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COHERENCE, LOCAL INDICABILITY AND NONPOSITIVE IMMERSIONS

DANIEL T. WISE

Dept. of Math. & Stats., McGill University, Montreal, Quebec H3A 2K6, Canada
(wise@math.mcgill.ca)

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Abstract We examine 2-complexes X with the property that for any compact connected Y, and immersion $Y \to X$, either $\chi(Y) \leqslant 0$ or $\pi_1 Y = 1$. The mapping torus of an endomorphism of a free group has this property. Every irreducible 3-manifold with boundary has a spine with this property. We show that the fundamental group of any 2-complex with this property is locally indicable. We outline evidence supporting the conjecture that this property implies coherence. We connect the property to asphericity. Finally, we prove coherence for 2-complexes with a stricter form of this property. As a corollary, every one-relator group with torsion is coherent.

Keywords: coherent groups; locally indicable groups

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1. Nonpositive immersions

The goal of this paper is to examine implications of the following notion.

Definition 1.1 (Nonpositive immersions). A map $Y \to X$ is an *immersion* if it is a local injection. A 2-complex X has *nonpositive immersions* if for any immersion $Y \to X$ where Y is a compact connected 2-complex, either $\chi(Y) \leqslant 0$ or $\pi_1 Y$ is trivial.

Definition 1.2 (Variants). There are a number of variations of the above definition that primarily focus on the default case where $\chi(Y) > 0$.

We say X has *weak nonpositive immersions* if $\chi(Y) \leqslant 1$ for any immersion $Y \to X$ with Y a compact connected complex. We say X has *collapsing nonpositive immersions* if $\chi(Y) \geqslant 1$ implies that Y collapses to a point. We say X has *contracting nonpositive immersions* if $\chi(Y) \geqslant 1$ implies that Y is contractible.

We say X has *negative immersions* if there is a constant $c > 0$ such that for each immersion $Y \to X$ where Y is compact and has no free faces, either Y is a single vertex or $\chi(Y) \leqslant -c\mathsf{Area}(Y)$, where $\mathsf{Area}(Y)$ denotes the number of 2-cells in Y. A similar definition requires that $\chi(Y) \leqslant -c|Y^0|$ whenever Y is not a single vertex and has no free face or

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cutpoint. Here $|Y^0|$ denotes the number of 0-cells in Y. I do not know whether negative immersions are equivalent to the assumption that $\chi(Y) < 0$ whenever Y is nontrivial, has no free faces, and has no cutpoint.

A *tower map* is a composition of covers and subcomplexes (see Definition 2.1). In a slightly different direction, we say that X has *nonpositive towers* if for each tower map $Y \to X$ with Y compact and connected, either $\chi(Y) \leqslant 0$ or $\pi_1 Y = 1$. One can similarly refine the default case, as above. Most of the proofs function under the tower map hypothesis, and I know of no 2-complex with nonpositive towers but without nonpositive immersions.

A group G is *coherent* if every finitely generated subgroup of G is finitely presented. G is *locally indicable* if every nontrivial finitely generated subgroup of G has an infinite cyclic quotient. Our main results are the following.

Theorem 1.3. *If the 2-complex X has nonpositive immersions, then $\pi_1 X$ is locally indicable.*

Theorem 1.4. *Let X be a locally finite 2-complex with negative immersions. Then $\pi_1 X$ is coherent.*

Corollary 1.5. *Every one-relator group with torsion is coherent.*

Theorem 1.6. *If the 2-complex X has weak nonpositive immersions, then X is aspherical.*

An advantage of Theorem 1.6 over the most common method of proving asphericity is that it does not prove the strong form of asphericity known as 'diagrammatic reducibility'. It thus has the potential of organizing a route for establishing Whitehead's asphericity conjecture. In particular, we pose the following.

Conjecture 1.7. *Every contractible 2-complex has (weak) nonpositive immersions.*

Note that Whitehead's asphericity conjecture would follow from Conjecture 1.7 since if X is aspherical, then so is $\widetilde{X}$, but a subcomplex $Y \subset X$ would be covered by a component $\widehat{Y}$ of its preimage in $\widetilde{X}$, and $\widehat{Y}$ is aspherical by Theorem 1.6.

We now describe some classes of 2-complexes with nonpositive immersions: the mapping torus of any injective endomorphisms of a free group is shown to have nonpositive immersions in § 6. Their fundamental groups, which are *ascending HNN extensions* of free groups, were proven to be coherent by Feighn and Handel in [2].

In § 7, we show that a two-dimensional (2D) spine of an irreducible 3-manifold with boundary has nonpositive immersions. Fundamental groups of three-dimensional manifolds were shown to be coherent by Scott [16] and Shalen (unpublished).

In [15], we showed that any 2-complex satisfying the 'perimeter hypothesis' was coherent, and we applied this to prove that a variety of small-cancellation groups are coherent. In § 8, we describe a related class of small-cancellation complexes with nonpositive immersions. I expect that the perimeter hypothesis always implies nonpositive immersions, but it seems this will require more sophisticated counting arguments.

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I discuss several additional classes of examples with nonpositive immersions in other papers. An intriguing class consists of the 2-complexes with *nonpositive sectional curvature*, which satisfy a strong local version of the nonpositive immersion hypothesis arising from the combinatorial Gauss–Bonnet theorem [21]. This class contains a rich collection of examples including standard 2-complexes of Adian groups and canonical spines of finite volume hyperbolic 3-manifolds. Some further classes of 2-complexes with nonpositive sectional curvature are described in [23].

In [19], we examine the connection between the nonpositive immersion property and the vanishing of the second L^2-betti number of a 2-complex – an intriguing connection suggested by Gromov. Several applications are given there towards small-cancellation groups, and the method promises wider applications.

