statement on nilpotent groups: G has polynomial growth if and only if it has a nilpotent subgroup of finite index.

A second problem of Milnor's—whether every group has either exponential or polynomial growth—was solved in the negative by Grigorchuk in the 1980s [**7**]. Grigorchuk's example of a group of intermediate growth and other similar examples later constructed by Gupta and Sidki are often expressed as groups of automorphisms of rooted trees. All these topics, and more, are covered in the final three hefty chapters of the book.

I do not see this book being usable as a course textbook, for example, because of the combination of style and choice of material. Nevertheless, anyone working seriously in the area of geometric group theory will want to have access to it as a handy reference manual, and its appearance is greatly to be welcomed.

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Palmer, Theodore W. *Banach algebras and the general theory of ∗-algebras*, vol. II, *∗-Algebras* (Cambridge University Press, 2001), 834 pp., 0 521 36638 0 (hardback) £90.00.

This book is the second, and final, volume of a great work on Banach ∗-algebras. The first volume [**4**], which was published in 1994, was devoted to the general theory of algebras and Banach algebras. This volume was reviewed by me in [**1**]. This second volume is devoted to Banach algebras with an involution. It has already been reviewed by George Willis in [**7**].

It is immediately obvious that this is a mammoth work; the total number of pages in the two volumes is almost 1600. The bibliography at the end of this second volume contains over 2000 items, with about 25% dated after 1990; some references have been found in rather obscure sources. The author has a truly remarkable knowledge of the literature, and this has been assimilated into the text. The history of the results in the book is given in great detail.

It seems that the author has typeset the whole text by himself; this is a tremendous achievement in itself. The details seem to be very accurate, and throughout the author is careful to point out possible ambiguities and pitfalls to be avoided. But the details need to be accurate because a large amount of information in this text is carried by subscripts and superscripts. For an example, see Proposition 10.1.2, where $\mathcal{A}^{\dagger}_{R,1}$ denotes the set of representable positive linear functionals of norm 1 on an algebra $\mathcal{A}$; it is a challenge to the memory of this reader to recall all the notation, but there is a clear 'symbol index' on pp. 1612–1616. (In this index, $\mathcal{A}^{\dagger}_{R,1}$ is listed under the letter 'R'). Also a large amount of information is displayed in impressive charts of relationships between various classes of algebras, for example. My small criticisms in this area include the following: my eyesight prefers $\widetilde{T}$ to $\tilde{T}$ and $\mathbb{C}^2$ to $\mathbb{C}^2$; sometimes the braces around expressions are too small; a few expressions are awkwardly split between lines; and, in general, a few more equations should have been displayed.

There seem to be very few errors; the author is collecting those that he knows about on his website.

Naturally, this work is a very complete exposition of the topics that it covers; in the earlier chapters of this second volume there are only a few points at which reference is made to original papers for proofs that are omitted, and so the whole development of the subject is before our eyes.

At one stage the author planned two further chapters, on the cohomology of Banach algebras, following the style of the major contributions of Barry Johnson, and on the K-theory of Banach algebras. In the end, it was not possible to include these chapters in either the present or a future volume. In fact, work in these two areas has advanced very rapidly recently (in particular, Viktor Losert has recently resolved a long-standing open question by showing that $\mathcal{H}^1(L^1(G), M(G)) = \{0\}$ for every locally compact group G), and so perhaps this is not a great loss; we may hope for a new synthesis, taking account of recent developments, in these two areas sometime in the fairly near future. Some references to these lost chapters remain in Palmer's bibliography.

Earlier works on Banach $*$-algebras include the classic text of Rickart [**6**]. Major examples of Banach $*$-algebras are, of course, C^*-algebras, and there is a multitude of books on these examples; the present text subsumes some of this theory, but it is primarily a book on the more general class. Another recent and long book on Banach algebras is that of the reviewer [**2**]; however, the overlap between that book and the present volume is very small. (Note that the involution in a Banach $*$-algebra is required to be an isometry in [**2**], but in the present work the involution in such an algebra is not necessarily continuous.)

