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HIGHER DERIVATIONS AND THE JORDAN CANONICAL FORM OF THE COMPANION MATRIX

BY
LESLIE G. ROBERTS

The purpose of this note is to give a basis with respect to which the companion matrix of an equation (over a field of any characteristic) is in Jordan canonical form.

Let k be a field. Define a k-linear map $D_i: k[X] \to k[X]$ by $D_i X^n = C_i^n X^{n-i}$, where the integer C_i^n is the binomial coefficient $n!/i!\,(n-i)!$. We adopt the usual convention that $C_i^n = 0$ if $i > n$, or $i < 0$. Then $D = (D_0, D_1, D_2, \ldots)$ is a higher derivation (see [1, p. 192]). Thus if $f, g \in k[X]$ we have

$$D_i(fg) = \sum_{j+k=i} D_j(f) D_k(g).$$

From this one can prove by induction on n that $D_i(X-\alpha)^n = C_i^n (X-\alpha)^{n-i}$. Applying the last formula to $f = (X-\alpha)^i g$, where $g(\alpha) \neq 0$, we see that α is an i-fold root of f if and only if $f(\alpha) = 0$, $(D_1 f)(\alpha) = 0, \ldots, (D_{i-1} f)(\alpha) = 0$, but $(D_i f)(\alpha) \neq 0$.

Consider the n-dimensional vector space k^n over k. For convenience of notation we will write elements of k^n as row vectors, with the matrix of a linear transformation acting on the right. Let $f(X) = \prod_{i=1}^r (X-\alpha_i)^{n_i}$, where $\sum_{i=1}^r n_i = n$, and the $\alpha_i \in k$ are distinct. Then

$$f(X) = X^n + p_1 X^{n-1} + \cdots + p_n.$$

A row vector with co-ordinates in $k[X]$ gives a function from k to k^n, by substituting $\alpha \in k$ in place of X. The D_i act on such row vectors co-ordinatewise. Let $\mathbf{X} = (1, X, X^2, \ldots, X^{n-1})$, and let $V_{ij} = (D_j \mathbf{X})(\alpha_i)$, $(1 \le i \le r;\ 0 \le j \le n_i - 1)$. I claim that the V_{ij} form a basis of k^n, and that with regard to this basis, the companion matrix A of f is in Jordan canonical form.

The matrix A has ones in the diagonal below the main diagonal, and its last column is the transpose of $(-p_n, -p_{n-1}, \ldots, -p_1)$. We can write

$$\mathbf{X} = (X^{k-1}), \quad 1 \le k \le n,$$

thus

$$V_{ij} = (D_j(X^{k-1})(\alpha_i)) = (C_j^{k-1} \alpha_i^{k-j-1}), \quad 1 \le k \le n.$$

A straightforward calculation shows that $V_{ij} A = (C_j^k \alpha_i^{k-j})$, $1 \le k \le n$. The initial remark that α_i is a root of $D_j f = \sum_{k=0}^n p_{n-k} C_j^k X^{k-j}$ is used in calculating the nth co-ordinate. Then we make use of the relation $C_j^k = C_{j-1}^{k-1} + C_j^{k-1}$ $(k \geq 1,\ j \geq 0)$ to show that $V_{ij}A = \alpha_i V_{ij} + V_{i,j-1}$ if $j \geq 1$, and $V_{i0}A = \alpha_i V_{i0}$. The V_{ij}, for fixed i and variable j, are linearly independent since their first nonzero co-ordinate, the $(j+1)$st, is equal to one. The above equations then show that the V_{ij} (fixed i) lie in V_i = kernel of $(A - \alpha_i)^{n_i}$. They form a basis of V_i since they are linearly independent, and there are $n_i = \dim V_i$ of them. Since $k^n = \bigoplus_{i=1}^r V_i$, the V_{ij} $(1 \leq i \leq r,\ 0 \leq j \leq n_i - 1)$ form a basis of k^n. The matrix for the linear transformation A with regard to this basis is in Jordan canonical form, again because of the calculation of $V_{ij}A$.

If each n_i equals one, the matrix with the V_{i0} as rows is the Vandermonde matrix. As another example, suppose $n_1 = 1$, $n_2 = 4$ and $n = 5$. Then the basis is $(1, \alpha_1, \alpha_1^2, \alpha_1^3, \alpha_1^4)$, $(1, \alpha_2, \alpha_2^2, \alpha_2^3, \alpha_2^4)$, $(0, 1, 2\alpha_2, 3\alpha_2^2, 4\alpha_2^3)$, $(0, 0, 1, 3\alpha_2, 6\alpha_2^2)$, $(0, 0, 0, 1, 4\alpha_2)$.

Bibliography

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Queen's University,
Kingston, Ontario