As the examples above suggest, nonpositive immersions is the unifying rubric that subsumes all known coherence results for groups of cohomological dimension 2. However, it seems overly optimistic that the nonpositive immersion property characterizes coherent groups of cohomological dimension 2:

Problem 1.8. Find a finitely generated coherent group G with $\mathsf{cd}(G) = 2$, but $\chi(G) \geqslant 1$.

Our proofs that various 2-complexes have nonpositive immersions are rather ad-hoc:

Problem 1.9. Does there exist an algorithm to recognize whether or not a compact 2-complex has nonpositive immersions?

Finally, the larger agenda of this paper is the following:

Conjecture 1.10. *Let X have nonpositive immersions. Then $\pi_1 X$ is coherent.*

This paper is a revised version of a paper that I circulated in 2003. In the original version, I gave an incorrect proof of Conjecture 1.10, and I am grateful to Mladen Bestvina for finding a flaw in my proof. I continue to believe the conjecture is correct. The immediate motivation of Conjecture 1.10 was to provide a proof of Baumslag's conjecture that every one-relator group is coherent, by coupling it with a conjecture about nonpositive immersions for 2-complexes with a single 2-cell. The latter conjecture was proven recently by Louder–Wilton and Helfer–Wise [5, 11] using methods that ultimately rely on orderability. The proof of Corollary 1.5 is a consequence of combining Theorem 1.4 with the strong form of nonpositive immersions that was obtained for one-relator groups. I was initially drawn to this subject by Baumslag's conjecture, and my initial work with McCammond proved that $\langle a, b \mid W^n \rangle$ is coherent when $n \geqslant |W|$. I am hopeful the subject will attract further investigators, both to broaden the class of examples, and to understand further ramifications of the definitions.

2. Background on towers and disc diagrams

2.1. Towers

We recall here some background on towers which is due to Howie [6]. A map $X \to Y$ between CW-complexes is *combinatorial* provided that its restriction to each open cell of

X is a homeomorphism onto an open cell of Y. A CW-complex is *combinatorial* provided that the attaching map of each of its cells is a combinatorial map (after a suitable subdivision). Unless otherwise indicated, the spaces in this paper will be 2D combinatorial complexes, and the maps between these spaces will be cellular.

Definition 2.1 (Tower). A map $A \to B$ of connected CW-complexes is a *tower* provided that it can be expressed as the following composition where the maps are alternately inclusions of subcomplexes and covering maps.

$$A = B_n \hookrightarrow \widehat{B}_{n-1} \to B_{n-1} \hookrightarrow \cdots \hookrightarrow \widehat{B}_2 \to B_2 \hookrightarrow \widehat{B}_1 \to B_1 = B.$$

Let $C \to B$ be a map of connected CW-complexes. A map $C \to A$ is a *tower lift* of $C \to B$ if there is a tower $A \to B$ such that the following diagram commutes:

$$\begin{array}{ccc} & & A \\ & \nearrow & \downarrow \\ C & \to & B \end{array}$$

The tower lift $C \to A$ is *maximal* if, for any tower lift $C \to D$ of $C \to A$, the map $D \to A$ is an isomorphism.

There is a more restrictive notion of *cyclic tower* which is the composition of subcomplexes and regular covers with infinite cyclic covering groups. We define *cyclic tower lifts* and *maximal cyclic tower lifts* analogously.

The following was proven by Howie for combinatorial maps in [6].

Lemma 2.2. *Let S be a compact connected CW-complex and $S \to K$ be a cellular map. Then $S \to K$ has a maximal [cyclic] tower lift.*

Proof. Our proof is similar to Howie's but uses the following notion of complexity: we let $\mathsf{Cells}(X)$ denote the number of cells in a complex X. Let M be an upper bound on the number of cells in the image of any maximal tower lift of $C \to K$, where C varies over the closed cells of S. Let $K_0 = K$ and let $S \to K_{i+1}$ be a surjective [cyclic] tower lift of $S \to K_i$ for each $i \geqslant 0$. Observe that $\mathsf{Cells}(K_i) \leqslant M \cdot \mathsf{Cells}(S)$. Furthermore, if $K_{i+1} \to K_i$ is not an isomorphism, then $\mathsf{Cells}(K_{i+1}) > \mathsf{Cells}(K_i)$. Therefore, the number of times that $K_{i+1} \to K_i$ fails to be an isomorphism is bounded by $M \cdot \mathsf{Cells}(S)$. Consequently, a maximal [cyclic] tower lift exists. □

2.2. Disc diagrams

We now briefly summarize the basic definitions concerning disc diagrams. We refer the reader to [4, 12, 14] for more detailed accounts.

Definition 2.3. A *disc diagram* D is a compact contractible 2-complex with a fixed embedding in the 2-sphere S^2. The *boundary cycle* of D is the attaching map of the open 2-cell $S^2 - D$ so that S^2 is a 2-complex. Choosing a starting vertex and an orientation, we can regard it as a closed path $\partial_{\mathsf{p}} D$ called the *boundary path* of D. Observe that $\partial_{\mathsf{p}} D \to \partial D$

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is surjective, and $\partial_{\mathsf{p}} D$ traverses a 1-cell e of ∂D once if e lies on the boundary of a 2-cell and twice otherwise.

Let $P \to X$ be a closed path factoring as $P \cong \partial_{\mathsf{p}} D \to D \to X$ where D is a disc diagram. We then say $D \to X$ is a *disc diagram for* $P \to X$. A theorem of van Kampen's asserts that a disc diagram $D \to X$ exists for each null-homotopic closed path $P \to X$.

Let $D \to X$ be a map from a disc diagram to a 2-complex. A *cancellable pair* consists of a pair of distinct 2-cells R_1, R_2 in D with the following property: there is a 1-cell e in D that is traversed by the boundary paths $\partial_{\mathsf{p}} R_1, \partial_{\mathsf{p}} R_2$ of two 2-cells, so $\partial_{\mathsf{p}} R_i = eA_i$ for each i, and eA_1, eA_2 project to the same path in X. The map $D \to X$ is *reduced* if it has no cancellable pair. If there is a cancellable pair, then one can remove the open cells $R_1 \cup e \cup R_2$ and glue to obtain a smaller area diagram. Considering disc diagrams with a minimal number of 2-cells, we see that every null-homotopic path $P \to X$ actually bounds a reduced disc diagram.