The second important class of Banach $*$-algebras is that of group algebras (see below). The natural class that contains both C^*-algebras and group algebras is exactly that of Banach $*$-algebras; the author seeks and indeed finds a substantial theory of this general class. Each Banach algebra has an algebraic and a topological structure, and the interest and beauty of the subject arises from the interplay between these two structures; nevertheless, Palmer's heart lies in the algebraic theory, and he first develops the purely algebraic theory of $*$-algebras, adding extra hypotheses that we shall later see are satisfied by all Banach $*$-algebras. Thus his development is initially very algebraic, and so this book should be of interest to algebraists as well as specialists in Banach algebra theory.

This is not a book in which one can find quick summaries of 'the state of the art' in, say, C^*-algebra theory, and it is disjoint from some of the major advances of the last decade. Rather, the author has reworked many of the foundations of the subject, and has given an account of results mainly of the 1970s and 1980s; some of the text consists of his own unpublished results of that time; often the results are given for a wider class of algebras than heretofore. Thus this book is what the title implies: a general theory of algebras with an involution. It will be the definitive text, and an essential reference, for the material that it covers for many years to come.

I shall now describe the contents of the four chapters of the present volume. Quite often the author includes summaries of topics in areas that seem somewhat distant from the main theme, and so there is much more in this volume than is immediately apparent from the titles of the sections. This is a tribute to the author's wide-ranging scholarship.

The first chapter (Chapter 9) deals with the algebraic theory of $*$-algebras with no initial topological structure. It certainly has more details and is cast in greater generality than any earlier work.

In § 9.1, we find the basic definitions; often the author uses the language of categories (but the reader does not require prior knowledge of this terminology). Some basic examples, including group algebras for a discrete group and C^*-algebras, are introduced. Topics discussed include finite-rank operators, the numerical range of an operator on a Hilbert space, and Hilbert–Schmidt and trace class operators. There is a very detailed description that I have not seen elsewhere of low-dimensional $*$-algebras, given in a very careful and full table on pp. 857–859, and a complete list of all finite-dimensional $*$-simple $*$-algebras. It is good to know that such a list exists, even if it is not really light reading.

Algebras are fundamentally studied through the theory of representations, and we begin this study in § 9.2. In this work we are discussing $*$-representations of $*$-algebras on a Hilbert space H, regarding $\mathcal{B}(H)$ as a C^*-algebra. In his quest for generality, the author discusses pre-$*$-representations on pre-Hilbert spaces as well as the better-known $*$-representations on a Hilbert space. We learn that many of the differences between the theory of representations of algebras and the theory of $*$-representations, which is our major theme, follow from the fact that every closed T-invariant subspace of a $*$-representation on a Hilbert space automatically decomposes, in the sense that orthogonal complements are also T-invariant (Proposition 9.2.4).

The author says that he will omit any systematic study of C^*-algebras, but he does give a discussion of the various categories in which we can view these algebras ('their category is fundamentally algebraic'), and he does prove the von Neumann and Kaplansky density theorems in § 9.3. He discusses very carefully the definitions of W^*-algebras and von Neumann algebras, but does not prove the famous abstract characterization of von Neumann algebras by Sakai. The spectral theorem for normal operators is proved.

In § 9.4, Palmer discusses the relationship between topologically cyclic $*$-representations and representable and admissible positive linear functionals. The basic idea follows the seminal one of Gelfand and Naimark, with contributions of Segal, Rickart and Sebestyen, but the treatment is the most general possible, and is based on unpublished results of the author from the 1970s.

The main theorem of § 9.5 is the author's version of the Gelfand–Naimark theorem: an algebra semi-norm which satisfies the C^*-condition is specified by a $*$-representation. The first proof is classical and the second is based on numerical range considerations, giving the remarkable fact that a norm-unital Banach algebra is isometrically isomorphic to a C^*-algebra if and only if the shape of its unit ball near the identity is the same as that of the unit ball of a C^*-algebra in its unique C^*-norm. This is a consequence of the Vidav–Palmer theorem. The theory of topologically irreducible $*$-representations in § 9.6 is again developed for a wider class of $*$-algebras than heretofore; very general structure spaces are defined and their relationships are discussed carefully.

