

## DP-MINIMAL VALUED FIELDS

FRANZISKA JAHNKE, PIERRE SIMON, AND ERIK WALSBERG

**Abstract.** We show that dp-minimal valued fields are henselian and give classifications of dp-minimal ordered abelian groups and dp-minimal ordered fields without additional structure.

**§1. Introduction.** Very little is known about NIP fields. It is widely believed that an NIP field is either real closed, separably closed, or admits a definable henselian valuation. Note that even the stable case of this conjecture is open.

In this paper we study a very special case of this question: that of a valued or ordered dp-minimal field. Dp-minimality is a combinatorial generalization of o-minimality and C-minimality. A dp-minimal structure can be thought of as a *one-dimensional* NIP structure. The main result of this paper is that a dp-minimal valued field is henselian. As a consequence, we show that an  $\aleph_1$ -saturated dp-minimal ordered field admits a henselian valuation with residue field  $\mathbb{R}$ . These results can be seen either as a special case of the conjecture on NIP fields or as a generalization of what is known in the C-minimal and weakly o-minimal cases.

Our proof has two parts. We first establish some facts about dp-minimal topological structures. In Section 3, we generalize statements and proofs of Goodrick [6] and Simon [17] on ordered dp-minimal structures to the more general setting of a dp-minimal structure admitting a definable *uniform topology*. This seems to be the most general framework to which the proofs apply. Only afterwards will we assume that the topology comes either from a valuation or an order. Once we have established the necessary facts about dp-minimal uniform structures, the remainder of the proof, given in Section 4, follows part of the proof that weakly o-minimal fields are real closed. The proof that weakly o-minimal fields are real closed is surprisingly more complicated than the proof that o-minimal fields are real closed. It involves first finding some henselian valuation and then showing that the residue field is real closed and the value group is divisible, from which the result follows. This argument was extended by Guingona [7] to dp-small structures, a strengthening of dp-minimality. We follow again the same path, but assuming only dp-minimality, the value group is not necessarily divisible.

In Section 5, we use the Gurevich–Schmitt quantifier elimination for ordered abelian groups to show that an ordered abelian group  $\Gamma$  without additional structure is dp-minimal if and only if it is *nonsingular*, that is if  $|\Gamma/p\Gamma| < \infty$  holds for

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all primes  $p$ . It follows that an  $\aleph_1$ -saturated dp-minimal ordered field admits a henselian valuation with residue field  $\mathbb{R}$  and nonsingular value group. In fact, this is best possible: Chernikov and Simon [3] show that any such field is dp-minimal. This gives rise to our classification of dp-minimal ordered fields in Section 6. Finally, in Proposition 6.5, we show that an ordered field which is not dense in its real closure admits a definable convex valuation. This result may be of independent interest.

After finishing this paper, we learnt from Will Johnson that he simultaneously proved more general results on dp-minimal fields using different methods. Most of our results can be deduced from his paper [9], where he gives a complete classification of dp-minimal fields up to elementary equivalence. He proves in particular that if such a field is neither algebraically closed nor real closed, then it admits a nontrivial definable henselian valuation. His proof can be roughly broken down as follows:

STEP 1. Finding a  $V$  topology;

STEP 2. Building from the  $V$  topology a henselian valuation;

STEP 3. Deducing the classification using extra input from [8] and [10].

Step 1 is completely absent from our work since we assume from the start that we have a definable valuation. Step 2 of Johnson's paper has a very large overlap with our Sections 3 and 4. The context however is slightly different: in Section 3, we start with a general definable topology, the key result being Proposition 3.9 which says that images of balls by finite-to-one definable functions have nonempty interior. Johnson deals with a specific  $V$  topology, which however is not known to be genuinely definable at this point of the paper. He proves the same statement in his Proposition 5.3. He then deduces that a definable valuation in a dp-minimal field must be henselian (our Proposition 4.5) with an argument different from ours. His argument uses crucially a result about bounded germs (Theorem 4.7). This is similar to our Lemma 3.5, although again neither implies the other, because the assumptions are different. Finally, his third step is also absent from our paper. We only manage to classify *ordered* dp-minimal fields (Theorem 6.2). That theorem is not explicitly stated in Johnson's work, but follows from it at once: if a valued field is orderable, then its residue field must also be orderable, and the only orderable residue field in Johnson's classification is  $\mathbb{R}$ .

**§2. Dp-minimality.** Let  $L$  be a multisorted language with a distinguished home sort and let  $T$  be a theory in  $L$ . Throughout this section,  $\mathcal{M}$  is an  $|L|^+$ -saturated model of  $T$  with distinguished home sort  $M$ . We recall the definitions of inp- and dp-minimality. These definitions are usually stated for one-sorted structures, we define them in our multisorted setting.

**DEFINITION.** The theory  $T$  is *not inp-minimal* with respect to the home sort if there are two formulas  $\phi(x; \bar{y})$  and  $\psi(x; \bar{z})$ , where  $x$  is a single variable of sort  $M$ , two sequences  $(\bar{b}_i : i < \omega)$  and  $(\bar{c}_i : i < \omega)$  in  $\mathcal{M}$  and  $k \in \mathbb{N}$  such that:

- the sets  $\{\phi(x; \bar{b}_i) : i < \omega\}$  and  $\{\psi(x; \bar{c}_i) : i < \omega\}$  are each  $k$ -inconsistent;
- $\phi(x; \bar{b}_i) \wedge \psi(x; \bar{c}_j)$  is consistent for all  $i, j < \omega$ .

The data consisting of two sequences  $(\bar{b}_i : i < \omega)$  and  $(\bar{c}_i : i < \omega)$  and formulas  $\phi(x; \bar{y})$ ,  $\psi(x; \bar{z})$  with properties as above is called an inp-pattern (of depth 2). If  $T$  is inp-minimal and one-sorted then  $T$  is  $\text{NTP}_2$ .

DEFINITION. The theory  $T$  is *not dp-minimal* with respect to the home sort if there are two formulas  $\phi(x; \bar{y})$  and  $\psi(x; \bar{z})$ , where  $x$  is a single variable of sort  $M$ , two sequences  $(\bar{b}_i : i < \omega)$  and  $(\bar{c}_i : i < \omega)$  in  $\mathcal{M}$  such that:

- for any  $i_*, j_* \in \omega$ , the conjunction

$$\phi(x; \bar{b}_{i_*}) \wedge \left[ \bigwedge_{i \neq i_*} \neg \phi(x; \bar{b}_i) \right] \wedge \psi(x; \bar{c}_{j_*}) \wedge \left[ \bigwedge_{j \neq j_*} \neg \psi(x; \bar{c}_j) \right]$$

is consistent.

If  $T$  is one-sorted then  $T$  is dp-minimal if and only if  $T$  is both inp-minimal and NIP (see for example [17]). It follows that if  $T$  is NIP and inp-minimal with respect to the home sort then  $T$  is dp-minimal with respect to the home sort.

