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Stability of line solitons for the KP-II equation in $\mathbb{R}^2$. II

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The KP-II equation was derived by Kadmotsev and Petviashvili to explain stability of line solitary waves of shallow water. Recently, Mizumachi proved nonlinear stability of 1-line solitons for exponentially localized perturbations. In this paper, we prove stability of 1-line solitons for perturbations in $(1+x^2)^{-1/2-0}H^1(\mathbb{R}^2)$ and perturbations in $H^1(\mathbb{R}^2)\cap\partial_x L^2(\mathbb{R}^2)$.

Keywords: KP-II; line soliton; transverse stability

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1. Introduction

The KP-II equation,

$$\partial_x(\partial_t u + \partial_x^3 u + 3\partial_x(u^2)) + 3\partial_y^2 u = 0 \quad \text{for } t > 0 \text{ and } (x, y) \in \mathbb{R}^2, \tag{1.1}$$

is a generalization to two spatial dimensions of the Korteweg–de Vries (KdV) equation,

$$\partial_t u + \partial_x^3 u + 3\partial_x(u^2) = 0, \tag{1.2}$$

and has been derived as a model in the study of the transverse stability of solitary wave solutions to the KdV equation with respect to two-dimensional perturbations when the surface tension is weak or absent. See [15] for the derivation of (1.1).

The global well-posedness of (1.1) in $H^s(\mathbb{R}^2)$ ($s \geqslant 0$) around line solitons has been studied by Molinet *et al*. [31], the proof of which is based on the work of Bourgain [6]. For the other contributions to the Cauchy problem for the KP-II equation, see, for example, [10–12, 14, 36–39] and the references therein.

Let

$$\varphi_c(x) \equiv c\cosh^{-2}\left(\sqrt{\tfrac{1}{2}c}x\right), \quad c > 0.$$

Then $\varphi_c(x - 2ct)$ is a solitary wave solution of the KdV equation (1.2) and a line soliton solution of (1.1) as well.

Let us briefly explain known results on stability of 1-solitons for the KdV equation first. Stability of the 1-soliton $\varphi_c(x-2ct)$ of (1.2) was proved in [2,4,41] using the fact that φ_c is a minimizer of the Hamiltonian on the manifold $\{u \in H^1(\mathbb{R}) \mid \|u\|_{L^2(\mathbb{R})} = \|\varphi_c\|_{L^2(\mathbb{R})}\}$. As is well known, a solitary wave of the KdV equation travels at a speed faster than the maximum group velocity of linear waves and the larger solitary wave moves faster to the right. Using this property, Pego and Weinstein [33] proved asymptotic stability of solitary wave solutions of (1.2) in an exponentially weighted space. Later, Martel and Merle established the Liouville theorem for the generalized KdV equations by using a virial-type identity, and proved the asymptotic stability of solitary waves in $H^1_{\mathrm{loc}}(\mathbb{R})$ (see, for example, [21]). For stability of multi-solitons of the generalized KdV equations, see [22].

For the KP-II equation, its Hamiltonian is infinitely indefinite and the variational approach given in, for example, [9] is not applicable. Hence, it seems natural to study stability of line solitons using strong linear stability of line solitons. Spectral transverse stability of line solitons of (1.1) has been studied in [1, 7]. See also [13] for transverse linear stability of cnoidal waves. Alexander *et al*. [1] proved that the spectrum of the linearized operator in $L^2(\mathbb{R}^2)$ consists of the entire imaginary axis. On the other hand, in an exponentially weighted space where the size of perturbation is biased in the direction of motion, the spectrum of the linearized operator consists of a curve of resonant continuous eigenvalues that goes through 0 and the set of the continuous spectrum that is located in the stable half-plane and is away from the imaginary axis (see [7, 24]). The former appears because line solitons are not localized in the transversal direction, and 0, which is related to the symmetry of line solitons, cannot be an isolated eigenvalue of the linearized operator. Such a situation is common with planer travelling wave solutions for the heat equation; see, for example, [16, 19, 42].

Using the inverse scattering method, Villarroel and Ablowitz [40] studied solutions around line solitons for (1.1). Recently, Mizumachi [24] proved transversal stability of line soliton solutions of (1.1) for exponentially localized perturbations. The idea was to use the exponential decay property of the linearized equation satisfying a secular-term condition and to describe variations of local amplitudes and local inclinations of line solitons by a system of Burgers equations.

The purpose of the present paper is to prove transverse stability of the line soliton solutions for perturbations that are the x-derivative of $L^2(\mathbb{R}^2)$ functions and for polynomially localized perturbations. Let us now introduce our results.

THEOREM 1.1. *Let $c_0 > 0$ and $u(t, x, y)$ be a solution of (1.1) satisfying that $u(0, x, y) = \varphi_{c_0}(x) + v_0(x, y)$. There exist positive constants ε_0 and C satisfying the following: if $v_0 \in \partial_x L^2(\mathbb{R}^2)$ and*

$$\|v_0\|_{L^2(\mathbb{R}^2)} + \||D_x|^{1/2} v_0\|_{L^2(\mathbb{R}^2)} + \||D_x|^{-1/2}|D_y|^{1/2} v_0\|_{L^2(\mathbb{R}^2)} < \varepsilon_0,$$

then there exist C^1-functions $c(t, y)$ and $x(t, y)$ such that, for every $t \geqslant 0$ and $k \geqslant 0$,

$$\|u(t, x, y) - \varphi_{c(t,y)}(x - x(t, y))\|_{L^2(\mathbb{R}^2)} \leqslant C\|v_0\|_{L^2}, \tag{1.3}$$

$$\|c(t, \cdot) - c_0\|_{H^k(\mathbb{R})} + \|\partial_y x(t, \cdot)\|_{H^k(\mathbb{R})} + \|x_t(t, \cdot) - 2c(t, \cdot)\|_{H^k(\mathbb{R})} \leqslant C\|v_0\|_{L^2}, \tag{1.4}$$

$$\lim_{t\to\infty} (\|\partial_y c(t, \cdot)\|_{H^k(\mathbb{R})} + \|\partial_y^2 x(t, \cdot)\|_{H^k(\mathbb{R})}) = 0, \tag{1.5}$$

and, for any $R > 0$,

$$\lim_{t\to\infty} \|u(t, x + x(t, y), y) - \varphi_{c(t,y)}(x)\|_{L^2((x>-R)\times\mathbb{R}_y)} = 0. \tag{1.6}$$

Theorem 1.2. *Let $c_0 > 0$ and $s > 1$. Suppose that u is a solution of (1.1) satisfying $u(0,x,y) = \varphi_{c_0}(x) + v_0(x,y)$. Then there exist positive constants ε_0 and C such that if $\|\langle x\rangle^s v_0\|_{H^1(\mathbb{R}^2)} < \varepsilon_0$, there exist $c(t,y)$ and $x(t,y)$ satisfying (1.5), (1.6) and*

$$\|u(t,x,y) - \varphi_{c(t,y)}(x - x(t,y))\|_{L^2(\mathbb{R}^2)} \leqslant C\|\langle x\rangle^s v_0\|_{H^1(\mathbb{R}^2)}, \tag{1.7}$$

$$\|c(t,\cdot) - c_0\|_{H^k(\mathbb{R})} + \|\partial_y x(t,\cdot)\|_{H^k(\mathbb{R})} + \|x_t(t,\cdot) - 2c(t,\cdot)\|_{H^k(\mathbb{R})} \leqslant C\|\langle x\rangle^s v_0\|_{H^1(\mathbb{R}^2)} \tag{1.8}$$

for every $t \geqslant 0$ and $k \geqslant 0$.

Remark 1.3. By (1.4) and (1.5),

$$\lim_{t\to\infty} \sup_{y\in\mathbb{R}} (|c(t,y) - c_0| + |x_y(t,y)|) = 0,$$

and as $t \to \infty$ the modulating line soliton $\varphi_{c(t,y)}(x - x(t,y))$ converges to a y-independent modulating line soliton $\varphi_{c_0}(x - x(t,0))$ in $L^2(\mathbb{R}_x \times (|y| \leqslant R))$ for any $R > 0$. Hence, it follows from (1.6) that

$$\lim_{t\to\infty} \|u(t, x + x(t,0), y) - \varphi_{c_0}(x)\|_{L^2((x>-R)\times(|y|\leqslant R))} = 0.$$

We remark that the phase shift $x(t,y)$ in (1.3) and (1.6) cannot be uniform in y because of the variation of the local phase shift around $y = \pm 2\sqrt{2c_0}t + O(\sqrt{t})$. See [24, theorems 1.4 and 1.5].

Remark 1.4. The KP-II equation has no localized solitary waves (see [5, 8]). On the other hand, the KP-I equation has stable ground states (see [5, 20]) and line solitons of the KP-I equation are unstable (see [34, 35, 43]). See, for example, [18] and the references therein for numerical studies of KP-type equations.

Remark 1.5. Following the idea of Merle and Vega [23], Mizumachi and Tzvetkov [26] used the Miura transformation to prove stability of line soliton solutions to the perturbations that are periodic in the transverse directions. They prove that the Miura transformation gives a local isomorphism between solutions around a 1-line soliton and solutions around the null solution of KP-II via solutions around a kink of MKP-II (the modified Kadmotsev–Petviashvili-II equation).

The argument in [26] fails for localized perturbations because, in view of the resonant continuous eigenvalues of MKP-II in $L^2(\mathbb{R}^2; \mathrm{e}^{2\alpha x}\,\mathrm{d}x\,\mathrm{d}y)$ with $\alpha \in (0, \sqrt{2c_0})$ (see [24, lemma 2.5]), the motion of waves along the crest of the modulating line kink of MKP-II is expected to be unilateral, whereas the wave motion along the crest of a modulating line soliton for the KP-II equation is bidirectional (see [24, theorem 1.5]).

Now let us explain our strategy for the proof. To prove stability of line solitons in [24], we relied on the fact that solutions of the linearized equation decay exponentially in exponentially weighted norm as $t \to \infty$ if the data are orthogonal to the adjoint resonant continuous eigenmodes. To describe the behaviour of solutions around a line soliton, we represent them by using an ansatz

$$u(t,x,y) = \varphi_{c(t,y)}(z) - \psi_{c(t,y)}(z + 3t) + v(t,z,y), \quad z = x - x(t,y), \tag{1.9}$$

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where $c(t,y)$ and $x(t,y)$ are the local amplitude and the local phase shift of the modulating line soliton $\varphi_{c(t,y)}(x - x(t,y))$ at time t along the line parallel to the x-axis, and $\psi_{c(t,y)}$ is an auxiliary function such that

$$\int_{\mathbb{R}} v(t,z,y)\,\mathrm{d}z = \int_{\mathbb{R}} v(0,z,y)\,\mathrm{d}z \quad \text{for any } y \in \mathbb{R}.$$

One of the key steps is to prove that $\|v(t)\|_{L^2_{\mathrm{loc}}}$ is square integrable in time. In [24], we imposed a non-secular condition on $v(t)$ such that the perturbation $v(t)$ is orthogonal to the adjoint resonant eigenfunctions in order to apply the strong linear stability property of line solitons (see proposition 2.2) to v. Since the adjoint resonant eigenfunctions grow exponentially as $x \to \infty$, the secular-term condition is not feasible for a $v(t)$ that is not exponentially localized as $x \to \infty$. Following the idea of [25, 26, 28], we split the perturbation $v(t)$ into the sum of a small solution $v_1(t)$ of (1.1) satisfying $v_1(0) = v_0$, and the remainder part $v_2(t)$. As is the case with other long wave models, the solitary wave part moves faster than the freely propagating perturbations, and the localized L^2-norms of v_1 are square integrable in time thanks to the virial identity. The remainder part $v_2(t)$ is exponentially localized as $x \to \infty$ and is mainly driven by the interaction between v_1 and the line soliton. We impose the secular-term condition on v_2 in order to apply the linear stability estimate. Using the linear stability estimate as well as a virial-type identity, we have the square integrability of $\|\mathrm{e}^{\alpha z} v_2(t)\|_{L^2}$ in time for small $\alpha > 0$.

For Boussinesq equations, Pedersen [32] heuristically observed that the modulation of line solitary waves is described by a system of Burgers equations. We expect that the method presented in this paper is applicable to the other two-dimensional long wave models.

Our plan of the present paper is as follows. In § 2 we recollect the strong linear stability property of line solitons that was proved in [24]. In § 3 we decompose a solution around line solitons into a sum of the modulating line soliton $\varphi_{c(t,y)}(z)$, a small freely propagating part v_1, an exponentially localized remainder part v_2 and an auxiliary function $\psi_{c(t,y)}$. In § 4 we compute the time derivative of the secular-term condition on v_2 and derive a system of Burgers equations that describe the local amplitude $c(t,y)$ and the local phase shift $x(t,y)$. In § 5 we estimate $\tilde{c}(t) := c(t) - c_0$ and $x_y(t)$. In this paper $\tilde{c}(t)$ and $x_y(t)$ are not necessarily pseudo-measures and we are not able to apply $\mathcal{F}^{-1}L^\infty$–L^2 estimates to $\tilde{c}$ and x_y. Instead, we use the monotonicity formula to obtain time global bounds for $\tilde{c}(t)$ and $x_y(t)$. Since the terms related to $v_1(t)$ are merely square integrable in time and cubic terms that appear in the energy identity are not necessarily integrable in time, we use a change of variables to eliminate these terms to obtain time global estimates. In § 6 we estimate the L^2-norm of the remainder term v. In § 7 we introduce several estimates for v_1, which is a small solution of (1.1). First, we show that a virial identity [8] ensures that a localized norm of v_1 is square integrable in time. We then explain that the nonlinear scattering theory in [12] gives a time global bound for L^p-norms with $p > 2$ if $v_1(0) = v_0 \in |D_x|^{1/2}L^2(\mathbb{R}^2)$ and v_0 is sufficiently smooth. In § 8 we estimate the exponentially weighted norm of v_2 following the procedure of [24]. We use the semigroup estimate introduced in § 2 to estimate the low frequencies in y and apply a virial-type estimate to estimate high frequencies in y to avoid a loss of derivatives. Since we split the perturbation v into two parts

v_1 and v_2, we cannot cancel the derivative of the nonlinear term by integration by parts and we need a time global bound of $\|v_1(t)\|_{L^3}$ to estimate the exponentially localized energy norm of $v_2(t)$ using the virial identity. In §§ 9 and 10 we prove theorems 1.1 and 1.2.

Finally, let us introduce some notation. For Banach spaces V and W, let $B(V,W)$ be the space of all linear continuous operators from V to W and let $\|T\|_{B(V,W)} = \sup_{\|x\|_V=1} \|Tu\|_W$ for $T \in B(V,W)$. We abbreviate $B(V,V)$ to $B(V)$. For $f \in \mathcal{S}(\mathbb{R}^n)$ and $m \in \mathcal{S}'(\mathbb{R}^n)$, let

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(x)\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}x,$$

$$(\mathcal{F}^{-1}f)(x) = \check{f}(x) = \hat{f}(-x), \qquad (m(D_x)f)(x) = (2\pi)^{-n/2}(\check{m} * f)(x).$$

We use $a \lesssim b$ and $a = O(b)$ to mean that there exists a positive constant such that $a \leqslant Cb$. Various constants will be simply denoted by C and C_i ($i \in \mathbb{N}$) in the course of the calculations. We define $\langle x\rangle = \sqrt{1+x^2}$ for $x \in \mathbb{R}$.

2. Preliminaries

In this section we recollect decay estimates of the semigroup generated by the linearized operator around a 1-line soliton in exponentially weighted spaces.

Since (1.1) is invariant under the scaling $u \mapsto \lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y)$, we may assume that $c_0 = 2$ in theorems 1.1 and 1.2 without loss of generality. Let

$$\varphi = \varphi_2, \qquad \mathcal{L} = -\partial_x^3 + 4\partial_x - 3\partial_x^{-1}\partial_y^2 - 6\partial_x(\varphi\cdot).$$

We remark that $\mathrm{e}^{t\mathcal{L}}$ is a C^0-semigroup on $X := L^2(\mathbb{R}^2; \mathrm{e}^{2\alpha x}\,\mathrm{d}x\,\mathrm{d}y)$ for any $\alpha > 0$ because $\mathcal{L}_0 := -\partial_x^3+4\partial_x-3\partial_x^{-1}\partial_y^2$ is m-dissipative on X and $\mathcal{L}-\mathcal{L}_0$ is infinitesimally small with respect to $\mathcal{L}_0$.

We have the following exponential decay estimates for $\mathrm{e}^{t\mathcal{L}_0}$ on X.

Lemma 2.1 (Mizumachi [24, lemma 3.4]). *Suppose that $\alpha > 0$. Then there exists a positive constant C such that, for every $f \in C_0^\infty(\mathbb{R}^2)$ and $t > 0$,*

$$\|\mathrm{e}^{t\mathcal{L}_0}f\|_X \leqslant C\mathrm{e}^{-\alpha(4-\alpha^2)t}\|f\|_X,$$

$$\|\mathrm{e}^{t\mathcal{L}_0}\partial_x f\|_X + \|\mathrm{e}^{t\mathcal{L}_0}\partial_x^{-1}\partial_y f\|_X \leqslant C(1+t^{-1/2})\mathrm{e}^{-\alpha(4-\alpha^2)t}\|f\|_X,$$

$$\|\mathrm{e}^{t\mathcal{L}_0}\partial_x f\|_X \leqslant C(1+t^{-3/4})\mathrm{e}^{-\alpha(4-\alpha^2)t}\|\mathrm{e}^{ax}f\|_{L_x^1L_y^2}.$$

Solutions of $\partial_t u = \mathcal{L}u$ satisfying a *secular-term condition* decay like solutions to the free equation $\partial_t u = \mathcal{L}_0 u$. To be more precise, let us introduce a family of continuous resonant eigenvalues near 0 and the corresponding continuous eigenfunctions of the linearized operator $\mathcal{L}$. Let

$$\beta(\eta) = \sqrt{1+\mathrm{i}\eta}, \qquad \lambda(\eta) = 4\mathrm{i}\eta\beta(\eta),$$

$$g(x,\eta) = \frac{-\mathrm{i}}{2\eta\beta(\eta)}\partial_x^2(\mathrm{e}^{-\beta(\eta)x}\operatorname{sech}x), \qquad g^*(x,\eta) = \partial_x(\mathrm{e}^{\beta(-\eta)x}\operatorname{sech}x).$$

Then

$$\mathcal{L}(\eta)g(x,\pm\eta) = \lambda(\pm\eta)g(x,\pm\eta), \qquad \mathcal{L}(\eta)^*g^*(x,\pm\eta) = \lambda(\mp\eta)g^*(x,\pm\eta).$$

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Now we define a spectral projection to the resonant eigenmodes $\{g_\pm(x,\eta)\}$. Let

$$g_1(x,\eta)=2\operatorname{Re} g(x,\eta),\qquad g_2(x,\eta)=-2\eta\operatorname{Im} g(x,\eta),$$
$$g_1^*(x,\eta)=\operatorname{Re} g^*(x,\eta),\qquad g_2^*(x,\eta)=-\eta^{-1}\operatorname{Im} g^*(x,\eta),$$

and let $P_0(\eta_0)$ be a projection to resonant modes defined by

$$P_0(\eta_0)f(x,y)=\frac{1}{2\pi}\sum_{k=1,2}\int_{-\eta_0}^{\eta_0}a_k(\eta)g_k(x,\eta)\mathrm{e}^{\mathrm{i}y\eta}\,\mathrm{d}\eta,$$
$$\begin{aligned}a_k(\eta)&=\int_{\mathbb{R}}\lim_{M\to\infty}\left(\int_{-M}^{M}f(x_1,y_1)\mathrm{e}^{-\mathrm{i}y_1\eta}\,\mathrm{d}y_1\right)\overline{g_k^*(x_1,\eta)}\,\mathrm{d}x_1\\&=\sqrt{2\pi}\int_{\mathbb{R}}(\mathcal{F}_yf)(x,\eta)\overline{g_k^*(x,\eta)}\,\mathrm{d}x.\end{aligned}$$

For η_0 and M satisfying $0<\eta_0\leqslant M\leqslant\infty$, let

$$P_1(\eta_0,M)u(x,y):=\frac{1}{2\pi}\int_{\eta_0\leqslant|\eta|\leqslant M}\int_{\mathbb{R}}u(x,y_1)\mathrm{e}^{\mathrm{i}\eta(y-y_1)}\,\mathrm{d}y_1\,\mathrm{d}\eta,$$
$$P_2(\eta_0,M):=P_1(0,M)-P_0(\eta_0).$$

Then we have the following.

Proposition 2.2 (Mizumachi [24, proposition 3.2, corollary 3.3]). *Let $\alpha\in(0,2)$ and let η_1 be a positive number satisfying $\operatorname{Re}\beta(\eta_1)-1<\alpha$. Then there exist positive constants K and b such that, for any $\eta_0\in(0,\eta_1]$, $M\geqslant\eta_0$, $f\in X$ and $t\geqslant 0$,*

$$\|\mathrm{e}^{t\mathcal{L}}P_2(\eta_0,M)f\|_X\leqslant K\mathrm{e}^{-bt}\|f\|_X.$$

Moreover, there exist positive constants K' and b' such that, for $t>0$,

$$\|\mathrm{e}^{t\mathcal{L}}P_2(\eta_0,M)\partial_xf\|_X\leqslant K'\mathrm{e}^{-b't}t^{-1/2}\|\mathrm{e}^{ax}f\|_X,$$
$$\|\mathrm{e}^{t\mathcal{L}}P_2(\eta_0,M)\partial_xf\|_X\leqslant K'\mathrm{e}^{-b't}t^{-3/4}\|\mathrm{e}^{ax}f\|_{L^1_xL^2_y}.$$

3. Decomposition of the perturbed line soliton

Let us decompose a solution around a line soliton solution $\varphi(x-4t)$ into the sum of a modulating line soliton and a non-resonant dispersive part plus a small wave that is caused by amplitude changes of the line soliton, namely,

$$u(t,x,y)=\varphi_{c(t,y)}(z)-\psi_{c(t,y),L}(z+3t)+v(t,z,y),\quad z=x-x(t,y),\tag{3.1}$$

where $\psi_{c,L}(x)=2(\sqrt{2c}-2)\psi(x+L)$, $\psi(x)$ is a non-negative function such that $\psi(x)=0$ if $|x|\geqslant 1$, $\int_{\mathbb{R}}\psi(x)\,\mathrm{d}x=1$, and $L>0$ is a large constant to be fixed later. The modulation parameters $c(t_0,y_0)$ and $x(t_0,y_0)$ denote the maximum height and the phase shift of the modulating line soliton $\varphi_{c(t,y)}(x-x(t,y))$ along the line $y=y_0$ at the time $t=t_0$, and $\psi_{c,L}$ is an auxiliary function such that

$$\int_{\mathbb{R}}\psi_{c,L}(x)\,\mathrm{d}x=\int_{\mathbb{R}}(\varphi_c(x)-\varphi(x))\,\mathrm{d}x.\tag{3.2}$$

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Since a localized solution to KP-type equations satisfies $\int_{\mathbb{R}} u(t,x,y)\,\mathrm{d}x = 0$ for any $y \in \mathbb{R}$ and $t > 0$ (see [29]), it is natural to expect small perturbations to appear in the rear of the solitary wave if the solitary wave is amplified.

In order to use exponential linear stability of line solitons for solutions that are not exponentially localized in space, we further decompose v into a small solution of (1.1) and an exponentially localized part, following the idea of [25] (see also [27,28]). Let $\tilde{v}_1$ be a solution of

$$\left.\begin{aligned}\partial_t\tilde{v}_1 + \partial_x^3\tilde{v}_1 + 3\partial_x(\tilde{v}_1^2) + 3\partial_x^{-1}\partial_y^2\tilde{v}_1 = 0,\\ \tilde{v}_1(0,x,y) = v_0(x,y),\end{aligned}\right\} \tag{3.3}$$

and

$$v_1(t,z,y) = \tilde{v}_1(t, z + x(t,y), y), \quad v_2(t,z,y) = v(t,z,y) - v_1(t,z,y). \tag{3.4}$$

Obviously, we have $v_2(0) = 0$ and $v_2(t) \in X := L^2(\mathbb{R}^2; \mathrm{e}^{2\alpha z}\,\mathrm{d}z\,\mathrm{d}y)$ for $t \geqslant 0$ as long as the decomposition (3.1) persists. Indeed, we have the following.

Lemma 3.1. *Let $v_0 \in H^{1/2}(\mathbb{R}^2)$ and let $\tilde{v}_1(t)$ be a solution of (3.3). Suppose that $u(t)$ is a solution of (1.1) satisfying $u(0,x,y) = \varphi(x) + v_0(x,y)$. Let $w(t,x,y) = u(t,x+4t,y) - \varphi(x) - \tilde{v}_1(t,x+4t,y)$. Then, for any $\alpha \in [0,1)$,*

$$w \in C([0,\infty); X), \tag{3.5}$$

$$\partial_x w, \partial_x^{-1}\partial_y w \in L^2(0,T;X) \quad \text{for every } T > 0. \tag{3.6}$$

Moreover, if, in addition, $v_0 \in \partial_x L^2(\mathbb{R}^2)$, then

$$\partial_x^{-1}(u(t,x,y) - \varphi(x-4t)) \in C([0,\infty); L^2(\mathbb{R}^2)). \tag{3.7}$$

We remark that, by [31], $\partial_x w, \partial_x^{-1}\partial_y w \in L_x^\infty L^2([-T,T]\times\mathbb{R}_y)$ for any $T > 0$ provided that $v_0 \in L^2(\mathbb{R}^2)$. To prove lemma 3.1, we use the following embedding inequalities.

Claim 3.2. *Let $p_n(x) = \mathrm{e}^{2\alpha n x}(1 + \tanh\alpha(x-n))$. There exists a positive constant C such that, for every $n \in \mathbb{N}$,*

$$\begin{aligned}&\int_{\mathbb{R}^2} p_n'(x)^3 w^6(s,x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &\qquad\leqslant C\left[\int_{\mathbb{R}^2} p_n'(x)\{(\partial_x w)^2 + (\partial_x^{-1}\partial_y w)^2 + w^2\}(s,x,y)\,\mathrm{d}x\,\mathrm{d}y\right]^3.\end{aligned} \tag{3.8}$$

Moreover, for any $p \in [2,6]$,

$$\|\mathrm{e}^{\alpha x}u\|_{L^p} \leqslant C_1\|u\|_X^{3/p-1/2}(\|\partial_x u\|_X + \|\partial_x^{-1}\partial_y u\|_X + \|u\|_X)^{3/2-3/p}. \tag{3.9}$$

Proof. First, we remark that

$$0 < p_n'(x) \leqslant 2\alpha p_n(x) \leqslant 4\alpha\mathrm{e}^{2\alpha x}, \qquad |p_n''(x)| \leqslant 2\alpha p_n'(x), \qquad |p_n'''(x)| \leqslant 4\alpha^2 p_n'(x). \tag{3.10}$$

Using (3.10), we have (3.8) in the same way as in the proof of [30, lemma 2] and [26, claim 5.1].