3. Local indicability and asphericity

Lemma 3.1. *Let H be a finitely generated group. There exists a finitely presented group K and a surjection $K \twoheadrightarrow H$ such that the map $\mathsf{H}_1(K) \to \mathsf{H}_1(H)$ is an isomorphism.*

Proof. Let H be generated by $\langle h_1, \dots, h_r \rangle$. Since every finitely generated abelian group is finitely presented, we can choose a finite presentation $\langle h_1, \dots, h_r \mid R_1, \dots, R_s, [h_i, h_j] : i < j \rangle$ for the abelianization $\mathsf{H}_1(H)$ of H in terms of the original generators of H. For each i we have $R_i =_H W_i$, where W_i is a product of commutators. Regard $R_i(h_1, \dots, h_r)$ and $W_i(h_1, \dots, h_r)$ as words in $h_i^{\pm 1}$, and rewrite these in terms of new generators $\{k_1, \dots, k_r\}$ to obtain a group K presented by:

$$\langle k_1, \dots, k_r \mid R_1(k_1, \dots, k_r) = W_1(k_1, \dots, k_r), \dots, R_s(k_1, \dots, k_r) = W_s(k_1, \dots, k_r) \rangle.$$

There is an obvious surjection $K \twoheadrightarrow H$ induced by $k_i \mapsto h_i$, and it induces an isomorphism $\mathsf{H}_1(K) \to \mathsf{H}_1(H)$ by construction. □

Definition 3.2. The 2-complex X has *nonpositive cyclic towers* if for each cyclic tower map $Y \to X$ with Y compact and connected, either $\chi(Y) \leqslant 0$ or $\pi_1 Y = 1$.

Theorem 3.3. *Let X have nonpositive cyclic towers. Then $\pi_1 X$ is locally indicable.*

Note that Theorem 3.3 implies Theorem 1.3 since nonpositive immersions implies nonpositive cyclic towers, and likewise Theorem 3.4 implies Theorem 1.6.

We use the notation β_i for the ith betti number.

Proof. Let H be a finitely generated subgroup of $\pi_1 X$ with $\beta_1(H) = 0$. By Lemma 3.1, there is a finitely presented group K with $\beta_1(K) = 0$ and a surjection $K \twoheadrightarrow H$.

Let Y be the standard 2-complex of K, and let $Y \to X$ be a based cellular map such that $\pi_1 Y$ maps surjectively to H. Let $Y \to T$ be a maximal tower lift of $Y \to X$, as indicated by the following diagram:

$$\begin{array}{ccc} & & T \\ & \nearrow & \downarrow \\ Y & \to & X \end{array}$$

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By maximality of the tower lift, $Y \to T$ is surjective, and hence T is compact. Also, by maximality of the tower lift, $Y \to T$ is π_1-surjective, and hence $\pi_1 T$ maps surjectively to H. Since $\pi_1 Y \to \pi_1 T$ is surjective, we see that $\beta_1(T) = 0$. But then $\chi(T) \geqslant 1 - \beta_1(T) \geqslant 1$, so $\chi(T) \geqslant 1$. But then $\pi_1 T$ is trivial by Definition 1.1, and hence H is trivial since the surjection $K \twoheadrightarrow H$ factors through the trivial group $\pi_1 T$. □

We close this section by noting that while the 2-sphere has nonpositive immersions, in practice, 2-complexes with nonpositive immersions tend to be aspherical because of the following:

Theorem 3.4. *Suppose that $\chi(Y) \leqslant 1$ for any cyclic tower map $Y \to X$ with Y compact and connected. Then X is aspherical.*

Proof. Let $S \to X$ be a cellular map of a sphere. Let $S \to T$ be a maximal tower lift. Since $\mathsf{H}^1(T) = 0$ we have $\beta_1(T) = 0$. Thus $\beta_2(T) = 0$ since $1 - \beta_1(T) + \beta_2(T) = \chi(T) \leqslant 1$, and hence $\mathsf{H}_2(T) = 0$. By Hurewicz's theorem we have $\pi_2 T = \mathsf{H}_2(T) = 0$. Therefore, $S \to T$ and hence $S \to X$ is null-homotopic. □

4. Negative immersions and coherence

The goal of this section is to prove coherence in the case of negative immersions. While the result is broader, the proof closely follows the argument proving coherence for negative generalized sectional curvature. We borrow an additional idea from the proof of the coherence of 3-manifold groups to easily ensure our immersions have no free faces or isolated edges.

A 1-cell of a 2-complex is *isolated* if it is not traversed by the boundary path of any 2-cell.

Lemma 4.1. *Let $Y \to X$ be a combinatorial map of connected 2-complexes that is not π_1-injective. Then there is a complex Y^+, and an immersion $Y^+ \to X$, such that $Y \to X$ factors as $Y \to Y^+ \to X$ and the following hold:*

1. *$Y \to Y^+$ is π_1-surjective but not π_1-injective;*
2. *all but finitely many cells of Y^+ are in the image of $Y \to Y^+$;*
3. *each isolated 1-cell of Y^+ is the image of an isolated 1-cell of Y;*
4. *each free face of Y^+ is the image of a free face or isolated 1-cell of Y.*

Proof. By the failure of π_1-injectivity, we may choose a closed immersed path $P \to Y$ whose image in X is null-homotopic. Let $D \to X$ be a reduced disc diagram whose boundary path is P. Form the complex $Y \cup_P D$ which is obtained by gluing Y and D together along the images of edges of P. The induced map $Y \to Y \cup_P D$ is not π_1-injective since $P \to Y \cup_P D$ is null-homotopic.