We define the *Shelah expansion*  $\mathcal{M}^{Sh}$  of  $\mathcal{M}$ . Fix an  $|\mathcal{M}|^+$ -saturated elementary expansion  $\mathcal{M} \prec \mathcal{N}$ . We say that a subset of  $M$  is *externally definable* if it is of the form  $\{a \in M : \mathcal{N} \models \phi(a; b)\}$  for some  $L$ -formula  $\phi(x; y)$  and  $b \in \mathcal{N}^{|\mathcal{V}|}$ . A straightforward application of  $|\mathcal{M}|^+$ -saturation shows that the collection of externally definable sets does not depend on the choice of  $\mathcal{N}$ . Note that the externally definable sets form a boolean algebra. The Shelah expansion  $\mathcal{M}^{Sh}$  is the expansion of  $\mathcal{M}$  by predicates for all externally definable subsets of  $M$ . More precisely, for each formula  $\phi(x; y) \in L$  and each  $b \in \mathcal{N}^{|\mathcal{V}|}$ , we have a relation  $R_{\phi(x;b)}(x)$  which is interpreted as  $\{a \in M : \mathcal{N} \models \phi(a; b)\}$ . Shelah proved the following, see e.g., [19, Chapter 3].

PROPOSITION 2.1 (Shelah). *If  $\mathcal{M}$  is NIP then  $\mathcal{M}^{Sh}$  admits elimination of quantifiers and is consequently also NIP.*

We make extensive use of the next fact which was observed in [14].

PROPOSITION 2.2. *If  $T$  is NIP and dp-minimal with respect to the home sort then  $\text{Th}(\mathcal{M}^{Sh})$  is also dp-minimal (relative to the same home sort).*

PROOF. Since  $T$  is dp-minimal, it is NIP, hence the structure  $\mathcal{M}^{Sh}$  has elimination of quantifiers. We suppose towards a contradiction that  $\text{Th}(\mathcal{M}^{Sh})$  is not dp-minimal. Then there are two formulas  $\phi(x; \bar{y}), \psi(x; \bar{z})$  such that for any  $n < \omega$ , there are finite sequences  $(a_i : i < n)$  and  $(b_i : i < n)$  of elements of  $M$  with the property as in the definition of dp-minimality. As  $\mathcal{M}^{Sh}$  admits quantifier elimination there are tuples  $d, d' \in \mathcal{N}$  and  $L$ -formulas  $\phi'(x; \bar{y}; d), \psi'(x; \bar{z}; d')$  such that for any  $(a, b) \in \mathcal{M}^{|\mathcal{X}|} \times \mathcal{M}^{|\mathcal{V}|}$ , we have  $\mathcal{M}^{Sh} \models \phi(a; b) \iff \mathcal{N} \models \phi'(a; b; d)$  likewise for  $\psi$  and  $\psi'$ . Using compactness, we see that the formulas  $\phi'(x; \bar{y}; d)$  and  $\psi'(x; \bar{z}; d')$  contradict the dp-minimality of  $T$ .  $\dashv$

If  $M$  admits a definable linear order, then every convex subset of  $M$  is externally definable, so the expansion of an ordered dp-minimal structure by a convex set is always dp-minimal. We end the section with a proof of the well-known fact that dp-minimal fields are perfect (which appears also in [9, Observation 2.2]). The proof in fact works more generally for fields of finite dp-rank.

LEMMA 2.3. *Every dp-minimal field is perfect.*

PROOF. Let  $K$  be a nonperfect field of characteristic  $p > 0$ . We let  $K^p$  be the set of  $p$ th powers. Fix  $z \in K \setminus K^p$ . Let  $\tau : K \times K \rightarrow K$  be given by  $\tau(x, y) = x^p + zy^p$ .

We show that  $\tau$  is injective, it follows that  $K$  is not dp-minimal. Fix  $x_0, x_1, y_0, y_1 \in K$  and suppose that  $\tau(x_0, y_0) = \tau(x_1, y_1)$  that is:

$$x_0^p + zy_0^p = x_1^p + zy_1^p.$$

If  $y_0 \neq y_1$  then  $y_0^p \neq y_1^p$ , so:

$$z = \frac{x_1^p - x_0^p}{y_0^p - y_1^p} \in K^p,$$

so we must have  $y_0 = y_1$ . Then  $x_0^p = x_1^p$  and so also  $x_0 = x_1$ . Thus  $\tau$  is injective.  $\dashv$

**§3. Dp-minimal uniform spaces.** Many dp-minimal structures of interest admit natural “definable topologies”. Typically, the topology is given by a definable linear order or a definable valuation. In this section, we develop a framework designed to encompass many of these examples, dp-minimal structures equipped with definable uniform structures. We prove some results of Goodrick [6] and Simon [17] in this setting. The proofs are essentially the same. We first recall the notion of a uniform structure on  $M$ . We let  $\Delta$  be the diagonal  $\{(x, x) : x \in M\}$ . Given  $U, V \subseteq M^2$ , we let

$$U \circ V = \{(x, z) \in M^2 : (\exists y \in M)(x, y) \in U, (y, z) \in V\}.$$

A *basis* for a uniform structure on  $M$  is a collection  $\mathcal{B}$  of subsets of  $M^2$  satisfying the following:

- (1) the intersection of the elements of  $\mathcal{B}$  is equal to  $\Delta$ ;
- (2) if  $U \in \mathcal{B}$  and  $(x, y) \in U$  then  $(y, x) \in U$ ;
- (3) for all  $U, V \in \mathcal{B}$  there is a  $W \in \mathcal{B}$  such that  $W \subseteq U \cap V$ ;
- (4) for all  $U \in \mathcal{B}$  there is a  $V \in \mathcal{B}$  such that  $V \circ V \subseteq U$ .

The *uniform structure* on  $M$  generated by  $\mathcal{B}$  is defined as

$$\tilde{\mathcal{B}} = \{U \subseteq M^2 : (\exists V \in \mathcal{B}) V \subseteq U\}.$$

Elements of  $\tilde{\mathcal{B}}$  are called *entourages*, and elements of  $\mathcal{B}$  are called *basic entourages*. Given  $U \in \mathcal{B}$  and  $x \in M$ , we let  $U[x] = \{y : (x, y) \in U\}$ . As usual, one defines a topology on  $M$  by declaring that a subset  $A \subseteq M$  is open if for every  $x \in A$ , there is  $U \in \mathcal{B}$  such that  $U[x] \subseteq A$ . The first condition above ensures that this topology is Hausdorff. The collection  $\{U[x] : U \in \mathcal{B}\}$  forms a basis of neighborhoods of  $x$ . We will refer to them as *balls* with center  $x$ . We say that  $\mathcal{B}$  is a *definable uniform structure* if there is a formula  $\varphi(x, y, \bar{z})$  such that

$$\mathcal{B} = \{\varphi(M^2, \bar{c}) \mid \bar{c} \in D\}$$

for some definable set  $D$ . (This is a slight abuse of terminology, since  $\mathcal{B}$  is only a definable basis for a uniform structure.) Note that the conditions above are first order conditions on  $\varphi$ . We give some examples of definable uniform structures.

