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Abstract

Let $U_q'(\mathfrak{g})$ be a quantum affine algebra with an indeterminate q, and let $\mathscr{C}_\mathfrak{g}$ be the category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules. We write $\mathscr{C}_\mathfrak{g}^0$ for the monoidal subcategory of $\mathscr{C}_\mathfrak{g}$ introduced by Hernandez and Leclerc. In this paper, we associate a simply laced finite-type root system to each quantum affine algebra $U_q'(\mathfrak{g})$ in a natural way and show that the block decompositions of $\mathscr{C}_\mathfrak{g}$ and $\mathscr{C}_\mathfrak{g}^0$ are parameterized by the lattices associated with the root system. We first define a certain abelian group $\mathcal{W}$ (respectively $\mathcal{W}_0$) arising from simple modules of $\mathscr{C}_\mathfrak{g}$ (respectively $\mathscr{C}_\mathfrak{g}^0$) by using the invariant Λ^∞ introduced in previous work by the authors. The groups $\mathcal{W}$ and $\mathcal{W}_0$ have subsets Δ and Δ_0 determined by the fundamental representations in $\mathscr{C}_\mathfrak{g}$ and $\mathscr{C}_\mathfrak{g}^0$, respectively. We prove that the pair $(\mathbb{R} \otimes_\mathbb{Z} \mathcal{W}_0, \Delta_0)$ is an irreducible simply laced root system of finite type and that the pair $(\mathbb{R} \otimes_\mathbb{Z} \mathcal{W}, \Delta)$ is isomorphic to the direct sum of infinite copies of $(\mathbb{R} \otimes_\mathbb{Z} \mathcal{W}_0, \Delta_0)$ as a root system.

Contents

1. Introduction

Let q be an indeterminate and let $U_q'(\mathfrak{g})$ be a quantum affine algebra. The category $\mathscr{C}_\mathfrak{g}$ of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules has a rich structure. For example, the category $\mathscr{C}_\mathfrak{g}$ is not semi-simple and has a rigid monoidal category structure. Because of its rich structure, it has been studied actively in various research areas of mathematics and physics (see [AK97, CP94, FR99, Kas02, Nak01] for examples).

The category $\mathscr{C}_\mathfrak{g}$ has been studied from the viewpoint of cluster algebras. Suppose that $\mathfrak{g}$ is of simply laced affine ADE type. In [HL10], Hernandez and Leclerc defined the full subcategory $\mathscr{C}_\mathfrak{g}^0$ of $\mathscr{C}_\mathfrak{g}$ such that all simple subquotients of its objects are obtained via simple subquotients of tensor products of certain fundamental representations. They then introduced certain monoidal subcategories $\mathscr{C}_\ell$ ($\ell \in \mathbb{Z}_{>0}$) and studied their Grothendieck rings using cluster algebras. As any simple module in $\mathscr{C}_\mathfrak{g}$ can be obtained from a tensor product of suitable parameter shifts of simple modules in $\mathscr{C}_\mathfrak{g}^0$, the category $\mathscr{C}_\mathfrak{g}^0$ has an essential position in $\mathscr{C}_\mathfrak{g}$. Note that an algorithm for computing q-characters of Kirillov–Reshetikhin modules for any untwisted quantum affine algebras was described in [HL16], by studying the cluster algebra structure of the Grothendieck ring of the subcategory $\mathscr{C}_\mathfrak{g}^-$ of $\mathscr{C}_\mathfrak{g}^0$. On the other hand, Hernandez and Leclerc introduced another abelian monoidal subcategory $\mathscr{C}_\mathfrak{g}^Q$ which categorifies the coordinate ring $\mathbb{C}[N]$ of the unipotent group associated with the finite-dimensional simple Lie algebra $\mathfrak{g}_0$ inside $\mathfrak{g}$ [HL15]. For each Dynkin quiver Q, they defined an abelian subcategory $\mathscr{C}_\mathfrak{g}^Q$ of $\mathscr{C}_\mathfrak{g}^0$ which contains some fundamental representations parameterized by the coordinates of vertices of the Auslander–Reiten quiver of Q, and proved that $\mathscr{C}_\mathfrak{g}^Q$ is stable under taking tensor products and that its complexified Grothendieck ring $\mathbb{C} \otimes_\mathbb{Z} K(\mathscr{C}_\mathfrak{g}^Q)$ is isomorphic to the coordinate ring $\mathbb{C}[N]$. Moreover, under this isomorphism, the set of isomorphism classes of simple modules in $\mathscr{C}_\mathfrak{g}^Q$ corresponds to the upper global basis of $\mathbb{C}[N]$.