The map $Y \to Y \cup_P D$ is π_1-surjective is proven inductively as follows: by ignoring all 1-cells of D that do not lie in ∂D, we may pretend that D is a disc diagram D' obtained from (larger) 2-cells that are glued together with each other and with isolated 1-cells along their boundaries. Each arc of isolated 1-cells of D' is associated to a pair P_i, P_i' of

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subpaths of P mapping to the same path in Y. We choose a pair of P that are innermost in the sense that P has a subpath $P_i Q P_i^{-1}$ where Q does not contain a paired path, so Q bounds a single 2-cell D_i of D. We glue D_i to Y along Q after identifying its endpoints, and then fold P_i and P_i' to obtain a complex $\bar{Y}$. Note that the map $Y \to \bar{Y}$ is π_1-surjective since D_i kills the new element obtained by identifying the endpoints of Q, and the folding is π_1-surjective. Finally, note that $\bar{Y}$ bounds a diagram containing fewer pretend 2-cells. In the base case where D is nonsingular, π_1-surjectivity is immediate since the map between 1-skeleta is an isomorphism.

Let $Y^+ \to X$ be the top of a maximal tower lift of $Y \cup_P D \to X$. Note that the map $Y \to Y^+$ is π_1-surjective since $Y \cup_P D \to X$, and not π_1-injective since Y factors through $Y \cup_P D$.

Finally, note that under a combinatorial map $A \to B$, the preimage of an isolated 1-cell consists of isolated 1-cells, so this holds for the map $Y \cup_P D \to Y^+$. But all isolated 1-cells of $Y \cup_P D$ must lie in the image of Y. Let e be an edge of Y^+ that is a free face of a 2-cell R. Then for any edge $\hat{e}$ in the preimage of e, either $\hat{e}$ is isolated or a free face, or $\hat{e}$ lies in the boundary of two 2-cells $\hat{R}$, $\hat{R}'$ that fold together along $\hat{e}$ as they map to R. For the map $Y \cup_P D \to Y^+$, no such 1-cell can lie in the interior of D, since we have assumed D is reduced. Thus all free face 1-cells arise from the image of 1-cells of Y, which must themselves be isolated or free faces since $Y \to X$ is an immersion. □

The following is repeated from [21].

Lemma 4.2 (No self-immersion). *Let $\phi\colon Z \to Z$ be a combinatorial immersion where Z is a compact connected n-complex. Then ϕ is an isomorphism.*

Proof. For $n \geqslant 0$, let $Z_n = \phi^n(Z)$ and observe that for each n, ϕ restricts to a map $\phi\colon Z_n \to Z_{n+1}$. Since Z has finitely many subcomplexes, for some $q > p \geqslant 1$ we have $Z_q = Z_p$, and so the restriction of ϕ^{q-p} to Z_p is an isomorphism onto Z_p. Furthermore, as Z_p is finite the group $\mathrm{Aut}(Z_p)$ is finite, so for some n, the restriction of $(\phi^{(q-p)})^n$ to Z_p is the identity. It follows that $\phi^{n(q-p)}$ is a retraction of Z onto the subcomplex Z_p.

If ϕ is not an isomorphism, then since Z is finite and ϕ is combinatorial, ϕ is not surjective and so $Z_p = \phi^p(Z)$ is a proper subcomplex of Z. Therefore, since Z is connected, some cell c in $Z - Z_p$ is adjacent to a 0-cell v of Z_p. Since both c and $\phi^{n(q-p)}(c)$ are adjacent to $v = \phi^{n(q-p)}(v)$ we see that $\phi^{n(q-p)}$ is not an immersion at v, which contradicts the hypothesis that ϕ is an immersion. □

The following consequence of Grushko's theorem is a result arising in the proof of the coherence of 3-manifolds [16, 18]. A group is *indecomposable* if it is not isomorphic to the free product of two nontrivial groups.

Lemma 4.3 (Indecomposable). *Let H be a finitely generated indecomposable group. Suppose each subgroup J generated by fewer elements than H is finitely presented. Then there is a finitely presented group K and a surjective homomorphism $K \to H$ with the following property: whenever $K \to H$ factors as $K \to L \to H$ with $K \to L$ surjective, the group L is indecomposable.*

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Lemma 4.4 (Boundedly many immersions). *Let X be a locally finite 2-complex. For each $A, B \geqslant 0$ there exists $f(A, B) \geqslant 0$ such that the following holds: there are at most $f(A, B)$ distinct compressed based immersions $Y \to X$ such that* $\mathsf{Area}(Y) \leqslant A$ *and Y has at most B isolated 1-cells.*

Note that we regard $Y_1 \to X$ and $Y_2 \to X$ as *distinct* if there is no isomorphism $Y_1 \to Y_2$ such that $Y_1 \to X$ factors as $Y_1 \to Y_2 \to X$.

Proof. Each 0-cell and nonisolated 1-cell of Y lies on a 2-cell of Y, so Y can be built by gluing together at most A 2-cells and B 1-cells mapping to X. By local finiteness, there is an upper bound C on the number of 1-cells and 2-cells of X, that can be reached by a path which is the concatenation of at most $A + B$ subpaths, each of which is a single edge or lies on a single 2-cell of X. Hence the 2-cells of Y map to at most C possible 2-cells of X, and the isolated 1-cells map to at most C possible 1-cells of X. Finally, there is a uniform number of ways of gluing these 2-cells together (depending on their sides lengths). And in particular, there are finitely many ways of gluing them together so that there is an induced map to X. □

The following was already asserted in [21, § 14] as a generalization of the negative sectional curvature case:

Theorem 4.5. *Let X be a locally finite 2-complex with negative immersions. Then $\pi_1 X$ is coherent.*