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Equation (3.9) is obvious if $p = 2$. For $p = 6$, we have (3.9) with $p = 6$ by passing to the limit $n \to \infty$ in (3.8) because $p_n'(x) > 0$ for every $x \in \mathbb{R}$ and $p_n'(x)$ is monotone increasing in n. Thus, we have (3.9) by interpolation. □

Proof of lemma 3.1. First, we prove (3.5) assuming that $v_0 \in H^3(\mathbb{R}^2)$ and $v_0 \in \partial_x H^2(\mathbb{R}^2)$. It then follows from [6,31] that $\tilde{v}_1, w \in C(\mathbb{R}; H^3(\mathbb{R}^2))$ and $\partial_x^{-1}\tilde{v}_1, \partial_x^{-1} w \in C(\mathbb{R}; H^2(\mathbb{R}^2))$. Since $\mathcal{L}_0\varphi = 3\partial_x\varphi^2$ and u and $\tilde{v}_1$ are solutions of (1.1),

$$\left.\begin{aligned} \partial_t w = \mathcal{L}_0 w - \partial_x \mathfrak{N}_1, \\ w(0, x, y) = 0, \end{aligned}\right\} \tag{3.11}$$

where $\mathfrak{N}_1 = 6\varphi(w + \bar{v}_1) + 3w(w + 2\bar{v}_1)$. Multiplying (3.11) by $2p_n(x)w(t, x, y)$ and integrating the resulting equation by parts, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} p_n(x) w^2(t,x,y)\,\mathrm{d}x\,\mathrm{d}y &+ \int_{\mathbb{R}^2} p_n'(x)\{\mathcal{E}(w) - 4w^3\}(t,x,y)\,\mathrm{d}x\,\mathrm{d}y \\ = 6\int_{\mathbb{R}^2} &\{p_n'(x)(\bar{v}_1(t,x,y) + \varphi(x)) - p_n(x)(\partial_x\bar{v}_1(t,x,y) + \varphi'(x))\} w(t,x,y)^2\,\mathrm{d}x\,\mathrm{d}y \\ &- 12\int_{\mathbb{R}^2} p_n(x) w(t,x,y)\partial_x(\varphi(x)\bar{v}_1(t,x,y))\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^2} p_n'''(x) w^2(t,x,y)\,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{3.12}$$

where $\mathcal{E}(w) = 3(\partial_z w)^2 + 3(\partial_z^{-1}\partial_y w)^2 + 4w^2$. By claim 3.2,

$$\begin{aligned} &\left|\int_{\mathbb{R}^2} p_n'(x) w^3(t,x,y)\,\mathrm{d}x\,\mathrm{d}y\right| \\ &\qquad \lesssim \|w(t)\|_{L^2}\left(\int_{\mathbb{R}^2} p_n'(x) w^2(t,x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/4}\left(\int_{\mathbb{R}^2} p_n'(x)\mathcal{E}(w)(t,x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{3/4} \end{aligned}$$

and it follows from (3.10) and the above that there exist positive constants ν and C_1 such that, for any $n \in \mathbb{N}$, $T \geqslant 0$ and $t \in [0, T]$,

$$\begin{aligned} \int_{\mathbb{R}^2} p_n(x) w^2(t,x,y)\,\mathrm{d}x\,\mathrm{d}y &+ \nu\int_0^t\int_{\mathbb{R}^2} p_n'(x)\mathcal{E}(w)(s,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\ &\leqslant C_1 T \sup_{t\in[0,T]}\|\bar{v}_1(t)\|_{H^1}^2 \\ &\qquad + C_1 \sup_{t\in[0,T]}(1 + \|\bar{v}_1(t)\|_{H^3} + \|w(t)\|_{L^2}^4)\int_0^t\int_{\mathbb{R}^2} p_n(x) w^2(s,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

By Gronwall's inequality, we have, for $t \in [0, T]$,

$$\int_{\mathbb{R}^2} p_n(x) w^2(t,x,y)\,\mathrm{d}x\,\mathrm{d}y \leqslant C_2 \sup_{t\in[0,T]}\|\bar{v}_1(t)\|_{H^1}^2,$$

where C_2 is a constant independent of n. By passing to the limit $n \to \infty$, we have

$$\|w(t)\|_X^2 \leqslant C_2 \sup_{t\in[0,T]}\|\bar{v}_1(t)\|_{H^1}^2 \quad \text{for } t \in [0, T],$$

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since $0 < p_n(x) \uparrow 2e^{2\alpha x}$ as $n \to \infty$. Thus, we prove that $w \in L^\infty(0,T;X)$ and $\partial_x w, \partial_x^{-1}\partial_y w \in L^2(0,T;X)$ for every $T \geqslant 0$ provided that $v_0 \in H^3(\mathbb{R}^2) \cap \partial_x H^2(\mathbb{R}^2)$.

Let $p(x) = e^{2\alpha x}$. Integrating by parts the second and the third terms of the right-hand side of (3.12), integrating the result over $[0,t]$, and passing to the limit $n \to \infty$, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} p(x)w^2(t,x,y)\,\mathrm{d}x\,\mathrm{d}y &+ \int_0^t\int_{\mathbb{R}^2} p'(x)\{\mathcal{E}(w) - 4w^3\}(s,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
&= 12\int_{\mathbb{R}^2} (\bar{v}_1(s,x,y) + \varphi(x))\{p'(x)w^2(s,x,y) + p(x)(w\partial_x w)(s,x,y)\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 12\int_0^t\int_{\mathbb{R}^2} \partial_x\{p(x)w(s,x,y)\}\varphi(x)\bar{v}_1(s,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \int_0^t\int_{\mathbb{R}^2} p'''(x)w^2(s,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

By the Hölder inequality and claim 3.2,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}\{p'(x)w^2(s,x,y) + p(x)(w\partial_x w)(s,x,y)\}(\bar{v}_1(s,x,y) + \varphi(x))\,\mathrm{d}x\,\mathrm{d}y\right| \\
&\qquad\qquad\qquad \lesssim (\|\partial_x w(s)\|_X + \|w(s)\|_X)\|e^{\alpha x}w(s)\|_{L^4}\|\tilde{v}_1(s) + \varphi\|_{L^4} \\
&\qquad\qquad\qquad \lesssim \|\tilde{v}_1(s) + \varphi\|_{L^4}\|w(s)\|_X^{1/4}\|\mathcal{E}(w(s))^{1/2}\|_X^{7/4}, \\
&\left|\int_{\mathbb{R}^2}\partial_x\{p(x)w(s,x,y)\}\varphi(x)\bar{v}_1(s,x,y)\,\mathrm{d}x\,\mathrm{d}y\right| \lesssim \|\bar{v}_1(s)\|_{L^2}\|\mathcal{E}(w(s))^{1/2}\|_X,
\end{aligned}$$

and

$$\left|\int_{\mathbb{R}^2} p(x)w^3(s,x,y)\,\mathrm{d}x\,\mathrm{d}y\right| \lesssim \|w(s)\|_{L^2}\|w(s)\|_X^{1/2}\|\mathcal{E}(w(s))^{1/2}\|_X^{3/2}.$$

Combining the above, we have, for $t \in [0,T]$,

$$\begin{aligned}
&\|w(t)\|_X^2 + \nu\int_0^t \|\mathcal{E}(w(s))^{1/2}\|_X^2\,\mathrm{d}s \\
&\lesssim \int_0^t \{\|\bar{v}_1(s)\|_{L^2}^2 + (\|\bar{v}_1(s)\|_{L^2}^2 + \|w(s)\|_{L^2}^4 + \|\varphi + \bar{v}_1(s)\|_{L^4}^8)\|w(s)\|_X^2\}\,\mathrm{d}s, \quad (3.13)
\end{aligned}$$

where ν is a positive constant independent of T. Since $\|\bar{v}_1(t)\|_{L^2} = \|v_0\|_{L^2}$ for every $t \in \mathbb{R}$ and $H^{1/2}(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$, it follows from Gronwall's inequality that

$$\|w(t)\|_X^2 \leqslant C_3 T e^{C_4 t}\|v_0\|_{L^2}^2 \quad \text{for } t \in [0,T], \tag{3.14}$$

where C_3, C_4 are positive constants depending only on $\|v_1(t)\|_{H^{1/2}}$ and $\|w(t)\|_{L^2}$. By a standard limiting argument, we have (3.14) and (3.6) for every $v_0 \in H^{1/2}(\mathbb{R}^2)$.

Next, we will show that $w \in C([0,\infty);X)$. By claim 3.2, (3.14) and (3.6), we have that

$$\begin{gathered}
\|e^{ax}w\|_{L^4} \lesssim \|w\|_X^{1/4}\|\mathcal{E}(w)^{1/2}\|_X^{3/4} \in L^{8/3}(0,T;X), \\
\|\mathfrak{N}_1\|_X \lesssim \|w\|_X + \|\bar{v}_1\|_{L^2} + (\|w\|_{L^4} + \|\bar{v}_1\|_{L^4})\|e^{ax}w\|_{L^4} \in L^{8/3}(0,T;X).
\end{gathered}$$

By the variation of constants formula,

$$w(t) = -\int_0^t \mathrm{e}^{(t-s)\mathcal{L}_0}\partial_x \mathfrak{N}_1. \tag{3.15}$$

By lemma 2.1, (3.15) and the fact that $\mathfrak{N}_1 \in L^{8/3}(0, T; X)$, we have, for $h > 0$,

$$\|w(t+h) - w(t)\|_X \leqslant \left\|(\mathrm{e}^{h\mathcal{L}_0} - I)\int_0^t \mathrm{e}^{(t-s)\mathcal{L}_0}\partial_x \mathfrak{N}_1(s)\,\mathrm{d}s\right\|_X + O(h^{1/8}).$$

Since $\mathrm{e}^{t\mathcal{L}_0}$ is a C^0-semigroup on X, it follows that $w \in C([0,\infty); X)$.

Finally, we will show that (3.7) holds. Let $\bar{u}(t,x,y) := u(t, x+4t, y) - \varphi(x)$. Then, by the variation of constants formula,

$$\bar{u}(t) = \mathrm{e}^{t\mathcal{L}_0}v_0 - 3\partial_x \int_0^t \mathrm{e}^{(t-s)\mathcal{L}_0}(2\varphi\bar{u}(s) + \bar{u}^2(s))\,\mathrm{d}s. \tag{3.16}$$

Since $\mathrm{e}^{t\mathcal{L}_0}$ is unitary on $L^2(\mathbb{R}^2)$, $\partial_x^{-1}v_0 \in L^2(\mathbb{R}^2)$ and $\bar{u}(t) \in C(\mathbb{R}; H^{1/2}(\mathbb{R}^2))$, we easily see that (3.7) follows from (3.16). Thus, we complete the proof. □

Next, we will show the continuity of $H^{1/2}(\mathbb{R}^2) \ni v_0 \mapsto u - \tilde{v}_1 - \varphi(x-4t) \in X$.

Lemma 3.3. *Let $v_0 \in H^{1/2}(\mathbb{R}^2)$ and $v_{0,n} \in H^{1/2}(\mathbb{R}^2)$ for $n \in \mathbb{N}$. Suppose that $\tilde{v}_1$, $\tilde{v}_{1,n}$, u and u_n are solutions of (1.1) satisfying*

$$\begin{aligned} \tilde{v}_1(0,x,y) &= v_0(x,y), & u(0,x,y) &= \varphi(x) + v_0(x,y),\\ \tilde{v}_{1,n}(0,x,y) &= v_{0,n}(x,y), & u_n(0,x,y) &= \varphi(x) + v_{0,n}(x,y). \end{aligned}$$

If $\lim_{n\to\infty}\|v_{0,n} - v_0\|_{H^{1/2}(\mathbb{R}^2)} = 0$, then, for any $T \in (0,\infty)$,

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|u(t) - \tilde{v}_1(t) - u_n(t) + \tilde{v}_{1,n}(t)\|_X = 0.$$

Proof. Let $\bar{v}_{1,n}(t,x,y) = \tilde{v}_{1,n}(t, x+4t, y)$, $w_n(t,x,y) = u_n(t,x+4t,y) - \varphi(x) - \bar{v}_{1,n}(t,x,y)$ and $\tilde{w}_n = w - w_n$. Then

$$\left.\begin{aligned} \partial_t \tilde{w}_n = \mathcal{L}_0\tilde{w}_n - \partial_x(\mathfrak{N}_2 + \mathfrak{N}_3),\\ \tilde{w}_n(0,x,y) = 0, \end{aligned}\right\} \tag{3.17}$$

where

$$\begin{aligned} \mathfrak{N}_2(t) &= 3(2\varphi + 2\bar{v}_{1,n}(t) + w(t) + w_n(t))\tilde{w}_n(t),\\ \mathfrak{N}_3(t) &= 6(\varphi + w(t))(\bar{v}_{1,n}(t) - \bar{v}_1(t)). \end{aligned}$$

Multiplying (3.17) by $2\mathrm{e}^{2\alpha x}\tilde{w}_n$ and integrating the resulting equation over $\mathbb{R}^2 \times [0,t]$, we have

$$\begin{aligned} \|\tilde{w}_n(t)\|_X^2 &+ 2\alpha\int_0^t \|\mathcal{E}(\tilde{w}_n(s))^{1/2}\|_X^2\,\mathrm{d}s\\ &= -2\int_0^t\int_{\mathbb{R}^2}\mathrm{e}^{2\alpha x}\tilde{w}_n(s)\partial_x(\mathfrak{N}_2(s) + \mathfrak{N}_3(s))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{3.18}$$

Using claim 3.2 and the fact that $L^4(\mathbb{R}^2) \subset H^{1/2}(\mathbb{R}^2)$, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} \mathrm{e}^{2\alpha x} \tilde{w}_n \partial_x \mathfrak{N}_2 \, \mathrm{d}x \, \mathrm{d}y \right| \\
&\qquad \lesssim \|\mathrm{e}^{ax} \tilde{w}_n\|_{L^4} (\|\partial_x \tilde{w}_n\|_X + \|\tilde{w}_n\|_X)(1 + \|\bar{v}_{1,n}\|_{H^{1/2}} + \|w + w_n\|_{H^{1/2}}) \\
&\qquad \lesssim (1 + \|\bar{v}_{1,n}\|_{H^{1/2}} + \|w + w_n\|_{H^{1/2}}) \|\tilde{w}_n\|_X^{1/4} \|\mathcal{E}(\tilde{w}_n)^{1/2}\|_X^{7/4}, \\
&\left| \int_{\mathbb{R}^2} \mathrm{e}^{2\alpha x} \tilde{w}_n \partial_x \mathfrak{N}_3 \, \mathrm{d}x \, \mathrm{d}y \right| \lesssim (1 + \|\mathrm{e}^{ax} w\|_{L^4}) \|\bar{v}_{1,n} - \bar{v}_1\|_{L^4} (\|\tilde{w}_n\|_X + \|\partial_x \tilde{w}_n\|_X) \\
&\qquad\qquad\qquad\qquad \lesssim (1 + \|\mathcal{E}(w)^{1/2}\|_X) \|\bar{v}_{1,n} - \bar{v}_1\|_{H^{1/2}} \|\mathcal{E}(\tilde{w}_n)^{1/2}\|_X.
\end{aligned}$$

Combining the above with (3.18), we have

$$\begin{aligned}
\|\tilde{w}_n(t)\|_X^2 &\lesssim (T + \|\mathcal{E}(w_n)^{1/2}\|_{L^2(0,T;X)}) \sup_{t \in [0,T]} \|v_{1,n}(t) - v_1(t)\|_{H^{1/2}}^2 \\
&\quad + \sup_{t \in [0,T]} (1 + \|\bar{v}_{1,n}(t)\|_{H^{1/2}} + \|w_n(t) + w(t)\|_{H^{1/2}})^8 \int_0^t \|\tilde{w}_n(s)\|_X^2 \, \mathrm{d}s.
\end{aligned} \tag{3.19}$$

Thanks to the well-posedness of (1.1) (see, for example, [6, 31]),

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \|v_{1,n}(t) - v_1(t)\|_{H^{1/2}} = 0, \qquad \lim_{n \to \infty} \sup_{t \in [0,T]} \|\tilde{w}_n(t)\|_{H^{1/2}} = 0.$$

Thus, by (3.14), (3.6) and (3.19), we have, for $t \in [0, T]$,

$$\|\tilde{w}_n(t)\|_X^2 \leqslant C_1 \sup_{t \in [0,T]} \|v_{1,n}(t) - v_1(t)\|_{H^{1/2}}^2 + C_2 \int_0^t \|\tilde{w}_n(s)\|_X^2 \, \mathrm{d}s, \tag{3.20}$$

where C_1 and C_2 are positive constants independent of n. Applying Gronwall's inequality to (3.19), we obtain lemma 3.3. Thus, we complete the proof. □

To fix the decomposition (3.1), we impose that $v_2(t, z, y)$ is symplectically orthogonal to low frequency resonant modes. More precisely, we impose the constraint that, for $k = 1, 2$,

$$\lim_{M \to \infty} \int_{-M}^{M} \int_{\mathbb{R}} v_2(t, z, y) \overline{g_k^*(z, \eta, c(t, y))} \mathrm{e}^{-\mathrm{i}y\eta} \, \mathrm{d}z \, \mathrm{d}y = 0 \quad \text{in } L^2(-\eta_0, \eta_0), \tag{3.21}$$

where $g_1^*(z, \eta, c) = c g_1^*(\sqrt{c/2} z, \eta)$ and $g_2^*(z, \eta, c) = \frac{1}{2} c g_2^*(\sqrt{c/2} z, \eta)$.

We will show that the decomposition (3.1) with (3.4) and (3.21) is well defined as long as v_2 remains small in the exponentially weighted space X.

Now let us introduce the subspaces of $L^2(\mathbb{R})$ in order to analyse modulation parameters $c(t, y)$ and $x(t, y)$. For an $\eta_0 > 0$, let Y and Z be closed subspaces of $L^2(\mathbb{R})$ defined by

$$Y = \mathcal{F}_\eta^{-1} Z, \qquad Z = \{f \in L^2(\mathbb{R}) \mid \operatorname{supp} f \subset [-\eta_0, \eta_0]\}.$$

Let $Y_1 = \mathcal{F}_\eta^{-1} Z_1$ and $Z_1 = \{f \in Z \mid \|f\|_{Z_1} := \|f\|_{L^\infty} < \infty\}$.

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Remark 3.4. We have

$$\|f\|_{\dot H^s} \leqslant \eta_0^s\|f\|_{L^2} \quad \text{for any } s \geqslant 0 \text{ and } f \in Y, \tag{3.22}$$

since $\hat f$ is 0 outside of $[-\eta_0,\eta_0]$. We have $\|f\|_{L^\infty} \lesssim \|f\|_{L^2}$ for any $f \in Y$.

Let $\tilde P_1$ be a projection defined by $\tilde P_1 f = \mathcal{F}_\eta^{-1}\mathbf{1}_{[-\eta_0,\eta_0]}\mathcal{F}_y f$, where $\mathbf{1}_{[-\eta_0,\eta_0]}(\eta) = 1$ for $\eta \in [-\eta_0,\eta_0]$ and $\mathbf{1}_{[-\eta_0,\eta_0]}(\eta) = 0$ for $\eta \notin [-\eta_0,\eta_0]$. Then $\|\tilde P_1 f\|_{Y_1} \leqslant (2\pi)^{-1/2}\|f\|_{L^1}(\mathbb{R})$ for any $f \in L^1(\mathbb{R})$. In particular, for any $f, g \in Y$,

$$\|\tilde P_1(fg)\|_{Y_1} \leqslant (2\pi)^{-1/2}\|fg\|_{L^1} \leqslant (2\pi)^{-1/2}\|f\|_Y\|g\|_Y. \tag{3.23}$$

Next, we introduce functionals to prove the existence of the representation (3.1), (3.4) that satisfies the orthogonality condition (3.21). For $\tilde u \in X$, $\gamma, \tilde c \in Y$ and $L \geqslant 0$, let $c(y) = 2 + \tilde c(y)$ and

$$\begin{aligned} F_k[\tilde u,\tilde c,\gamma,L](\eta) := \mathbf{1}_{[-\eta_0,\eta_0]}(\eta) \lim_{M\to\infty}\int_{-M}^{M}\int_{\mathbb{R}}\{\tilde u(x,y) + \varphi(x) - \varphi_{c(y)}(x-\gamma(y)) \\ + \psi_{c(y),L}(x-\gamma(y))\}\overline{g_k^*(x-\gamma(y),\eta,c(y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

The mapping $F = (F_1, F_2)$ maps $X \times Y \times Y \times \mathbb{R}$ into $Z \times Z$.

Lemma 3.5 (Mizumachi [24, lemma 5.1]). *Let $\alpha \in (0,2)$, $\tilde u \in X$, $\tilde c, \gamma \in Y$ and $L \geqslant 0$. Then there exists a $\delta > 0$ such that if $\|\tilde c\|_Y + \|\gamma\|_Y \leqslant \delta$, then $F_k[\tilde u,\tilde c,\gamma,L] \in Z$ for $k = 1, 2$.*

Lemma 3.6 (Mizumachi [24, lemma 5.2]). *Let $\alpha \in (0,2)$. There exist positive constants δ_0, δ_1 and L_0 such that if $\|\tilde u\|_X < \delta_0$ and $L \geqslant L_0$, then there exists a unique $(\tilde c,\gamma)$ with $c = 2+\tilde c$ satisfying*

$$\|\tilde c\|_Y + \|\gamma\|_Y < \delta_1, \tag{3.24}$$

$$F_1[\tilde u,\tilde c,\gamma,L] = F_2[\tilde u,\tilde c,\gamma,L] = 0. \tag{3.25}$$

Moreover, the mapping $\{\tilde u \in X \mid \|u\|_X < \delta_0\} \ni \tilde u \mapsto (\tilde c,\gamma) =: \Phi(\tilde u)$ is C^1.

Remark 3.7. Let u be a solution of (1.1) satisfying $u(0,x,y) = \varphi(x) + v_0(x,y)$ and let $\tilde v_1$ be a solution of (3.3). Suppose that $v_0 \in H^{1/2}(\mathbb{R}^2)$. Since $\tilde v \in C([0,T];X)$ by lemma 3.1, and $\|\tilde v(0)\|_X$ is small, we see from lemma 3.6 that there exists a $T > 0$ such that

$$(v_2,\tilde c,\tilde x) \in C([0,T];X \times Y \times Y).$$

Moreover, replacing u in [24, remark 5.3] by $\tilde u = u - \tilde v_1$ and using lemma 3.1, we can see that there exists a $T > 0$ such that

$$(\tilde c(t),\tilde x(t)) = \Phi(\tilde v(t)) \in C([0,T];Y \times Y) \cap C^1((0,T);Y \times Y),$$

where $\tilde v(t,x,y) = \tilde u(t,x+4t,y) - \varphi(x)$. Moreover, we have $v_2 \in C([0,T];X)$ and $(\tilde v(0),\tilde c(0),\tilde x(0)) = (0,0,0)$.

Remark 3.8. Let u, $\tilde v_1$, $\tilde c$ and $\tilde x$ be as in remark 3.7 and let u_n and $\tilde v_{1,n}$ be as in lemma 3.3. By lemmas 3.1 and 3.3,

$$\tilde v_n(t,x,y) := u_n(t,x+4t,y) - \tilde v_{1,n}(t,x+4t,y) - \varphi(x) \in C([0,\infty);X),$$
$$\lim_{n\to\infty}\|\tilde v_n(t) - \tilde v(t)\|_X = 0,$$

and it follows from lemma 3.6 that there exists a $T > 0$ such that

$$(\tilde{c}_n(t), \tilde{x}_n(t)) := \Phi(\tilde{v}_n(t)) \in C([0,T]; Y \times Y) \cap C^1((0,T); Y \times Y),$$
$$\lim_{n\to\infty} \sup_{t\in[0,T]} (\|\tilde{c}_n(t) - \tilde{c}(t)\|_Y + \|\tilde{x}_n(t) - \tilde{x}(t)\|_Y) = 0.$$

Following the argument of [24, remark 5.3], we also have

$$\lim_{n\to\infty} \sup_{t\in[0,T]} (\|\partial_t\tilde{c}_n(t) - \partial_t\tilde{c}(t)\|_Y + \|\partial_t\tilde{x}_n(t) - \partial_t\tilde{x}(t)\|_Y) = 0.$$

We use a continuation principle that ensures the existence of (3.1) as long as $\|v_2(t)\|_X$ and $\|\tilde{c}(t)\|_Y$ remain small.

Proposition 3.9. *Let $\alpha \in (0,1)$, let δ_0 and L be the same as in lemma 3.6, and let $u(t)$ and $\tilde{v}_1(t)$ be as in lemma 3.1. Then there exists a constant $\delta_2 > 0$ such that if (3.1), (3.4) and (3.21) hold for $t \in [0,T)$, and $v_2(t,z,y)$, $\tilde{c}(t,y) := c(t,y) - 2$ and $\tilde{x}(t,y) := x(t,y) - 4t$ satisfy*

$$(\tilde{c}, \tilde{x}) \in C([0,T); Y \times Y) \cap C^1((0,T); Y \times Y), \tag{3.26}$$

$$\sup_{t\in[0,T)} \|v_2(t)\|_X \leqslant \frac{\delta_0}{2}, \qquad \sup_{t\in[0,T)} \|\tilde{c}(t)\|_Y < \delta_2, \qquad \sup_{t\in[0,T)} \|\tilde{x}(t)\|_Y < \infty, \tag{3.27}$$

then either $T = \infty$ or T is not the maximal time of the decomposition (3.1) satisfying (3.21), (3.26) and (3.27).