The notion of the categories $\mathscr{C}_\mathfrak{g}^0$ and $\mathscr{C}_\mathfrak{g}^Q$ has been extended to all untwisted and twisted quantum affine algebras [KKKO16, KO19, OS19a, OS19b]. Let $\sigma(\mathfrak{g}) := I_0 \times \mathbf{k}^\times / \sim$, where the equivalence relation is given by $(i, x) \sim (j, y)$ if and only if $V(\varpi_i)_x \simeq V(\varpi_j)_y$. The set $\sigma(\mathfrak{g})$ has a quiver structure determined by the pole of R-matrices between tensor products of fundamental representations $V(\varpi_i)_x$ ($(i, x) \in \sigma(\mathfrak{g})$). Let $\sigma_0(\mathfrak{g})$ be a connected component of $\sigma(\mathfrak{g})$. Then the category $\mathscr{C}_\mathfrak{g}^0$ is defined to be the smallest full subcategory of $\mathscr{C}_\mathfrak{g}$ that has the following properties:

(a) $\mathscr{C}_\mathfrak{g}^0$ contains $V(\varpi_i)_x$ for all $(i, x) \in \sigma_0(\mathfrak{g})$;
(b) $\mathscr{C}_\mathfrak{g}^0$ is stable by taking subquotients, extensions and tensor products.

The subcategory $\mathscr{C}_\mathfrak{g}^Q$ was introduced in [KKKO16] for twisted affine type $A^{(2)}$ and $D^{(2)}$, in [KO19] for untwisted affine types $B^{(1)}$ and $C^{(1)}$, and in [OS19a, OS19b] for exceptional affine type. For a Dynkin quiver Q of a certain type with additional data, a finite subset $\sigma_Q(\mathfrak{g})$ of $\sigma_0(\mathfrak{g})$

was determined. Then the category $\mathscr{C}_{\mathfrak{g}}^Q$ is defined to be the smallest full subcategory of $\mathscr{C}_{\mathfrak{g}}^0$ for which the following hold:

(a) $\mathscr{C}_{\mathfrak{g}}^Q$ contains $\mathbf{1}$ and $V(\varpi_i)_x$ for all $(i, x) \in \sigma_Q(\mathfrak{g})$;
(b) $\mathscr{C}_{\mathfrak{g}}^Q$ is stable by taking subquotients, extensions and tensor products

(see §§ 2.3 and 2.4 for more details).

We can summarize the results of this paper as follows:

(i) we associate a simply laced root system to each quantum affine algebra $U_q'(\mathfrak{g})$ in a natural way;
(ii) we give the block decomposition of $\mathscr{C}_{\mathfrak{g}}$ parameterized by a lattice $\mathcal{W}$ associated with the root system.

Let $U_q'(\mathfrak{g})$ be a quantum affine algebra of *arbitrary* type. We first consider certain subgroups $\mathcal{W}$ and $\mathcal{W}_0$ of the abelian group $\operatorname{Hom}(\sigma(\mathfrak{g}), \mathbb{Z})$ arising from simple modules of $\mathscr{C}_{\mathfrak{g}}$ and $\mathscr{C}_{\mathfrak{g}}^0$, respectively (see (4.2)). The subgroups $\mathcal{W}$ and $\mathcal{W}_0$ have subsets Δ and Δ_0 determined by the fundamental representations in $\mathscr{C}_{\mathfrak{g}}$ and $\mathscr{C}_{\mathfrak{g}}^0$, respectively. Let $\mathcal{E} := \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{W}$ and $\mathcal{E}_0 := \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{W}_0$. Let $\mathfrak{g}_{\mathrm{fin}}$ be the simply laced finite-type Lie algebra corresponding to the affine type of $\mathfrak{g}$ in table (4.5). When $\mathfrak{g}$ is of untwisted affine type ADE, $\mathfrak{g}_{\mathrm{fin}}$ coincides with $\mathfrak{g}_0$. We prove that the pair $(\mathcal{E}_0, \Delta_0)$ is the irreducible root system of the Lie algebra $\mathfrak{g}_{\mathrm{fin}}$ and the pair $(\mathcal{E}, \Delta)$ is isomorphic to the direct sum of infinite copies of $(\mathcal{E}_0, \Delta_0)$ as a root system (see Theorem 4.6 and Corollary 4.7). Interestingly enough, the quantum affine algebra $U_q'(\mathfrak{g})$ and its Langlands dual $U_q'(^L\mathfrak{g})$, whose Cartan matrix is the transpose of that of $\mathfrak{g}$, yield the same simply laced root system. This coincidence can also be viewed in terms of the mysterious duality between $U_q'(\mathfrak{g})$ and its Langlands dual $U_q'(^L\mathfrak{g})$ (see [FH11a, FH11b, FR98]). We conjecture that the categories of representations of two quantum affine algebras are equivalent if and only if their associated root systems are the same. From this viewpoint, the simply laced finite-type root system plays the role of an invariant for the representation categories of quantum affine algebras. For each simply laced finite-type root system, the corresponding untwisted quantum affine algebra, the one of twisted type (if it exists) and its Langland dual have the same categorical structure.