- (1) Suppose that  $\Gamma$  is an  $\mathcal{M}$ -definable ordered abelian group and  $d$  is a definable  $\Gamma$ -valued metric on  $M$ . We can take  $\mathcal{B}$  to be the collection of sets of the form  $\{(x, y) \in X^2 : d(x, y) < t\}$  for  $t \in \Gamma$ . The typical case is when  $\Gamma = M$  and  $d(x, y) = |x - y|$ .
- (2) Suppose that  $\Gamma$  is a definable linear order with minimal element and that  $d$  is a definable  $\Gamma$ -valued ultrametric on  $M$ . Then we can put a definable uniform

structure on  $M$  in the same way as above. The typical case is when  $M$  is a valued field.

- (3) Suppose that  $M$  expands a group. Let  $D$  be a definable set and suppose that  $\{U_{\bar{z}} : \bar{z} \in D\}$  is a definable family of subsets of  $M$  which forms a neighborhood basis at the identity for the topology on  $M$  under which  $M$  is a topological group. Then sets of the form  $\{(x, y) \in M^2 : x^{-1}y \in U_{\bar{z}}\}$  for  $\bar{z} \in D$  give a definable uniform structure on  $M$ .

*For the remainder of this section, we assume that  $\mathcal{B}$  is a definable uniform structure on  $M$  and that  $\mathcal{M}$  is inp-minimal with respect to the home sort. We further suppose that  $M$  does not have any isolated points.*

LEMMA 3.1. *Every infinite definable subset of  $M$  is dense in some ball.*

PROOF. We suppose towards a contradiction that  $X \subseteq M$  is infinite, definable, and nowhere dense. Let  $(a_i : i < \omega)$  be a sequence of distinct points in  $X$ . Applying  $\aleph_1$ -saturation, we let  $U_0 \in \mathcal{B}$  be such that  $U_0[a_i] \cap U_0[a_j] = \emptyset$  when  $i \neq j$ . Now inductively construct elements  $U_n \in \mathcal{B}$  and  $x_{i,n} \in U_0[a_i]$  such that  $a_i \in U_n[x_{i,n}]$  and  $U_{n+1}[x_{i,n}] \cap X = \emptyset$  for all  $i, n < \omega$ . Assume we have defined  $U_m$  for  $m \leq n$  and  $x_{i,m}$  for  $m \leq n - 1$ . By assumption,  $X$  is not dense in the ball  $U_n[a_i]$  for any  $i$ , hence we can find some  $U_{n+1} \in \mathcal{B}$  and points  $x_{i,n} \in U_n[a_i]$  such that  $U_{n+1}[x_{i,n}] \cap X = \emptyset$  for all  $i$ . If  $U_n[x] = \pi(x, \bar{b}_n)$ , then the formulas

$$\phi(x, a_i) \equiv x \in U_0[a_i] \text{ and } \psi(x, \bar{b}_n, \bar{b}_{n+1}) \equiv (U_{n+1}[x] \cap X = \emptyset \wedge U_n[x] \cap X \neq \emptyset)$$

witness that  $T$  is not inp-minimal with respect to the home sort. This contradiction shows that the lemma holds. ⊥

We leave the simple topological proof of the following to the reader:

COROLLARY 3.2. *Any definable closed subset of  $M$  is the union of an open set and finitely many points. Moreover, the closure of a definable open set is equal to itself plus finitely many points.*

The next lemma is a version of Lemma 3.19 of [6]. Given  $R \subseteq M^2$  and  $a \in M$  we define  $R(a) = \{b \in M : (a, b) \in R\}$ .

LEMMA 3.3. *Let  $R \subseteq M^2$  be a definable relation such that for every  $a \in M$ , there is  $V \in \mathcal{B}$  satisfying  $V[a] \subseteq R(a)$ . Then there are  $U, V \in \mathcal{B}$  and a point  $a \in M$  such that  $V[b] \subseteq R(b)$  for every  $b \in U[a]$ .*

PROOF. We suppose otherwise towards a contradiction. We suppose that for every  $a \in M$  and  $U, V \in \mathcal{B}$ , there is a  $b \in U[a]$  such that  $V[b] \not\subseteq R(b)$ . Let  $(a_i : i < \omega)$  be a sequence of pairwise distinct elements of  $M$  and fix  $U \in \mathcal{B}$  such that the balls  $U[a_i]$  are pairwise disjoint. For each  $i < \omega$ , pick some  $x_{i,0} \in U[a_i]$ . Then choose  $U_1 \in \mathcal{B}$  such that  $U_1[x_{i,0}] \subseteq R(x_{i,0})$  for all  $i$ . Next pick points  $x_{i,1} \in U[a_i]$  such that  $U_1[x_{i,1}] \not\subseteq R(x_{i,1})$  and choose  $U_2 \subset U_1$  such that  $U_2[x_{i,1}] \subseteq R(x_{i,1})$  holds for all  $i$ . Iterating this, we obtain a decreasing sequence  $(U_k : 1 \leq k < \omega)$  of elements of  $\mathcal{B}$  such that for each  $i, k < \omega$ , there is a  $x_{i,k} \in U[a_i]$  such that  $U_{k+1}[x_{i,k}] \subseteq R(x_{i,k})$  but  $U_k[x_{i,k}] \not\subseteq R(x_{i,k})$ . Then the formulas

$$x \in U[a_i] \text{ and } U_{k+1}[x] \subseteq R(x) \wedge U_k[x] \not\subseteq R(x)$$

form an inp-pattern of depth 2 and contradict inp-minimality. ⊥

LEMMA 3.4. *Let  $X$  be a definable set and  $a \in M$ . Suppose that  $X \cap U[a]$  is infinite for every  $U \in \mathcal{B}$ . Then  $X$  does not divide over  $a$ .*

PROOF. Let  $\bar{b}$  be the parameters defining  $X$  and let  $\phi$  be a formula such that  $X = \phi(M, \bar{b})$ . Let  $X_{\bar{c}} = \phi(M, \bar{c})$ . Assume that  $X$  divides over  $a$ , so there is an  $a$ -indiscernible sequence  $\bar{b} = \bar{b}_0, \bar{b}_1, \dots$ , such that the intersection  $\bigcap_{i < \omega} X_{\bar{b}_i}$  is empty. We build by induction a decreasing sequence  $(U_j : j < \omega)$  of elements of  $\mathcal{B}$  such that for any  $i, j < \omega$ , the intersection  $(U_j[a] \setminus U_{j+1}[a]) \cap X_{\bar{b}_i}$  is nonempty. Suppose we have  $U_0, \dots, U_n$ . For each  $i < \omega$ , there is a point in  $U_n[a] \cap X_{\bar{b}_i}$  other than  $a$ . Thus for all  $i < \omega$  there is a  $V \in \mathcal{B}$  such that  $U_n[a] \setminus V[a]$  intersects  $X_{\bar{b}_i}$ . An application of  $\aleph_1$ -saturation gives a  $U_{n+1} \in \mathcal{B}$  such that  $U_{n+1} \subseteq U_n$  and  $U_n[a] \setminus U_{n+1}[a]$  intersects  $X_{\bar{b}_i}$  for all  $i < \omega$ . We obtain an inp-pattern of depth 2 by considering the formulas  $x \in X_{\bar{b}_i}$  and  $x \in U_j[a] \setminus U_{j+1}[a]$ . This contradicts inp-minimality.  $\dashv$

For the remainder of this section, we assume that  $\mathcal{M}$  is NIP and hence dp-minimal. We say that  $X, Y \subseteq M$  have the same germ at  $a \in M$  if there is a  $U \in \mathcal{B}$  such that  $U[a] \cap X = U[a] \cap Y$ .