Proof. Since $u(t,x,y) - \varphi(x-4t) - \tilde{v}_1(t,x,y) \in C([0,\infty); X)$ by lemma 3.1, we can prove proposition 3.9 in the same way as [24, proposition 5.5]. □

4. Modulation equations

In this section we will derive a system of partial differential equations (PDEs) that describe the motion of the modulation parameters $c(t,y)$ and $x(t,y)$. Substituting $\tilde{v}_1(t,x,y) = v_1(t,z,y)$ with $z = x - x(t,y)$ into (1.1), we have

$$\partial_t v_1 - 2c\partial_z v_1 + \partial_z^3 v_1 + 3\partial_z^{-1}\partial_y^2 v_1 = \partial_z(N_{1,1} + N_{1,2}) + N_{1,3}, \tag{4.1}$$

where $N_{1,1} = -3v_1^2$, $N_{1,2} = \{x_t - 2c - 3(x_y)^2\}v_1$ and $N_{1,3} = 6\partial_y(x_y v_1) - 3x_{yy}v_1$. Substituting the ansatz (3.1) into (1.1), we have

$$\partial_t v = \mathcal{L}_c v + \ell + \partial_z(N_1 + N_2) + N_3, \tag{4.2}$$

where $\mathcal{L}_c v = -\partial_z(\partial_z^2 - 2c + 6\varphi_c)v - 3\partial_z^{-1}\partial_y^2$, $\ell = \ell_1 + \ell_2$, $\ell_k = \ell_{k1} + \ell_{k2} + \ell_{k3}$ $(k = 1,2)$, $\tilde{\psi}_c(z) = \psi_{c,L}(z + 3t)$ and

$$\begin{aligned}
\ell_{11} &= (x_t - 2c - 3(x_y)^2)\varphi_c' - (c_t - 6c_y x_y)\partial_c\varphi_c,\\
\ell_{12} &= 3x_{yy}\varphi_c,\\
\ell_{13} &= 3c_{yy}\int_z^\infty \partial_c\varphi_c(z_1)\,\mathrm{d}z_1 + 3(c_y)^2\int_z^\infty \partial_c^2\varphi_c(z_1)\,\mathrm{d}z_1,\\
\ell_{21} &= (c_t - 6c_y x_y)\partial_c\tilde{\psi}_c - (x_t - 4 - 3(x_y)^2)\tilde{\psi}_c',
\end{aligned}$$

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$$\begin{aligned}
\ell_{22} &= (\partial_z^3-\partial_z)\tilde\psi_c - 3\partial_z(\tilde\psi_c^2) + 6\partial_z(\varphi_c\tilde\psi_c) - 3x_{yy}\tilde\psi_c,\\
\ell_{23} &= -3c_{yy}\int_z^\infty \partial_c\tilde\psi_c(z_1)\,\mathrm{d}z_1 - 3(c_y)^2\int_z^\infty \partial_c^2\tilde\psi_c(z_1)\,\mathrm{d}z_1,\\
N_1 &= -3v^2,\\
N_2 &= \{x_t-2c-3(x_y)^2\}v + 6\tilde\psi_c v,\\
N_3 &= 6x_y\partial_y v + 3x_{yy}v = 6\partial_y(x_y v) - 3x_{yy}v.
\end{aligned}$$

Here we use the fact that φ_c is a solution of

$$\varphi_c'' - 2c\varphi_c + 3\varphi_c^2 = 0. \tag{4.3}$$

We slightly change the definition of $\tilde\psi$ from that in [24] in order to apply the virial identity to $\int_{\mathbb{R}^2}\tilde\psi_c(z)v_1^2(t,z,y)\,\mathrm{d}z\,\mathrm{d}y$.

Subtracting (4.1) from (4.2), we have

$$\partial_t v_2 = \mathcal{L}_c v_2 + \ell + \partial_z(N_{2,1}+N_{2,2}+N_{2,4}) + N_{2,3}, \tag{4.4}$$

where

$$\begin{aligned}
N_{2,1} &= -3(2v_1v_2+v_2^2), & N_{2,2} &= \{x_t-2c-3(x_y)^2\}v_2 + 6\tilde\psi_c v_2,\\
N_{2,3} &= 6\partial_y(x_yv_2) - 3x_{yy}v_2, & N_{2,4} &= 6(\tilde\psi_c-\varphi_c)v_1.
\end{aligned}$$

Let

$$\begin{aligned}
\mathbb{M}_{c,x}(T) &= \sup_{[0,T]}(\|\tilde c(t)\|_Y + \|x_y(t)\|_Y) + \|c_y\|_{L^2(0,T;Y)} + \|x_{yy}\|_{L^2(0,T;Y)},\\
\mathbb{M}_1(T) &= \sup_{t\in[0,T]}\|v_1(t)\|_{L^2} + \|\mathcal{E}(v_1)^{1/2}\|_{L^2(0,T;W(t))},\\
\mathbb{M}_1'(T) &= \sup_{t\in[0,T]}\|\tilde v_1(t)\|_{L^3},\\
\mathbb{M}_2(T) &= \sup_{0\leqslant t\leqslant T}\|v_2(t)\|_X + \|\mathcal{E}(v_2)^{1/2}\|_{L^2(0,T;X)},\\
\mathbb{M}_v(T) &= \sup_{t\in[0,T]}\|v(t)\|_{L^2},
\end{aligned}$$

where $\|v\|_{W(t)} = \|(\mathrm{e}^{-\alpha|z|/2} + \mathrm{e}^{-\alpha|z+3t+L|})v\|_{L^2(\mathbb{R}^2)}$, L is a large positive constant and

$$\partial_z^{-1}\partial_y v(t,z,y) := \mathcal{F}^{-1}_{\xi,\eta}\Bigl(\frac{\eta}{\xi}\mathcal{F}_{z,y}v(t,\xi,\eta)\Bigr).$$

By lemma 3.1, we have $v_2(t)\in X$ and

$$\partial_x^{-1}v_2(t,z,y) = -\int_z^\infty v_2(t,z_1,y)\,\mathrm{d}z_1 \in X$$

if $x(t,\cdot)\in L^\infty(\mathbb{R})$.

Now we will derive modulation equations for $c(t,y)$ and $x(t,y)$ from the orthogonality condition (3.21) assuming the smallness of $\mathbb{M}_{c,x}(T)$, $\mathbb{M}_1(T)$ and $\mathbb{M}_2(T)$. It follows from [31] and lemma 3.2 of [17] that $\tilde v_1(t),\tilde v(t)\in C(\mathbb{R};L^2(\mathbb{R}^2))$ and $\partial_x^{-1}\partial_y\tilde v_1, \partial_x^{-1}\partial_y\tilde v \in L_x^\infty L^2([-T,T]\times\mathbb{R}_y)$ for any $T>0$. Moreover, lemma 3.1 implies

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that $\tilde{v}(t) \in C([0,\infty);X)$ and $\partial_x^{-1}\partial_y\tilde{v} \in L^2(0,T;X)$. If $\mathbb{M}_{c,x}(T)$ and $\mathbb{M}_2(T)$ are sufficiently small, then we see from remark 3.7 and proposition 3.9 that the decomposition (3.1) satisfying (3.21) and (3.26) exists for $t \in [0,T]$. Since $Y \subset \bigcap_{s\geqslant 0} H^s(\mathbb{R})$, we have

$$\begin{aligned} v_2(t,z,y) &- \tilde{v}(t,z+\tilde{x}(t,y),y) \\ &= \varphi(z+\tilde{x}(t,y)) - \varphi_{c(t,y)}(z) + \tilde{\psi}_{c(t,y)}(z) \in L^2(\mathbb{R}^2)\cap X, \end{aligned} \tag{4.5}$$

and we easily see that $v_2(t) \in C([0,T];X\cap L^2(\mathbb{R}^2))$. Moreover, since

$$\int_{\mathbb{R}} \{\varphi(z+\tilde{x}(t,y)) - \varphi_{c(t,y)}(z) + \tilde{\psi}_{c(t,y)}(z)\}\,\mathrm{d}z = 0$$

for any $y \in \mathbb{R}$ by (3.2), and its integrand decays exponentially as $z \to \pm\infty$, we have

$$(\partial_z^{-1}\partial_y v_2)(t,z,y) \in L^2(0,T;X) \cap L_x^\infty L^2([-T,T]\times\mathbb{R}_y).$$

Approximating $g_k^*(z,\eta)$ by $C_0^4(\mathbb{R})$-functions in $L^2(\mathbb{R};\mathrm{e}^{-2\alpha z}\,\mathrm{d}z)$ and using proposition 3.9 and remark 3.7, we can justify that the mapping

$$t \mapsto \int_{\mathbb{R}^2} v_2(t,z,y)\overline{g_k^*(z,\eta,c(t,y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y \in Z$$

is C^1 for $t \in [0,T]$ if we have (3.26) and (3.27). Differentiating (3.21) with respect to t and substituting (4.4) into the resulting equation, we have, in $L^2(-\eta_0,\eta_0)$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}&\int_{\mathbb{R}^2} v_2(t,z,y)\overline{g_k^*(z,\eta,c(t,y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y \\ &= \int_{\mathbb{R}^2} \ell\overline{g_k^*(z,\eta,c(t,y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y + \sum_{j=1}^{6}\mathrm{II}_k^j(t,\eta) = 0, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} \mathrm{II}_k^1 &= \int_{\mathbb{R}^2} v_2(t,z,y)\mathcal{L}_{c(t,y)}^*(\overline{g_k^*(t,z,c(t,y))\mathrm{e}^{iy\eta}})\,\mathrm{d}z\,\mathrm{d}y, \\ \mathrm{II}_k^2 &= -\int_{\mathbb{R}^2} N_{2,1}\overline{\partial_z g_k^*(z,\eta,c(t,y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y, \\ \mathrm{II}_k^3 &= \int_{\mathbb{R}^2} N_{2,3}\overline{g_k^*(z,\eta,c(t,y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y \\ &\quad + 6\int_{\mathbb{R}^2} v_2(t,z,y)c_y(t,y)x_y(t,y)\overline{\partial_c g_k^*(z,\eta,c(t,y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y, \\ \mathrm{II}_k^4 &= \int_{\mathbb{R}^2} v_2(t,z,y)(c_t - 6c_y x_y)(t,y)\overline{\partial_c g_k^*(z,\eta,c(t,y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y, \\ \mathrm{II}_k^5 &= -\int_{\mathbb{R}^2} N_{2,2}\overline{\partial_z g_k^*(z,\eta,c(t,y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y, \\ \mathrm{II}_k^6 &= -\int_{\mathbb{R}^2} N_{2,4}\overline{\partial_z g_k^*(z,\eta,c(t,y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y. \end{aligned}$$

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The modulation PDEs of $c(t,y)$ and $x(t,y)$ can be obtained by computing the inverse Fourier transform of (4.6) in η. The leading term of

$$\frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2}\ell_1\overline{g_k^*(z,\eta,c(t,y_1))}\mathrm{e}^{\mathrm{i}\eta(y-y_1)}\,\mathrm{d}z\,\mathrm{d}y_1\,\mathrm{d}\eta$$

is

$$G_k(t,y)=\int_{\mathbb{R}}\ell_1\overline{g_k^*(z,0,c(t,y))}\,\mathrm{d}z. \tag{4.7}$$

Since

$$g_1^*(z,0,c)=\varphi_c(z)\quad\text{and}\quad g_2^*(z,0,c)=(\tfrac{1}{2}c)^{3/2}\int_{-\infty}^{z}\partial_c\varphi_c,$$

we can compute G_1 and G_2 explicitly.

LEMMA 4.1 (Mizumachi [24, lemma 6.1]). *Let* $\mu_1=1/2-\pi^2/12$ *and* $\mu_2=\pi^2/32-3/16$. *Then*

$$\begin{aligned}
G_1&=16x_{yy}(\tfrac{1}{2}c)^{3/2}-2(c_t-6c_yx_y)(\tfrac{1}{2}c)^{1/2}+6c_{yy}-\frac{3}{c}(c_y)^2,\\
G_2&=-2(x_t-2c-3(x_y)^2)(\tfrac{1}{2}c)^2+6x_{yy}(\tfrac{1}{2}c)^{3/2}-\tfrac{1}{2}(c_t-6c_yx_y)(\tfrac{1}{2}c)^{1/2}\\
&\quad+\mu_1c_{yy}+\mu_2(c_y)^2(\tfrac{1}{2}c)^{-1}.
\end{aligned}$$

We remark that (G_1,G_2) are the dominant parts of the modulation equations for c and x. Now we will write the remainder part of

$$\int_{\mathbb{R}^2}\ell_1\overline{g_k^*(z,\eta,c(t,y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y$$

in the same way as [24]. For $q_c=\varphi_c,\varphi_c',\partial_c\varphi_c$ and

$$\partial_z^{-1}\partial_c^m\varphi_c(z)=-\int_z^\infty\partial_c^m\varphi_c(z_1)\,\mathrm{d}z_1\quad(m\geqslant 1),$$

let $S_k^1[q_c]$ and $S_k^2[q_c]$ be operators defined by

$$\begin{aligned}
S_k^1[q_c](f)(t,y)&=\frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2}f(y_1)q_2(z)\overline{g_{k1}^*(z,\eta,2)}\mathrm{e}^{\mathrm{i}(y-y_1)\eta}\,\mathrm{d}y_1\,\mathrm{d}z\,\mathrm{d}\eta,\\
S_k^2[q_c](f)(t,y)&=\frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2}f(y_1)\tilde{c}(t,y_1)\overline{g_{k2}^*(z,\eta,c(t,y_1))}\mathrm{e}^{\mathrm{i}(y-y_1)\eta}\,\mathrm{d}y_1\,\mathrm{d}z\,\mathrm{d}\eta,
\end{aligned}$$

where

$$\begin{aligned}
g_{k1}^*(z,\eta,c)&=\frac{g_k^*(z,\eta,c)-g_k^*(z,0,c)}{\eta^2},\qquad \delta q_c(z)=\frac{q_c(z)-q_2(z)}{c-2},\\
g_{k2}^*(z,\eta,c)&=g_{k1}^*(z,\eta,2)\delta q_c(z)+\frac{g_{k1}^*(z,\eta,c)-g_{k1}^*(z,\eta,2)}{c-2}q_c(z).
\end{aligned}$$

Note that $S_k^1 \in B(Y)$ and S_k^1 is independent of $c(t,y)$ whereas $\|S_k^2\|_{B(Y,Y_1)} \lesssim \|\tilde{c}\|_Y$; see [24, claims B.1 and B.2]. Using S_k^j $(j,k=1,2)$, we have

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2} & \ell_1(\overline{g_k^*(z,\eta,c(t,y)) - g_k^*(z,0,c(t,y))})\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}\eta \\
&= -\sum_{j=1,2}\partial_y^2(S_k^j[\varphi_c'](x_t - 2c - 3(x_y)^2) - S_k^j[\partial_c\varphi_c](c_t - 6c_y x_y)) \\
&\quad - \partial_y^2(R_k^1 + R_k^2),
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
R_k^1 &= 3S_k^1[\varphi_c](x_{yy}) - 3S_k^1[\partial_z^{-1}\partial_c\varphi_c](c_{yy}),\\
R_k^2 &= 3S_k^2[\varphi_c](x_{yy}) - 3S_k^2[\partial_z^{-1}\partial_c\varphi_c](c_{yy}) - 3\sum_{j=1,2}S_k^j[\partial_z^{-1}\partial_c^2\varphi_c](c_y^2).
\end{aligned}$$

We rewrite the linear term R_k^1 as

$$\begin{pmatrix} R_1^1 \\ R_2^1 \end{pmatrix} = \tilde{S}_0 \begin{pmatrix} c_{yy} \\ x_{yy} \end{pmatrix}, \quad \tilde{S}_0 = 3\begin{pmatrix} -S_1^1[\partial_z^{-1}\partial_c\varphi_c] & S_1^1[\varphi_c] \\ -S_2^1[\partial_z^{-1}\partial_c\varphi_c] & S_2^1[\varphi_c] \end{pmatrix}.$$

Next, we deal with

$$\frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2}\ell_2\overline{g_k^*(z,\eta,c(t,y_1))}\mathrm{e}^{\mathrm{i}(y-y_1)\eta}\,\mathrm{d}z\,\mathrm{d}y_1\,\mathrm{d}\eta.$$

Let $S_k^3[p]$ and $S_k^4[p]$ be operators defined by

$$\begin{aligned}
S_k^3[p](f)(t,y) &= \frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2} f(y_1)p(z+3t+L)\overline{g_k^*(z,\eta)}\mathrm{e}^{\mathrm{i}(y-y_1)\eta}\,\mathrm{d}y_1\,\mathrm{d}z\,\mathrm{d}\eta,\\
S_k^4[p](f)(t,y) &= \frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2} f(y_1)\tilde{c}(t,y_1)p(z+3t+L)\\
&\qquad\times\overline{g_{k3}^*(z,\eta,c(t,y_1))}\mathrm{e}^{\mathrm{i}(y-y_1)\eta}\,\mathrm{d}y_1\,\mathrm{d}z\,\mathrm{d}\eta,
\end{aligned}$$

where $g_{k3}^*(z,\eta,c) = (c-2)^{-1}(g_k^*(z,\eta,c) - g_k^*(z,\eta))$. By the definition of $\tilde{\psi}_c$,

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2} & \ell_{21}\overline{g_k^*(z,\eta,c(t,y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}\eta\\
&= (S_k^3[\psi] + S_k^4[\psi])(\sqrt{2/c}(c_t - 6c_y x_y))\\
&\quad - 2\sqrt{2}(S_k^3[\psi'] + S_k^4[\psi'])(\sqrt{c} - \sqrt{2})(x_t - 4 - 3(x_y)^2).
\end{aligned} \tag{4.9}$$

The operator norms of $S_k^j[\psi]$, $S_k^j[\psi']$ $(j=3,4,\ k=1,2)$ decay exponentially as $t \to \infty$ because $g_k^*(z,\eta)$ and $g_k^*(z,\eta,c)$ are exponentially localized as $z \to -\infty$ and $\psi \in C_0^\infty(\mathbb{R})$; see (A 3) and (A 4) in appendix A.

Next, we decompose

$$(2\pi)^{-1}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2}(\ell_{22} + \ell_{23})\overline{g_k^*(z,\eta,c(t,y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}\eta$$

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into a linear part and a nonlinear part with respect to $\tilde{c}$ and $\tilde{x}$. The linear part can be written as

$$\frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2}\ell_{2,\mathrm{lin}}(t,z,y_1)\overline{g_k^*(z,\eta)}\mathrm{e}^{\mathrm{i}(y-y_1)\eta}\,\mathrm{d}y_1\,\mathrm{d}z\,\mathrm{d}\eta =: \tilde{a}_k(t,D_y)\tilde{c}, \tag{4.10}$$

where

$$\begin{aligned}\ell_{2,\mathrm{lin}}(t,z,y) &= \tilde{c}(t,y)\partial_z\{\partial_z^2-1+6\varphi(z)\}\psi(z+3t+L)\\ &\quad - 3c_{yy}(t,y)\int_z^\infty \psi(z_1+3t+L)\,\mathrm{d}z_1,\\ \tilde{a}_k(t,\eta) &= \biggl[\int_{\mathbb{R}}\{\partial_z(\partial_z^2-1+6\varphi(z))\psi(z+3t+L)\}\overline{g_k^*(z,\eta)}\,\mathrm{d}z\\ &\quad + 3\eta^2\int_{\mathbb{R}}\biggl(\int_z^\infty \psi(z_1+3t+L)\,\mathrm{d}z_1\biggr)\overline{g_k^*(z,\eta)}\,\mathrm{d}z\biggr]\mathbf{1}_{[-\eta_0,\eta_0]}(\eta),\end{aligned} \tag{4.11}$$

and the nonlinear part is

$$\begin{aligned}R_k^3(t,y) := \frac{1}{2\pi}&\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}}(\ell_{22}+\ell_{23})\overline{g_k^*(z,\eta,c(t,y_1))}\mathrm{e}^{\mathrm{i}(y-y_1)\eta}\,\mathrm{d}z\,\mathrm{d}y_1\,\mathrm{d}\eta\\ &- \frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}}\ell_{2,\mathrm{lin}}\overline{g_k^*(z,\eta)}\mathrm{e}^{\mathrm{i}(y-y_1)\eta}\,\mathrm{d}z\,\mathrm{d}y_1\,\mathrm{d}\eta.\end{aligned} \tag{4.12}$$

Next, we deal with II_k^j $(j=1,\dots,6)$ in (4.6). Let

$$\begin{aligned}\mathrm{II}_{k1}^3 &= -3\int_{\mathbb{R}^2} v_2(t,z,y)x_{yy}(t,y)\overline{g_k^*(z,\eta,c(t,y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y,\\ \mathrm{II}_{k2}^3 &= 6\int_{\mathbb{R}^2} v_2(t,z,y)x_y(t,y)\overline{g_k^*(z,\eta,c(t,y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y\end{aligned}$$

so that $\mathrm{II}_k^3 = \mathrm{II}_{k1}^3 + \mathrm{i}\eta\mathrm{II}_{k2}^3$. For $k=1$ and 2, let

$$\left.\begin{aligned}R_k^4(t,y) &= \frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\{\mathrm{II}_k^1(t,\eta)+\mathrm{II}_k^2(t,\eta)+\mathrm{II}_{k1}^3(t,\eta)\}\mathrm{e}^{\mathrm{i}y\eta}\,\mathrm{d}\eta,\\ R_k^5(t,y) &= \frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\mathrm{II}_{k2}^3(t,\eta)\mathrm{e}^{\mathrm{i}y\eta}\,\mathrm{d}\eta.\end{aligned}\right\} \tag{4.13}$$

Let S_k^5 and S_k^6 be operators defined by

$$\begin{aligned}S_k^5(f)(t,y) &= \frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2} v_2(t,z,y_1)f(y_1)\overline{\partial_c g_k^*(z,\eta,c(t,y_1))}\mathrm{e}^{\mathrm{i}(y-y_1)\eta}\,\mathrm{d}z\,\mathrm{d}y_1\,\mathrm{d}\eta,\\ S_k^6(f)(t,y) &= -\frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2} v_2(t,z,y_1)f(y_1)\overline{\partial_z g_k^*(z,\eta,c(t,y_1))}\mathrm{e}^{\mathrm{i}(y-y_1)\eta}\,\mathrm{d}z\,\mathrm{d}y_1\,\mathrm{d}\eta,\end{aligned}$$

and

$$R_k^6 = -\frac{3}{\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2}\psi_{c(t,y_1),L}(z+3t)v_2(t,z,y_1)\overline{\partial_z g_k^*(z,\eta,c(t,y_1))}\mathrm{e}^{\mathrm{i}(y-y_1)\eta}\,\mathrm{d}y_1\,\mathrm{d}z\,\mathrm{d}\eta.$$

Then

$$\left.\begin{aligned}\mathbf{1}_{[-\eta_0,\eta_0]}(\eta)\mathrm{II}_k^4(t,\eta) &= \sqrt{2\pi}\mathcal{F}_y(S_k^5(c_t-6c_yx_y)),\\ \mathbf{1}_{[-\eta_0,\eta_0]}(\eta)\mathrm{II}_k^5(t,\eta) &= \sqrt{2\pi}\mathcal{F}_y\{S_k^6(x_t-2c-3(x_y)^2)+R_k^6\}.\end{aligned}\right\} \tag{4.14}$$

Let $R^{v_1}=(R_1^{v_1},R_2^{v_1})^{\mathrm{T}}$ and

$$R_k^{v_1}(t,y)=\frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\mathrm{II}_k^6(t,\eta)\mathrm{e}^{\mathrm{i}y\eta}\,\mathrm{d}\eta \quad \text{for } k=1 \text{ and } 2. \tag{4.15}$$

Using (4.7)–(4.15), we can rewrite (4.6) as

$$\begin{aligned}\tilde{P}_1\begin{pmatrix}G_1\\G_2\end{pmatrix}-(\partial_y^2(\tilde{S}_1+\tilde{S}_2)-\tilde{S}_3-\tilde{S}_4-\tilde{S}_5)\begin{pmatrix}c_t-6c_yx_y\\x_t-2c-3(x_y)^2\end{pmatrix}&\\ +\widetilde{\mathcal{A}}_1(t)\begin{pmatrix}\tilde{c}\\x\end{pmatrix}-\partial_y^2R^1+\tilde{R}^1+\partial_y\tilde{R}^2+R^{v_1}&=0,\end{aligned} \tag{4.16}$$

where $R^j=(R_1^j,R_2^j)^{\mathrm{T}}$ for $j=1,\dots,6,v_1$ and

$$\tilde{S}_j=\begin{pmatrix}-S_1^j[\partial_c\varphi_c] & S_1^j[\varphi_c']\\ -S_2^j[\partial_c\varphi_c] & S_2^j[\varphi_c']\end{pmatrix}\quad\text{for } j=1,\,2,\qquad \tilde{S}_3=\begin{pmatrix}S_1^3[\psi] & 0\\ S_2^3[\psi] & 0\end{pmatrix},$$

$$\tilde{S}_4=\begin{pmatrix}S_1^3[\psi]((\sqrt{2/c}-1)\cdot)+S_1^4[\psi](\sqrt{2/c}\cdot) & -2(S_1^3[\psi']+S_1^4[\psi'])((\sqrt{2c}-2)\cdot)\\ S_2^3[\psi]((\sqrt{2/c}-1)\cdot)+S_2^4[\psi](\sqrt{2/c}\cdot) & -2(S_2^3[\psi']+S_2^4[\psi'])((\sqrt{2c}-2)\cdot)\end{pmatrix},$$

$$\tilde{S}_5=\begin{pmatrix}S_1^5 & S_1^6\\ S_2^5 & S_2^6\end{pmatrix},\qquad \widetilde{\mathcal{A}}_1(t)=\begin{pmatrix}\tilde{a}_1(t,D_y) & 0\\ \tilde{a}_2(t,D_y) & 0\end{pmatrix},$$

$$\tilde{R}^1=R^3+R^4+R^6+\tilde{S}_4\begin{pmatrix}0\\2\tilde{c}\end{pmatrix},\qquad \tilde{R}^2=R^5-\partial_yR^2.$$

To translate the nonlinear terms $6(c/2)^{1/2}c_yx_y$ and $16x_{yy}\{(c/2)^{3/2}-1\}$ in G_1 into a divergence form, we will make use of the following change of variables. Let

$$\begin{gathered}b(t,\cdot)=\tfrac{1}{3}\tilde{P}_1\{\sqrt{2}c(t,\cdot)^{3/2}-4\},\qquad \mathcal{C}_1=\tfrac{1}{2}\tilde{P}_1\{c(t,\cdot)^2-4\}\tilde{P}_1,\\ \widetilde{\mathcal{C}}_1=\begin{pmatrix}0&0\\0&\mathcal{C}_1\end{pmatrix},\qquad B_1=\begin{pmatrix}2&0\\ \frac{1}{2}&2\end{pmatrix},\qquad B_2=\begin{pmatrix}6&16\\ \mu_1&6\end{pmatrix}.\end{gathered} \tag{4.17}$$

We remark that $b\simeq\tilde{c}=c-2$ if c is close to 2 (see [24, claim D.6]). By (4.17), we have $b_t=\tilde{P}_1(c/2)^{1/2}c_t$, $b_y=\tilde{P}_1(c/2)^{1/2}c_y$ and it follows from lemma 4.1 that

$$\tilde{P}_1\begin{pmatrix}G_1\\G_2\end{pmatrix}=-(B_1+\widetilde{\mathcal{C}}_1)\tilde{P}_1\begin{pmatrix}b_t-6(bx_y)_y\\x_t-2c-3(x_y)^2\end{pmatrix}+B_2\begin{pmatrix}c_{yy}\\x_{yy}\end{pmatrix}+\tilde{P}_1R^7, \tag{4.18}$$

where $R^7=(R_1^7,R_2^7)^{\mathrm{T}}$ and

$$\left.\begin{aligned}R_1^7&=\{4\sqrt{2}c^{3/2}-16-12b\}x_{yy}-6(2b_y-(2c)^{1/2}c_y)x_y-3c^{-1}(c_y)^2,\\ R_2^7&=6\{(\tfrac{1}{2}c)^{3/2}-1\}x_{yy}+3(\tfrac{1}{2}c)^{1/2}c_yx_y-3(bx_y)_y+\mu_2\frac{2}{c}(c_y)^2\\ &\quad+\tfrac{3}{2}(c^2-4)(I-\tilde{P}_1)(x_y)^2.\end{aligned}\right\} \tag{4.19}$$

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Let

$$\mathcal{C}_2 = \tilde{P}_1\left\{\left(\frac{c(t,\cdot)}{2}\right)^{1/2} - 1\right\}\tilde{P}_1, \qquad \widetilde{\mathcal{C}}_2 = \begin{pmatrix} \mathcal{C}_2 & 0 \\ 0 & 0 \end{pmatrix},$$
$$\bar{S}_j = \tilde{S}_j(I + \widetilde{\mathcal{C}}_2)^{-1} \quad \text{for } 1 \leqslant j \leqslant 5$$

and

$$B_3 = B_1 + \widetilde{\mathcal{C}}_1 + \partial_y^2(\bar{S}_1 + \bar{S}_2) - \bar{S}_3 - \bar{S}_4 - \bar{S}_5. \tag{4.20}$$

Note that $I + \widetilde{\mathcal{C}}_2$ is invertible as long as $\tilde{c}(t,\cdot)$ remains small in Y and that B_3 is a bounded operator on $Y \times Y$ depending on $\tilde{c}$ and v. Substituting (4.18) into (4.16), we have

$$\begin{aligned} B_3\tilde{P}_1 &\begin{pmatrix} b_t - 6(bx_y)_y \\ x_t - 2c - 3(x_y)^2 \end{pmatrix} \\ &= \{(B_2 - \partial_y^2\tilde{S}_0)\partial_y^2 + \widetilde{\mathcal{A}}_1(t)\}\begin{pmatrix} b \\ x \end{pmatrix} + \tilde{P}_1 R^7 + \tilde{R}^1 + \tilde{R}^3 + \partial_y(\tilde{R}^2 + \tilde{R}^4) + R^{v_1}, \end{aligned}$$

where $\tilde{R}^3 = R^9 + R^{11}$, $\tilde{R}^4 = R^8 + R^{10}$ and

$$R^8 = 6\partial_y(\bar{S}_1 + \bar{S}_2)\begin{pmatrix} (I + \mathcal{C}_2)(c_y x_y) - (bx_y)_y \\ 0 \end{pmatrix},$$
$$R^9 = -6\sum_{3\leqslant j\leqslant 5}\bar{S}_j\begin{pmatrix} (I + \mathcal{C}_2)(c_y x_y) - (bx_y)_y \\ 0 \end{pmatrix},$$
$$R^{10} = (\partial_y^2\tilde{S}_0 - B_2)\begin{pmatrix} b_y - c_y \\ 0 \end{pmatrix}, \qquad R^{11} = \widetilde{\mathcal{A}}_1(t)\begin{pmatrix} \tilde{c} - b \\ 0 \end{pmatrix}.$$

We have the following.