Topologically irreducible $*$-representations are plentiful whenever there are enough $*$-representations. In the classical theory a $*$-algebra A is reduced if the reducing ideal A_R, which is the intersection of the kernels of the $*$-representations, is the zero ideal. This ideal defines a radical in certain categories of $*$-algebras, but not in the category of all $*$-algebras, as examples make clear; in this category one naturally considers pre-$*$-representations, which give the pre-reducing ideal A_{pR}. (The latter ideal is called the $*$-radical and is denoted by $*\text{-rad}\,A$ in [**2**]). It seems to be open whether or not A_{pR} always contains the radical of A. The author also introduces

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the $*$-representation topology, not explicitly defined elsewhere. This is all explored in § 9.7. The author studies particular classes of $*$-algebras in this section; these are the regular, very proper, proper, semiproper, quasi-proper, and ordered $*$-algebras. In a diagram, he shows all the known relations between these properties, and then, in a series of carefully worked-out examples, gives counterexamples to nearly all the other possible relationships. The most interesting example is the $*$-algebra $\mathcal{B}$ consisting of all $\mathbb{N} \times \mathbb{N}$-matrices that have only finitely many non-zero entries in each row and in each column; it seems to be open whether or not the spectrum of $b \in \mathcal{B}$ is contained in $\mathbb{R}$ whenever $b = b^*$ in $\mathcal{B}$.

The remaining sections of this first chapter deal with hermitian, symmetric and completely symmetric $*$-algebras, and with various linear maps between $*$-algebras; again the results are given in a general algebraic setting extending earlier more special results.

It is the author's credo that the objects of primary interest, Banach $*$-algebras, are best studied by defining various additional hypotheses that general $*$-algebras might satisfy. For example, he defines in Definition 10.1.1 the Gelfand–Naimark semi-norm γ on a $*$-algebra $\mathcal{A}$ by setting $\gamma(a)$, for $a \in \mathcal{A}$, to be the supremum of $\|T_a\|$ when T ranges through the class of all $*$-representations of $\mathcal{A}$ on a Hilbert space; he coins the term 'G^*-algebra' for the class of $*$-algebras for which $\gamma(a)$ is finite for each $a \in \mathcal{A}$. The justification for this approach is as follows: first, the various classes of $*$-algebras which he defines have an intrinsic interest; more importantly, the classes lend themselves to particularly simple proofs of the main theorems because they isolate exactly the property which is required; furthermore, the categories that the properties define are much better behaved than the rather awkward category of Banach $*$-algebras, and so constructions and proofs are facilitated. Most of the categories have been defined by the author himself, and this is their first connected exposition. In my opinion, the case for each particular definition is well made, and the author's personal experience and very detailed calculations have provided a coherent and attractive setting for the various theorems. The important point to remember when one is seeking to apply the results is that essentially all the classes contain the class of Banach $*$-algebras, or at least the class of hermitian Banach $*$-algebras.

The notion of a G^*-algebra is natural, inter alia, because many simple conditions on a $*$-algebra are equivalent to it being a G^*-algebra. The Gelfand–Naimark semi-norm γ is the largest C^*-seminorm on a $*$-algebra $\mathcal{A}$; its kernel is $\mathcal{A}_R$, and the completion of $(\mathcal{A}/\mathcal{A}_R, \gamma)$ is the enveloping C^*-algebra $C^*(\mathcal{A})$ of $\mathcal{A}$. However, there is a problem with the class of G^*-algebras: little of the $*$-representation theory of Banach $*$-algebras can be carried through for this class. Thus the author defines a slightly smaller class, that of BG^*-algebras, and then shows that essentially all the features of the known theory of Banach $*$-algebras can be carried through for BG^*-algebras. In particular, every $*$-representation of a $*$-ideal in a BG^*-algebra can be extended to a $*$-representation of the whole algebra on the same Hilbert space (Theorem 10.1.21).