LEMMA 3.5. *Let  $\phi(x; \bar{y})$  be a formula,  $x$  of sort  $M$ , and  $a \in M$ . There is a finite family  $(\bar{b}_i : i < n)$  of parameters such that for any  $\bar{b} \in M^{|\bar{y}|}$  there is an  $i < n$ , such that  $\phi(M; \bar{b})$  and  $\phi(M; \bar{b}_i)$  have the same germ at  $a$ .*

PROOF. Let  $M_0$  be a small submodel of  $\mathcal{M}$  containing  $a$ . Let  $\bar{b}_0$  and  $\bar{b}_1$  have the same type over  $M_0$ . In NIP theories, a global type does not fork over a model  $M$  if and only if it is  $M$ -invariant. The formula  $\phi(x; \bar{b}_0) \triangle \phi(x; \bar{b}_1)$  does not extend to an  $M_0$ -invariant type, therefore it forks over  $M_0$ . Since forking equals dividing over models in NIP theories, it divides over  $M$ . Lemma 3.4 shows that

$$U[a] \cap (\phi(x; \bar{b}_0) \triangle \phi(x; \bar{b}_1)) = \emptyset \quad \text{for some } U \in \mathcal{B}.$$

This means that  $\phi(x; \bar{b}_0)$  and  $\phi(x; \bar{b}_1)$  have the same germ at  $a$ . The lemma follows by  $|L|^+$ -saturation.  $\dashv$

We now assume that  $M$  admits a definable abelian group operation. We assume that the group operations are continuous and that the basic entourages are invariant under the group action, i.e.,  $U[0] + a = U[a]$  for all  $U \in \mathcal{B}$  and  $a \in M$ . We make the second assumption without loss of generality. If we only assume that the group operations are continuous then we can define an invariant uniform structure whose entourages are of the form

$$\{(x, y) \in M \times M : x - y \in U[0] \ \& \ y - x \in U[0]\} \quad \text{for } U \in \mathcal{B}.$$

We also assume that  $M$  is divisible and, more precisely, assume that for every  $U \in \mathcal{B}$  and  $n$  there is a  $V \in \mathcal{B}$  such that for all  $y \in V[0]$  there is an  $x \in U[0]$  such that  $nx = y$ . This assumption will hold if  $M$  is a field,  $+$  is the field addition, and the uniform structure on  $M$  is given by a definable order or valuation. Under these assumptions, we prove:

PROPOSITION 3.6. *Every infinite definable subset of  $M$  has nonempty interior.*

PROOF. Let  $X$  be an infinite definable subset of  $M$ . Consider the family of translates

$$\{X - b : b \in M\}.$$



By Lemma 3.5, there are finitely many elements  $(b_i : i < n)$  such that for any  $b \in M$ , there is an  $i < n$  such that  $X - b$  and  $X - b_i$  have the same germ at 0. For each  $i < n$ , let  $X_i$  be the set of  $b \in X$  such that  $X - b_i$  and  $X - b$  have the same germ at 0. Fix  $i < n$  such that  $X_i$  is infinite. Let  $b \in X_i$  and let  $V \in \mathcal{B}$  be such that  $X - b$  and  $X - b_i$  agree on  $V[0]$ . Notice that if  $c \in V[0] \cap (X_i - b)$  then  $c \in X_i - b_i$  (Why?  $X - b - c$  and  $X - b_i - c$  agree on a small neighborhood of 0, also  $X - b - c$  and  $X - b_i$  agree on a small neighborhood of 0, hence  $X - b_i - c$  and  $X - b_i$  agree on a small neighborhood of 0). Likewise, if  $c \in V[0]$  and  $c \in X_i - b_i$  then  $c \in X_i - b$ . Thus,  $X_i - b$  has the same germ at 0 as  $X_i - b_i$  for all  $b \in X_i$ . Replacing  $X$  by  $X_i$ , we may assume  $X - b$  and  $X - b'$  have the same germ at 0 for all  $b, b' \in X$ .

By Lemma 3.1,  $X$  is dense in some ball. It follows that  $X$  has no isolated points. Translating  $X$ , we may assume that  $0 \in X$ . For any  $b \in X$ , there is a  $U \in \mathcal{B}$  such that  $X$  and  $X - b$  coincide on  $U[0]$ , equivalently  $X$  and  $X + b$  coincide on  $U[b]$ . Let  $R$  be the relation given by

$$R(x, y) := (y \in X \leftrightarrow y \in X + x).$$

We have shown that for each  $b \in X$ , there is a  $U \in \mathcal{B}$  such that  $U[b] \subseteq R(b, M)$ . For the moment take  $X$  to be the distinguished home sort of  $\mathcal{M}$ . As  $X$  has no isolated points and has dp-rank 1 we can apply Lemma 3.3 to get a  $V \in \mathcal{B}$  and an open  $W \subseteq M$  such that  $X$  is dense in  $W$  and  $V[x] \subseteq R(x, M)$  for all  $x \in W \cap X$ . Translating again, we may assume that there is a  $U \in \mathcal{B}$  such that  $X$  is dense in  $U[0]$  and  $V[x] \subseteq R(x, M)$  for every  $x \in U[0] \cap X$ . Finally, we may replace both  $U$  and  $V$  by some  $U' \subseteq U \cap V$ . Hence to summarize, we have the following assumption on  $X$ :

⊠  $X$  is dense in  $U[0]$  and  $X$  and  $X - b$  coincide on  $U[0]$  for any  $b \in X \cap U[0]$ .

Pick  $V \in \mathcal{B}$  such that  $V[0] - V[0] \subseteq U[0]$ .

CLAIM: If  $g, h \in X \cap V[0]$ , then  $-h, g + h$  are in  $X$ .

PROOF OF CLAIM. Since  $0 \in X$  by assumption, it is enough to show that  $g - h \in X$ . As  $-h \in (X - h) \cap U[0]$ , by ⊠, we also have  $-h \in X \cap U[0]$ . Then from  $g \in X \cap U[0]$ , we deduce  $g \in X + h \cap U[0]$  and hence  $g - h \in X$  as required.

Suppose that the family  $\{X - b : b \in M\}$  has strictly less than  $n$  distinct germs at 0. Fix  $W \in \mathcal{B}$  such that the sum of any  $n!$  elements from  $W[0]$  falls in  $V[0]$ . ⊣

CLAIM: For any  $g \in W[0]$ , we have  $k \cdot g \in X$  for some  $k \leq n$ .