Proposition 4.2. *There exists a $\delta_3 > 0$ such that if $\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T) + \eta_0 + \mathrm{e}^{-\alpha L} < \delta_3$ for a $T \geqslant 0$, then*

$$\begin{pmatrix} b_t \\ \tilde{x}_t \end{pmatrix} = \mathcal{A}(t)\begin{pmatrix} b \\ \tilde{x} \end{pmatrix} + \sum_{i=1}^{5}\mathcal{N}^i, \tag{4.21}$$

where $B_4 = B_1 + \partial_y^2\tilde{S}_1 - \tilde{S}_3 = B_3|_{\tilde{c}=0, v_2=0}$,

$$\mathcal{A}(t) = B_4^{-1}(B_2 - \partial_y^2\tilde{S}_0)\partial_y^2 + B_3^{-1}\widetilde{\mathcal{A}}_1(t) + \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},$$
$$\mathcal{N}^1 = \tilde{P}_1\begin{pmatrix} 6(b\tilde{x}_y)_y \\ 2(\tilde{c} - b) + 3(\tilde{x}_y)^2 \end{pmatrix}, \qquad \mathcal{N}^2 = \mathcal{N}^{2a} + \mathcal{N}^{2b},$$
$$\mathcal{N}^{2a} = B_3^{-1}\left(\tilde{P}_1\begin{pmatrix} R_1^7 \\ 0 \end{pmatrix} + \tilde{R}^1 + \tilde{R}^3\right), \qquad \mathcal{N}^{2b} = B_3^{-1}\tilde{P}_1\begin{pmatrix} 0 \\ R_2^7 \end{pmatrix},$$
$$\mathcal{N}^3 = B_3^{-1}\partial_y(\tilde{R}^2 + \tilde{R}^4), \qquad \mathcal{N}^4 = (B_3^{-1} - B_4^{-1})(B_2 - \partial_y^2\tilde{S}_0)\partial_y\begin{pmatrix} b_y \\ x_y \end{pmatrix},$$
$$\mathcal{N}^5 = B_3^{-1}R^{v_1}.$$

Moreover, if $v_2(0) = 0$,

$$b(0,\cdot) = 0, \qquad x(0,\cdot) = 0. \tag{4.22}$$

Proof. Proposition 3.9 implies (3.1) persists on $[0,T]$ if δ_3 is sufficiently small. Moreover, claims 4.3–4.5 below imply that B_3, B_4 and $I+\widetilde{\mathcal{C}}_k$ are invertible if $\|\tilde{c}(t)\|_Y$, $\|v(t)\|_X$, η_0 and $\mathrm{e}^{-\alpha L}$ are sufficiently small. Thus, we have (4.21). Since $v_2(0) = 0$, we have (4.22) from lemma 3.6. This completes the proof of proposition 4.2. $\square$

Claim 4.3. *There exist positive constants δ and C such that if $\mathbb{M}_{c,x}(T) \leqslant \delta$, then, for $s \in [0,T]$ and $k = 1,2$,*

$$\sup_{t\in[0,T]} \|\widetilde{\mathcal{C}}_k(t)\|_{B(Y)} + \|\widetilde{\mathcal{C}}_k\|_{L^4(0,T;B(Y))} \leqslant C\mathbb{M}_{c,x}(T), \tag{4.23}$$

$$\sup_{t\in[0,T]} \|\widetilde{\mathcal{C}}_k(t)\|_{B(Y,Y_1)} \leqslant C\mathbb{M}_{c,x}(T), \tag{4.24}$$

$$\|(I+\widetilde{\mathcal{C}}_k)^{-1}\|_{B(Y)} + \|(I+\widetilde{\mathcal{C}}_k)^{-1}\|_{B(Y_1)} \leqslant C.$$

Claim 4.3 follows from [24, claim B.6] and the definition of $\mathbb{M}_{c,x}(T)$.

Claim 4.4. *There exist positive constants C and δ such that if $\eta_0^2 + \mathrm{e}^{-\alpha L} \leqslant \delta$, then*

$$\|B_4^{-1}\|_{B(Y)} + \|B_4^{-1}\|_{B(Y_1)} \leqslant C.$$

Claim 4.5. *There exist positive constants δ and C such that if $\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T) + \eta_0^2 + \mathrm{e}^{-\alpha L} \leqslant \delta$, then, for $t \in [0,T]$,*

$$\|B_3 - B_4\|_{B(Y)} + \|B_3 - B_4\|_{B(Y_1)} \leqslant C(\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T)),$$
$$\|B_3^{-1}\|_{B(Y)} + \|B_3^{-1}\|_{B(Y_1)} \leqslant C.$$

The proof of claims 4.4 and 4.5 is exactly the same as the proof of claims 6.2 and 6.3 in [24].

5. *A priori* estimates for the local speed and the local phase shift

In this section we will estimate $\mathbb{M}_{c,x}(T)$ assuming the smallness of $\mathbb{M}_{c,x}(T)$, $\mathbb{M}_i(T)$ $(i = 1,2)$, η_0 and $\mathrm{e}^{-\alpha L}$.

Lemma 5.1. *There exist positive constants δ_4 and C such that if $\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T) + \eta_0 + \mathrm{e}^{-\alpha L} \leqslant \delta_4$, then*

$$\mathbb{M}_{c,x}(T) \leqslant C(\|v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2). \tag{5.1}$$

Before we start to prove lemma 5.1, we estimate the upper bounds of c_t and $x_t - 2c - 3(x_y)^2$.

Lemma 5.2. *Let δ_3 be as in proposition 4.2. Suppose that $\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T) + \eta_0 + \mathrm{e}^{-\alpha L} < \delta_3$ for a $T \geqslant 0$. Then*

$$\begin{aligned}\|c_t\|_{L^\infty(0,T;Y)\cap L^2(0,T;Y)} &+ \|x_t - 2c - 3(x_y)^2\|_{L^\infty(0,T;L^2(\mathbb{R}))\cap L^2(0,T;L^2(\mathbb{R}))} \\ &\lesssim \eta_0^{-1/2}\mathbb{M}_{c,x}(T)^2 + \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2.\end{aligned}$$

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To begin with, we will estimate the nonlinear terms of (4.21).

CLAIM 5.3. *Let* $\mathbb{M}_{c,x}(T)$*,* $\mathbb{M}_1(T)$*,* $\mathbb{M}_2(T)$*,* η_0 *and* L *be as in lemma 5.2. Then*

$$\sup_{t\in[0,T]}\|b\tilde{x}_y\|_Y+\|(b\tilde{x}_y)_y\|_{L^2(0,T;Y)}\lesssim\mathbb{M}_{c,x}(T)^2, \tag{5.2}$$

$$\sup_{t\in[0,T]}\|\mathcal{N}^{2a}(t)\|_Y+\|\mathcal{N}^{2a}\|_{L^1(0,T;Y)}\lesssim\mathbb{M}_{c,x}(T)^2, \tag{5.3}$$

$$\sup_{t\in[0,T]}\|\mathcal{N}^{2b}(t)\|_Y+\|\mathcal{N}^{2b}\|_{L^2(0,T;Y)}\lesssim\mathbb{M}_{c,x}(T)^2+\mathbb{M}_1(T)^2+\mathbb{M}_2(T)^2, \tag{5.4}$$

$$\sup_{t\in[0,T]}\|\mathcal{N}^{3}(t)\|_Y+\|\mathcal{N}^{3}\|_{L^2(0,T;Y)}\lesssim\mathbb{M}_{c,x}(T)^2+\mathbb{M}_{c,x}(T)\mathbb{M}_2(T), \tag{5.5}$$

$$\sup_{t\in[0,T]}\|\mathcal{N}^{4}(t)\|_Y+\|\mathcal{N}^{4}\|_{L^2(0,T;Y)}\lesssim\mathbb{M}_{c,x}(T)^2+\mathbb{M}_{c,x}(T)\mathbb{M}_2(T), \tag{5.6}$$

$$\sup_{t\in[0,T]}\|\mathcal{N}^{5}(t)\|_Y+\|\mathcal{N}^{5}\|_{L^2(0,T;Y)}\lesssim\mathbb{M}_1(T). \tag{5.7}$$

Proof of claim 5.3. Equation (5.2) follows from [24, claim D.6] and the fact that $Y\subset H^1(\mathbb{R})$. Equations (5.3)–(5.5) follow from claims 4.5, B.1, B.2, B.4–B.6, (A 3) and (A 4).

Next, we will estimate $\mathcal{N}^4$. Let $\bar{S}'=\partial_y^2\bar{S}_2+\bar{S}_4+\bar{S}_5$ and $\bar{S}''=\partial_y^2(\tilde{S}_1-\bar{S}_1)+\tilde{S}_3-\bar{S}_3$. Then $B_3^{-1}-B_4^{-1}=B_3^{-1}(\bar{S}'+\bar{S}'')B_4^{-1}$ and

$$\sup_{t\in[0,T]}\|\bar{S}'\|_{B(Y,Y_1)}\lesssim\mathbb{M}_{c,x}(T)+\mathbb{M}_2(T) \tag{5.8}$$

by (A 2), (A 6) and (A 7), and

$$\sup_{t\in[0,T]}\|\bar{S}''\|_{B(Y,Y_1)}\lesssim(\eta_0^2+\mathrm{e}^{-\alpha L})\mathbb{M}_{c,x}(T) \tag{5.9}$$

by (A 1), (A 6) and claim 4.3. Combining (5.8), (5.9) with claims 4.4 and 4.5, we have (5.6). We can prove (5.7) in the same way as (B 14) of claim B.7 in appendix B. □

Proof of lemma 5.2. Claims 5.3 and B.3, (4.21) and [24, (D.12)] imply that

$$\begin{aligned}
&\|c_t\|_{L^\infty(0,T;Y)\cap L^2(0,T;Y)}+\|x_t-2c-3\tilde{P}_1(x_y)^2\|_{L^\infty(0,T;Y)\cap L^2(0,T;Y)}\\
&\qquad\lesssim\|b_{yy}\|_Y+\|x_{yy}\|_Y+\|\widetilde{\mathcal{A}}_1(t)(b,\tilde{x})\|_Y+\|(b\tilde{x}_y)_y\|_Y+\sum_{2\leqslant i\leqslant 5}\|\mathcal{N}^i\|_Y\\
&\qquad\lesssim\mathbb{M}_{c,x}(T)+\mathbb{M}_1(T)+\mathbb{M}_2(T)^2.
\end{aligned}$$

Since $\mathcal{F}_y\{(I-\tilde{P}_1)(x_y^2)\}(t,\eta)=0$ for $\eta\in[-\eta_0,\eta_0]$, we have

$$\|(I-\tilde{P}_1)(x_y)^2\|_{L^2}\leqslant\eta_0^{-1}\|\partial_y(x_y)^2\|_{L^2}\lesssim\eta_0^{-1/2}\|x_y\|_Y\|x_{yy}\|_Y, \tag{5.10}$$

whence $\|(I-\tilde{P}_1)(x_y)^2\|_{L^\infty(0,T;L^2)\cap L^2(0,T;L^2)}\lesssim\eta_0^{-1/2}\mathbb{M}_{c,x}(T)^2$. Thus, we complete the proof. □

To prove lemma 5.1, we need the following.

CLAIM 5.4. *There exist positive constants η_1, δ and C such that if $\eta_0 \in (0, \eta_1]$ and $\mathbb{M}_{c,x}(T) \leqslant \delta$, then $[\partial_y, B_4] = 0$,*

$$\|[\partial_y, B_3]f\|_{L^2(0,T;Y_1)} \leqslant C(\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T)) \sup_{t\in[0,T]} \|f(t)\|_Y,$$

$$\|[\partial_y, B_3]f\|_{L^1(0,T;Y_1)} \leqslant C(\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T))\|f\|_{L^2(0,T;Y)}.$$

The proof is given in appendix A.

Proof of lemma 5.1. Let us translate (4.21) into a system of b and x_y. Let

$$\begin{aligned} A(t) &= \operatorname{diag}(1, \partial_y)\mathcal{A}(t) \operatorname{diag}(1, \partial_y^{-1}), \qquad B_5 = B_1 + \partial_y^2 \tilde{S}_1, \\ A_0 &= \operatorname{diag}(1, \partial_y) \left\{ B_5^{-1}(B_2 - \partial_y^2 \tilde{S}_0)\partial_y^2 + \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \right\} \operatorname{diag}(1, \partial_y)^{-1}, \\ A_1(t, D_y) &= \operatorname{diag}(1, \partial_y)(B_4^{-1} - B_5^{-1})(B_2 - \partial_y^2 \tilde{S}_0) \operatorname{diag}(\partial_y^2, \partial_y) \\ &\quad + \operatorname{diag}(1, \partial_y)B_3^{-1}\widetilde{\mathcal{A}}_1(t), \end{aligned}$$

where $\partial_y^{-1} = \mathcal{F}_\eta^{-1}(\mathrm{i}\eta)^{-1}\mathcal{F}_y$. Then $A(t) = A_0(D_y) + A_1(t, D_y)$. Note that $\widetilde{\mathcal{A}}_1(t) = \widetilde{\mathcal{A}}_1(t) \operatorname{diag}(1, \partial_y^{-1})$. Multiplying (4.21) by $\operatorname{diag}(1, \partial_y)$ from the left, we can transform (4.21) into

$$\left.\begin{aligned} \partial_t \begin{pmatrix} b \\ x_y \end{pmatrix} &= A(t) \begin{pmatrix} b \\ x_y \end{pmatrix} + \sum_{i=1}^{5} \operatorname{diag}(1, \partial_y)\mathcal{N}^i, \\ b(0, \cdot) &= 0, \qquad x_y(0, \cdot) = 0. \end{aligned}\right\} \tag{5.11}$$

Let $A_0(\eta)$ be the Fourier transform of the operator A_0. Then

$$\begin{aligned} A_0(\eta) &= \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i}\eta \end{pmatrix} (B_1^{-1} + O(\eta^2))(B_2 + O(\eta^2)) \begin{pmatrix} -\eta^2 & 0 \\ 0 & \mathrm{i}\eta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2\mathrm{i}\eta & 0 \end{pmatrix} \\ &= A_*(\eta) + \begin{pmatrix} O(\eta^4) & O(\eta^3) \\ O(\eta^5) & O(\eta^4) \end{pmatrix}, \end{aligned} \tag{5.12}$$

where

$$A_*(\eta) = \begin{pmatrix} -3\eta^2 & 8\mathrm{i}\eta \\ \mathrm{i}\eta(2 + \mu_3\eta^2) & -\eta^2 \end{pmatrix} \quad \text{and} \quad \mu_3 = -\frac{\mu_1}{2} + \frac{3}{4} = \frac{1}{2} + \frac{\pi^2}{24} > \frac{1}{8}.$$

Next, we will diagonalize $A_*(\eta)$, a lower-order part of $A_0(\eta)$. Let

$$\omega(\eta) = \sqrt{16 + (8\mu_3 - 1)\eta^2}, \qquad \lambda_*^{\pm}(\eta) = -2\eta^2 \pm \mathrm{i}\eta\omega(\eta),$$
$$\mathit{\Pi}_*(\eta) = \frac{1}{4\mathrm{i}} \begin{pmatrix} 8\mathrm{i} & 8\mathrm{i} \\ \eta + \mathrm{i}\omega(\eta) & \eta - \mathrm{i}\omega(\eta) \end{pmatrix}.$$

Then $\mathit{\Pi}_*(\eta)^{-1}A_*(\eta)\mathit{\Pi}_*(\eta) = \operatorname{diag}(\lambda_*^+(\eta), \lambda_*^-(\eta))$. We remark that if μ_3 is replaced by $\frac{1}{8}$, then $\omega(\eta) = 4$ and $\mathrm{e}^{tA_*(D_y)}$ is a composition of the wave and heat kernels. In our setting,

$$|\omega(\eta) - 4| \lesssim \eta^2. \tag{5.13}$$

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By the change of variables

$$\boldsymbol{b}(t,y)=\begin{pmatrix}b_1(t,y)\\ b_2(t,y)\end{pmatrix},\qquad \begin{pmatrix}b(t,\cdot)\\ x_y(t,\cdot)\end{pmatrix}=\Pi_*(D_y)\begin{pmatrix}b_1(t,\cdot)\\ b_2(t,\cdot)\end{pmatrix},$$

we have

$$\partial_t\boldsymbol{b}=\{2\partial_y^2 I+\partial_y\omega(D_y)\sigma_3+A_2(D_y)+A_3(t,D_y)\}\boldsymbol{b}+\Pi_*^{-1}(D_y)\sum_{i=1}^{5}\operatorname{diag}(1,\partial_y)\mathcal{N}^i, \tag{5.14}$$

where

$$\sigma_3=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad A_2(\eta)=\Pi_*(\eta)^{-1}(A_0(\eta)-A_*(\eta))\Pi_*(\eta),$$
$$A_3(t,\eta)=\Pi_*(\eta)^{-1}A_1(t,\eta)\Pi_*(\eta).$$

For $\eta\in[-\eta_0,\eta_0]$,

$$\left|\Pi_*(\eta)-\begin{pmatrix}2&2\\1&-1\end{pmatrix}\right|+\left|\Pi_*(\eta)^{-1}-\frac{1}{4}\begin{pmatrix}1&2\\1&-2\end{pmatrix}\right|\lesssim|\eta|. \tag{5.15}$$

Hence, $\Pi_*(D_y)$ and $\Pi_*^{-1}(D_y)$ are bounded operators on Y for sufficiently small η_0. By (5.15) and Plancherel's theorem,

$$\left\|\begin{pmatrix}b(t,\cdot)\\ x_y(t,\cdot)\end{pmatrix}-\begin{pmatrix}2&2\\1&-1\end{pmatrix}\boldsymbol{b}(t,\cdot)\right\|_Y\lesssim\|\partial_y\boldsymbol{b}(t,\cdot)\|_Y. \tag{5.16}$$

By (5.12) and (5.15),

$$A_2(\eta)=O(\eta^3). \tag{5.17}$$

Since $\|A_1(t,D_y)\|_{B(Y)}\lesssim \mathrm{e}^{-\alpha(3t+L)}$ for $t\geqslant 0$ by claim B.3,

$$\|A_3(t,D_y)\|_{B(Y)}\lesssim \mathrm{e}^{-\alpha(3t+L)}\quad\text{for } t\geqslant 0. \tag{5.18}$$

To obtain the energy estimate for b_1 and b_2, we translate the nonlinear term as

$$\Pi_*^{-1}(D_y)\sum_{i=1}^{5}\operatorname{diag}(1,\partial_y)\mathcal{N}^i=\mathcal{N}'+\partial_y(\mathcal{N}^0+\mathcal{N}'')-\partial_t K(t,y) \tag{5.19}$$

such that $\mathcal{N}_0$ is quadratic in b_1 and b_2, that $\lim_{t\to\infty}\|K(t,\cdot)\|_Y=0$ and that

$$\left.\begin{aligned}&\sup_{t\in[0,T]}\|\mathcal{N}'(t)\|_Y+\|\mathcal{N}'(t)\|_{L^1(0,T;Y)}\\ &\qquad\lesssim(\mathrm{e}^{-\alpha L}+\mathbb{M}_{c,x}(T))\mathbb{M}_{c,x}(T)+\mathbb{M}_1(T)^2+\mathbb{M}_2(T)^2,\\ &\sup_{t\in[0,T]}\|\mathcal{N}''(t)\|_Y+\|\mathcal{N}''\|_{L^2(0,T;Y)}\\ &\qquad\lesssim\mathbb{M}_1(T)+\mathbb{M}_{c,x}(T)(\mathbb{M}_{c,x}(T)+\mathbb{M}_2(T)).\end{aligned}\right\} \tag{5.20}$$

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To begin with, we will translate the dominant part of $\varPi_*^{-1}(D_y)\,\mathrm{diag}(1,\partial_y)\mathcal{N}^1$ in terms of b_1 and b_2. Let

$$\widetilde{\mathcal{N}}^0 = \varPi_*^{-1}(D_y)\tilde{P}_1\begin{pmatrix} 6(bx_y) \\ 3(x_y)^2 - \frac{1}{4}b^2 \end{pmatrix}, \qquad \widetilde{\mathcal{N}}^1 = \varPi_*^{-1}(D_y)\tilde{P}_1\begin{pmatrix} 0 \\ \frac{1}{4}b^2 - 2(b-\tilde{c}) \end{pmatrix},$$

$$\mathcal{N}^0 = \tilde{P}_1\begin{pmatrix} 4b_1^2 - 4b_1b_2 - 2b_2^2 \\ 2b_1^2 + 4b_1b_2 - 4b_2^2 \end{pmatrix}, \qquad \widetilde{\mathcal{N}}^2 = \widetilde{\mathcal{N}}^0 - \mathcal{N}^0.$$

Then

$$\varPi_*^{-1}(D_y)\,\mathrm{diag}(1,\partial_y)\mathcal{N}^1 = \partial_y(\mathcal{N}^0 + \widetilde{\mathcal{N}}^1 + \widetilde{\mathcal{N}}^2).$$

By [24, (D.16)] and the Sobolev inequality $\|f\|_{L^\infty(\mathbb{R})}^2 \leqslant 2\|f\|_{L^2(\mathbb{R})}\|f'\|_{L^2(\mathbb{R})}$,

$$\sup_{t\in[0,T]}\|\widetilde{\mathcal{N}}^1(t)\|_Y + \|\widetilde{\mathcal{N}}^1\|_{L^2(0,T;Y)} \lesssim \mathbb{M}_{c,x}(T)^3. \tag{5.21}$$

It follows from (5.15) and (5.16) that $\|\widetilde{\mathcal{N}}^2(t,\cdot)\|_Y \lesssim \|\boldsymbol{b}(t,\cdot)\|_Y\|\partial_y\boldsymbol{b}(t,\cdot)\|_Y$ and that

$$\sup_{t\in[0,T]}\|\widetilde{\mathcal{N}}^2(t,\cdot)\|_Y + \|\widetilde{\mathcal{N}}^2\|_{L^2(0,T;Y)} \lesssim \mathbb{M}_{c,x}(T)^2. \tag{5.22}$$

Next, we will decompose $\mathrm{diag}(1,\partial_y)\mathcal{N}^2$ into a sum of an $L^1(0,T;Y)$ function and a y-derivative of $L^2(0,T;Y)$ as follows:

$$\left.\begin{aligned} \mathrm{diag}(1,\partial_y)\mathcal{N}^2 &= \mathrm{diag}(1,\partial_y)\mathcal{N}^{21} + \partial_y\mathcal{N}^{22}, \\ \sup_{t\in[0,T]}\|\mathcal{N}^{21}\|_Y + \|\mathcal{N}^{21}\|_{L^1(0,T;Y)} &\lesssim \mathbb{M}_{c,x}(T)^2 + \mathbb{M}_1(T)^2 + \mathbb{M}_2(T)^2, \\ \sup_{t\in[0,T]}\|\mathcal{N}^{22}\|_Y + \|\mathcal{N}^{22}\|_{L^2(0,T;Y)} &\lesssim \mathbb{M}_{c,x}(T)^2. \end{aligned}\right\} \tag{5.23}$$

By (4.20),

$$B_3^{-1} = B_1^{-1} - B_1^{-1}\left(\widetilde{\mathcal{C}}_1 + \partial_y^2\sum_{j=1,2}\bar{S}_j - \sum_{3\leqslant j\leqslant 5}\bar{S}_j\right)B_3^{-1}. \tag{5.24}$$

Let $E_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Since

$$\mathrm{diag}(1,\partial_y)B_1^{-1}E_2 = \tfrac{1}{2}\partial_y E_2, \qquad \mathrm{diag}(1,\partial_y)B_1^{-1}\widetilde{\mathcal{C}}_1 = \tfrac{1}{2}\partial_y\widetilde{\mathcal{C}}_1, \tag{5.25}$$

we have $\mathrm{diag}(1,\partial_y)\mathcal{N}^{2b} = \partial_y\mathcal{N}^{2b1} + \mathrm{diag}(1,\partial_y)\mathcal{N}^{2b2}$, where

$$\mathcal{N}^{2b1} = \left\{\tfrac{1}{2}(E_2 - \widetilde{\mathcal{C}}_1B_3^{-1}) + \mathrm{diag}(\partial_y,\partial_y^2)\sum_{j=1,2}B_1^{-1}\bar{S}_jB_3^{-1}\right\}\begin{pmatrix} 0 \\ R_2^7 \end{pmatrix},$$

$$\mathcal{N}^{2b2} = -\sum_{3\leqslant j\leqslant 5}B_1^{-1}\bar{S}_jB_3^{-1}\begin{pmatrix} 0 \\ R_2^7 \end{pmatrix}.$$

By (B 5), (A 6) and (A 7),

$$\sup_{t\in[0,T]}\|\mathcal{N}^{2b1}\|_Y + \|\mathcal{N}^{2b1}\|_{L^2(0,T;Y)} \lesssim \mathbb{M}_{c,x}(T)^2,$$

$$\sup_{t\in[0,T]}\|\mathcal{N}^{2b2}\|_Y + \|\mathcal{N}^{2b2}\|_{L^1(0,T;Y)} \lesssim (\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T))\mathbb{M}_{c,x}(T)^2,$$

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and it follows from claim 5.3 and the above that

$$\mathcal{N}^{21} := \mathcal{N}^{2a} + \mathcal{N}^{2b2} \quad \text{and} \quad \mathcal{N}^{22} := \mathcal{N}^{2b1}$$

satisfy (5.23).

