_{\mathbb{R}} = \mathbb{R}^7$ with Hodge numbers $\mathbf{h} = (1, 1, 1, 1, 1, 1, 1)$. Kerr and Pearlstein have shown that $\text{bd}(D)$ contains seven $G_{\mathbb{R}}$ -orbits, three of which are polarizable [KP14, § 6.1.3].

Here $\mathcal{W}^0 = \{\mathbb{1}\}$. Applying Theorem 5.9, we find that Υ_D consists of three elements; they are given by:

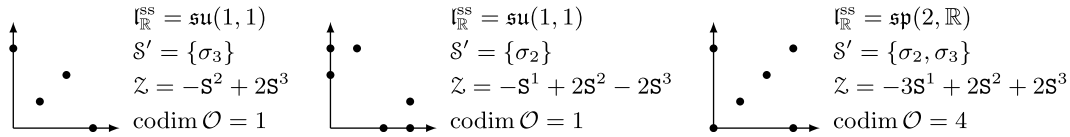


Example 5.33 (Period domain for $\mathbf{h} = (1, 2, 2, 1)$). We have $G_{\mathbb{R}} = \text{Sp}(3, \mathbb{R})$ and $E_{\varphi} = \mathbf{S}^1 + \mathbf{S}^3$. In this case $\mathcal{W}^0 = \{\mathbb{1}, (2)\}$. Applying Theorem 5.9, we find that Υ_D contains seven elements. The corresponding data are:

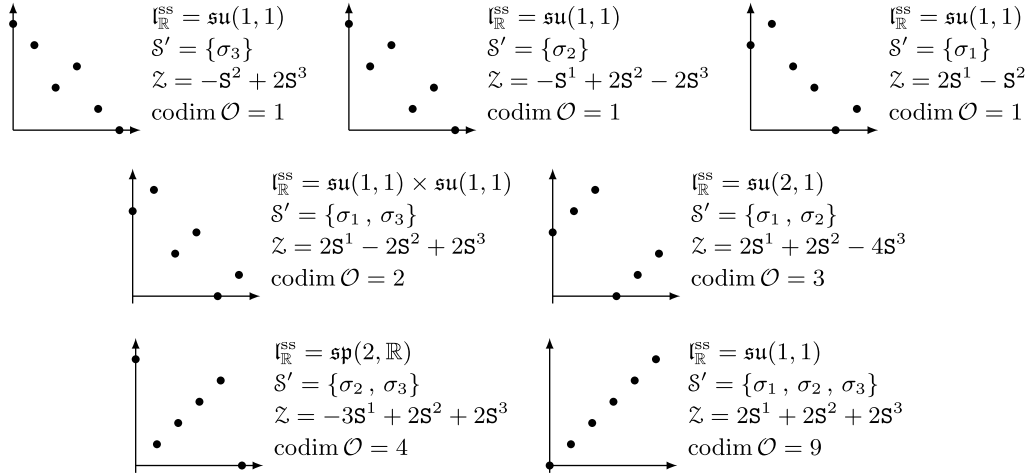


Example 5.34 (Period domain for $\mathbf{h} = (2, 1, 1, 2)$). We have $G_{\mathbb{R}} = \text{Sp}(3, \mathbb{R})$ and $E_{\varphi} = \mathbf{S}^2 + \mathbf{S}^3$. In this case $\mathcal{W}^0 = \{\mathbb{1}, (1)\}$. Applying Theorem 5.5, we find that there are three (conjugacy classes of) horizontal $\text{SL}(2)$ s on the period domain D . The corresponding data are:

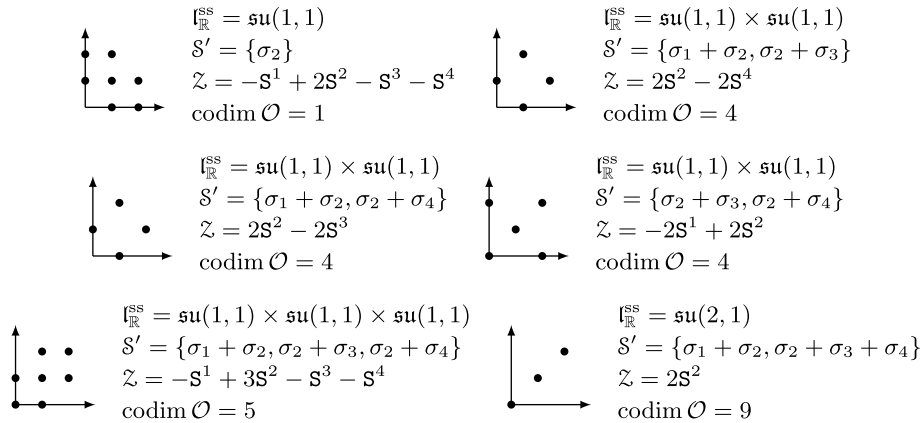
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Example 5.35 (Period domain for $\mathbf{h} = (1, 1, 1, 1, 1, 1)$). We have $G_{\mathbb{R}} = Sp(3, \mathbb{R})$ and $E_{\varphi} = S^1 + S^2 + S^3$. In this case $\mathcal{W}^0 = \{\mathbb{1}\}$. Applying Theorem 5.5, we find that there are seven (conjugacy classes of) horizontal $SL(2)$ s on the period domain D . The corresponding data are:

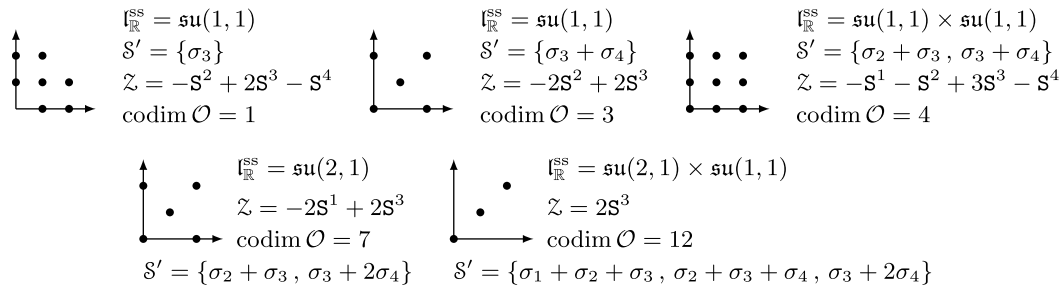


Example 5.36 (Period domain for $\mathbf{h} = (2, 4, 2)$). We have $G_{\mathbb{R}} = SO(4, 4)^{\circ}$, $E_{\varphi} = S^2$, and $\mathcal{W}^0 = \{(1), (3), (4)\}$. There are six ($G_{\mathbb{R}}$ -conjugacy classes of) horizontal $SL(2)$ s:



Remark 5.37. Note that the second and third ($G_{\mathbb{R}}$ -conjugacy classes of) horizontal $SL(2)$ s in (the first row of) Example 5.36 are *not* distinguished by their Hodge diamonds.

Example 5.38 (Cattani–Kaplan). In [CK78, §4] Cattani and Kaplan consider the case that D is the period domain for Hodge numbers $\mathbf{h} = (3, 3, 3)$, and find that there are five conjugacy classes of horizontal $SL(2)$ s. In the notation of Theorems 5.5 and 5.9 (and the introduction to §5.5), those conjugacy classes are enumerated as follows. We have $G_{\mathbb{R}} = SO(3, 6)$. The grading element is $E_{\varphi} = S^3$, and \mathcal{W}^0 is generated by the simple reflections $\{(1), (2), (4)\}$.



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Appendix. Noncompact real forms

The classical noncompact simple real forms $\mathfrak{g}_{\mathbb{R}}$ that contain a compact Cartan subalgebra are listed in Table A.1 along with their maximal compact subalgebras; there $a, b > 0$. Recall that

$$\mathfrak{so}(2) \simeq \mathbb{R}, \quad \mathfrak{sp}(1) \simeq \mathfrak{su}(2), \quad \mathfrak{sp}(2) \simeq \mathfrak{so}(5), \quad \mathfrak{su}(4) \simeq \mathfrak{so}(6).$$

Table A.2 lists those noncompact real forms $\mathfrak{g}_{\mathbb{R}}$ of the exceptional simple complex Lie algebras \mathfrak{g} that contain a compact Cartan subalgebra. The table also lists the maximal compact Lie subalgebra $\mathfrak{k}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$, and the real rank $\text{rank}_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}$. In the first column we give the two common notations for the real forms; in the case of the second, the notation $X_n(s)$ indicates the complex form X_n of the algebra, and $s = \dim \mathfrak{k}_{\mathbb{R}}^{\perp} - \dim \mathfrak{k}_{\mathbb{R}}$.

TABLE A.1. The classical real forms.

$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{su}(a, b)$	$\mathfrak{sp}(a, b)$	$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{so}(2a, b)$	$\mathfrak{so}^*(2n)$
$\mathfrak{k}_{\mathbb{R}}$	$\mathfrak{s}(\mathfrak{u}(a) \oplus \mathfrak{u}(b))$	$\mathfrak{sp}(a) \oplus \mathfrak{sp}(b)$	$\mathfrak{u}(n)$	$\mathfrak{so}(2a) \oplus \mathfrak{so}(b)$	$\mathfrak{u}(n)$

TABLE A.2. The exceptional real forms.

\mathfrak{g}	$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{k}_{\mathbb{R}}$	$\text{rank}_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}$
\mathfrak{e}_6	E II = $E_6(2)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	4
	E III = $E_6(-14)$	$\mathfrak{so}(10) \oplus \mathbb{R}$	2
	E V = $E_7(7)$	$\mathfrak{su}(8)$	7
\mathfrak{e}_7	E VI = $E_7(-5)$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	4
	E VII = $E_7(-25)$	$\mathfrak{e}_6 \oplus \mathbb{R}$	3
\mathfrak{e}_8	E VIII = $E_8(8)$	$\mathfrak{so}(16)$	8
	E IX = $E_8(-24)$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	4
\mathfrak{f}_4	F I = $F_4(4)$	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	4
	F II = $F_4(-20)$	$\mathfrak{so}(9)$	1
\mathfrak{g}_2	G = $G_2(2)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	2

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