PROOF OF CLAIM. By Lemma 3.5 and choice of  $n$ , there are distinct  $k, k' < n$  such that  $X - kg$  and  $X - k'g$  have the same germ at 0. We suppose that  $k < k'$ . As  $X$  is dense in  $U[0]$ , there is an  $h \in V[0]$  such that  $h \in X - kg$  and  $h \in X - k'g$  and  $kg + h, k'g + h \in V[0]$ . As  $k'g + h, kg + h \in X \cap V[0]$ , we have  $(k' - k)g \in X$ . This proves the claim. ⊣

Applying the assumptions on  $M$ , we let  $W' \in \mathcal{B}$  be such that  $W'[0] \subseteq W[0]$  and if  $y \in W'[0]$  then there is an  $x \in W[0]$  such that  $n! \cdot x = y$ . Now pick some  $g \in W'[0]$ . Let  $h \in W[0]$  such that  $n! \cdot h = g$ . For some  $k \leq n, k \cdot h \in X$ . But then  $(n!/k)(kh) = g \in X$ , using the first claim. Hence  $W'[0] \subseteq X$ . Thus  $X$  has nonempty interior. ⊣

We leave the easy topological proof of the following corollary to the reader:

**COROLLARY 3.7.** *Any definable subset of  $M$  is the union of a definable open set and finitely many points.*

For this one can show:

**COROLLARY 3.8.** *A definable subset of  $M^n$  has dp-rank  $n$  if and only if it has nonempty interior.*

**PROOF.** Proposition 3.6 above shows that the Corollary holds in the case  $n = 1$ . Proposition 3.6 of [18] shows that the Corollary holds if  $M$  expands a dense linear order and every definable subset of  $M$  is a union of an open set and finitely many points. The proof of Proposition 3.6 of [18] goes through *mutatis mutandis* in our setting.  $\dashv$

The following Proposition is crucial in the proof that a dp-minimal valued field is henselian.

**PROPOSITION 3.9.** *Let  $X \subseteq M^n$  be a definable set with nonempty interior. Let  $f : M^n \rightarrow M^n$  be a definable finite-to-one function. Then  $f(X)$  has nonempty interior.*

**PROOF.** As finite-to-one definable functions preserve dp-rank, the proposition follows immediately from Corollary 3.8.  $\dashv$

**§4. Dp-minimal valued fields.** *Throughout this section,  $(F, v)$  is a valued field. We assume that the reader has some familiarity with valuation theory. We let  $F_v$  the residue field of  $(F, v)$ ,  $\mathcal{O}_v$  be the valuation ring and  $\mathcal{M}_v$  be the maximal ideal of  $\mathcal{O}_v$ . We denote the henselization of  $(F, v)$  by  $(F^h, v^h)$ . Given a function  $p : F^n \rightarrow F^m$  such that  $p = (p_1, \dots, p_m)$  for some  $p_1, \dots, p_m \in F[X_1, \dots, X_n]$ , we let  $J_p(a)$  be the Jacobian of  $p$  at  $a \in F^n$ .*

**LEMMA 4.1.** *Let  $p \in F[X_1, \dots, X_n]^m$  and let  $B \subseteq (F^h)^n$  be an open polydisc. Suppose that  $J_p(a) \neq 0$  for some  $a \in (B \cap F^n)$ . There is an open polydisc  $U \subseteq B$  with  $a \in U$  such that the restriction of  $p$  to  $U$  is injective.*

**PROOF.** This follows immediately from [16, Theorem 7.4].  $\dashv$

**PROPOSITION 4.2.** *Suppose that  $(F, v)$  is dp-minimal. Let  $p_1, \dots, p_n \in F[X_1, \dots, X_n]$  and  $p = (p_1, \dots, p_n)$ . If  $J_p(\bar{a}) \neq 0$  at  $\bar{a} \in F^n$  then  $p(U)$  has nonempty interior for all nonempty open neighborhoods  $U$  of  $\bar{a}$ .*

**PROOF.** This follows immediately from the previous lemma along with Proposition 3.9.  $\dashv$

The following lemma is included in the proof of the weakly o-minimal case in [13] and stated for arbitrary fields in [7]. This lemma goes back to [12].

**LEMMA 4.3.** *Let  $K$  be a field extension of  $F$ , and let  $\alpha \in K \setminus F$  be algebraic over  $F$ . Let  $\alpha = \alpha_1, \dots, \alpha_n$  be the conjugates of  $\alpha$  over  $F$ , and let  $g$  be given by:*

$$g(X_0, \dots, X_{n-1}, Y) := \prod_{i=1}^n \left( Y - \sum_{j=0}^{n-1} \alpha_i^j X_j \right).$$

*Then  $g \in F[X_0, \dots, X_{n-1}, Y]$  and there are  $G_0, \dots, G_{n-1} \in F[X_0, \dots, X_{n-1}]$  such that*

$$g(X_0, \dots, X_{n-1}, Y) = \sum_{j=0}^{n-1} G_j(X_0, \dots, X_{n-1}) Y^j + Y^n.$$



Letting  $G = (G_0, \dots, G_{n-1})$  we have:

- (1) If  $\bar{c} = (c_0, \dots, c_{n-1}) \in F^n$  and  $c_j \neq 0$  for some then  $g(\bar{c}, Y)$  has no roots in  $F$ ;
- (2) There is a  $\bar{d} = (d_0, \dots, d_{n-1}) \in F^n$  such that  $d_j \neq 0$  for some  $j$  and  $J_G(\bar{d}) \neq 0$ .

In the proposition below,  $\Gamma$  is the value group of  $(F, v)$ . In the ordered case, this proposition is a consequence of Proposition 3.6 of [7], which shows that a dp-minimal ordered field is closed in its real closure.

**PROPOSITION 4.4.** *Suppose that  $(F, v)$  is dp-minimal and let  $(F^h, v^h)$  be the henselization of  $(F, v)$ . Let  $\alpha \in F^h$  such that for any  $\gamma \in \Gamma$  there is some  $\beta \in F$  such that  $v^h(\beta - \alpha) \geq \gamma$ . Then  $\alpha \in F$ .*

**PROOF.** The proof is essentially the same as those of [13, 5.4] and [7, 3.6]. We give slightly less details. We suppose that  $\alpha$  has degree  $n$  over  $F$  and let  $g$  and  $G$  be as in Lemma 4.3. Let  $\bar{d}$  be as in (2) above. By Lemma 4.1, there is an open set  $U \subseteq F^n$  containing  $\bar{d}$  such that the restriction of  $G$  to  $U$  is injective. By Proposition 4.2,  $G(U)$  has nonempty interior. As  $J_G$  is continuous we may assume, after shrinking  $U$  if necessary, that  $J_G$  is nonzero on  $U$ . In the same manner, we may suppose that for all  $(x_0, \dots, x_{n-1}) \in U$  there is a  $j$  such that  $x_j \neq 0$ . After changing the point  $\bar{d}$  if necessary, we may also assume that  $\bar{e} := G(\bar{d})$  lies in the interior of  $G(U)$ . Let  $V$  be an open neighborhood of  $\bar{e}$  inside  $G(U)$ . We define a continuous function  $f : F^h \setminus \{0\} \rightarrow F^h$  by

$$f(y) := - \left( y^n + \sum_{j=0}^{n-2} e_j y^j \right) / y^{n-1}.$$

Thus for every  $y \neq 0$  we have:

$$y^n + f(y)y^{n-1} + \sum_{j=0}^{n-2} e_j y^j = 0.$$

We define

$$h(x) := \sum_{j=0}^{n-1} d_j x^j.$$

Then  $h(\alpha)$  is a zero of  $g(\bar{d}, Y)$ , so  $h(\alpha) \neq 0$  as  $g(\bar{d}, y)$  has no roots in  $F$ . As  $h(\alpha)$  is a zero of  $g(\bar{d}, y)$  we also have  $f(h(\alpha)) = e_{n-1}$ . If  $\beta \in F$  is sufficiently close to  $\alpha$  then  $h(\beta) \neq 0$  and

$$(e_0, \dots, e_{n-2}, (f \circ h)(\beta)) \in V.$$

There is thus a  $\bar{c} \in U$  with

$$G(\bar{c}) = (e_0, \dots, e_{n-2}, (f \circ h)(\beta)).$$

Now by our choice of  $U$ , there is a  $j$  such that  $c_j \neq 0$  and so  $g(\bar{c}, Y)$  has no root in  $F$ . On the other hand,  $h(\beta)$  is a root of  $g(\bar{c}, Y)$ . Contradiction. ⊥