Let

$$\mathcal{N}^{31} = [B_3^{-1}, \partial_y](\tilde{R}^2 + \tilde{R}^4), \qquad \mathcal{N}^{32} = B_3^{-1}(\tilde{R}^2 + \tilde{R}^4).$$

Then $\mathcal{N}^3 = \mathcal{N}^{31} + \partial_y \mathcal{N}^{32}$ and we have

$$\left.\begin{aligned} \sup_{t\in[0,T]} \|\mathcal{N}^{31}(t)\|_{Y_1} + \|\mathcal{N}^{31}\|_{L^1(0,T;Y_1)} &\lesssim \mathbb{M}_{c,x}(T)(\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T))^2, \\ \sup_{t\in[0,T]} \|\mathcal{N}^{32}(t)\|_{Y} + \|\mathcal{N}^{32}\|_{L^2(0,T;Y)} &\lesssim \mathbb{M}_{c,x}(T)(\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T)) \end{aligned}\right\} \tag{5.26}$$

in exactly the same way as the proof of (5.5). To prove the estimate for $\mathcal{N}^{31}$, we use claim 5.4.

Secondly, we estimate $\mathcal{N}^4$. Using (4.20), we write $\mathcal{N}^4$ as

$$\mathcal{N}^4 = B_4^{-1}\left\{\widetilde{\mathcal{C}}_1 + \sum_{j=1,2} \partial_y^2(\bar{S}_j - \tilde{S}_j) - \sum_{3\leqslant j\leqslant 5}(\bar{S}_j - \tilde{S}_j)\right\} B_3^{-1}(\partial_y^2\tilde{S}_0 - B_2)\begin{pmatrix} b_{yy} \\ x_{yy}\end{pmatrix}.$$

Using the fact that

$$B_4^{-1} = B_1^{-1} - B_1^{-1}\tilde{S}_3 B_4^{-1} + \partial_y^2 B_1^{-1}\tilde{S}_1 B_4^{-1}, \qquad \mathrm{diag}(1, \partial_y) B_1^{-1}\widetilde{\mathcal{C}}_1 = \tfrac{1}{2}\partial_y \widetilde{\mathcal{C}}_1,$$

we have

$$\mathrm{diag}(1,\partial_y)\mathcal{N}^4 = \mathrm{diag}(\mathcal{N}^{41} + \partial_y\mathcal{N}^{42}) + \partial_y \mathcal{N}^{43},$$

$$\mathcal{N}^{41} = \left\{B_1^{-1}\tilde{S}_3 B_4^{-1}\widetilde{\mathcal{C}}_1 + B_4^{-1}\sum_{3\leqslant j\leqslant 5}(\bar{S}_j - \tilde{S}_j)\right\} B_3^{-1}(B_2 - \partial_y^2\tilde{S}_0)\begin{pmatrix} b_{yy} \\ x_{yy}\end{pmatrix},$$

$$\mathcal{N}^{42} = \left\{B_1^{-1}\partial_y\tilde{S}_1 B_4^{-1}\widetilde{\mathcal{C}}_1 + B_4^{-1}\sum_{j=1,2}\partial_y(\bar{S}_j - \tilde{S}_j)B_3^{-1}\right\} B_3^{-1}(\partial_y^2\tilde{S}_0 - B_2)\begin{pmatrix} b_{yy} \\ x_{yy}\end{pmatrix},$$

$$\mathcal{N}^{43} = \tfrac{1}{2}\widetilde{\mathcal{C}}_1 B_3^{-1}(\partial_y^2\tilde{S}_0 - B_2)\begin{pmatrix} b_{yy} \\ x_{yy}\end{pmatrix}.$$

Note that $[B_4, \partial_y] = 0$ and $[\tilde{S}_0, \partial_y] = 0$. By claim 4.3, we have

$$\|\bar{S}_j - \tilde{S}_j\|_{B(Y)} \lesssim \|\tilde{c}\|_{L^\infty}\|\tilde{S}_j\|_{B(Y)} \quad \text{for } 1 \leqslant j \leqslant 5. \tag{5.27}$$

By [24, claim B.1], we have $\|\tilde{S}_0\|_{B(Y)} \lesssim 1$. Using claims 4.4 and 4.5, (A 6)–(A 7), (5.27) and the above, we have

$$\sup_{t\in[0,T]} \|\mathcal{N}^{41}(t)\|_Y + \|\mathcal{N}^{41}\|_{L^1(0,T;Y)} \lesssim \mathbb{M}_{c,x}(T)(\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T)). \tag{5.28}$$

By claim 4.4, (A 1), (A 2) and (5.27),

$$\sup_{t\in[0,T]} (\|\mathcal{N}^{42}(t)\|_Y + \|\mathcal{N}^{43}(t)\|_Y) + \|\mathcal{N}^{42}\|_{L^2(0,T;Y)} + \|\mathcal{N}^{43}\|_{L^2(0,T;Y)} \lesssim \mathbb{M}_{c,x}(T)^2. \tag{5.29}$$

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A crude estimate $\|\mathcal{N}^5\|_{L^2(0,T;Y)} \lesssim \mathbb{M}_1(T)$ is insufficient to obtain upper bounds of $\mathbb{M}_{c,x}(T)$. We decompose II^6_1 as $\mathrm{II}^6_1 = \mathrm{II}^6_{11} + \eta^2\mathrm{II}^6_{12} - \mathrm{II}^6_{13}$, where

$$\mathrm{II}^6_{11} = 6\int_{\mathbb{R}^2} v_1(t,z,y)\varphi_{c(t,y)}\overline{\partial_z g_1^*(z,0,c(t,y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y,$$
$$\mathrm{II}^6_{12} = 6\int_{\mathbb{R}^2} v_1(t,z,y)\varphi_{c(t,y)}\overline{\partial_z g_{11}^*(z,\eta,c(t,y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y,$$
$$\mathrm{II}^6_{13} = 6\int_{\mathbb{R}^2} v_1(t,z,y)\tilde{\psi}_{c(t,y)}(z)\overline{\partial_z g_1^*(z,\eta,c(t,y))}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y.$$

By the fact that $g_1^*(z,0,c) = \frac{1}{2}\varphi_c$ and (4.3),

$$\mathrm{II}^6_{11} = \frac{1}{2}\int_{\mathbb{R}^2}\{(\partial_z^3 - 2c(t,y)\partial_z)v_1(t,z,y)\}\varphi_{c(t,y)}(z)\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y.$$

Substituting (4.1) into the above, we have

$$\begin{aligned}\mathrm{II}^6_{11} &+ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} v_1(t,z,y)\varphi_{c(t,y)}(z)\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y\\
&= -\frac{3}{2}\int_{\mathbb{R}^2}\partial_z^{-1}\partial_y^2 v_1(t,z,y)\varphi_{c(t,y)}(z)\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y\\
&\quad - \frac{1}{2}\int_{\mathbb{R}^2}(N_{1,1}+N_{1,2})\varphi'_{c(t,y)}(z)\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2} N_{1,3}\varphi_{c(t,y)}(z)\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2} v_1(t,z,y)c_t(t,y)\partial_c\varphi_{c(t,y)}(z)\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y.\end{aligned}$$

Let

$$\begin{aligned}S_1^7[q_c](f)(t,y) &= \frac{1}{4\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2} v_1(t,z,y_1)f(y_1)q_{c(t,y_1)}(z)\mathrm{e}^{\mathrm{i}(y-y_1)\eta}\,\mathrm{d}z\,\mathrm{d}y_1\,\mathrm{d}\eta,\\
k(t,y) &= \frac{1}{4\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2} v_1(t,z,y_1)\varphi_{c(t,y_1)}(z)\mathrm{e}^{\mathrm{i}(y-y_1)\eta}\,\mathrm{d}z\,\mathrm{d}y_1\,\mathrm{d}\eta.\end{aligned}\tag{5.30}$$

By integration by parts, we have

$$\begin{aligned}&\mathbf{1}_{[-\eta_0,\eta_0]}(\eta)\left\{\mathrm{II}^6_{11}(t,\eta) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} v_1(t,z,y)\varphi_{c(t,y)}(z)\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y\right\}\\
&\quad = \sqrt{2\pi}\mathcal{F}_y\{S_1^7[\partial_c\varphi_c](c_t) - S_1^7[\varphi'_c](x_t - 2c - 3(x_y)^2)\} + \mathrm{II}^6_{111}(t,\eta) + \mathrm{i}\eta\mathrm{II}^6_{112}(t,\eta),\end{aligned}\tag{5.31}$$

where

$$\begin{aligned}\mathrm{II}^6_{111}(t,\eta) &= \frac{3}{2}\int_{\mathbb{R}^2} v_1(t,z,y)^2\varphi'_{c(t,y)}(z)\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y\\
&\quad + \frac{3}{2}\int_{\mathbb{R}^2}(\partial_z^{-1}\partial_y v_1)(t,z,y)c_y(t,y)\partial_c\varphi_{c(t,y)}(z)\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&- \frac{3}{2}\int_{\mathbb{R}^2} v_1(t,z,y)\{x_{yy}(t,y)\varphi_{c(t,y)}(z) \\
&\qquad\qquad + 2(c_y x_y)(t,y)\partial_c\varphi_{c(t,y)}(z)\}\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y,\\
\mathrm{II}^6_{112}(t,\eta) &= -\frac{3}{2}\int_{\mathbb{R}^2}(\partial_z^{-1}\partial_y v_1)(t,z,y)\varphi_{c(t,y)}(z)\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y\\
&\quad + 3\int_{\mathbb{R}^2} v_1(t,z,y)x_y(t,y)\varphi_{c(t,y)}(z)\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y.
\end{aligned}
$$

Let

$$
\begin{aligned}
R^{v_1}_{11} &= \frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\{\mathrm{II}^6_{111}(t,\eta) - \mathrm{II}^6_{13}(t,\eta)\}\mathrm{e}^{\mathrm{i}y\eta}\,\mathrm{d}\eta,\\
R^{v_1}_{12} &= \frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\{\mathrm{II}^6_{112}(t,\eta) - \mathrm{i}\eta\mathrm{II}^6_{12}(t,\eta)\}\mathrm{e}^{\mathrm{i}y\eta}\,\mathrm{d}\eta.
\end{aligned}
$$

Then

$$
\begin{aligned}
R^{v_1}_1 &= \frac{1}{2\pi}\int_{-\eta_0}^{\eta_0}\mathrm{II}^6_1(t,\eta)\mathrm{e}^{\mathrm{i}y\eta}\,\mathrm{d}\eta\\
&= S^7_1[\partial_c\varphi_c](c_t) - S^7_1[\varphi_c'](x_t - 2c - 3(x_y)^2) - \partial_t k + R^{v_1}_{11} + \partial_y R^{v_1}_{12}.
\end{aligned}
$$

Combining the above with (5.24) and (5.25), we have

$$
\begin{aligned}
\mathrm{diag}(1,\partial_y)\mathcal{N}^5 &= \mathrm{diag}(1,\partial_y)(\mathcal{N}^{51} + \partial_y\mathcal{N}^{52}) + \partial_y\mathcal{N}^{53},\\
\mathcal{N}^{51} &= B_3^{-1}\begin{pmatrix} R^{v_1}_{11} + S^7_1[\partial_c\varphi_c](c_t) - S^7_1[\varphi_c'](x_t - 2c - 3(x_y)^2)\\ 0\end{pmatrix}\\
&\quad + [B_3^{-1},\partial_y]\begin{pmatrix} R^{v_1}_{12}\\ 0\end{pmatrix} + B_1^{-1}\sum_{3\leqslant i\leqslant 5}\bar{S}_j B_3^{-1}\begin{pmatrix} 0\\ R^{v_1}_2\end{pmatrix} + [\partial_t, B_3^{-1}]\begin{pmatrix} k\\ 0\end{pmatrix},\\
\mathcal{N}^{52} &= B_3^{-1}\begin{pmatrix} R^{v_1}_{12}\\ 0\end{pmatrix} - B_1^{-1}\partial_y(\bar{S}_1 + \bar{S}_2)B_3^{-1}\begin{pmatrix} 0\\ R^{v_1}_2\end{pmatrix},\\
\mathcal{N}^{53} &= \tfrac{1}{2}(E_2 - \widetilde{\mathcal{C}}_1 B_3^{-1})\begin{pmatrix} 0\\ R^{v_1}_2\end{pmatrix}.
\end{aligned}
$$

Then

$$
\mathrm{diag}(1,\partial_y)\mathcal{N}^5 = \mathrm{diag}(1,\partial_y)\left\{\mathcal{N}^{51} + \partial_y\mathcal{N}^{52} - \partial_t B_3^{-1}\begin{pmatrix} k\\ 0\end{pmatrix}\right\} + \partial_y\mathcal{N}^{53}.
$$

By lemma 5.2 and claim A.1,

$$
\begin{aligned}
&\|S^7_1[\partial_c\varphi_c](c_t)\|_{L^1(0,T;Y_1)} + \|S^7_1[\varphi_c'](x_t - 2c - 3(x_y)^2)\|_{L^1(0,T;Y_1)}\\
&\qquad \lesssim \|v_1\|_{L^2(0,T;W(t))}(\|c_t\|_{L^2(0,T;L^2(\mathbb{R}))} + \|x_t - 2c - 3(x_y)^2\|_{L^2(0,T;L^2(\mathbb{R}))})\\
&\qquad \lesssim \mathbb{M}_1(T)(\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2),
\end{aligned}
$$

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and

$$\sup_{t\in[0,T]}\|S_1^7[\partial_c\varphi_c](c_t)\|_{Y_1}+\sup_{t\in[0,T]}\|S_1^7[\varphi_c'](x_t-2c-3(x_y)^2)\|_{Y_1}$$
$$\lesssim \sup_{t\in[0,T]}\{\|v_1(t)\|_{L^2(R^2)}(\|c_t\|_{L^2(0,T;L^2(\mathbb{R}))}+\|x_t-2c-3(x_y)^2\|_{L^2(0,T;L^2(\mathbb{R}))})\}$$
$$\lesssim \mathbb{M}_1(T)(\mathbb{M}_{c,x}(T)+\mathbb{M}_1(T)+\mathbb{M}_2(T)^2).$$

Combining the above with claims 4.5, 5.4, A.2, B.7, B.8, (A 6) and (A 7), we have

$$\sup_{t\in[0,T]}\|\mathcal{N}^{51}\|_{Y_1}+\|\mathcal{N}^{51}\|_{L^1(0,T;Y_1)} \lesssim (\mathrm{e}^{-\alpha L}+\mathbb{M}_{c,x}(T)+\mathbb{M}_1(T)+\mathbb{M}_2(T))\mathbb{M}_1(T), \tag{5.32}$$

$$\sup_{t\in[0,T]}(\|\mathcal{N}^{52}\|_{Y}+\|\mathcal{N}^{53}\|_{Y})+\|\mathcal{N}^{52}\|_{L^2(0,T;Y)}+\|\mathcal{N}^{53}\|_{L^2(0,T;Y)}\lesssim \mathbb{M}_1(T). \tag{5.33}$$

Let

$$\begin{aligned}
\mathcal{N}' &= \mathit{\Pi}_*^{-1}(D_y)\operatorname{diag}(1,\partial_y)\sum_{2\leqslant j\leqslant 5}\mathcal{N}^{j1},\\
\mathcal{N}'' &= \widetilde{\mathcal{N}}^1+\widetilde{\mathcal{N}}^2+\mathit{\Pi}_*^{-1}(D_y)\operatorname{diag}(1,\partial_y)(\mathcal{N}^{32}+\mathcal{N}^{42}+\mathcal{N}^{52})\\
&\quad+\mathit{\Pi}_*^{-1}(D_y)(\mathcal{N}^{22}+\mathcal{N}^{43}+\mathcal{N}^{53}),\\
K &= \begin{pmatrix}K_1\\K_2\end{pmatrix}=\mathit{\Pi}_*^{-1}(D_y)\operatorname{diag}(1,\partial_y)B_3^{-1}\begin{pmatrix}k\\0\end{pmatrix},\\
\widetilde{\mathcal{N}}'' &= \mathcal{N}''+\{\omega(D_y)\sigma_3-4+\partial_y^{-1}A_2(D_y)\}\boldsymbol{b}.
\end{aligned}$$

Then, from (5.14) and (5.19), we have

$$\partial_t(\boldsymbol{b}+K)=2\partial_y^2\boldsymbol{b}+4\partial_y\sigma_3\boldsymbol{b}+A_3(t,D_y)\boldsymbol{b}+\mathcal{N}'+\partial_y(\mathcal{N}^0+\widetilde{\mathcal{N}}''). \tag{5.34}$$

Equation (5.20) follows from (5.21)–(5.23), (5.26), (5.28), (5.29), (5.32) and (5.33). Claims 4.5 and B.8 imply that

$$\sup_{t\in[0,T]}\|K(t,\cdot)\|_Y+\|K\|_{L^2(0,T;Y)}\lesssim \mathbb{M}_1(T),\qquad \lim_{t\to\infty}\|K(t,\cdot)\|_Y=0. \tag{5.35}$$

By (5.13), (5.17) and (5.20),

$$\sup_{t\in[0,T]}\|\widetilde{\mathcal{N}}''(t)\|_Y+\|\widetilde{\mathcal{N}}''\|_{L^2(0,T;Y)}\lesssim \eta_0\mathbb{M}_{c,x}(T)+\mathbb{M}_1(T)+\mathbb{M}_{c,x}(T)^2+\mathbb{M}_2(T)^2. \tag{5.36}$$

The time global bound for $\|\boldsymbol{b}(t)\|_Y$ does not follow directly from the energy identity of (5.34) because the $L^2(\mathbb{R})$-inner product of $\partial_y\mathcal{N}^0$ and $\boldsymbol{b}$ is not necessarily integrable globally in time for a v_0 that is not strongly localized in space. To eliminate cubic nonlinear terms in the energy identity, we make use of the following change of variables:

$$\boldsymbol{d}=\begin{pmatrix}d_1\\d_2\end{pmatrix}=\boldsymbol{b}-\tfrac{1}{2}(b_1+K_1)(b_2+K_2)\boldsymbol{e}_1+K,\quad \boldsymbol{e}_1=\begin{pmatrix}1\\1\end{pmatrix}. \tag{5.37}$$

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By (5.37), (5.34) can be rewritten as

$$\partial_t \boldsymbol{d} = 2\partial_y^2 \boldsymbol{b} + 4\partial_y \sigma_3 \boldsymbol{b} + A_3(t, D_y)\boldsymbol{b} + \mathcal{N}' + \partial_y(\mathcal{N}^0 + \widetilde{\mathcal{N}}'') \\ - \{2\langle \partial_y \sigma_3 \boldsymbol{b}, \sigma_1 \boldsymbol{b}\rangle + \mathcal{R}_1 + \mathcal{R}_2\}\boldsymbol{e}_1, \tag{5.38}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^2$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{R}_1 = \tfrac{1}{2}\langle \sigma_1 \boldsymbol{b}, \partial_y(\mathcal{N}^0 + \widetilde{\mathcal{N}}'')\rangle,$$
$$\mathcal{R}_2 = \tfrac{1}{2}\partial_t\{(b_1 + K_1)(b_2 + K_2)\} - 2\langle \partial_y \sigma_3 \boldsymbol{b}, \sigma_1 \boldsymbol{b}\rangle - \mathcal{R}_1.$$

Taking the $L^2(\mathbb{R})$-inner product of (5.38) and $\boldsymbol{d}$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{d}(t)\|_{L^2(\mathbb{R})}^2 + 2\|\partial_y \boldsymbol{b}(t)\|_{L^2(\mathbb{R})}^2 \\ = \int_{\mathbb{R}} \langle \partial_y \sigma_3 \boldsymbol{b}, 4\boldsymbol{b} - 2b_1 b_2 \boldsymbol{e}_1\rangle \,\mathrm{d}y - 2\int_{\mathbb{R}} \langle \partial_y \sigma_3 \boldsymbol{b}, \sigma_1 \boldsymbol{b}\rangle \langle \boldsymbol{b}, \boldsymbol{e}_1\rangle \,\mathrm{d}y \\ + \int_{\mathbb{R}} \langle \partial_y \mathcal{N}^0, \boldsymbol{b}\rangle \,\mathrm{d}y + \mathfrak{R}_1 + \mathfrak{R}_2 + \mathfrak{R}_3, \tag{5.39}$$

where

$$\mathfrak{R}_1 = \int_{\mathbb{R}} \{2\langle \partial_y \boldsymbol{d}, \partial_y(\boldsymbol{b} - \boldsymbol{d})\rangle + 4\langle \partial_y \sigma_3 \boldsymbol{b}, K\rangle + \partial_y \langle \partial_y \boldsymbol{b}, \sigma_1 \boldsymbol{b}\rangle\} \,\mathrm{d}y,$$
$$\mathfrak{R}_2 = \int_{\mathbb{R}} \langle A_3(t, D_y)\boldsymbol{b} + \mathcal{N}' - \mathcal{R}_2 \boldsymbol{e}_1, \boldsymbol{d}\rangle \,\mathrm{d}y,$$
$$\mathfrak{R}_3 = \int_{\mathbb{R}} \{\langle \partial_y \mathcal{N}^0, \boldsymbol{d} - \boldsymbol{b}\rangle + \langle \partial_y \widetilde{\mathcal{N}}'', \boldsymbol{d}\rangle - 2\langle \partial_y \sigma_3 \boldsymbol{b}, \sigma_1 \boldsymbol{b}\rangle \langle \boldsymbol{d} - \boldsymbol{b}, \boldsymbol{e}_1\rangle - \mathcal{R}_1 \langle \boldsymbol{e}_1, \boldsymbol{d}\rangle\} \,\mathrm{d}y.$$

Since

$$\langle \partial_y \sigma_3 \boldsymbol{b}, 4\boldsymbol{b} - 2b_1 b_2 \boldsymbol{e}_1\rangle - 2\langle \partial_y \sigma_3 \boldsymbol{b}, \sigma_1 \boldsymbol{b}\rangle \langle \boldsymbol{e}_1, \boldsymbol{b}\rangle + \langle \partial_y \mathcal{N}^0, \boldsymbol{b}\rangle \\ = 2\partial_y \langle \sigma_3 \boldsymbol{b}, \boldsymbol{b}\rangle + \partial_y \langle \mathcal{N}^0, \boldsymbol{b}\rangle - \tfrac{4}{3}\partial_y(b_1^3 - b_2^3),$$

it follows from (5.39) that

$$\sup_{t\in[0,T]} \|\boldsymbol{d}(t)\|_{L^2}^2 + 4\int_0^T \|\partial_y \boldsymbol{b}(t)\|_Y^2 \,\mathrm{d}t \lesssim \|v_0\|_{L^2}^2 + \sum_{1\leqslant j\leqslant 3} \|\mathfrak{R}_j\|_{L^1(0,T)}. \tag{5.40}$$

Here we used the fact that $\boldsymbol{b}(0, \cdot) \equiv 0$ and $\|\boldsymbol{d}(0)\|_Y = O(\|K(0)\|_Y) = O(\|v_0\|_{L^2})$.