Chapter 10 is devoted to $*$-algebras that satisfy some extra conditions. In § 10.2, the author introduces T^*-algebras, S^*-algebras and Sq^*-algebras; they seem destined to be minor players in the ongoing drama, but they have a role in the process of staging the full and important proof that each Banach $*$-algebra is a BG^*-algebra, and their introduction clarifies the logic of the proof of this latter theorem. The Raikov–Ptak functional $\tau : a \mapsto \rho(a^*a)^{1/2}$ is important here (where ρ denotes the spectral radius). All known proofs that Banach $*$-algebras are BG^*-algebras depend on finding square roots of certain hermitian elements by using some form of the Shirali–Ford square-root lemma; this result is captured by the definition of Sq^*-algebras. By Theorem 10.2.8, every $*$-ideal in a Banach $*$-algebra is an Sq^*-algebra.

A U^*-algebra, from § 10.3, is a $*$-algebra $\mathcal{A}$ such that the unitization of $\mathcal{A}$ is the linear span of its unitary group. Each Sq^*-algebra is a U^*-algebra. This notion was devised by the author around 1972; many of the properties of BG^*-algebras given here for the first time were originally proved for U^*-algebras. A γS^*-algebra, from § 10.4, is the natural generalization of a hermitian Banach $*$-algebra; the seminorms γ and τ coincide on a γS^*-algebra; furthermore, the Jacobson radical of a γS^*-algebra coincides with its reducing ideal. It is open whether or not γS^*-algebras are always BG^*-algebras.

In Chapter 11, the author turns to Banach $*$-algebras themselves. First, § 11.1 helpfully summarizes, and sometimes reproves, much of the earlier theory as it applies to Banach $*$-algebras. Similarly, § 11.4 summarizes the satisfying theory of hermitian Banach $*$-algebras; the seminal work on these algebras is due to Raikov, to Ptak, and to the author in the 1970s, and it was the knowledge of this theory that led the author to write the first version of his magnum opus at that time. In § 11.2, the author discusses the unitary structure of Banach $*$-algebras; the development using the Mobius–Potapov–Harris transform is still very attractive; strengthened forms of the Russo–Dye theorem are obtained. In § 11.3, there is a brief account of the automatic continuity of positive linear functionals on a Banach $*$-algebra $\mathcal{A}$ (it is still not known whether or not all such functionals are continuous whenever $\mathcal{A}^2$ is closed and of finite codimension in $\mathcal{A}$) and of homomorphisms from $\mathcal{A}$; on this topic, the account in [**2**] is much more comprehensive.

The remaining sections of this chapter deal with the ideal structure of Banach $*$-algebras, with minimal ideals, with H^*-algebras (showing in Theorem 11.6.18 that essentially the only examples are $*$-algebras of Hilbert–Schmidt matrices on an index set), and with Hilbert and Tomita algebras (the treatment is based on the approach of Rieffel in 1969). These algebras fit into the present chapter because a full left Hilbert algebra is a hermitian Banach $*$-algebra, and so earlier results apply. In fact we even see a study of quasi-left Hilbert algebras; it is comforting to see that left Hilbert algebras obtain their 'fulfillment' in § 11.7, even if British eyes would prefer a fulfilment. There is rather a long discussion of Tomita–Takesaki theory, including substantial preliminaries on unbounded operators which are defined on a dense subspace of a Hilbert space.

As we stated, the two most important classes of Banach $*$-algebras are those of C^*-algebras and of group algebras. Indeed, if it were not for the enormous importance, in both general theory and applications, of these particular classes of algebra, it would be hard to justify the study of Banach $*$-algebras, their common generalization. It is in the final chapter, Chapter 12, of this work that the author applies his general theory to the study of group algebras. These are algebras of the form $L^1(G)$, where G is a locally compact group with left Haar measure λ. The product that makes $L^1(G)$ into a Banach algebra is convolution, which is denoted by $*$, so that

$$(f * g)(u) = \int f(v) g(v^{-1}u) \, \mathrm{d}v \quad (u \in G)$$

for $f, g \in L^1(G)$, and the involution $*$ is defined by

$$f^*(u) = \Delta(u^{-1}) \overline{f(u^{-1})} \quad (u \in G)$$

for $f \in L^1(G)$, where Δ is the modular function. (There are small differences of notation between the present work, [**2**] and [**3**]; a comprehensive table of notation is given on p. 1484. Palmer uses $*$ for both convolution product and the involution; in [**2**], I use $\star$ for the convolution product.) A closely related Banach $*$-algebra is $M(G)$, the measure algebra of G. The algebra $L^1(G)$ is a closed $*$-ideal in $M(G)$, and $M(G)$ is the multiplier, or double centralizer, algebra of $L^1(G)$. However, the complicated structure of the character space of $M(G)$ in the case where G is abelian is not discussed in this book; a modern text on this topic is a major gap in the literature.