**PROPOSITION 4.5.** *If  $(F, v)$  is dp-minimal then  $v$  is henselian.*

PROOF. Suppose that  $(F, v)$  is dp-minimal. Let  $\mathcal{O}_v$  be the valuation ring and  $\Gamma$  be the value group. This proof follows the proofs of [13, 5.12] and [7, 3.12]: We suppose towards a contradiction that  $v$  is not henselian. There is a polynomial

$$p(X) = X^n + aX^{n-1} + \sum_{i=0}^{n-2} c_i X^i \in \mathcal{O}_v[X],$$

such that  $v(a) = 0$ ,  $v(c_i) > 0$  for all  $i$  and such that  $p$  has no root in  $F$ . Let  $(F^h, v^h)$  be a henselization of  $(F, v)$  and take some  $\alpha \in F^h$  be such that  $p(\alpha) = 0$ ,  $v(\alpha - a) > 0$ , and  $v(p'(\alpha)) = 0$ . Consider the subset

$$S := \{v^h(b - \alpha) \in \Gamma \mid b \in F, v^h(b - \alpha) > 0\}$$

of  $\Gamma$ , and let  $\Delta$  be the convex subgroup of  $\Gamma$  generated by  $S$ . Note that Proposition 4.4 implies  $\Delta \neq \Gamma$  as otherwise we would have  $\alpha \in F$ .

CLAIM:  $S$  is cofinal in  $\Delta$ .

PROOF OF CLAIM. Identical with that of [13, Claim 5.12.1]. ⊢

Let  $w$  be the coarsening of  $v$  with value group  $\Gamma/\Delta$  and let  $w^h$  be the corresponding coarsening of  $v^h$ . As  $\Delta$  is externally definable,  $w$  is definable in the Shelah expansion of  $(F, v)$ . Then  $(F, v, w)$  is dp-minimal and so the residue field  $Fw$  of  $w$  is also dp-minimal. We let  $\bar{v}$  be the nontrivial valuation induced on  $Fw$  by  $v$ .

CLAIM: There is a  $\beta \in F$  such that  $p(\beta) \in \mathcal{M}_w$ .

PROOF OF CLAIM. By the definition of  $\Delta$ , the residue  $\alpha w^h$  is approximated arbitrarily well in the residue field  $Fw$  (with respect to the valuation  $\bar{v}$ ). We show that  $(F^h w^h, \bar{v}^h)$  is a henselization of  $(Fw, \bar{v})$ : Since  $w$  is a coarsening of  $v$ , the valued field  $(F^h, v^h)$  is an extension of the henselization of  $(F, w)$ . If these henselizations coincide,  $(Fw, \bar{v})$  is henselian by [5, 4.1.4]. If the extension is proper, [5, 4.1.4] implies once more that  $(F^h w^h, \bar{v}^h)$  is a henselization of  $(Fw, \bar{v})$ , as desired. Thus Proposition 4.4 gives  $\alpha w^h \in Fw$ . Take some  $\beta \in F$  with the same residue (with respect to  $w$ ) as  $\alpha$ . In particular,  $\beta$  is a root of the polynomial  $\bar{p}(x)$  (that is  $p(x)$  considered in  $Kw$ ), i.e., we have  $p(\beta) \in \mathcal{M}_w$ . This proves the claim. ⊢

We declare

$$J := \{b \in F \mid v(b - a) > 0\}.$$

Then, as  $\beta - \alpha \in \mathcal{M}_w \subseteq \mathcal{M}_v$  holds, we have  $\beta \in J$ .

CLAIM: For all  $b \in J$ , we have  $v(b - \alpha) = v(p(b))$ .

PROOF OF CLAIM. This is shown in the first part of the proof of [13, Claim 5.12.2]. ⊢

However, by the definition of  $\Delta$ ,  $w(p(b)) = 0$  for any  $b \in J$ . This contradicts  $p(\beta) \in \mathcal{M}_w$ , and hence finishes the proof. ⊢

**§5. Dp-minimal ordered abelian groups.** In this section,  $(\Gamma, +, \leq)$  is an  $\aleph_1$ -saturated ordered abelian group with no additional structure. In this section, we describe dp-minimal ordered abelian groups without additional structure. Let  $M$  be a first order structure expanding a linear order in a language  $L$  and suppose that  $M$  is  $|L|^+$ -saturated. Then  $M$  is weakly quasi-o-minimal if every definable subset of  $M$  is a boolean combination of convex sets and  $\emptyset$ -definable sets. This notion was introduced in [11]. We say that  $\Gamma$  is nonsingular if  $\Gamma/p\Gamma$  is finite for all primes  $p$ .

Proposition 5.27 of [2] implies that a nonsingular torsion free abelian group without additional structure is dp-minimal.

PROPOSITION 5.1. *The following are equivalent:*

- (1)  $(\Gamma, +, \leq)$  is nonsingular.
- (2)  $(\Gamma, +, \leq)$  is dp-minimal.
- (3) There is a definitional expansion of  $(\Gamma, +, \leq)$  by countably many formulas which is weakly quasi-o-minimal.

PROOF. Theorem 6.8 of [1] implies that a weakly quasi-o-minimal structure is dp-minimal so (3) implies (2). We show that (2) implies (1). Suppose that  $(\Gamma, +, \leq)$  is dp-minimal. The subgroup  $p\Gamma$  is cofinal in  $\Gamma$  for any prime  $p$ . A cofinal subgroup of a dp-minimal ordered group has finite index, see [17, Lemma 3.2]. Thus  $\Gamma$  is nonsingular.

It remains to show that (1) implies (3). Suppose that  $\Gamma$  is nonsingular. We apply the quantifier elimination for ordered abelian groups given in [4]. We use the notation of that paper. We let  $L_{qe}$  be the language described in [4]. Given an abelian group  $G$  and  $x, y \in G$ , we say that  $x \equiv_m y$  if  $x - y \in mG$ .