Finally, we will estimate $\|\mathfrak{R}_j\|_{L^1(0,T)}$. Taking into account claim B.8 and the fact that $\operatorname{supp} \widehat{b}_i(t, \eta) \subset [-\eta_0, \eta_0]$, we have

$$\sup_{t\in[0,T]} \|\boldsymbol{b}(t) - \boldsymbol{d}(t)\|_{L^2(\mathbb{R})} \lesssim \sup_{t\in[0,T]} (\|\boldsymbol{b}(t)\|_Y^2 + \|K(t)\|_Y) \\ \lesssim \mathbb{M}_{c,x}(T)^2 + \mathbb{M}_1(T) \tag{5.41}$$

and, for $k \geqslant 1$,

$$\|\partial_y^k \boldsymbol{b} - \partial_y^k \boldsymbol{d}\|_{L^2(0,T;L^2(\mathbb{R}))} \lesssim \|\boldsymbol{b}\|_{L^\infty(0,T;Y)} \|\partial_y \boldsymbol{b}\|_{L^2(0,T;Y)} + \|K(t)\|_{L^2(0,T;Y)} \\ \lesssim \mathbb{M}_{c,x}(T)^2 + \mathbb{M}_1(T). \tag{5.42}$$

In view of (5.35) and (5.42),

$$\left\| \int_{\mathbb{R}} \langle \partial_y \boldsymbol{d}, \partial_y(\boldsymbol{d}-\boldsymbol{b})\rangle \,\mathrm{d}y \right\|_{L^1(0,T)} \lesssim \mathbb{M}_{c,x}(T)^3 + \mathbb{M}_{c,x}(T)\mathbb{M}_1(T) + \mathbb{M}_1(T)^2,$$
$$\|\langle \partial_y \sigma_3 \boldsymbol{b}, K\rangle\|_{L^1(0,T;Y)} \lesssim \mathbb{M}_{c,x}(T)\mathbb{M}_1(T),$$

and it follows that

$$\|\mathfrak{R}_1\|_{L^1(0,T)} \lesssim \mathbb{M}_{c,x}(T)^3 + \mathbb{M}_{c,x}(T)\mathbb{M}_1(T) + \mathbb{M}_1(T)^2. \tag{5.43}$$

Substituting (5.34) into $\mathcal{R}_2$, we see that

$$\begin{aligned}\|\mathcal{R}_2\|_{Y_1} \lesssim{}& \|\partial_y \boldsymbol{b}\|_Y^2 + \|\boldsymbol{b}\|_Y(\|A_3(t,D_y)\boldsymbol{b}\|_Y + \|\mathcal{N}'\|_Y) \\ &+ \|K\|_Y(\|\partial_y \boldsymbol{b}\|_Y + \|A_3(t,D_y)\boldsymbol{b}\|_Y + \|\mathcal{N}^0\|_Y + \|\mathcal{N}'\|_Y + \|\widetilde{\mathcal{N}}''\|_Y).\end{aligned}$$

Combining the above with (5.18), (5.20), (5.35) and (5.36), we have

$$\|\mathcal{R}_2\|_{L^1(0,T;Y_1)} \lesssim \mathbb{M}_{c,x}(T)^2 + \mathbb{M}_1(T)^2 + (\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T))\mathbb{M}_2(T)^2$$

and

$$\|\mathfrak{R}_2\|_{L^1(0,T)} \lesssim (\mathrm{e}^{-\alpha L} + \mathbb{M}_{c,x}(T))\mathbb{M}_{c,x}(T)^2 + \mathbb{M}_1(T)^2 + (\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T))\mathbb{M}_2(T)^2. \tag{5.44}$$

Using the Sobolev inequality, we have, for $j_1, j_2, j_3, j_4 = 1, 2$,

$$\left\| \int_{\mathbb{R}} \partial_y b_{j_1} b_{j_2} b_{j_3} b_{j_4} \,\mathrm{d}y \right\|_{L^1(0,T)} \lesssim \|\partial_y \boldsymbol{b}\|_{L^2(0,T)}^2 \|\boldsymbol{b}\|_{L^\infty(0,T;Y)}^2 \lesssim \mathbb{M}_{c,x}(T)^4. \tag{5.45}$$

By (5.35) and (5.45),

$$\left\| \int_{\mathbb{R}} \langle \partial_y \mathcal{N}^0, \boldsymbol{d}-\boldsymbol{b}\rangle \,\mathrm{d}y \right\|_{L^1(0,T)} \lesssim \mathbb{M}_{c,x}(T)^4 + \mathbb{M}_{c,x}(T)^2\mathbb{M}_1(T). \tag{5.46}$$

By (5.35) and (5.36),

$$\begin{aligned}\left\| \int_{\mathbb{R}} \langle \partial_y \widetilde{\mathcal{N}}'', \boldsymbol{d}\rangle \,\mathrm{d}y \right\|_{L^1(0,T)} &= \left\| \int_{\mathbb{R}} \langle \widetilde{\mathcal{N}}'', \partial_y \boldsymbol{d}\rangle \,\mathrm{d}y \right\|_{L^1(0,T)} \\ &\lesssim \{\mathbb{M}_1(T) + (\eta_0 + \mathbb{M}_{c,x}(T))\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T)^2\} \\ &\quad \times (\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T))\end{aligned} \tag{5.47}$$

and

$$\begin{aligned}&\left\| \int_{\mathbb{R}} \mathcal{R}_1 \langle \boldsymbol{e}_1, \boldsymbol{d}\rangle \,\mathrm{d}y \right\|_{L^1(0,T)} \\ &\quad \lesssim \{\mathbb{M}_1(T) + (\eta_0 + \mathbb{M}_{c,x}(T))\mathbb{M}_{c,x}(T) + \mathbb{M}_2(T)^2\}\mathbb{M}_{c,x}(T)(\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T)).\end{aligned} \tag{5.48}$$

By (5.35) and (5.41),

$$\left\| \int_{\mathbb{R}} \langle \partial_y \sigma_3 \boldsymbol{b}, \sigma_1 \boldsymbol{b}\rangle \langle \boldsymbol{d}-\boldsymbol{b}, \boldsymbol{e}_1\rangle \,\mathrm{d}y \right\|_{L^1(0,T)} \lesssim \mathbb{M}_{c,x}(T)^2(\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T)). \tag{5.49}$$

It follows from (5.46)–(5.49) that

$$\begin{aligned}\|\mathfrak{R}_3\|_{L^1(0,T)} \lesssim (\mathrm{e}^{-\alpha L} + \mathbb{M}_{c,x}(T))\mathbb{M}_{c,x}(T)^2 + \mathbb{M}_{c,x}(T)\mathbb{M}_1(T) + \mathbb{M}_1(T)^2 \\ + (\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T))\mathbb{M}_2(T)^2. \end{aligned} \tag{5.50}$$

Combining (5.40) with (5.41), (5.43), (5.44) and (5.50), we obtain (5.1). This completes the proof of lemma 5.1. □

6. The $L^2(\mathbb{R}^2)$ estimate

In this section we will estimate $\mathbb{M}_v(T)$ assuming smallness of $\mathbb{M}_{c,x}(T)$, $\mathbb{M}_1(T)$ and $\mathbb{M}_2(T)$.

Lemma 6.1. *Let $\alpha \in (0,1)$ and δ_4 be as in lemma 5.1. Then there exists a positive constant C such that*

$$\mathbb{M}_v(T) \leqslant C(\|v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)).$$

To prove lemma 6.1, we will show a variant of the L^2 conservation law on v as in [24, lemma 8.1].

Lemma 6.2. *Let $\alpha \in (0,2)$ and $T > 0$. Suppose that $v_1 \in C([0,T]; L^2(\mathbb{R}^2))$, $v_2 \in C([0,T]; X \cap L^2(\mathbb{R}^2))$ and that $v_2(t)$, $c(t)$ and $x(t)$ satisfy (3.21), (3.26) and (3.27). Then*

$$Q(t,v) := \int_{\mathbb{R}^2} \{v(t,z,y)^2 - 2\psi_{c(t,y),L}(z+3t)v(t,z,y)\}\,\mathrm{d}z\,\mathrm{d}y$$

satisfies, for $t \in [0,T]$,

$$\begin{aligned} Q(t,v) = Q(0,v) &+ 2\int_0^t\int_{\mathbb{R}^2} (\ell_{11} + \ell_{12} + 6\varphi'_{c(s,y)}(z)\tilde{\psi}_{c(s,y)}(z))v(s,z,y)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s \\ &- 6\int_0^t\int_{\mathbb{R}^2} (\partial_z^{-1}\partial_y v)(s,z,y)c_y(s,y)\partial_c\varphi_{c(t,y)}(z)\,\mathrm{d}z\,\mathrm{d}y \\ &- 6\int_0^t\int_{\mathbb{R}^2} \varphi'_{c(s,y)}(z)v(s,z,y)^2\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s \\ &- 2\int_0^t\int_{\mathbb{R}^2} \ell\psi_{c(s,y),L}(z+3s)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

Proof. Let

$$\ell_{13}^* = c_{yy}(s,y)\int_{-\infty}^z \partial_c\varphi_{c(s,y)}(z_1)\,\mathrm{d}z_1 + c_y(s,y)^2\int_{-\infty}^z \partial_c^2\varphi_{c(s,y)}(z_1)\,\mathrm{d}z_1.$$

If, in addition, $v_0 \in X$, then

$$\int_{\mathbb{R}^2} v(t,z,y)\ell_{13}^*\,\mathrm{d}z\,\mathrm{d}y = \int_{\mathbb{R}^2} (\partial_z^{-1}\partial_y v)(t,z,y)c_y(t,y)\varphi_{c(t,y)}\,\mathrm{d}z\,\mathrm{d}y.$$

Thus, we can conclude lemma 6.2 from [24, lemma 8.2] by way of a limiting argument. □

Now we are in position to prove lemma 6.1.

Proof of lemma 6.1. By remark 3.7 and proposition 3.9, we can apply lemma 6.2 for $t \in [0,T]$ if $\mathbb{M}_{c,x}(T)$ and $\mathbb{M}_2(T)$ are sufficiently small.

Since we have, for $j,k \geqslant 0$ and $z \in \mathbb{R}$,

$$\partial_z^j \partial_c^k \varphi_c(z) \lesssim \mathrm{e}^{-2\alpha|z|}, \qquad \int_{-\infty}^{z} \partial_c^j \varphi_c(z_1)\,\mathrm{d}z_1 \lesssim \min(1, \mathrm{e}^{2\alpha z}), \tag{6.1}$$

it follows that

$$\begin{aligned}
&\sup_{t\in[0,T]} \left| \int_0^t \int_{\mathbb{R}^2} (\ell_{11}+\ell_{12}) v \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s \right| \\
&\quad \lesssim (\|c_t - 6c_y x_y\|_{L^2((0,T)\times\mathbb{R})} + \|x_t - 2c - 3(x_y)^2\|_{L^2((0,T)\times\mathbb{R})} + \|x_{yy}\|_{L^2((0,T)\times\mathbb{R})}) \\
&\qquad \times (\|v_1\|_{L^2(0,T;W(t))} + \|v_2\|_{L^2(0,T;X)}),
\end{aligned} \tag{6.2}$$

$$\begin{aligned}
&\sup_{t\in[0,T]} \left| \int_0^t \int_{\mathbb{R}^2} c_y(s,y)\partial_c \varphi_{c(s,y)} (\partial_z^{-1}\partial_y v)(s,z,y) \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s \right| \\
&\quad \lesssim \|c_y\|_{L^2((0,T)\times\mathbb{R})} (\|\partial_z^{-1}\partial_y v_1\|_{L^2(0,T;W(t))} + \|\partial_z^{-1}\partial_y v_2\|_{L^2(0,T;X)}),
\end{aligned} \tag{6.3}$$

$$\sup_{t\in[0,T]} \left| \int_0^t \int_{\mathbb{R}^2} \varphi'_{c(s,y)}(z) v^2(s,z,y) \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s \right| \lesssim (\|v_1\|_{L^2(0,T;W(t))} + \|v_2\|_{L^2(0,T;X)})^2. \tag{6.4}$$

In view of the definition of $\tilde{\psi}$,

$$\left.\begin{aligned}
\|\tilde{\psi}_{c(t,y)}\|_X &\lesssim \|\tilde{c}\|_Y \mathrm{e}^{-\alpha(3t+L)}, \\
\|\tilde{\psi}_{c(t,y)}\|_{L^2(\mathbb{R}^2)} &= 2\sqrt{2}\|\sqrt{c}-\sqrt{2}\|_{L^2(\mathbb{R})}\|\psi\|_{L^2(\mathbb{R})} \lesssim \|\tilde{c}\|_Y.
\end{aligned}\right\} \tag{6.5}$$

By (6.1) and (6.5),

$$\begin{aligned}
&\sup_{[0,T]} \left| \int_0^t \int_{\mathbb{R}^2} \varphi'_{c(s,y)}(z)\tilde{\psi}_{c(s,y)}(z) v(s,z,y) \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s \right| \\
&\qquad \lesssim \|\tilde{\psi}_{c(t,y)}\|_{L^2(0,T;X)} \|\mathrm{e}^{-\alpha|z|} v(t)\|_{L^2(0,T;L^2(\mathbb{R}^2))} \\
&\qquad \lesssim \mathrm{e}^{-\alpha L} \sup_{t\in[0,T]} \|\tilde{c}(t)\|_Y (\|v_1\|_{L^2(0,T;W(t))} + \|v_2\|_{L^2(0,T;X)}),
\end{aligned} \tag{6.6}$$

$$\begin{aligned}
&\sup_{[0,T]} \left| \int_0^t \int_{\mathbb{R}^2} (\ell_{11}+\ell_{12})\tilde{\psi}_{c(s,y)}(z) \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s \right| \\
&\quad \leqslant \sup_{t\in[0,T]} \|\mathrm{e}^{-\alpha z}(\ell_{11}+\ell_{12})\|_{L^2_{yz}} \|\tilde{\psi}_{c(t,y)}\|_{L^1(0,T;X)} \\
&\quad \lesssim \mathrm{e}^{-\alpha L} \sup_{t\in[0,T]} \{\|\tilde{c}\|_Y (\|c_t - 6c_y x_y\|_{L^2} + \|x_t - 2c - 3(x_y)^2\|_{L^2} + \|x_{yy}\|_{L^2})\}.
\end{aligned} \tag{6.7}$$

By integration by parts, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} (\ell_{21}+\ell_{22})\tilde{\psi}_{c(t,y)}(z) \,\mathrm{d}z\,\mathrm{d}y \\
&\qquad = \int_{\mathbb{R}^2} (c_t(t,y)\tilde{\psi}_{c(t,y)}(z)\partial_c\tilde{\psi}_{c(t,y)}(z) + 3\varphi'_{c(t,y)}(z)\tilde{\psi}^2_{c(t,y)}(z)) \,\mathrm{d}z\,\mathrm{d}y,
\end{aligned}$$

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and it follows that

$$\sup_{t\in[0,T]}\left|\int_0^t\int_{\mathbb{R}^2}(\ell_{21}+\ell_{22})\tilde{\psi}_{c(s,y)}(s,z,y)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s-\frac{1}{2}\left[\int_{\mathbb{R}^2}\tilde{\psi}^2_{c(s,y)}(z)\,\mathrm{d}z\,\mathrm{d}y\right]_{s=0}^{t}\right|$$
$$\begin{aligned}&\leqslant 3\|\varphi'_{c(t,y)}(z)\tilde{\psi}_{c(t,y)}(z)\|_{L^1(0,T;L^1(\mathbb{R}^2))}\\ &\lesssim \mathrm{e}^{-\alpha L}\sup_{t\in[0,T]}\|\tilde{c}(t)\|_Y^2.\end{aligned}\tag{6.8}$$

By integration by parts,

$$\int_{\mathbb{R}^2}(\ell_{13}+\ell_{23})\tilde{\psi}_{c(t,y)}(z)\,\mathrm{d}z\,\mathrm{d}y$$
$$=-3\int_{\mathbb{R}^2}c_y^2(t,y)\partial_c\tilde{\psi}_{c(t,y)}(z)\left\{\int_z^\infty\partial_c\varphi_{c(t,y)}(z_1)-\partial_c\tilde{\psi}_{c(t,y)}(z_1)\,\mathrm{d}z_1\right\}\mathrm{d}z\,\mathrm{d}y.$$

Since $\int_z^\infty(\partial_c\varphi_c-\partial_c\tilde{\psi}_c)$ and $\|\partial_c\tilde{\psi}_c\|_{L^1(\mathbb{R})}$ are uniformly bounded for $c\in[1/2,3/2]$,

$$\sup_{t\in[0,T]}\left|\int_0^t\int_{\mathbb{R}^2}(\ell_{13}+\ell_{23})\tilde{\psi}_{c(s,y)}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s\right|\lesssim\|c_y\|^2_{L^2(0,T;Y)}.\tag{6.9}$$

Combining (6.2)–(6.4) and (6.6)–(6.9) with lemmas 5.2 and 6.2, we see that, for $t\in(0,T]$,

$$[Q(s,v)+8\|\psi\|^2_{L^2}\|\sqrt{c(s)}-\sqrt{2}\|^2_{L^2(\mathbb{R})}]_{s=0}^{s=t}\lesssim(\mathbb{M}_{c,x}(T)+\mathbb{M}_1(T)+\mathbb{M}_2(T))^2.\tag{6.10}$$

Since $c(0,\cdot)=2$ and

$$Q(t,v)=\|v(t)\|^2_{L^2(\mathbb{R}^2)}+O(\|\tilde{c}(t)\|_Y\|v(t)\|_{L^2(\mathbb{R}^2)}),$$

lemma 6.1 follows immediately from (6.10). Thus, we complete the proof. □

7. Estimates for v_1

In this section we will give upper bounds of $\mathbb{M}_1(\infty)$ and $\mathbb{M}'_1(\infty)$.

Lemma 7.1. *There exist positive constants C and δ_5 such that if $\|v_0\|_{L^2}<\delta_5$, then $\mathbb{M}_1(\infty)\leqslant C\|v_0\|_{L^2}$.*

Lemma 7.2. *There exist positive constants C and δ'_5 such that if $\||D_x|^{-1/2}v_0\|_{L^2}+\||D_x|^{1/2}v_0\|_{L^2}+\||D_x|^{-1/2}|D_y|^{1/2}v_0\|_{L^2}<\delta'_5$, then*

$$\mathbb{M}'_1(\infty)\leqslant C(\||D_x|^{-1/2}v_0\|_{L^2}+\||D_x|^{1/2}v_0\|_{L^2}+\||D_x|^{-1/2}|D_y|^{1/2}v_0\|_{L^2}).$$

7.1. Virial estimates for v_1

The virial identity for L^2-solutions of the KP-II equation (1.1) was shown in [8]. It ensures that $v_1(t)\in L^2([0,\infty);L^2_{\mathrm{loc}}(\mathbb{R}^2))$. Let $\chi_{+,\varepsilon}(x)=1+\tanh\varepsilon x$, let $\tilde{x}_1(t)$ be a C^1 function and let

$$I_{x_0}(t)=\int_{\mathbb{R}^2}\chi_{+,\varepsilon}(x-\tilde{x}_1(t)-x_0,y)\tilde{v}_1^2(t,x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Then we have the following.

LEMMA 7.3. *Let $\tilde v_1(t)$ be a solution of (1.1) satisfying $\tilde v_1(0) = v_0 \in L^2(\mathbb{R}^2)$. Then for any $c_1 > 0$ there exist positive constants ε_0 and δ such that if $\inf_t \tilde x_1'(t) \geqslant c_1$, $\varepsilon \in (0, \varepsilon_0)$ and $\|v_0\|_{L^2} < \delta$, then, for any $x_0 \in \mathbb{R}$,*

$$I_{x_0}(t) + \nu \int_0^t \int_{\mathbb{R}^2} \chi'_{+,\varepsilon}(x - \tilde x_1(s) - x_0)\mathcal{E}(\tilde v_1)(s, x, y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \leqslant I_{x_0}(0),$$

where $\nu = \frac{1}{2}\min\{3, c_1\}$. Moreover,

$$\lim_{t\to\infty} I_{x_0}(t) = 0 \quad \textit{for any } x_0 \in \mathbb{R}. \tag{7.1}$$

See, for example, [26, lemma 5.3] for the proof. Lemma 7.1 follows from lemma 7.3 and the L^2-conservation law of the KP-II equation.

7.2. The L^3-estimate of v_1

In order to estimate the L^3-norm of v_1, we apply the small data scattering result for the KP-II equation from [12].

In the interest of making the present work self-contained, we introduce some notation from [12]. Let $\mathcal{Z}$ be a set of finite partitions $-\infty = t_0 < t_1 < \cdots < t_K = \infty$. We denote by V^p $(1 \leqslant p < \infty)$ the set of all functions $v\colon \mathbb{R} \to L^2(\mathbb{R}^2)$ such that $\lim_{t\to\pm\infty} v(t)$ exists and for which the norm

$$\|v\|_{V^p} = \left\{ \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|^p_{L^2(\mathbb{R}^2)} \right\}^{1/p}$$

is finite, where $v(-\infty) := \lim_{t\to-\infty} v(t)$ and $v(\infty) := 0$. We denote by $V^p_{-,rc}$ the closed subspace of every right-continuous function $v \in V^p$ satisfying $\lim_{t\to-\infty} v(t) = 0$. Let $V^p_S := \mathrm{e}^{\cdot S} V^p$ and $V^p_{-rc,-,S} := \mathrm{e}^{\cdot S} V^p$ with $S = -\partial_x^3 - 3\partial_x^{-1}\partial_y^2$.

Let $\chi \in C_0^\infty(-2, 2)$ be an even non-negative function such that $\chi(\eta) = 1$ for $\eta \in [-1, 1]$. Let $\bar\chi(t) = \chi(t) - \chi(2t)$ and let P_N be a projection defined by

$$\widehat{P_N u}(\tau, \xi, \eta) = \bar\chi(N^{-1}\xi)\hat u(\tau, \xi, \eta) \quad \text{for } N = 2^n \text{ and } n \in \mathbb{Z}.$$

For $s \leqslant 0$, we denote by $\dot Y^s$ the closure of $C(\mathbb{R}; H^1(\mathbb{R}^2)) \cap V^2_{-,rc}$ with respect to the norm

$$\|u\|_{\dot Y^s} = \left(\sum_N N^{2s} \|P_N u\|^2_{V^2_S} \right)^{1/2}.$$

We denote by $\dot Y^s(0, T)$ the restriction of $\dot Y^s$ to the time interval $[0, T]$ with the norm

$$\|u\|_{\dot Y^s(0,T)} = \inf\{\|\tilde u\|_{\dot Y^s} \mid \tilde u \in \dot Y^s,\ \tilde u(t) = u(t) \text{ for } t \in [0, T]\}.$$

Proposition 3.1 and theorem 3.2 in [12] ensure that higher-order Sobolev norms of a solution to (4.1) remain small provided that v_0 is small in the higher-order Sobolev spaces. Let $T \geqslant 0$ and

$$I_T(u_1, u_2)(t) = \int_0^t \mathbf{1}_{[0,T]}(s)\mathrm{e}^{(t-s)S}\partial_x(u_1 u_2)(s)\,\mathrm{d}s.$$

Then we have the following.

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Lemma 7.4. *Let $s \geqslant 0$ and u_1, $u_2 \in \dot{Y}^{-1/2}$. Then there exists a positive constant C such that for any $T \in (0, \infty)$,*

$$\||D_x|^s I_T(u_1, u_2)\|_{\dot{Y}^{-1/2}} \leqslant C\||D_x|^s u_1\|_{\dot{Y}^{-1/2}}\|u_2\|_{\dot{Y}^{-1/2}}, \tag{7.2}$$

$$\|\langle D_y\rangle^s I_T(u_1, u_2)\|_{\dot{Y}^{-1/2}} \leqslant C \prod_{j=1,2} \|\langle D_y\rangle^s u_j\|_{\dot{Y}^{-1/2}}. \tag{7.3}$$

Proof. We have (7.2) in exactly the same way as in the proof of [12, theorem 3.2]. Note that (7.2) and (7.3) are the same as in [12, corollary 3.4] when $s = 0$. Using the fact that $1 + \eta_3^2 \lesssim (1 + \eta_1^2)(1 + \eta_2^2)$ for η_1, η_2 and η_3 satisfying $\eta_1 + \eta_2 + \eta_3 = 0$, we can prove (7.3) in the same way as in [12, proposition 3.1 and theorem 3.2]. □

Thanks to lemma 7.4, we have the following.

Proposition 7.5. *There exists a positive constant δ_5' such that if*

$$\||D_x|^{-1/2} v_0\|_{L^2} + \||D_x|^{-1/2}|D_y|^{1/2} v_0\|_{L^2} \leqslant \delta_5',$$

then a solution $\tilde{v}_1$ of (3.3) satisfies

$$\left.\begin{aligned} \|\partial_x \tilde{v}_1\|_{\dot{Y}^{-1/2}} &\lesssim \||D_x|^{1/2} v_0\|_{L^2}, \\ \|\langle D_y\rangle^{1/2} \tilde{v}_1\|_{\dot{Y}^{-1/2}} &\lesssim \||D_x|^{-1/2} v_0\|_{L^2} + \||D_x|^{-1/2}|D_y|^{1/2} v_0\|_{L^2}. \end{aligned}\right\} \tag{7.4}$$

Proof. Using the variation of constants formula, we have

$$\tilde{v}_1(t) = \mathrm{e}^{tS} v_0 - 3I_T(\tilde{v}_1(s), \tilde{v}_1(s))\,\mathrm{d}s \quad \text{for } t \in [0, T].$$

By lemma 7.4 and the fact that $\|\mathrm{e}^{tS} v_0\|_{\dot{Y}^{-1/2}(0,T)} \lesssim \||D_x|^{-1/2} v_0\|_{L^2}$,

$$\begin{aligned} \|\tilde{v}_1\|_{\dot{Y}^{-1/2}(0,T)} &\lesssim \||D_x|^{-1/2} v_0\|_{L^2} + \|\tilde{v}_1\|^2_{\dot{Y}^{-1/2}(0,T)}, \\ \|\partial_x \tilde{v}_1\|_{\dot{Y}^{-1/2}(0,T)} &\lesssim \||D_x|^{1/2} v_0\|_{L^2} + \|\partial_x \tilde{v}_1\|_{\dot{Y}^{-1/2}(0,T)}\|\tilde{v}_1\|_{\dot{Y}^{-1/2}(0,T)}, \\ \|\langle D_y\rangle^{1/2} \tilde{v}_1\|_{\dot{Y}^{-1/2}(0,T)} &\lesssim \||D_x|^{-1/2}\langle D_y\rangle^{1/2} v_0\|_{L^2} + \|\langle D_y\rangle^{1/2} \tilde{v}_1\|^2_{\dot{Y}^{-1/2}(0,T)}. \end{aligned}$$

If δ is sufficiently small, it follows from the above that

$$\begin{aligned} \|\tilde{v}_1\|_{\dot{Y}^{-1/2}(0,T)} &\leqslant C_1\||D_x|^{-1/2} v_0\|_{L^2} + C_2\|\tilde{v}_1\|^2_{\dot{Y}^{-1/2}(0,T)}, \\ \|\langle D_y\rangle^{1/2} \tilde{v}_1\|_{\dot{Y}^{-1/2}(0,T)} &\leqslant C_1\||D_x|^{-1/2}\langle D_y\rangle^{1/2} v_0\|_{L^2} + C_2\|\langle D_y\rangle^{1/2} \tilde{v}_1\|^2_{\dot{Y}^{-1/2}(0,T)}, \\ \||D_x|^{1/2} \tilde{v}_1\|_{\dot{Y}^0(0,T)} &\leqslant C_1\||D_x|^{1/2} v_0\|_{L^2} + \|\tilde{v}_1\|_{\dot{Y}^{-1/2}(0,T)}\||D_x|^{1/2} \tilde{v}_1\|_{\dot{Y}^0(0,T)}, \end{aligned}$$

where C_1 and C_2 are positive constants independent of T. Suppose that $v_0 \in H^2(\mathbb{R}^2)$. Then

$$\|I_T(\tilde{v}_1, \tilde{v}_1)\|_{\dot{Y}^{-1/2}(0,T)}, \qquad \|I_T(\tilde{v}_1, \tilde{v}_1)\|_{\dot{Y}^0(0,T)}, \qquad \|\langle D_y\rangle^{1/2} \tilde{v}_1\|_{\dot{Y}^{-1/2}(0,T)}$$

are continuous in T because $\tilde{v}_1 \in C(\mathbb{R}; H^2(\mathbb{R}^2))$ and

$$\partial_t(\mathrm{e}^{-tS} I_T(\tilde{v}_1, \tilde{v}_1)(t)) = \begin{cases} \mathrm{e}^{-tS} \partial_x \tilde{v}_1^2(t) & \text{for } t \in [0, T], \\ 0 & \text{otherwise.} \end{cases}$$

Taking the limit $T \to \infty$, we have (7.4) for any $v_0 \in H^2(\mathbb{R}^2)$ satisfying the assumption in proposition 7.5. For general v_0, we have (7.4) by approximating v_0 by $H^3(\mathbb{R}^2)$ functions. Thus, we complete the proof. □

Proposition 7.5 implies the L^3-bound of v_1.