Proof. Let $H \subset \pi_1 X$ be a finitely generated subgroup, and let $\widehat{X} \to X$ be the based cover with $\pi_1 \widehat{X} = H$. The proof is by induction on the minimal number of generators of H. In the base case, H is finitely presented since it is trivial. Suppose H is generated by n elements, and suppose each finitely generated subgroup with fewer than n elements is finitely presented. We may assume that H is indecomposable, since if $H \cong A * B$, then A and B are finitely presented so H is finitely presented. By Lemma 4.3, there is a finitely presented group K and a surjective homomorphism $K \to H$ such that whenever $K \to H$ factors as $K \to L \to H$ with $K \to L$ surjective, the group L is indecomposable. Let Z be a compact connected based 2-complex with $\pi_1 Z \cong K$, and consider a based cellular map $Z \to \widehat{X}$ such that $\pi_1 Z \to \pi_1 \widehat{X}$ induces the homomorphism $K \to H$. By Lemma 2.2, there is a lift $Z \to T$ of $Z \to X$ to a maximal tower $T \to \widehat{X}$. Note that $T \to \widehat{X}$ is a π_1-surjective immersion, and T is compact.

We claim that T contains a subcomplex Y_1 such that Y_1 has no free faces or isolated 1-cells, and $Y_1 \to \widehat{X}$ is π_1-surjective using a different basepoint. Indeed, let Y_1 be obtained from T by collapsing free faces, and note that we allow the collapse of a free face at the basepoint, which merely results in a conjugation of the image by the corresponding edge in the fundamental groupoid. We now use the property of K to see that by possibly passing to a proper subcomplex, we may also assume that Y_1 has no isolated 1-cell e. If e separates Y_1, then let $Y_1 - e = A \sqcup B$, and observe that $\pi_1 Y_1 \cong \pi_1 A * \pi_1 B$, and hence since $K \to \pi_1 Y_1$ is surjective, we see that either $\pi_1 A$ or $\pi_1 B$ is trivial, and we may pass to a smaller subcomplex that is still π_1-surjective. In the nonseparating case, $Y_1 = C \cup e$, and $\pi_1 Y_1 \cong C * \mathbb{Z}$, and hence $\pi_1 C$ must be trivial, and this violates our assumption that H is not cyclic or trivial.

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We will construct below a sequence $Y_1 \to Y_2 \to Y_3 \to \cdots$, where each Y_i is a compact complex with a π_1-surjective map $Y_i \to \widehat{X}$. We will show that this sequence must terminate at a complex Y_t such that $\pi_1 Y_t \to \pi_1 \widehat{X}$ is π_1-injective, and so H is finitely presented.

For each i, either $Y_i \to \widehat{X}$ is π_1-injective, or else, by Lemma 4.1 we can form a new based map $Y_{i+1} \to \widehat{X}$ and a map $Y_i \to Y_{i+1}$ such that $Y_i \to \widehat{X}$ factors as $Y_i \to Y_{i+1} \to \widehat{X}$, and $\pi_1 Y_i \to \pi_1 Y_{i+1}$ is surjective but not injective. Lemma 4.1 also ensures that each isolated edge of Y_{i+1} is the image of an isolated edge of Y_i, and each free face of Y_{i+1} is the image of an isolated edge or free face of Y_i. Since Y_1 has this property, we see that no Y_i has an isolated edge or free face.

By negative immersions, $\mathsf{Area}(Y_i) \leqslant -k\chi(Y_i)$. By π_1-surjectivity, $-\chi(Y_i) < b_1(Y_i) \leqslant b_1(Y_1)$ for each i. Hence, $\mathsf{Area}(Y_i) < kb_1(Y_1)$ is uniformly bounded.

By Lemma 4.4, there exists $i < j$ such that $Y_i \to \widehat{X}$ and $Y_j \to \widehat{X}$ are the same. However, the map $Y_i \to \widehat{X}$ factors as $Y_i \to Y_j \to \widehat{X}$, and the map $Y_i \to Y_j$ is not π_1-injective since the map $Y_i \to Y_{i+1}$ is not. There is thus an immersion $Y_i \to Y_i$ with Y_i compact, that is not an isomorphism since it is not π_1-injective. This is impossible by Lemma 4.2. □

5. Negative immersions and cyclometered complexes

In [21], we studied angled 2-complexes, and specifically examined *nonpositive* and *negative generalized sectional curvature*. These notions imply nonpositive and negative immersions, respectively, and we refer to [21] for a variety of examples.

Two stronger statements are proven there for negative generalized sectional curvature: first, there is a compact core for a finitely generated subgroup of $\pi_1 X$. Second, in the additional presence of a CAT(0) metric, one actually obtains local quasiconvexity. In [13] this is generalized to a setting that allows a proper action, which is quite natural since many examples arise from presentations whose relators are proper powers.

Conjecture 5.1. *Let X be a compact 2-complex with negative immersions. Then $\pi_1 X$ is a locally quasiconvex hyperbolic group.*

Even the hyperbolicity is not yet known. The conjecture is unknown in the stronger setting of negative generalized sectional curvature, unless all angles are nonnegative in which case hyperbolicity is a consequence of the Gersten–Pride weight test [4]. Some progress towards hyperbolicity is given in [3], where it was shown for many 2-complexes where all attaching maps of 2-cells are proper powers.

The connection between 'negative immersions' and torsion appears in [22, Remark 5.2], and though I had already understood that the coherence conclusion generalizes from the negative sectional curvature to the negative immersion framework [21, § 14], this did not seem to be the best route then since I thought that I had proven Conjecture 1.10. The local quasiconvexity assertion of Conjecture 5.1 remains an important objective.