For  $a \in \Gamma$  and prime  $p$ , we let  $H_{a,p}$  be the largest convex subgroup of  $\Gamma$  such that  $a \notin H_{a,p} + p\Gamma$ . In  $L_{qe}$  for each  $p$ , there is an auxiliary sort  $\mathcal{S}_p = \Gamma / \sim$  where  $a \sim b$  if and only if  $H_{a,p} = H_{b,p}$ . As  $H_{a,p}$  only depends on the class of  $a$  in  $\Gamma/p\Gamma$ ,  $\mathcal{S}_p$  is finite. The other auxiliary sorts  $\mathcal{T}_p$  and  $\mathcal{T}_p^+$  parametrize convex subgroups of  $\Gamma$  defined as unions or intersections of the  $H_{a,p}$ , hence they are also finite. Given an element  $\alpha$  of an auxiliary sort we let  $\Gamma_\alpha$  be the convex subgroup of  $\Gamma$  associated to  $\alpha$ . For  $k \in \mathbb{Z}$ , we let  $k_\alpha$  be the  $k$ th multiple of the minimal positive element of  $\Gamma/\Gamma_\alpha$  if  $\Gamma/\Gamma_\alpha$  is discrete and set  $k_\alpha = 0$  otherwise. Fix  $\alpha$  and let  $\pi : \Gamma \rightarrow \Gamma/\Gamma_\alpha$  be the quotient map. Given  $\diamond \in \{=, \leq, \equiv_m\}$  and  $a, b \in \Gamma$ , we say that  $a \diamond_\alpha b$  holds if and only if  $\pi(a) \diamond \pi(b)$  holds in  $\Gamma/\Gamma_\alpha$ . For each  $\alpha$  and  $m, m' \in \mathbb{N}$   $L_{qe}$  also has a binary relation denoted by  $\equiv_{m,\alpha}^{[m']}$ . We do not define this relation here, as for our purposes, it suffices to note that the truth value of  $a \equiv_{m,\alpha}^{[m']} b$  depends only on the classes of  $a$  and  $b$  in  $\Gamma/m\Gamma$ . As the auxiliary sorts are finite, it follows from the main theorem in [4] that every definable subset of  $\Gamma^k$  is a boolean combination of sets of the form

$$\{\bar{x} \in \Gamma^k : t(\bar{x}) \diamond_\alpha t'(\bar{x}) + k_\alpha\},$$

for  $\mathbb{Z}$ -linear functions  $t, t', \alpha$  from an auxiliary sort and  $\diamond \in \{=, \leq, \equiv_m, \equiv_m^{[m']}\}$ . If  $\diamond$  is  $\equiv_m^{[m']}$  then  $k_\alpha = 0$ . We claim that  $(\Gamma, + \leq)$  admits quantifier elimination in the language  $L_{short}$  containing:

- the constant 0, the symbols  $+$  and  $-$  and the order relation  $\leq$ ;
- for each  $n$  and each class  $\bar{a} \in \Gamma/n\Gamma$ , a unary predicate  $U_{n,\bar{a}}(x)$  naming the preimage of  $\bar{a}$  in  $\Gamma$ ;
- unary predicates naming each subgroup  $H_{a,p}$ ;
- constants naming a countable submodel  $\Gamma_0$ .

Having named a countable model, we can consider that the auxiliary sorts are in our structure, by identifying each one with a finite set of constants which projects onto it. We do not need to worry about the structure on those sorts since they are finite. Consider a 2-ary relation  $x_1 \diamond_\alpha x_2 + k_\alpha$ , where  $\diamond \in \{=, <, \equiv_m\}$ ,  $k \in \mathbb{Z}$ ,  $m \in \mathbb{N}$ ,  $\alpha$  from an auxiliary sort. If the symbol is equality then this is equivalent to

$x_2 - x_1 + c \in H_{a,n}$  for an appropriate  $a, n$  and constant  $c \in \Gamma_0$  projecting onto  $k_\alpha$ . If the symbol is  $<$  then we can rewrite it as

$$[x_1 < x_2 + c] \wedge [x_2 - x_1 + c \notin H_{a,n}].$$

Finally, if the symbol is a congruence relation, then its truth value depends only on the images of  $x_1$  and  $x_2$  in  $\Gamma/m\Gamma$ , hence the formula is equivalent to a boolean combination of atomic formulas  $U_{m,\bar{a}}(x_1)$  and  $U_{m,\bar{a}}(x_2)$ . The only relations left in  $L_{qe}$  are of the form  $x \equiv_{m,\alpha}^{[m']}$   $y$ , but again their truth value depends only on the image of  $x$  and  $y$  in  $\Gamma/m\Gamma$ , so they can be replaced by  $L_{short}$  quantifier-free formulas.

Weak quasi-o-minimality follows easily from quantifier elimination in  $L_{short}$ : an inequality of terms  $t(x) \leq t'(x)$  defines a convex set, any atomic formula  $t(x) \in H_{a,p}$  for  $t$  a term defines a convex set; an atomic formula of the form  $U_{m,\bar{a}}(t(x))$  defines a  $\emptyset$ -definable set. ⊢

**§6. Dp-minimal ordered fields.** *In this section,  $F$  is an ordered field with no additional structure. We make use of the following (which also follows from [9]):*

**PROPOSITION 6.1** ([3]). *Let  $(K, v)$  be a henselian valued field of equicharacteristic zero and residue field  $k$ . Assume that  $k^\times / (k^\times)^n$  is finite for all  $n$ . Then  $(K, v)$  is dp-minimal if and only if the residue field and value group of  $(K, v)$  are dp-minimal.*

Given a field  $k$  and an ordered abelian group  $\Gamma$ ,  $k((t^\Gamma))$  is the field of Hahn series with coefficients in  $k$  and exponents in  $\Gamma$ . By the Ax-Kochen/Ersov Theorem ([15, 4.6.4]), a field  $K$  admitting a henselian valuation with residue characteristic zero, residue field  $k$  and value group  $\Gamma$  is elementarily equivalent to  $k((t^\Gamma))$ .

**THEOREM 6.2.** *Let  $F$  be an ordered field and let  $\mathcal{L}_{of} = \mathcal{L}_{ring} \cup \{\leq\}$  be the language of ordered fields. Then, the  $\mathcal{L}_{of}$ -theory of  $F$  is dp-minimal if and only if  $F \equiv \mathbb{R}((t^\Gamma))$  as ordered fields for some nonsingular ordered abelian group  $\Gamma$ .*

**PROOF.** Suppose that  $F \equiv \mathbb{R}((t^\Gamma))$  for a nonsingular ordered abelian group  $\Gamma$ . Then Proposition 5.1 and Proposition 6.1 together show that  $F$  is dp-minimal as a valued field. Note that  $(F^\times)^2 / F^\times$  is finite. Given any order on  $F$ , each coset of  $(F^\times)^2$  is composed either only of positive elements, or only of negative elements. To specify an order on  $F$ , it is enough to say for each coset of  $(F^\times)^2$ , whether it is positive or negative. Thus, there are finitely many orders on  $F$  and there are all definable (with parameters) from the field structure. Therefore  $F$  is dp-minimal as an ordered field.