In § 12.1, the theory of locally compact groups is reviewed; for details and proofs that are omitted the reader is often referred to [**3**]. It is well known that the class of locally compact groups is a zoo; a plethora of subclasses has been introduced. Many of these classes are described by Palmer; there is a list on p. 1485 (where, for example, '5.13' means that the definition is given in subsection 12.5.13). The present section presents the details of several important examples, including the Heisenberg groups, and introduces the theory of group extensions in our setting; it shows that the study of topological groups reduces to the study of connected groups, of totally disconnected groups, and of group extensions.

The first of these classes of groups is studied in § 12.2; in fact, the author introduces the wider class of almost connected groups. Within this section, we find a reasonably detailed introduction to Lie groups and Lie algebras, to Hilbert's fifth problem, and to the Iwasawa decomposition. The structure of the special linear group $SL(2, \mathbb{R})$ is worked out in detail. The results of this section are rather classical. The second class, that of totally disconnected groups, is studied in § 12.3. We note that the compact, totally disconnected groups are just the pro-finite groups; we also see here a summary of Galois theory, its applications to topological groups, and some connections with number theory (especially using the p-adic integers); here one again realizes the author's substantial erudition on many diverse topics. Theorem 12.3.12 is a full and detailed classification of non-discrete, locally compact division rings, a theorem which descends from the work of van Dantzig in the 1930s. The section continues with some much more recent work: this is the theory of Willis 'that gives a whole new level of insight into the structure and classification of [totally disconnected, locally compact] groups'. The key notions are those of 'scale function' and 'tidy subgroup', which were introduced by Willis. As usual, the author gives the details of the key examples; this will surely be very valuable for future advances in the general theory.

It is well known and of fundamental importance that continuous, unitary representations of a locally compact group G correspond to $*$-representations of $L^1(G)$ and of $M(G)$ in a natural way. Full details of this correspondence are given in § 12.4. This leads to the Gelfand–Raikov theorem that there are enough topologically irreducible, continuous, unitary representations of each locally compact group to separate its points, and to a characterization of the positive-definite functions on the group. In this setting we are introduced to further classes of groups: these include the Moore groups [Moore], the maximally almost periodic groups [MAP], and the classes [CCR] and [Type I]. The relationships of these classes and the theory of their ideals is delineated with care and summarized in diagrams. This section includes a careful summary (with few proofs) of results about the Fourier algebra and the Fourier–Stieltjes algebra of an arbitrary locally compact group; this is at present a very fashionable area of research.

In §§ 12.5 and 12.6, the author defines 22 important classes of groups (and some less important classes) are defined, sometimes repeating earlier definitions. The relationships between the classes are determined; this work updates and expands the author's seminal survey paper [**5**]. A typical heading of a subsection is 'A compactly generated, totally disconnected, Takahashi group that is not a central group'. The account of amenable groups, a class that attracts much attention at present, is rather brief; many other texts give fuller accounts. I find the classes of groups determined by Banach $*$-algebraic properties of $L^1(G)$ to be particularly interesting. For example, G is said to be hermitian if the Banach $*$-algebra $L^1(G)$ is hermitian; no simple characterization of such groups is known. Is every hermitian group an amenable group? Very detailed references to the history of the various theorems are given. The tables at the end of § 12.6 give all known interrelationships between the defined classes of groups and list, for many explicit examples, the classes that the examples belong to; usually we see 'Y' or 'N' in the various columns, but sometimes there is a '?', and these question marks are obvious challenges for the future.

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