Definition 5.2 (Counting w-cycles). Let $A \to B$ be an immersion of directed graphs. Let $w \to B$ be an immersed aperiodic cycle. Let $A \otimes w$ denote the *fibre-product* of $A \to B$ and $w \to B$. Its vertices are pairs (u, v) of vertices $u \in A$ and $v \in w$ that map to the

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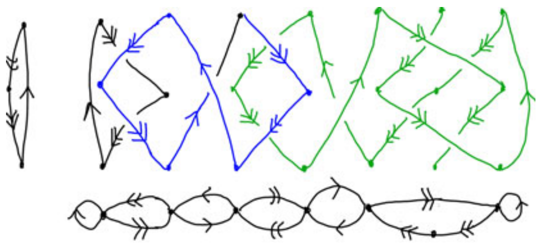

Figure 1. There are three components in $A \otimes w$, and $A \hat{\otimes} w$ contains the two circular components. They cover w with degrees 1 and 2.

same vertex in B. Its edges are pairs (d, e) where $d \in \text{Edges}(A)$ and $e \in \text{Edges}(w)$, and the initial and terminal vertices are $\iota(d, e) = (\iota(d), \iota(e))$ and $\tau(d, e) = (\tau(d), \tau(e))$. Note that there are immersions $A \otimes w \to A$ and $A \otimes w \to w$ that are induced by ignoring the right or left factor. Indeed, one can interpret $A \otimes w$ as a subspace of $A \times w$ and these are restrictions of projection maps. See Figure 1.

Let $A \hat{\otimes} w$ be the subgraph of $A \otimes w$ consisting of the union of components that are covers of w. Let $\#_w(A) = \deg(A \hat{\otimes} w \to w)$. For a set $\{w_i \to B\}$ of immersed cycles and an immersion $A \to B$, we let $\#(A) = \sum_i \#_{w_i}(A)$.

The strategy advocated in [22] for verifying nonpositive immersions focuses on the following stronger property:

Definition 5.3 (Cyclometered). Let $\{w_i \to B\}$ be a set of immersed circles in a graph. We say it is *cyclometered* if the following holds: let $A \to B$ be an immersion, and suppose each 1-cell e of A is either isolated or traversed at least twice by w-cycles, in the sense that e has at least two preimages under the map $\cup_i A \hat{\otimes} w_i \to A$. Then $\#(A) < \beta_1(A)$.

The 2-complex X is *cyclometered* if its set of 2-cell attaching maps $\{w_i \to X^1\}$ is cyclometered.

Remark 5.4. It is easy to check that X is cyclometered when X is an angled 2-complex with nonpositive generalized sectional curvature. It is also easy to check that any finite complex X has a finite branched cover that is cyclometered thus yielding another proof that $\langle a_1, a_2, \ldots \mid w_1^{n_1}, \ldots, w_m^{n_m} \rangle$ is coherent for n_i sufficiently large. In [5, Theorem 6.1], it is shown that X is cyclometered when X is 'bi-slim' which is a technical generalization of the notion of 'staggered' described in Definition 5.6. There are examples of 2-complexes with nonpositive immersions that are not cyclometered. One such example is the 2-complex associated to $\langle a, b \mid abb, a \rangle$.

Lemma 5.5. *Let X be a cyclometered 2-complex. Let $\dot{X} \to X$ be a branched cover, where the branching points are the centres of 2-cells, and where each branching degree is $\geqslant 2$. Then $\dot{X}$ has negative immersions.*

In practice, $\dot{X}$ arises from X as follows: first form a complex X' by replacing each 2-cell of X with attaching map w_i by a 2-cell with attaching map $w_i^{n_i}$ for some $n_i \geqslant 2$.

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Then pass to a finite regular cover $\widehat{X}'$ where each w_i has order n_i in $\mathsf{Aut}(\widehat{X}')$. Finally, we obtain $\dot{X}$ from $\widehat{X}'$, by either a quotient map identifying *equivalent* 2-cells having the same boundary cycle, or by taking a subcomplex where all but one 2-cell in each equivalence class is discarded.

Proof. Let $Y \to \dot{X}$ be an immersion with Y compact, collapsed, and connected. It suffices to consider the case where Y has no isolated edge, since $\chi(Y) \leqslant \sum \chi(Y_i)$ where removing arcs of isolated edges results in the disjoint union of components $\sqcup Y_i$. The induced map $Y^1 \to \dot{X}$ is also an immersion, and each w-cycle in Y^1 with respect to $\dot{X}$ is a w-cycle with respect to X whose multiplicity is the minimal branch degree n. As X is cyclometered, we have: $n\mathsf{Area}(Y) < \beta_1(Y^1)$. Consequently:

$$\chi(Y) = (1 - \beta_1(Y^1)) + \mathsf{Area}(Y) \leqslant (-n+1)\mathsf{Area}(Y) \leqslant -\mathsf{Area}(Y). \qquad \square$$

Definition 5.6. A 2-complex is *staggered* if there is a total ordering on its 1-cells, and a total ordering on its 2-cells, so that for 2-cells $\alpha < \beta$, we have $\mathsf{max}(\alpha) < \mathsf{max}(\beta)$ and $\mathsf{min}(\alpha) < \mathsf{min}(\beta)$, where $\mathsf{max}(\alpha)$ and $\mathsf{min}(\alpha)$ denote the highest and lowest 1-cells traversed by the boundary path of α. A group is *staggered* if it is the fundamental group of a staggered 2-complex. We use the term *staggered with torsion* to indicate that all attaching maps of 2-cells are proper powers, and we likewise use this term for a group with a staggered presentation and relators that are proper powers. Finally, we note that the order of a relator in the group is equal to the power. We refer to [8, 9, 12].

Theorem 5.7. *Every virtually torsion-free staggered group with torsion is coherent. In particular, every one-relator group with torsion is coherent.*

Note that one-relator groups with torsion are virtually torsion-free [12], and finitely presented staggered groups with torsion are residually finite and hence virtually torsion-free [20].