Proof of lemma 7.2. By (7.4),

$$\sup_{t \geqslant 0} \||D_x|^{1/2}\tilde{v}_1(t)\|_{L^2} \lesssim \||D_x|^{1/2}v_0\|_{L^2},$$
$$\sup_{t \geqslant 0} \||D_x|^{-1/2}|D_y|^{1/2}\tilde{v}_1(t)\|_{L^2} \lesssim \||D_x|^{-1/2}\langle D_y\rangle v_0\|_{L^2}.$$

Using an isotropic Sobolev embedding inequality

$$\|u\|_{L^3(\mathbb{R}^2)} \lesssim \||D_x|^{1/2}u\|_{L^2(\mathbb{R}^2)} + \||D_x|^{-1/2}|D_y|^{1/2}u\|_{L^2(\mathbb{R}^2)}, \tag{7.5}$$

we have

$$\|v_1(t)\|_{L^3} = \|\tilde{v}_1(t)\|_{L^3} \lesssim \||D_x|^{1/2}\tilde{v}_1(t)\|_{L^2} + \||D_x|^{-1/2}|D_y|^{1/2}\tilde{v}_1(t)\|_{L^2}.$$

Combining the above with (7.4), we have lemma 7.2. We remark that (7.5) follows by interpolating the embeddings $\mathrm{Id}\colon \dot{E}^1 \to L^6(\mathbb{R}^2)$ and $\mathrm{Id}\colon \dot{E}^0 \to L^2(\mathbb{R}^2)$ (see, for example, [3] and [30, lemma 2]), where $\dot{E}^t$ is a Banach space with the norm

$$\|u\|_{\dot{E}^t} = \left\|\left(\xi^2 + \frac{\eta^2}{\xi^2}\right)^{t/2}\hat{u}(\xi,\eta)\right\|_{L^2(\mathbb{R}^2)}.$$ □

8. Decay estimates in the exponentially weighted space

In this section we will estimate $\mathbb{M}_2(T)$ by following the argument of [24, ch. 8].

LEMMA 8.1. *Let η_0 and α be positive constants satisfying $\nu_0 < \alpha < 2$. Suppose that $\mathbb{M}_1'(\infty)$ is sufficiently small. Then there exist positive constants δ_6 and C such that if $\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T) + \mathbb{M}_v(T) \leqslant \delta_6$,*

$$\mathbb{M}_2(T) \leqslant C(\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T)). \tag{8.1}$$

Let $\chi \in C_0^\infty(-2,2)$ be an even non-negative function such that $\chi(\eta) = 1$ for $\eta \in [-1,1]$. Let $\chi_M(\eta) = \chi(\eta/M)$ and

$$P_{\leqslant M}u := \frac{1}{2\pi}\int_{\mathbb{R}^2} \chi_M(\eta)\hat{u}(\xi,\eta)\mathrm{e}^{\mathrm{i}(x\xi+y\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta, \quad P_{\geqslant M} = I - P_{\leqslant M}.$$

To prove lemma 8.1, we will use the linear stability property of line solitons (proposition 2.2) for the low frequency part $v_<(t) := P_{\leqslant M}v_2(t)$, and make use of a virial-type estimate for the high frequency part $v_>(t) := P_{\geqslant M}v_2(t)$.

8.1. Decay estimates for the low frequency part

LEMMA 8.2. *Let η_0 and α be positive constants satisfying $\nu_0 < \alpha < 2$. Suppose that $v_2(t)$ is a solution of (4.4) satisfying $v_2(0) = 0$. Then there exist positive constants*

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δ_6 and C such that if $\mathbb{M}_1(T)+\mathbb{M}_2(T)<\delta_6$ and $M\geqslant\eta_0$, then

$$\begin{aligned}&\|P_1(0,2M)v_2\|_{L^\infty(0,T;X)}+\|P_1(0,2M)v_2\|_{L^2(0,T;X)}\\&\qquad\leqslant C\{\mathbb{M}_{c,x}(T)+\mathbb{M}_1(T)+\mathbb{M}_2(T)(\mathbb{M}_2(T)+\mathbb{M}_v(T))\}.\end{aligned}\tag{8.2}$$

Proof of lemma 8.2. Let $\tilde v_2(t)=P_2(\eta_0,2M)v_2(t)$. Then

$$\left.\begin{aligned}\partial_t\tilde v_2=\mathcal{L}\tilde v_2+P_2(\eta_0,2M)\{\ell+\partial_x(N_{2,1}+N_{2,2}+N'_{2,2}+N_{2,4})+N_{2,3}\},\\ \tilde v_2(0)=0,\end{aligned}\right\}\tag{8.3}$$

where $N'_{2,2}=\{2\tilde c(t,y)+6(\varphi(z)-\varphi_{c(t,y)}(z))\}v_2(t,z,y)$. Hereafter, we abbreviate $P_2(\eta_0,2M)$ to P_2.

Applying proposition 2.2 to (8.3), we have

$$\begin{aligned}\|\tilde v_2(t)\|_X\lesssim&\int_0^t\mathrm{e}^{-b'(t-s)}(t-s)^{-3/4}\|\mathrm{e}^{\alpha z}P_2N_{2,1}(s)\|_{L^1_zL^2_y}\,\mathrm{d}s\\&+\int_0^t\mathrm{e}^{-b'(t-s)}(t-s)^{-1/2}(\|N_{2,2}(s)\|_X+\|N'_{2,2}(s)\|_X+\|N_{2,4}\|_X)\,\mathrm{d}s\\&+\int_0^t\mathrm{e}^{-b(t-s)}(\|\ell(s)\|_X+\|N_{2,3}(s)\|_X)\,\mathrm{d}s.\end{aligned}\tag{8.4}$$

Since $\|\mathrm{e}^{\alpha z}P_2N_{2,1}\|_{L^1_zL^2_y}\lesssim\sqrt{M}(\|v_1\|_{L^2}+\|v_2\|_{L^2})\|v_2\|_X$ (by [24, claim 9.1]), we have

$$\begin{aligned}&\sup_{t\in[0,T]}\|\mathrm{e}^{\alpha z}P_2N_{2,1}\|_{L^1_zL^2_y}+\|\mathrm{e}^{\alpha z}P_2N_{2,1}\|_{L^2(0,T;L^1_zL^2_y)}\\&\qquad\lesssim\sqrt{M}(\mathbb{M}_1(T)+\mathbb{M}_v(T))\mathbb{M}_2(T).\end{aligned}\tag{8.5}$$

By the definitions,

$$\begin{aligned}\|\ell_1\|_X&\lesssim\|x_t-2c-3(x_y)^2\|_{L^2}+\|c_t-6c_yx_y\|_{L^2}+\|x_{yy}\|_{L^2}+\|c_{yy}\|_{L^2}+\|c_y\|^2_{L^4},\\ \|\ell_2\|_X&\lesssim\mathrm{e}^{-\alpha(3t+L)}(\|c_t-6c_yx_y\|_{L^2}+\|x_t-2c-3(x_y)^2\|_{L^2}+\|\tilde c\|_{L^2}\\&\qquad+\|x_{yy}\|_{L^2}+\|c_{yy}\|_{L^2}+\|c_y\|^2_{L^4}),\\ \|N_{2,2}\|_X&\lesssim(\|x_t-2c-3(x_y)^2\|_{L^\infty}+\|\tilde c\|_{L^\infty})\|v_2\|_X,\\ \|N'_{2,2}\|_X&\lesssim\|\tilde c\|_{L^\infty}\|v_2(t)\|_X,\qquad\|N_{2,4}\|_X\lesssim\|v_1(t)\|_{W(t)}.\end{aligned}$$

Hence, it follows from lemma 5.2 and the definitions of $\mathbb{M}_{c,x}(T)$, $\mathbb{M}_1(T)$ and $\mathbb{M}_2(T)$ that

$$\sup_{t\in[0,T]}\|\ell\|_X+\|\ell\|_{L^2(0,T;X)}\lesssim\mathbb{M}_{c,x}(T)+\mathbb{M}_1(T)+\mathbb{M}_2(T)^2,\tag{8.6}$$

$$\sup_{t\in[0,T]}\|N_{2,2}\|_X+\|N_{2,2}\|_{L^2(0,T;X)}\lesssim(\mathbb{M}_{c,x}(T)+\mathbb{M}_1(T)+\mathbb{M}_2(T)^2)\mathbb{M}_2(T),\tag{8.7}$$

$$\begin{aligned}&\sup_{t\in[0,T]}(\|N'_{2,2}\|_X+\|N_{2,4}\|_X)+\|N'_{2,2}\|_{L^2(0,T;X)}+\|N_{2,4}\|_{L^2(0,T;X)}\\&\qquad\lesssim\mathbb{M}_{c,x}(T)\mathbb{M}_2(T)+\mathbb{M}_1(T).\end{aligned}\tag{8.8}$$

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Since $\|\partial_y P_2\|_{B(X)} \lesssim M$, we have $\|P_2 N_{2,3}\|_X \lesssim M(\|x_y\|_{L^\infty} + \|x_{yy}\|_{L^\infty})\|v_2\|_X$ and

$$\sup_{t\in[0,T]} \|N_{2,3}\|_X + \|N_{2,3}\|_{L^2(0,T;X)} \lesssim \mathbb{M}_{c,x}(T)\mathbb{M}_2(T). \tag{8.9}$$

Combining (8.5)–(8.9) with (8.4), we have

$$\sup_{t\in[0,T]} \|\tilde v_2(t)\|_X + \|\tilde v_2(t)\|_{L^2(0,T;X)} \lesssim \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + (\mathbb{M}_v(T) + \mathbb{M}_2(T))\mathbb{M}_2(T).$$

As long as $v_2(t)$ satisfies the orthogonality condition (3.21), and $\tilde c(t,y)$ remains small, we have

$$\|\tilde v_2(t)\|_X \lesssim \|P_1(0,2M)v_2(t)\|_X \lesssim \|\tilde v_2(t)\|_X$$

in exactly the same way as in the proof of lemma 9.2 in [24]. Thus, we have (8.2). This completes the proof of lemma 8.2. □

8.2. Virial estimates for v_2

We prove a virial-type estimate in the weighted space X in order to estimate the high frequency part of $v_>$, for which we require the smallness of $\sup_{t\geqslant 0} \|v_1(t)\|_{L^3(\mathbb{R}^2)}$.

LEMMA 8.3. *Let $\alpha \in (0,2)$ and let $v_2(t)$ be a solution to (4.4) satisfying $v_2(0) = 0$. Suppose that $\mathbb{M}_1'(\infty)$ is sufficiently small. Then there exist positive constants δ_6, M_1 and C such that if $\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T) + \mathbb{M}_v(T) < \delta_6$ and $M \geqslant M_1$, then, for $t \in [0,T]$,*

$$\|v_2(t)\|_X^2 \leqslant C\int_0^t \mathrm{e}^{-M\alpha(t-s)}(\|\ell(s)\|_X^2 + \|P_{\leqslant M}v_2(s)\|_X^2 + \|v_1(s)\|_{W(s)}^2)\,\mathrm{d}s.$$

Proof of lemma 8.3. Let $p(z) = \mathrm{e}^{2\alpha z}$. Multiplying (4.4) by $2p(z)v_2(t,z,y)$ and integrating the resulting equation by parts, we have, for $t \in [0,T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^2} p(z)v_2(t,z,y)^2\,\mathrm{d}z\,\mathrm{d}y\right) + \int_{\mathbb{R}^2} p'(z)(\mathcal{E}(v_2) - 4v_2^3)(t,z,y)\,\mathrm{d}z\,\mathrm{d}y$$
$$= \sum_{k=1}^5 \mathrm{III}_k(t). \tag{8.10}$$

where

$$\mathrm{III}_1 = 2\int_{\mathbb{R}^2} p(z)\ell v_2(s,z,y)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s,$$

$$\mathrm{III}_2 = -\int_{\mathbb{R}^2} p'(z)(\tilde x_t(t,y) - 3x_y(t,y)^2)v_2(t,z,y)^2\,\mathrm{d}z\,\mathrm{d}y,$$

$$\mathrm{III}_3 = \int_{\mathbb{R}^2}\left\{p'''(z) + 6p(z)^2\left(\frac{\varphi_{c(t,y)}(z) - \psi_{c(t,y),L}(z+3t)}{p(z)}\right)_z\right\}v_2(t,z,y)^2\,\mathrm{d}z\,\mathrm{d}y,$$

$$\mathrm{III}_4 = 12\int_{\mathbb{R}^2} p'(z)(v_1v_2^2)(t,z,y)\,\mathrm{d}z\,\mathrm{d}y + 12\int_{\mathbb{R}^2} p(z)(v_1v_2\partial_z v_2)(t,z,y)\,\mathrm{d}z\,\mathrm{d}y,$$

$$\mathrm{III}_5 = 12\int_{\mathbb{R}^2} \partial_z(p(z)v_2(t,z,y))(\varphi_{c(t,y)}(z) - \psi_{c(t,y),L}(z+3t))v_1(t,z,y)\,\mathrm{d}z\,\mathrm{d}y.$$

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Obviously,

$$|\mathrm{III}_1| \leqslant \int p'(z)v_2^2 \,\mathrm{d}z\,\mathrm{d}y + \frac{1}{2\alpha}\int p(z)\ell^2 \,\mathrm{d}z\,\mathrm{d}y,$$
$$|\mathrm{III}_3| \leqslant (M-1)\int_{\mathbb{R}^2} p'(z)v_2(t,z,y)^2 \,\mathrm{d}z\,\mathrm{d}y,$$

where

$$M = 1 + 4\alpha^2 + 6\sup_{t,y,z}\frac{p^2(z)}{p'(z)}\left|\left(\frac{\varphi_{c(t,y)}(z) - \psi_{c(t,y),L}(z+3t)}{p(z)}\right)_z\right|,$$

and

$$\mathrm{III}_5 \lesssim \left(\int_{\mathbb{R}^2} p'(z)\{(\partial_z v_2)^2 + v_2^2\}(t,z,y)\,\mathrm{d}z\,\mathrm{d}y\right)^{1/2}\|v_1(t)\|_{W(t)}.$$

Using claim 3.2 and the Hölder inequality, we have

$$\left|\int p'(z)v_2(t,z,y)^3 \,\mathrm{d}z\,\mathrm{d}y\right| \lesssim \|v_2(t)\|_{L^2}\int_{\mathbb{R}^2} p'(z)\mathcal{E}(v_2(t,z,y))\,\mathrm{d}z\,\mathrm{d}y,$$
$$\mathrm{III}_4 \lesssim \|v_1(t)\|_{L^3}\int_{\mathbb{R}^2} p'(z)((\partial_z v_2)^2 + (\partial_z^{-1}\partial_y v_2)^2 + v_2^2)(t,z,y)\,\mathrm{d}z\,\mathrm{d}y.$$

By lemma 5.2,

$$|\mathrm{III}_2| \lesssim (\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2)\int_{\mathbb{R}^2} p'(z)v_2(t,z,y)^2 \,\mathrm{d}z\,\mathrm{d}y$$

For high y-frequencies, the potential term can be absorbed into the left-hand side. Indeed, it follows from Plancherel's theorem and the Schwarz inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^2} p'(z)((\partial_z v_>)^2 + (\partial_z^{-1}\partial_y v_>)^2)(t,z,y)\,\mathrm{d}z\,\mathrm{d}y \\
&\quad = 2\alpha\int_{\mathbb{R}^2}\left(|\xi+\mathrm{i}\alpha|^2 + \frac{\eta^2}{|\xi+\mathrm{i}\alpha|^2}\right)|\mathcal{F}v_>(t,\xi+\mathrm{i}\alpha,\eta)|^2\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad \geqslant 2M\int_{\mathbb{R}^2} p'(z)v_>(t,z,y)^2\,\mathrm{d}z\,\mathrm{d}y.
\end{aligned}$$

Combining the above, we have, for $t \in [0,T]$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}&\int_{\mathbb{R}} p(z)v_2(t,z,y)^2\,\mathrm{d}z\,\mathrm{d}y + M\alpha\int_{\mathbb{R}} p(z)v_2(t,z,y)^2\,\mathrm{d}z\,\mathrm{d}y \\
&\leqslant \frac{1}{2\alpha}\int_{\mathbb{R}^2} p(z)\ell^2\,\mathrm{d}z\,\mathrm{d}y \\
&\quad + M\alpha\int_{\mathbb{R}^2} p(z)(v_<)^2(s,z,y)\,\mathrm{d}z\,\mathrm{d}y + O(\|v_1(t)\|_{W(t)}^2) \qquad (8.11)
\end{aligned}$$

provided that δ_6 is sufficiently small. Lemma 8.3 follows immediately from (8.11). Thus, we complete the proof. □

Now we are in a position to prove lemma 8.1.

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Proof of lemma 8.1. Since $\chi_M(\eta)=0$ for $\eta\notin[-2M,2M]$, we have

$$\|P_{\leqslant M}v_2(t)\|_X\leqslant\|P_1(0,2M)v_2(t)\|_X.$$

Combining lemmas 8.2 and 8.3 with (8.6) and the definition $\mathbb{M}_1(T)$, we have (8.1) provided that δ_6 is sufficiently small. This completes the proof of lemma 8.1. □

9. Proof of theorem 1.1

Now we are now in a position to complete the proof of theorem 1.1.

Proof of theorem 1.1. Let $\delta_*=\min_{0\leqslant i\leqslant 6}\delta_i/2$. Thanks to the scaling invariance of (1.1), we may assume that $c_0=2$ without loss of generality. Since $\tilde{c}(0)=\tilde{x}(0)\equiv 0$ in Y, and $v_1(0)=v_0$, $v_2(0)=0$, there exists a $T>0$ such that

$$\mathbb{M}_{\text{tot}}(T):=\mathbb{M}_{c,x}(T)+\mathbb{M}_1(T)+\mathbb{M}_2(T)+\mathbb{M}_v(T)\leqslant\tfrac{1}{2}\delta_*. \tag{9.1}$$

By proposition 3.9, we can extend the decomposition (3.1) satisfying (3.4) and (3.21) beyond $t=T$. Let $T_1\in(0,\infty]$ be the maximal time such that the decomposition (3.1) satisfying (3.4) and (3.21) exists for $t\in[0,T_1]$ and $\mathbb{M}_{\text{tot}}(T_1)\leqslant\delta_*$. Suppose that $T_1<\infty$. Then it follows from lemmas 5.1, 6.1, 7.1, 7.2 and 8.1 that if $\||D_x|^{-1/2}v_0\|_{L^2}+\||D_x|^{1/2}v_0\|_{L^2}+\||D_x|^{-1/2}|D_y|^{1/2}v_0\|_{L^2}$ is sufficiently small, then

$$\begin{aligned}
\mathbb{M}_1(T)&\lesssim\|v_0\|_{L^2},\\
\mathbb{M}_{c,x}(T)&\lesssim\|v_0\|_{L^2}+\mathbb{M}_1(T)+\mathbb{M}_2(T)^2\lesssim\|v_0\|_{L^2}+\mathbb{M}_2(T)^2,\\
\mathbb{M}_2(T)&\lesssim\mathbb{M}_{c,x}(T)+\mathbb{M}_1(T)\lesssim\|v_0\|_{L^2}+\mathbb{M}_2(T)^2,\\
\mathbb{M}_v(T)&\lesssim\|v_0\|_{L^2}+\mathbb{M}_{c,x}(T)+\mathbb{M}_1(T)+\mathbb{M}_2(T)\lesssim\|v_0\|_{L^2}+\mathbb{M}_2(T),
\end{aligned}$$

and $\mathbb{M}_{\text{tot}}(T_1)\lesssim\|v_0\|_{L^2(\mathbb{R}^2)}+\mathbb{M}_{\text{tot}}(T_1)^2$. If $\|v_0\|_{L^2(\mathbb{R}^2)}$ is sufficiently small, we have

$$\mathbb{M}_{\text{tot}}(T_1)\leqslant\tfrac{1}{2}\delta_*,$$

which contradicts the maximality of T_1. Thus, we prove that $T_1=\infty$ and

$$\mathbb{M}_{\text{tot}}(\infty)\lesssim\|v_0\|_{L^2(\mathbb{R}^2)}. \tag{9.2}$$

By (3.1), (6.5) and (9.2),

$$\begin{aligned}
\|u(t,x,y)-\varphi_{c(t,y)}(x-x(t,y))\|_{L^2(\mathbb{R}^2)}&\leqslant\|v(t)\|_{L^2(\mathbb{R}^2)}+\|\tilde{\psi}_{c(t,y)}\|_{L^2(\mathbb{R}^2)}\\
&\lesssim\mathbb{M}_v(\infty)+\mathbb{M}_{c,x}(\infty).
\end{aligned}$$

Since $H^k(\mathbb{R})\subset Y$ for any $k\geqslant 0$, we see that (1.4) follows immediately from (9.2) and lemma 5.2. Moreover, we have (1.5) because $c_y,x_{yy}\in L^2(0,\infty;Y)$ and $pd_tc_y,\partial_tx_{yy}\in L^\infty(0,\infty;Y)$.

Finally, we will prove (1.6). Since $\|f\|_{L^\infty}\lesssim\|f\|_Y^{1/2}\|\partial_yf\|_Y^{1/2}$ for any $f\in Y$, we have from (1.4) that

$$\|x_t(t)-2c(t)\|_{L^\infty}+\|c(t)-c_0\|_{L^\infty}\lesssim\|v_0\|_{L^2},$$

and, for any $y\in\mathbb{R}$,

$$x(t,y)=\int_0^t x_t(s,y)\,\mathrm{d}s\geqslant(2c_0+O(\|v_0\|_{L^2}))t.$$

Here we use $x(0, \cdot) = 0$. Let $\tilde{x}_1(t) = c_0 t$ and $x_0 = R$. Then, by lemma 7.3,

$$\lim_{t\to\infty} \|v_1(t, x + x(t, y), y)\|_{L^2(x>-c_0t/2, y\in\mathbb{R})} = 0. \tag{9.3}$$

Dividing the integral interval $[0, t]$ into $[0, t/2]$ and $[t/2, t]$ and using (8.4)–(8.9), we have

$$\lim_{t\to\infty} \|v_2(t)\|_X = 0.$$

Thus, we complete the proof of theorem 1.1. □

10. Proof of theorem 1.2

If $v_0(x, y)$ is polynomially localized, then at $t = 0$ we can decompose a perturbed line soliton into a sum of a locally amplified line soliton and a remainder part $v_*(x, y)$ satisfying $\int_{\mathbb{R}} v_*(x, y)\, \mathrm{d}x = 0$ for all $y \in \mathbb{R}$.

LEMMA 10.1. *Let $c_0 > 0$ and $s > 1$ be constants. There exists a positive constant ε_0 such that if $\varepsilon := \|\langle x\rangle^s v_0\|_{H^1(\mathbb{R}^2)} < \varepsilon_0$, then there exists $c_1(y) \in H^1(\mathbb{R})$ such that*

$$\int_{\mathbb{R}} (\varphi_{c_1(y)}(x) - \varphi_{c_0}(x))\, \mathrm{d}x = \int_{\mathbb{R}} v_0(x, y)\, \mathrm{d}x, \tag{10.1}$$

$$\left.\begin{aligned} \|c_1(\cdot) - c_0\|_{L^2(\mathbb{R})} &\lesssim \|\langle x\rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)}, \\ \|\partial_y c_1(\cdot)\|_{H^1(\mathbb{R})} &\lesssim \|\langle x\rangle^{s/2} v_0\|_{H^1(\mathbb{R}^2)}, \end{aligned}\right\} \tag{10.2}$$

$$\left.\begin{aligned} \|v_*\|_{L^2(\mathbb{R}^2)} &\lesssim \|\langle x\rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)}, \\ \|\partial_x^{-1} v_*\|_{L^2} + \|v_*\|_{H^1(\mathbb{R}^2)} &\lesssim \|\langle x\rangle^{s} v_0\|_{H^1(\mathbb{R}^2)}, \end{aligned}\right\} \tag{10.3}$$

where $v_*(x, y) = v_0(x, y) + \varphi_{c_0}(x) - \varphi_{c_1(y)}(x)$.

Proof. First, we will prove that

$$\sup_{y\in\mathbb{R}} \left| \int_{\mathbb{R}} v_0(x, y)\, \mathrm{d}x \right| \lesssim \|\langle x\rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)} + \|\langle x\rangle^{s/2} \partial_y v_0\|_{L^2(\mathbb{R}^2)}. \tag{10.4}$$

By the Schwarz inequality,

$$\left| \int_{\mathbb{R}} v_0(x, y)\, \mathrm{d}x \right| \lesssim \left\{ \int_{\mathbb{R}} \langle x\rangle^s v_0(x, y)^2\, \mathrm{d}x \right\}^{1/2}. \tag{10.5}$$

Substituting $\sup_y v_0^2(x, y) \leqslant \int_{\mathbb{R}} \{(\partial_y v_0)^2 + v_0^2\}(x, y)\, \mathrm{d}y$ into the right-hand side of (10.5), we have (10.4).

Let

$$c_1(y) = \left\{ \sqrt{c_0} + \frac{1}{2\sqrt{2}} \int_{\mathbb{R}} v_0(x, y)\, \mathrm{d}x \right\}^2.$$

Then we have (10.1) and $\int_{\mathbb{R}} v_*(x, y)\, \mathrm{d}x = 0$ for every $y \in \mathbb{R}$ because

$$\int_{\mathbb{R}} \{\varphi_{c_1(y)}(x) - \varphi_{c_0}(x)\}\, \mathrm{d}x = 2\sqrt{2}(\sqrt{c_1(y)} - \sqrt{c_0}). \tag{10.6}$$

Moreover, it follows from (10.4) that

$$\sup_{y\in\mathbb{R}}|c_1(y)-c_0|\lesssim\|\langle x\rangle^{s/2}v_0\|_{L^2(\mathbb{R}^2)}+\|\langle x\rangle^{s/2}\partial_y v_0\|_{L^2(\mathbb{R}^2)}.$$

By (10.1), (10.5) and (10.6),

$$\|c_1(y)-c_0\|_{L^2(\mathbb{R})}\lesssim\left\|\int_{\mathbb{R}}v_0(x,y)\,\mathrm{d}x\right\|_{L^2(\mathbb{R})}\lesssim\|\langle x\rangle^{s/2}v_0\|_{L^2(\mathbb{R}^2)}.$$

Using Minkowski's inequality, we have, for $j\geqslant 0$,

$$\begin{aligned}\|\partial_x^j\varphi_{c_1(y)}-\partial_x^j\varphi_{c_0}\|_{L^2(\mathbb{R}^2)}&\leqslant\left\|\int_{c_0}^{c_1(y)}\|\partial_x^j\partial_c\varphi_c\|_{L^2_x(\mathbb{R})}\,\mathrm{d}c\right\|_{L^2_y(\mathbb{R})}\\&\lesssim\|c_1(y)-c_0\|_{L^2(\mathbb{R})}\\&\lesssim\|\langle x\rangle^{s/2}v_0\|_{L^2(\mathbb{R}^2)},\end{aligned}$$

and $\|\partial_x^j v_*\|_{L^2(\mathbb{R}^2)}\lesssim\|\partial_x^j v_0\|_{L^2(\mathbb{R}^2)}+\|\langle x\rangle^{s/2}v_0\|_{L^2(\mathbb{R}^2)}$. Similarly, we have

$$\|\partial_y c_1\|_{L^2(\mathbb{R})}\lesssim\|\langle x\rangle^{s/2}\partial_y v_0\|_{L^2(\mathbb{R}^2)},$$
$$\|\partial_y v_*\|_{L^2(\mathbb{R}^2)}\lesssim\|\partial_y v_0\|_{L^2(\mathbb{R}^2)}+\|\partial_y c_1\|_{L^2(\mathbb{R})}\lesssim\|\langle x\rangle^{s/2}\partial_y v_0\|_{L^2(\mathbb{R}^2)}.$$

Since $\int_{\mathbb{R}}v_*(x,y)\,\mathrm{d}x=0$,

$$\partial_x^{-1}v_*(x,y)=\int_{\pm\infty}^x\{v_0(x_1,y)+\varphi_{c_0}(x_1)-\varphi_{c_1(y)}(x_1)\}\,\mathrm{d}x_1.$$

By the Schwarz inequality, we have, for $\pm x>0$,

$$\begin{aligned}|\partial_x^{-1}v_*(x,y)|&\lesssim(\|\langle x\rangle^s v_0(\cdot,y)\|_{L^2(\mathbb{R})}+\|\langle x\rangle^s(\varphi_{c_1(y)}-\varphi_{c_0})\|_{L^2(\mathbb{R})})\langle x\rangle^{-s+1/2}\\&\lesssim(\|\langle x\rangle^s v_0(\cdot,y)\|_{L^2(\mathbb{R})}+|c_1(y)-c_0|)\langle x\rangle^{-s+1/2}\end{aligned}$$

and

$$\|\partial_x^{-1}v_*\|_{L^2(\mathbb{R}^2)}\lesssim\|\langle x\rangle^s v_0\|_{L^2}+\|c_1-c_0\|_{L^2(\mathbb{R}^2)}\lesssim\|\langle x\rangle^s v_0\|_{L^2(\mathbb{R}^2)}.$$

Thus, we complete the proof. □

Now we are in a position to prove theorem 1.2.