We then show that there exist direct decompositions of $\mathscr{C}_{\mathfrak{g}}$ and $\mathscr{C}_{\mathfrak{g}}^0$ parameterized by elements of $\mathcal{W}$ and $\mathcal{W}_0$, respectively (Theorem 5.10), and we prove that each direct summand of the decompositions is a block (Theorem 5.14). This approach covers *all untwisted and twisted* quantum affine algebras in a uniform way and provides a transparent explanation of how the blocks of $\mathscr{C}_{\mathfrak{g}}^0$ exist from the perspective of the root system $(\mathcal{E}_0, \Delta_0)$ and the category $\mathscr{C}_{\mathfrak{g}}^Q$.

When $\mathfrak{g}$ is of untwisted type, the block decomposition was studied in [CM05, EM03, JM11]. Etingof and Moura [EM03] found the block decomposition of $\mathscr{C}_{\mathfrak{g}}$ whose blocks are parameterized by the *elliptic central characters* under the condition $|q| < 1$. Later, Chari and Moura [CM05] gave a different description of the block decomposition of $\mathscr{C}_{\mathfrak{g}}$ by using the quotient group $\mathscr{P}_q/\mathscr{Q}_q$ of the *ℓ-weight lattice* $\mathscr{P}_q$ by the *ℓ-root lattice* $\mathscr{Q}_q$. In the case of the quantum affine algebra $U_\xi(\mathfrak{g})$ at roots of unity, its block decomposition was studied in [JM11]. For affine Kac–Moody algebras, the block decomposition of the category of finite-dimensional modules was studied in [CM04, Sen10]. Note that the block decomposition for affine Kac–Moody algebras does not explain blocks for quantum affine algebras $U_q'(\mathfrak{g})$. We remark that in the untwisted-type case, the quotient group $\mathscr{P}_q/\mathscr{Q}_q$ given in [CM05] (and also the result of [EM03]) provides another group presentation of $\mathcal{W}$ (see Remark 5.16).

The main tools used to prove our results are the new invariants Λ, Λ^∞ and $\mathfrak{d}$ for a pair of modules in $\mathscr{C}_{\mathfrak{g}}$ introduced in [KKOP20]. For non-zero modules M and N in $\mathscr{C}_{\mathfrak{g}}$ such that $R_{M,N_z}^{\mathrm{univ}}$

is *rationally renormalizable*, the integers $\Lambda(M, N)$, $\Lambda^\infty(M, N)$ and $\mathfrak{d}(M, N)$ are defined by using the *renormalizing coefficient* $c_{M,N}(z)$ (see § 3 for details). These invariants are quantum affine algebra analogues of the invariants (with the same notation) for pairs of graded modules over quiver Hecke algebras arising from the grading of R-matrices. The new invariants play similar roles in the representation theory of quantum affine algebras to those for quiver Hecke algebras.

Let us explain our results more precisely. Let $U'_q(\mathfrak{g})$ be a quantum affine algebra of *arbitrary* type. For $M \in \mathscr{C}_\mathfrak{g}$ such that the universal R-matrix $R^{\mathrm{univ}}_{M,V(\varpi_i)_z}$ is rationally renormalizable for any $i \in I_0$, we define $\mathsf{E}(M) \in \mathrm{Hom}(\sigma(\mathfrak{g}), \mathbb{Z})$ by

$$\mathsf{E}(M)(i, a) := \Lambda^\infty(M, V(\varpi_i)_a) \quad \text{for } (i, a) \in \sigma(\mathfrak{g})$$

and investigate its properties (Lemma 4.1). For $(i, a) \in \sigma(\mathfrak{g})$, we set