Proof. Consider the staggered presentation with torsion $\langle a_1, a_2, \ldots \mid w_1^{n_1}, w_2^{n_2}, \ldots\rangle$ for $\dot{G}$. Let G be the associated group without torsion, so G is presented by $\langle a_1, a_2, \ldots \mid w_1, w_2, \ldots\rangle$ where no w_j is a proper power. Let X be the 2-complex associated to the presentation for G. Let $\dot{X} \to X$ be a finite branched cover of X corresponding to a finite index normal subgroup of $\dot{G}$ with the property that each w_j has order n_j in the associated quotient. Finally, $\dot{X}$ has negative immersions by Lemma 5.5. Thus coherence holds by Theorem 4.5. $\square$

6. Ascending HNN extensions of free groups have nonpositive immersions

Let F be the free group $\langle a_i : i \in I\rangle$, and let $\{A_i : i \in I\}$ be a set of words in $a_i^{\pm 1}$ which freely generate a subgroup. Let $\phi : F \to F$ be the monomorphism induced by $\phi(a_i) = A_i$. Let X be the standard 2-complex of the presentation $\langle t, a_1, \ldots \mid a_i^t = \phi(a_i) : i \in I\rangle$ for the ascending HNN extension associated to ϕ. In this section we show:

Theorem 6.1. *For any immersion $Y \to X$ with Y compact and connected, either $\chi(Y) \leqslant 0$ or Y collapses to a 0-cell.*

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Proof. Let $Y \to X$ be an immersion with Y compact and connected. Since collapsing along free faces does not change χ, it is sufficient to prove the theorem in the case that Y has no free faces. We first prove the theorem under the additional assumption that Y has no isolated 1-cell.

The HNN decomposition naturally corresponds to a graph of spaces structure on X, induced by a map $X \to \Gamma_X$, where in this case Γ_X is just a circle corresponding to the stable letter t.

The decomposition of $X \to \Gamma_X$ as a graph of spaces induces a decomposition $Y \to \Gamma_Y$ as a graph of spaces. Specifically, the vertex spaces of Y are the components of the preimage of the vertex space of X, and the edge spaces of Y correspond to the components of the preimage of the edge spaces of X. The graph Γ_Y is the quotient of Y obtained by identifying vertex spaces Y_v of Y to vertices $v \in \Gamma_Y$, and edge spaces $Y_e \times (0, 1)$ of Y to edges $e \in \Gamma_Y$. Consider the following formula where $\iota(e)$ and $\tau(e)$ denotes the initial and terminal vertices of the directed edge e:

$$\chi(Y) = \sum_{v\in\Gamma} \chi(Y_v) - \sum_{e\in\Gamma} \chi(Y_e) = \sum_{v\in\Gamma} \left[\chi(Y_v) - \sum_{\iota(e)=v} \chi(Y_e)\right].$$

Thus, to see that $\chi(Y) \leqslant 0$, it suffices to verify the following for each $v \in \Gamma_Y$

$$\chi(Y_v) - \sum_{\iota(e)=v} \chi(Y_e) \leqslant 0.$$

It is easy to see that for any subgraph B of a compact connected graph C with $\chi(C) \leqslant 0$, we have $\chi(C) - \chi(B) \leqslant 0$, so it is sufficient to check that $\chi(Y_v) \leqslant 0$. However, we will show that if Y_v is a tree then Y_v is all of Y and consists of a single 0-cell. Indeed, it follows that Y_e is a tree for each e with $\iota(e) = v$ or $\tau(e) = v$. This is because in each case the map $Y_e \to Y_v$ is π_1-injective since it projects to a map $X_e \to X_v$ which is π_1-injective. But if Y_e is a tree, then the corresponding edge space $Y_e \times I \subset Y$ either contains free faces corresponding to the leaves of Y_e, or consists of an isolated 1-cell if Y_e consists of a single 0-cell. It follows that there are no edge spaces attached to Y_v, and so $Y = Y_v$ by connectedness. Finally, Y is a tree with no free faces and therefore consists of a single 0-cell as claimed.

We now prove the general case by induction on the number of isolated 1-cells in Y. Suppose that Y has an isolated 1-cell e, then either e does not separate and $Y = Y_0 \cup e$, or e separates and $Y = Y_1 \cup e \cup Y_2$. In the former case, by induction on the number of isolated 1-cells, we can assume that $\chi(Y_0) \leqslant 1$, and so $\chi(Y) \leqslant 0$. In the latter case, again by induction on the number of isolated 1-cells, the theorem holds for Y_1 and Y_2, and hence $\chi(Y) \leqslant \chi(Y_1) - 1 + \chi(Y_2) \leqslant 1$ with equality if and only if both Y_1 and Y_2 are 0-cells, and hence Y collapses to a point. □

7. 3-manifold spines

In this section, we explain that if G is the fundamental group of a compact irreducible 3-manifold with nonempty boundary then $G \cong \pi_1 X$ where X is a 2-complex with nonpositive immersions. We note that fundamental groups of 3-manifolds are

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coherent [16], and are locally indicable provided that all the rational homology spheres in their prime decomposition are simply connected [7, § 6].

Lemma 7.1. *Let N be a compact connected aspherical* 3*-manifold with nonempty boundary. Then either $\pi_1 N$ is trivial, or $\chi(N) \leqslant 0$.*

Proof. Since $\bar{N} = N \cup_{\partial N} N$ is a closed 3-manifold, we have $2\chi(N) - \chi(\partial N) = \chi(\bar{N}) = 0$, and hence $\chi(N) = \frac{1}{2}\chi(\partial N)$. In particular, if no component of ∂N is a 2-sphere, then $\chi(N) \leqslant 0$. If some component S of ∂N is a 2-sphere, then $\pi_1 N$ is trivial. For if there is more than one component of the preimage of S in the universal cover $\widetilde{N}$, then each of these represents a nontrivial 2-cycle, and so $\widetilde{N}$ is not aspherical. Therefore, $\pi_1 N = 1$. □

A 2-complex X is *diagrammatically reducible* if there does not exist a combinatorial map $S \to X$ which is an immersion on $S - S^0$, where S is a 2-complex homeomorphic with the 2-sphere. This notion was introduced by Sieradski [17] and further studied by Gersten [4]. Note that a diagrammatically reducible 2-complex X is aspherical, and that if $Y \to X$ is an immersion then Y is also diagrammatically reducible.