Suppose that  $F$  is dp-minimal. We suppose without loss of generality that  $F$  is  $\aleph_1$ -saturated. Let  $\mathcal{O}$  be the convex hull of  $\mathbb{Z}$  in  $F$ . As  $\mathcal{O}$  is convex, it is externally definable. Thus  $(F, \mathcal{O})$  is dp-minimal. As  $\mathcal{O}$  is a valuation ring, Proposition 4.5 implies that the associated valuation is henselian. Let  $\Gamma$  be the value group of this valuation. Then  $\Gamma$  is dp-minimal hence nonsingular. The residue field is a subfield of  $\mathbb{R}$ . It follows from  $\aleph_1$ -saturation that the residue field is equal to  $\mathbb{R}$ . The Ax-Kochen/Ersov Theorem now implies  $F \equiv \mathbb{R}((t^\Gamma))$  (as valued fields). As above, we also get  $F \equiv \mathbb{R}((t^\Gamma))$  (as ordered fields). ⊢

**COROLLARY 6.3.** *Suppose that  $F$  is an ordered dp-minimal field. Then  $F$  has small absolute Galois group.*

**PROOF.** Suppose we have  $F \equiv \mathbb{R}((t^\Gamma))$  for a nonsingular  $\Gamma$ . For each prime  $p$  let  $r_p$  be the  $\mathbb{F}_p$ -dimension of  $\Gamma/p\Gamma$ . It follows from [5, Lemma 5.2.6] that the absolute

Galois group  $G_{\mathbb{R}((t^\Gamma))}$  of  $\mathbb{R}((t^\Gamma))$  is a semidirect product of the absolute Galois group  $\mathbb{Z}/2\mathbb{Z}$  of  $\mathbb{R}$  and the inertia group of the power series valuation on  $\mathbb{R}((t^\Gamma))$ . By [5, Theorem 5.3.3],  $G_{\mathbb{R}((t^\Gamma))}$  is isomorphic as a profinite group to

$$\left(\prod_p \mathbb{Z}_p^r\right) \rtimes \mathbb{Z}/2\mathbb{Z},$$

where the product is taken over all primes  $p$ . As  $G_{\mathbb{R}((t^\Gamma))}$  is small, we get that the absolute Galois group  $G_F$  of  $F$  is isomorphic to that of  $G_{\mathbb{R}((t^\Gamma))}$  and in particular also small. ⊣

To end this section, we give an explicit construction of an  $L_{\text{of}}$ -definable nontrivial valuation on a dp-minimal ordered field which is not real closed. Here  $L_{\text{of}}$  denotes the language of ordered fields, i.e.,  $L_{\text{of}} = L_{\text{ring}} \cup \{\leq\}$ . By Proposition 4.5, any such valuation is henselian.

The existence of such a valuation can be deduced from general arguments along the lines of those given by Johnson [9, 6.2], but making use of the ordering, we can give a simpler construction. We will actually prove that an ordered field is either dense in its real closure or admits a definable valuation.

**LEMMA 6.4.** *Let  $F$  be an ordered field with real closure  $R$ . Fix  $\alpha \in R$ , and suppose that for each  $\epsilon \in F$  with  $\epsilon > 0$ , there exists  $b \in F$  such that  $|\alpha - b| < \epsilon$ . Then,  $\alpha$  is in the closure of  $F$ .*

**PROOF.** We need to show that for each  $\epsilon \in R_{>0}$ , there is some  $\delta \in F_{>0}$  with  $\delta < \epsilon$ . Assume not, so there is some  $\epsilon \in R_{>0}$  such that for all  $\delta \in F_{>0}$  the inequality  $\epsilon < \delta$  holds. Then, if  $f(X) = \sum_{i \leq n} a_i X^i \in F[X]$  is irreducible we have  $a_0 \neq 0$  and  $-\delta < \sum_{0 < i \leq n} a_i \epsilon^i < \delta$  holds for all  $\delta \in F_{>0}$ . Thus,  $f(\epsilon) \neq 0$ . This contradicts the fact that  $R$  is algebraic over  $F$ . ⊣

We can now give the final results of this paper:

**PROPOSITION 6.5.** *Let  $F$  be an ordered field with real closure  $R$ . One of the following holds:*

- (1)  $F$  is dense in  $R$ ;
- (2)  $F$  admits an  $L_{\text{of}}$ -definable valuation.

**PROOF.** We suppose that  $F$  is not dense in  $R$ . Let  $\alpha$  be an element of  $R$  which is not in the closure of  $F$ . Without loss of generality, we may assume  $\alpha > 0$ . Then

$$D := \{a \in F \mid a < \alpha\}$$

is a definable subset of  $F$ . Hence,

$$A := \{y \in F_{\geq 0} \mid y + D \subseteq D\}$$

is also a definable subset of  $F$ .

**CLAIM:**  $\{0\} \subsetneq A \subsetneq F_{\geq 0}$  is a proper convex semigroup of  $F$ .

**PROOF OF CLAIM.** It is easy to see that  $A$  is convex. For any  $b \in F_{\geq 0}$  such that  $b > \alpha$ , we have  $b \notin A$ , thus we get  $A \neq F_{\geq 0}$ . Furthermore,  $A$  is closed under addition since for  $y_1, y_2 \in A$  and  $d \in D$  we have

$$(y_1 + y_2) + d = y_1 + \underbrace{(y_2 + d)}_{\in D} \in D.$$

Finally, assume  $A = \{0\}$ . Then, for all  $\epsilon \in F_{>0}$ , there is some  $d \in D$  with  $d + \epsilon > \alpha$ , hence,  $\alpha - d < \epsilon$ . Lemma 6.4 now implies that  $\alpha$  is in the closure of  $F$ , a contradiction. This proves the claim.  $\dashv$

Let  $\mathcal{O}^+ = \{a \in F : aA \subseteq A\}$  be the multiplicative stabilizer of  $A$ . It is easy to see that  $\mathcal{O} := \{a \in F : a \in \mathcal{O}^+ \text{ or } -a \in \mathcal{O}^+\}$  is a convex subring of  $F$ . For any  $b \in F_{>0}$  with  $b > a$  for all  $a \in A$  we have  $b \notin \mathcal{O}$ . Thus  $\mathcal{O}$  is nontrivial.  $\dashv$

**COROLLARY 6.6.** *Let  $F$  be a dp-minimal ordered field which is not real closed. Then, the definable valuation constructed in the proof of Proposition 6.5 is nontrivial and henselian.*

**PROOF.** Assume that  $F$  is a dp-minimal ordered field. By [7, Proposition 3.6],  $F$  is closed in its real closure  $R$ . Thus, if  $F$  is not real closed,  $F$  cannot be dense in  $R$  and thus admits a definable nontrivial valuation ring  $\mathcal{O}$  by Proposition 6.5. Now, Proposition 4.5 implies that  $\mathcal{O}$  is henselian.  $\dashv$

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INSTITUT FÜR MATHEMATISCHE LOGIK UND GRUNDLAGENFORSCHUNG  
WESTFÄLISCHE WILHELMS-UNIVERSITÄT MÜNSTER  
EINSTEINSTR. 62, 48149 MÜNSTER  
GERMANY  
*E-mail*: franziska.jahnke@uni-muenster.de

CNRS UMR 5208, INSTITUT CAMILLE JORDAN  
UNIV LYON, UNIVERSITÉ CLAUDE BERNARD LYON 1  
43 BLVD. F-69622 VILLEURBANNE CEDEX  
FRANCE  
*E-mail*: simon@math.univ-lyon1.fr

INSTITUTE OF MATHEMATICS  
THE HEBREW UNIVERSITY OF JERUSALEM  
GIVAT RAM, JERUSALEM  
91904, ISRAEL  
*E-mail*: erikw@math.ucla.edu