Proof of theorem 1.2. To prove theorem 1.2, we modify the definitions of $v_1(t,z,y)$, $v_2(t,z,y)$, $c(t,y)$ and $x(t,y)$ as follows. Let $\tilde v_1$ be a solution of (1.1) satisfying $\tilde v_1(0,x,y)=v_*(0,x,y)$. Then it follows from lemmas 7.1 and 10.1 that $\mathbb{M}_1(\infty)\lesssim\|\langle x\rangle^{s/2}v_0\|_{L^2(\mathbb{R}^2)}$. By (10.3),

$$\begin{aligned}\||D_x|^{-1/2}v_*\|_{L^2(\mathbb{R}^2)}+\||D_x|^{1/2}v_*\|_{L^2(\mathbb{R}^2)}+\||D_x|^{-1/2}|D_y|^{1/2}v_*\|_{L^2(\mathbb{R}^2)}&\\\lesssim\|v_*\|_{H^1(\mathbb{R}^2)}+\|\partial_x^{-1}v_*\|_{L^2(\mathbb{R}^2)}&\\\lesssim\|\langle x\rangle^s v_0\|_{H^1(\mathbb{R}^2)},&\end{aligned}$$

and $\mathbb{M}_1'(\infty)\lesssim\|\langle x\rangle^s v_0\|_{H^1(\mathbb{R}^2)}$ follows from lemma 7.2.

Let $\tilde{u}(t,x,y) = u(t,x,y) - \tilde{v}_1(t,x,y)$. Then $\tilde{u}(0,x,y) = \varphi_{c_1(y)}(x)$. By lemma 10.1,

$$\|\tilde{u}(0,x,y) - \varphi_{c_0}(x)\|_X \lesssim \|c_1(\cdot) - c_0\|_{L^2(\mathbb{R})} \lesssim \|\langle x\rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)},$$

and lemma 3.6 and remark 3.7 imply that there exist a $T > 0$ and $(v_2(t), \tilde{c}(t), \tilde{x}(t)) \in X \times Y \times Y$ satisfying (3.1), (3.4) and (3.21) for $t \in [0,T]$, where $\tilde{c}(t,y) = c(t,y) - c_0$ and $\tilde{x}(t,y) = x(t,y) - 2c_0 t$. Clearly, we have

$$\|v_2(0)\|_{X\cap L^2(\mathbb{R}^2)} + \|\tilde{c}(0)\|_Y + \|\tilde{x}(0)\|_Y \lesssim \|\langle x\rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)},$$

and following the proof of lemmas 5.1, 6.1 and 8.1, we can prove that

$$\begin{aligned}
\mathbb{M}_{c,x}(T) &\lesssim \|\langle x\rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_1(T) + \mathbb{M}_2(T)^2,\\
\mathbb{M}_v(T) &\lesssim \|\langle x\rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T),\\
\mathbb{M}_2(T) &\lesssim \|\langle x\rangle^{s/2} v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T).
\end{aligned}$$

Thus, we can prove theorem 1.2 in exactly the same way as theorem 1.1. □

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Appendix A. Proof of claim 5.4

By claims B.1 and B.2 in [24],

$$\|\tilde{S}_1\|_{B(Y)} + \|\tilde{S}_1\|_{B(Y_1)} \lesssim 1, \quad [\tilde{S}_1, \partial_y] = 0, \tag{A 1}$$

$$\|\tilde{S}_2\|_{B(Y_1,Y)} \lesssim \|\tilde{c}\|_Y, \quad \|\tilde{S}_2\|_{B(Y)} \lesssim \|\tilde{c}\|_{L^\infty}, \quad \|[\partial_y, \bar{S}_2]\|_{B(Y_1,Y)} \lesssim \|c_y\|_Y. \tag{A 2}$$

Following the proof of claims B.3–B.5 in [24], we can show that

$$\|S_k^3[p](f)(t,\cdot)\|_Y \leqslant C\mathrm{e}^{-a(3t+L)}\|\mathrm{e}^{\alpha z}p\|_{L^2}\|\tilde{P}_1 f\|_Y, \quad [S_k^3[p], \partial_y] = 0, \tag{A 3}$$

$$\|S_k^4[p](f)(t,\cdot)\|_{Y_1} \leqslant C\mathrm{e}^{-a(3t+L)}\|\mathrm{e}^{\alpha z}p\|_{L^2}\|\tilde{c}(t)\|_Y\|f\|_{L^2}, \tag{A 4}$$

$$\|S_k^5(f)(t,\cdot)\|_{Y_1} + \|S_k^6(f)\|_{Y_1} \leqslant C\|v_2(t,\cdot)\|_X\|f\|_{L^2} \tag{A 5}$$

in exactly the same way. By (A 1) and (A 3), we have $[\partial_y, B_4] = 0$.

Applying (A 3), (A 4) with $p(z) = \partial_z^j \psi(z)$ $(j \geqslant 0)$ and using (A 5) and claim 4.3, we have

$$\left.\begin{aligned}
\|\tilde{S}_3\|_{B(Y)} + \|\bar{S}_3\|_{B(Y)} &\lesssim \mathrm{e}^{-\alpha(3t+L)},\\
\|\tilde{S}_4\|_{B(Y,Y_1)} + \|\bar{S}_4\|_{B(Y,Y_1)} &\lesssim \|\tilde{c}(t)\|_Y \mathrm{e}^{-\alpha(3t+L)},
\end{aligned}\right\} \tag{A 6}$$

$$\|\tilde{S}_5\|_{L^2(0,T;B(Y,Y_1))} + \|\bar{S}_5\|_{L^2(0,T;B(Y,Y_1))} \lesssim \|v_2(t)\|_{L^2(0,T;X)}. \tag{A 7}$$

In view of (4.20),

$$[\partial_y, B_3] = [\partial_y, \widetilde{\mathcal{C}}_1] + \sum_{j=1,2} \partial_y^2[\partial_y, \bar{S}_j] - \sum_{j=3,4,5} [\partial_y, \bar{S}_j]. \tag{A 8}$$

We will estimate each term of the right-hand side following the proof of [24, claim 7.1]. By [24, claim B.7],

$$\|[\partial_y, \widetilde{\mathcal{C}}_k]\|_{B(Y,Y_1)} \lesssim \|c_y\|_Y \quad \text{for } k=1,2. \tag{A 9}$$

Applying [24, claims B.1–B.7] to $[\partial_y, \bar{S}_j] = \{[\partial_y, \tilde{S}_j] + \bar{S}_j[\widetilde{\mathcal{C}}_2, \partial_y]\}(I+\widetilde{\mathcal{C}}_2)^{-1}$, we have

$$\|[\partial_y, \bar{S}_j]\|_{B(Y,Y_1)} \lesssim \|c_y\|_Y \quad \text{for } 1 \leqslant j \leqslant 4. \tag{A 10}$$

By (A 5) and the fact that ∂_y is bounded on Y and Y_1,

$$\|[\partial_y, \bar{S}_5]\|_{B(Y,Y_1)} \lesssim \|v_2\|_X. \tag{A 11}$$

Combining (A 8)–(A 11), we obtain the first two estimates of claim 5.4. Thus, we complete the proof. □

Finally, we will estimate the operator norm of $S_1^7[q_c]$.

Claim A.1. *There exist positive constants C and δ such that if*

$$\sup_{t\in[0,T]} \|\tilde{c}(t)\|_{L^\infty} \leqslant \delta,$$

then

$$\|S^7[q_c](f)(t,\cdot)\|_{Y_1} \leqslant C\|v_1(t,\cdot)\|_{W(t)} \|\mathrm{e}^{\alpha|\cdot|} \sup_{c\in[2-\delta,2+\delta]} q_c\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}. \tag{A 12}$$

Proof. Applying the Schwarz inequality to the right-hand side of

$$\|S_1^7[q_c](f)(t,y)\|_{Y_1} = \frac{1}{2\sqrt{2\pi}} \left\| \int_{\mathbb{R}^2} v_1(t,z,y) f(y) q_{c(t,y)}(z) \mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y \right\|_{L^\infty[-\eta_0,\eta_0]},$$

we have (A 12). □

Using lemma 5.2, we can prove the following commutator estimate in the same way as claim 5.4.

Claim A.2. *There exist positive constants C and δ such that if $\mathbb{M}_{c,x}(T) \leqslant \delta$, then*

$$\|[\partial_t, B_3]\|_{B(L^2(0,T;Y),L^1(0,T;Y))} \leqslant C(\mathrm{e}^{-\alpha L} + \mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T)).$$

Appendix B. Estimates of R^k

Claim B.1. *There exist positive constants δ and C such that if $\mathbb{M}_{c,x}(T) \leqslant \delta$, then*

$$\|R_k^2\|_{L^2(0,T;Y)} \leqslant C\mathbb{M}_{c,x}(T)^2.$$

Proof. By [24, claims B.1 and B.2],

$$\|R_k^2\|_Y \lesssim \|\tilde{c}\|_{L^\infty}(\|x_{yy}\|_Y + \|c_{yy}\|_Y) + (1+\|\tilde{c}\|_{L^\infty})\|c_y\|_{L^\infty}\|c_y\|_Y.$$

Since $Y \subset H^1(\mathbb{R})$, we have claim B.1. □

Claim B.2. *There exist positive constants δ and C such that if $\mathbb{M}_1(T) \leqslant \delta$, then $\|R_k^3(t,\cdot)\|_Y \leqslant C\mathrm{e}^{-\alpha(3t+L)}\mathbb{M}_{c,x}(T)^2$ for $t \in [0,T]$.*

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Claim B.3. *There exist positive constants C and L_0 such that if $L \geqslant L_0$, then*

$$\|\widetilde{\mathcal{A}}_1(t)\|_{B(Y)} + \|\widetilde{\mathcal{A}}_1(t)\|_{B(Y_1)} + \|A_1(t)\|_{B(Y)} \leqslant C\mathrm{e}^{-\alpha(3t+L)} \quad \text{for every } t \geqslant 0.$$

Claims B.2 and B.3 can be shown in exactly the same way as [24, claims D.2 and D.3].

Claim B.4. *Suppose that $\alpha \in (0,1)$ and $\mathbb{M}_1(T) \leqslant \delta$. If δ is sufficiently small, then there exists a positive constant C such that*

$$\sup_{t\in[0,T]} \|R_k^4(t)\|_{Y_1} + \|R_k^4\|_{L^1(0,T;Y_1)} \leqslant C(\mathbb{M}_{c,x}(T) + \mathbb{M}_1(T) + \mathbb{M}_2(T))\mathbb{M}_2(T), \quad \text{(B 1)}$$

$$\sup_{t\in[0,T]} \|R_k^5(t)\|_{Y_1} + \|R_k^5\|_{L^2(0,T;Y_1)} \leqslant C\mathbb{M}_{c,x}(T)\mathbb{M}_2(T), \quad \text{(B 2)}$$

$$\|R_k^6\|_{Y_1} \leqslant C\mathrm{e}^{-\alpha(3t+L)}\mathbb{M}_{c,x}(T)\mathbb{M}_2(T). \quad \text{(B 3)}$$

Proof. Following the proof of claim D.5 in [24], we have

$$\begin{aligned}
\|\mathrm{II}_k^1(t,\cdot)\|_{Z_1} &\lesssim (\|c_y(t)\|_Y + \|c_{yy}\|_Y + \|c_y(t)\|_{L^4}^2)\|v_2(t)\|_X, \\
\|\mathrm{II}_k^2(t,\cdot)\|_{Z_1} &\lesssim (\|\mathrm{e}^{-\alpha|z|/2}v_1(t)\|_{L^2} + \|v_2(t)\|_X)\|v_2(t)\|_X, \\
\|\mathrm{II}_{k1}^3(t,\cdot)\|_{Z_1} &\lesssim \|x_{yy}(t)\|_Y\|v_2(t)\|_X, \\
\|\mathrm{II}_{k2}^3(t,\cdot)\|_{Z_1} &\lesssim \|x_y(t)\|_Y\|v_2(t)\|_X, \\
\|R_k^6\|_{Y_1} &\lesssim \|v_2(t)\|_X\|\tilde{\psi}_{c(t,y)}\|_X \\
&\lesssim \mathrm{e}^{-\alpha(3t+L)}\|\tilde{c}(t)\|_{L^2(\mathbb{R})}\|v_2(t)\|_X.
\end{aligned}$$

Claim B.4 follows immediately from the above. □

Claim B.5. *There exist positive constants δ and C such that if $\mathbb{M}_{c,x}(T) \leqslant \delta$, then*

$$\sup_{t\in[0,T]} \|\tilde{P}_1R_1^7\|_Y + \|\tilde{P}_1R_1^{71}\|_{L^1(0,T;Y)} \leqslant C\mathbb{M}_{c,x}(T)^2, \quad \text{(B 4)}$$

$$\sup_{t\in[0,T]} \|\tilde{P}_1R_2^7\|_Y + \|\tilde{P}_1R_2^7\|_{L^2(0,T;Y)} \leqslant C\mathbb{M}_{c,x}(T)^2. \quad \text{(B 5)}$$

Proof. Since $\|f\|_{L^\infty} \lesssim \|f\|_Y^{1/2}\|f_y\|_Y^{1/2}$ for $f \in Y$, it follows from [24, (D.11), (D.15)] that

$$\|(\tfrac{1}{2}c)^{1/2}c_y - b_y\|_{L^2} \lesssim \|(\tfrac{1}{2}c)^{1/2} - 1\|_{L^\infty}\|c_y\|_Y + \|b_y - c_y\|_Y \lesssim \|\tilde{c}\|_Y^{1/2}\|c_y\|_Y^{3/2}$$

and

$$\|(\tfrac{1}{2}c)^{3/2} - 1 - \tfrac{3}{4}b\|_{L^\infty} \lesssim \|(\tfrac{1}{2}c)^{3/2} - 1 - \tfrac{3}{4}b\|_{L^2}^{1/2}\|(\tfrac{1}{2}c)^{1/2}c_y - b_y\|_{L^2}^{1/2} \lesssim \|\tilde{c}\|_Y\|c_y\|_Y.$$

Combining the above with [24, (D.11), (D.13)], we have

$$\begin{aligned}
\|\tilde{P}_1R_1^7\|_Y &\lesssim \|(\tfrac{1}{2}c)^{3/2} - 1 - \tfrac{3}{4}b\|_{L^\infty}\|x_{yy}\|_{L^2} + \|x_y\|_{L^\infty}\|b_y - (\tfrac{1}{2}c)^{1/2}c_y\|_Y + \|c_y\|_Y^2 \\
&\lesssim \|x_{yy}\|_Y\|c_y\|_Y\|\tilde{c}\|_Y + \|x_y\|_Y^{1/2}\|x_{yy}\|_Y^{1/2}\|c_y\|_Y^{3/2}\|\tilde{c}\|_Y^{1/2} + \|c_y\|_Y^2.
\end{aligned}$$

Hence, by the definition of $\mathbb{M}_{c,x}(T)$, we have (B 4). We can prove (B 5) by using [24, claim D.6] and (5.10) in a similar way. Thus, we complete the proof. □

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CLAIM B.6. *There exist positive constants C and δ such that if $\mathbb{M}_{c,x}(T)+\mathbb{M}_2(T) < \delta$, then*

$$\sup_{t\in[0,T]} \|R^8(t)\|_Y + \|R^8\|_{L^2(0,T;Y)} \leqslant C\mathbb{M}_{c,x}(T)^2, \tag{B 6}$$

$$\sup_{t\in[0,T]} \|R^9(t)\|_Y + \|R^9\|_{L^1(0,T;Y)} \leqslant C\mathbb{M}_{c,x}(T)(\mathbb{M}_{c,x}(T)+\mathbb{M}_2(T)), \tag{B 7}$$

$$\sup_{t\in[0,T]} \|R^{10}(t)\|_Y + \|R^{10}\|_{L^2(0,T;Y)} \leqslant C\mathbb{M}_{c,x}(T)^2, \tag{B 8}$$

$$\sup_{t\in[0,T]} \|R^{11}(t)\|_Y + \|R^{11}\|_{L^1(0,T;Y)} \leqslant C\mathbb{M}_{c,x}(T)^2. \tag{B 9}$$

Proof. By (3.22) and the fact that $\|b\|_Y \lesssim \|\tilde{c}\|_Y$,

$$\|(I+\mathcal{C}_2)(c_yx_y)-(bx_y)_y\|_Y \lesssim (\|\tilde{c}\|_Y+\|x_y\|_Y)(\|c_y\|_Y+\|x_{yy}\|_Y),$$

whence

$$\|(I+\mathcal{C}_2)(c_yx_y)-(bx_y)_y\|_{L^2(0,T;Y)\cap L^\infty(0,T;Y)} \lesssim \mathbb{M}_{c,x}(T)^2. \tag{B 10}$$

Equation (B 6) follows from (B 10) and [24, (C.1), (C.2)]. Equation (B 7) follows from (B 10), (A 6) and (A 7).

By [24, claim B.1 and (D.11)], we have $\|\tilde{S}_0\|_{B(Y)} \lesssim 1$ and

$$\|R^{10}\|_Y \lesssim \|c_y\|_Y\|\tilde{c}\|_{L^\infty}. \tag{B 11}$$

By claim B.3 and [24, (D.10)],

$$\|R^{11}\|_Y \lesssim \mathrm{e}^{-\alpha(3t+L)}\|\tilde{c}\|_{L^\infty}\|\tilde{c}\|_Y. \tag{B 12}$$

The estimates (B 8) and (B 9) follow immediately from (B 11) and (B 12). □

CLAIM B.7. *There exist positive constants C and δ such that if $\mathbb{M}_{c,x}(T)\leqslant\delta$, then*

$$\|R_{11}^{v_1}\|_{L^1(0,T;Y_1)} \leqslant C\mathbb{M}_1(T)(\mathbb{M}_{c,x}(T)+\mathbb{M}_1(T)), \tag{B 13}$$

$$\|R_2^{v_1}\|_{L^2(0,T;Y)} + \|R_{12}^{v_1}\|_{L^2(0,T;Y)} \leqslant C\mathbb{M}_1(T). \tag{B 14}$$

Proof. By the assumption, there exists a $\delta'\in(0,2)$ such that $c(t,y)\in[2-\delta',2+\delta']$ for $t\in[0,T]$ and $y\in\mathbb{R}$. Since ψ has a compact support,

$$\begin{aligned}\|\mathrm{II}_{13}^6(t,\eta)\|_{L^\infty(-\eta_0,\eta_0)} &\lesssim \|v_1(t)\|_{L^2(\mathbb{R}^2)}\|\tilde{c}\|_Y \sup_{\substack{\eta\in[-\eta_0,\eta_0],\\ c\in[2-\delta',2+\delta']}} \|\psi(\cdot+3t)\partial_z g^*(\cdot,\eta,c)\|_{L^2(\mathbb{R})}\\ &\lesssim \mathrm{e}^{-\alpha(3t+L)}\|\tilde{c}(t)\|_Y\|v_1(t)\|_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{B 15}$$

By the Schwarz inequality,

$$\begin{aligned}\|\mathrm{II}_{111}^6(t,\eta)\|_{L^\infty(-\eta_0,\eta_0)} &\lesssim \|v_1(t)\|_{W(t)}^2 + \|\partial_x^{-1}\partial_y v_1(t)\|_{W(t)}\|c_y(t)\|_Y\\ &\quad + \|v_1(t)\|_{W(t)}(\|x_{yy}(t)\|_Y + \|(c_yx_y)(t)\|_{L^2(\mathbb{R})}).\end{aligned} \tag{B 16}$$

Combining (B 15) and (B 16), we have (B 13).

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Next, we will prove (B 14). We decompose II^6_{112} as $\mathrm{II}^6_{1121} + \mathrm{II}^6_{1122}$, where

$$\begin{aligned}
\mathrm{II}^6_{1121}(t,\eta) &= -\frac{3}{2}\int_{\mathbb{R}^2} (\partial_z^{-1}\partial_y v_1)(t,z,y)\varphi(z)\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y \\
&= -\frac{3\sqrt{2\pi}}{2}\int_{\mathbb{R}} \varphi(z)\mathcal{F}_y(\partial_z^{-1}\partial_y v_1)(t,z,\eta)\,\mathrm{d}z, \\
\mathrm{II}^6_{1122}(t,\eta) &= -\frac{3}{2}\int_{\mathbb{R}^2} (\partial_z^{-1}\partial_y v_1)(t,z,y)\tilde{c}(t,y)\delta\varphi_{c(t,y)}(z)\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y \\
&\quad + 3\int_{\mathbb{R}^2} v_1(t,z,y)x_y(t,y)\varphi_{c(t,y)}(z)\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z\,\mathrm{d}y.
\end{aligned}$$

By the Schwarz inequality and Plancherel's theorem,

$$\begin{aligned}
&\|\mathrm{II}^6_{1121}(t,\cdot)\|_{L^2(-\eta_0,\eta_0)} \\
&\quad\lesssim \left(\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}} \mathrm{e}^{-2\alpha|z|}|\mathcal{F}_y(\partial_z^{-1}\partial_y v_1)(t,z,\eta)|^2\,\mathrm{d}z\,\mathrm{d}\eta\right)^{1/2} \|\mathrm{e}^{\alpha|\cdot|}\varphi\|_{L^2(\mathbb{R})} \\
&\quad\lesssim \|v_1\|_{W(t)},
\end{aligned} \tag{B 17}$$

and

$$\|\mathrm{II}^6_{1122}(t,\eta)\|_{L^\infty(-\eta_0,\eta_0)} \lesssim (\|v_1\|_{W(t)} + \|\partial_z^{-1}\partial_y v_1\|_{W(t)})(\|\tilde{c}(t)\|_Y + \|x_y(t)\|_Y). \tag{B 18}$$

Similarly, we have

$$\|\mathrm{II}^6_2(t,\cdot)\|_{L^2(-\eta_0,\eta_0)} + \|\mathrm{II}^6_{12}(t,\cdot)\|_{L^2(-\eta_0,\eta_0)} \lesssim \|v_1(t)\|_{W(t)}. \tag{B 19}$$

Since $Y_1 \subset Y$, we have (B 14) from (B 17)–(B 19). Thus, we complete the proof. $\square$

Finally, we will estimate $k(t,y)$.

Claim B.8. *There exist positive constants C and δ such that if $\mathbb{M}_{c,x}(T) \leqslant \delta$, then*

$$\sup_{t\in[0,T]} \|k(t,\cdot)\|_Y + \|k\|_{L^2(0,T;Y)} \leqslant C\mathbb{M}_1(T). \tag{B 20}$$

Moreover,

$$\lim_{t\to\infty} \|k(t,\cdot)\|_Y = 0. \tag{B 21}$$

Proof. Let $\delta\varphi_c = (\varphi_c - \varphi)/\tilde{c}$ and

$$\begin{aligned}
k_1(t,y) &= \frac{1}{4\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2} v_1(t,z,y_1)\varphi(z)\mathrm{e}^{\mathrm{i}(y-y_1)\eta}\,\mathrm{d}z\,\mathrm{d}y_1\,\mathrm{d}\eta, \\
k_2(t,y) &= \frac{1}{4\pi}\int_{-\eta_0}^{\eta_0}\int_{\mathbb{R}^2} \tilde{c}(t,y_1)v_1(t,z,y_1)\delta\varphi_{c(t,y_1)}(z)\mathrm{e}^{\mathrm{i}(y-y_1)\eta}\,\mathrm{d}z\,\mathrm{d}y_1\,\mathrm{d}\eta.
\end{aligned}$$

By the definitions, we have $k = k_1 + k_2$. Using Plancherel's theorem and Minkowski's inequality, we have

$$\begin{aligned}\|k_1(t,\cdot)\|_Y &= \frac{1}{2\sqrt{2\pi}}\left\| \int_{\mathbb{R}} (\mathcal{F}_y v_1)(t,z,\cdot)\varphi(z)\,\mathrm{d}z \right\|_{L^2(-\eta_0,\eta_0)} \\ &\leqslant \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \|(\mathcal{F}_y v_1)(t,z,\cdot)\|_{L^2(-\eta_0,\eta_0)}\varphi(z)\,\mathrm{d}z \\ &\leqslant \|\mathrm{e}^{-\alpha|\cdot|}v_1(t,\cdot)\|_{L^2(\mathbb{R}^2)}\|\mathrm{e}^{\alpha|\cdot|}\varphi\|_{L^2(\mathbb{R})} \lesssim \|v_1(t)\|_{W(t)}. \end{aligned} \tag{B 22}$$

If $\mathbb{M}_{c,x}(T) \leqslant \delta$ and δ is sufficiently small, then there exists $\delta' \in (0, 2-\alpha)$ such that $|c(t,y) - 2| \leqslant \delta'$ for every $t \in [0,T]$ and $y \in \mathbb{R}$, and

$$\begin{aligned}\|k_2(t,\cdot)\|_{Y_1} &= \frac{1}{2\sqrt{2\pi}}\left\| \int_{\mathbb{R}} v_1(t,z,y)\tilde{c}(t,y)\delta\varphi_{c(t,y)}(z)\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}z \right\|_{L^\infty(-\eta_0,\eta_0)} \\ &\lesssim \|v_1(t)\|_{W(t)}\|\tilde{c}(t)\|_Y \quad \text{for } t\in[0,T]. \end{aligned} \tag{B 23}$$

Since $Y_1 \subset Y$, we see that (B 20) follows from (B 22) and (B 23). Moreover, we have (B 21) by combining (B 22) and (B 23) with (7.1). Thus, we complete the proof. □

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