Theorem 7.2. *Let M be a compact irreducible* 3*-manifold with nonempty boundary. Then M has a 2D spine with nonpositive immersions.*

Proof. It was shown in [1] that M has a strong deformation retraction onto a 2D spine X that is diagrammatically reducible.

Let $R(X)$ be a closed regular neighbourhood of X in M. Since $R(X)$ deformation retracts to the aspherical 2-complex X, we see that $R(X)$ is aspherical.

Consider a handle decomposition of $R(X)$ consisting of a closed B^3 neighbourhood for each 0-cell of X, a closed $B^2 \times B^1$ neighbourhood for each 1-cell of X, and a closed $B^1 \times B^2$ neighbourhood for each 2-cell of X.

For any immersion $Y \to X$, there is an induced 3-manifold thickening $R(Y)$ of Y, where the handle corresponding to each cell y of Y maps homeomorphically to the handle corresponding to the cell x, where x is the cell that y maps to.

Observe that since X is diagrammatically reducible, Y is diagrammatically reducible and hence aspherical. Consequently, $R(Y)$ is aspherical, since its deformation retracts to the aspherical 2-complex Y.

If Y is compact, then $R(Y)$ is a compact aspherical 3-manifold with nonempty boundary, and so we see from Lemma 7.1 that either $R(Y)$ and hence Y is contractible, or $\chi(Y) = \chi(R(Y)) \leqslant 0$ as claimed. □

Remark 7.3. It was shown in [10, Theorem E] that if X is a 2-complex associated with a handle decomposition with no 3-handles of an irreducible 3-manifold, then every subcomplex of X is aspherical. Hence, the proof of Theorem 7.2 shows that if X is any 2D spine of an irreducible 3-manifold, then X has nonpositive tower maps.

8. Small cancellation and perimeter

We refer the reader to [12] or [14] for the notions *pieces* and $C(p)$-$T(q)$ small-cancellation complexes. For simplicity, we will assume here that no 2-cell has a periodic attaching map, but we note however that the results presented here do generalize using an appropriate variant of the Euler characteristic.

A *piece weighting* on a 2-complex X is a choice of a real number $\mathrm{Wt}(R, P)$ for each 2-cell $R \to X$ and each piece $P \to X$ that factors through ∂R. This notion is more flexible than the *weighted perimeter* notion considered in [15] where 'sides of 2-cells' are weighted.

Theorem 8.1. *Let X be a $T(q)$ 2-complex with a piece weighting. Then X has nonpositive immersions provided that the following two conditions hold: for each 2-cell R, and for each expression of ∂R as the concatenation of pieces $P_1 P_2 \cdots P_k$, we have*

$$\sum_{i=1}^{k} \mathrm{Wt}(R, P_i) \geqslant 1. \tag{1}$$

And for each piece $P \to X$ we have

$$\frac{2}{q} + \sum_{\text{occurrences of } P \text{ in } \partial R_i} \mathrm{Wt}(R_i, P) \leqslant 1. \tag{2}$$

Proof. Consider an immersion $Y \to X$, and suppose that Y is compact and connected but does not consist of a single 0-cell. Without loss of generality, we may assume that Y does not have any free faces, and in particular, Y has no 0-cells of valence 1. If Y^1 is a circle, then either $\chi(Y) = 0$ or Y contains a 2-cell and hence $\pi_1 Y$ is trivial. Otherwise, Y^1 is the union of maximal arcs P beginning and ending at 0-cells of valence $\geqslant q$, and each such P maps to a piece in X. (The computation below focuses on the alternative complex obtained by ignoring valence 2 vertices of Y.) Let f, e, and v denote the numbers of 2-cells, maximal arcs, and 0-cells with valence $\geqslant q$ of Y. Now

$$\begin{aligned}
\chi(Y) = v - e + f &\leqslant \frac{2}{q} e - e + \sum_{R \text{ in } Y} \left(\sum_{P \text{ in } \partial R} \mathrm{Wt}(R, P) \right) \\
&= \frac{2}{q} e - e + \sum_{P \text{ in } Y} \left(\sum_{P \text{ in } \partial R} \mathrm{Wt}(R, P) \right) = \sum_{P \text{ in } Y} \left(\frac{2}{q} - 1 + \sum_{P \text{ in } \partial R} \mathrm{Wt}(R, P) \right) \leqslant 0. \qquad \square
\end{aligned}$$

The 2-complex X satisfies *the* (p, q, r) *condition* provided X is $C(p)$ and $T(q)$, and each 1-cell occurs at most r times along the boundaries of 2-cells.

Corollary 8.2. *If X is a (p, q, r) 2-complex and $\frac{2}{q} + \frac{r}{p} \leqslant 1$, then X has nonpositive immersions.*

Proof. Weight the occurrence of each piece P in ∂R by $\frac{1}{p}$. Then apply Theorem 8.1. $\square$

Corollary 8.3. *A $T(q)$ 2-complex X has nonpositive immersions provided the following condition holds: for each 2-cell R, and each piece P occurring in ∂R, the number of times P occurs as a piece in X is at most $\frac{q-2}{q} \frac{|\partial R|}{|P|}$.*

Proof. We assign $\mathrm{Wt}(R, P)$ to equal $\frac{|P|}{|\partial R|}$. Then equation (1) obviously holds, and equation (2) holds because of the following inequality, where R_0 is the smallest 2-cell

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containing P as a piece and K is the number of times P occurs as a piece in X.

$$\frac{2}{q} + \sum_{R} \frac{|P|}{|\partial R|} \leqslant \frac{2}{q} + K \frac{|P|}{|\partial R_0|} \leqslant \frac{2}{q} + \frac{q-2}{q} = 1. \qquad \square$$

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