$$\mathsf{s}_{i,a} := \mathsf{E}(V(\varpi_i)_a) \in \mathrm{Hom}(\sigma(\mathfrak{g}), \mathbb{Z})$$

and

$$\mathcal{W} := \{\mathsf{E}(M) \mid M \text{ is simple in } \mathscr{C}_\mathfrak{g}\}, \quad \Delta := \{\mathsf{s}_{i,a} \mid (i, a) \in \sigma(\mathfrak{g})\} \subset \mathcal{W},$$

$$\mathcal{W}_0 := \{\mathsf{E}(M) \mid M \text{ is simple in } \mathscr{C}^0_\mathfrak{g}\}, \quad \Delta_0 := \{\mathsf{s}_{i,a} \mid (i, a) \in \sigma_0(\mathfrak{g})\} \subset \mathcal{W}_0.$$

Then $\mathcal{W}$ and $\mathcal{W}_0$ are abelian subgroups of $\mathrm{Hom}(\sigma(\mathfrak{g}), \mathbb{Z})$. Moreover, we see in Lemma 4.2 that there exists a unique symmetric bilinear form $(-, -)$ on $\mathcal{W}$ such that $(\mathsf{E}(M), \mathsf{E}(N)) = -\Lambda^\infty(M, N)$ for any simple modules $M, N \in \mathscr{C}_\mathfrak{g}$; it induces a symmetric bilinear form on $\mathcal{E}$. Then we prove that the pair $(\mathcal{E}_0, \Delta_0)$ is an irreducible root system of the simply laced finite type given in (4.5) (Theorem 4.6) and that the pair $(\mathcal{E}, \Delta)$ is isomorphic to the direct sum of infinite copies of $(\mathcal{E}_0, \Delta_0)$ as a root system (Corollary 4.7). Furthermore, the bilinear form $(-, -)$ is invariant under the Weyl group action. Theorem 4.6 is proved in § 6 via a case-by-case approach, using the explicit descriptions of $\sigma_Q(\mathfrak{g})$ for $\mathscr{C}^Q_\mathfrak{g}$ given in § 2.4 and the denominator formulas in Appendix A.

We then consider the block decompositions of $\mathscr{C}_\mathfrak{g}$ and $\mathscr{C}^0_g$. For $\alpha \in \mathcal{W}$, let $\mathscr{C}_{\mathfrak{g},\alpha}$ be the full subcategory of $\mathscr{C}_\mathfrak{g}$ consisting of objects X such that $\mathsf{E}(S) = \alpha$ for any simple subquotient S of X. We show that there exist the direct decompositions

$$\mathscr{C}_\mathfrak{g} = \bigoplus_{\alpha \in \mathcal{W}} \mathscr{C}_{\mathfrak{g},\alpha} \quad \text{and} \quad \mathscr{C}^0_\mathfrak{g} = \bigoplus_{\alpha \in \mathcal{W}_0} \mathscr{C}_{\mathfrak{g},\alpha}$$

by proving that $\mathrm{Ext}^1_{U'_q(\mathfrak{g})}(M, N) = 0$ for $M \in \mathscr{C}_{\mathfrak{g},\alpha}$ and $N \in \mathscr{C}_{\mathfrak{g},\beta}$ with $\alpha \neq \beta$ (Theorem 5.10). We set $\mathcal{P} := \bigoplus_{(i,a) \in \sigma(\mathfrak{g})} \mathbb{Z}\mathsf{e}_{(i,a)}$ and $\mathcal{P}_0 := \bigoplus_{(i,a) \in \sigma_0(\mathfrak{g})} \mathbb{Z}\mathsf{e}_{(i,a)}$, where $\mathsf{e}_{(i,a)}$ is a symbol. Then we define a group homomorphism $p \colon \mathcal{P} \twoheadrightarrow \mathcal{W}$ by $p(\mathsf{e}_{(i,a)}) = \mathsf{s}_{i,a}$ and set $p_0 := p|_{\mathcal{P}_0} \colon \mathcal{P}_0 \to \mathcal{W}_0$. It turns out that the kernel $\ker p_0$ coincides with the subgroup $\mathcal{Q}_0$ of $\mathcal{P}_0$ generated by elements of the form $\sum_{k=1}^m \mathsf{e}_{(i_k, a_k)}$ $((i_k, a_k) \in \sigma_0(\mathfrak{g}))$ such that the trivial module $\mathbf{1}$ appears in $V(i_1)_{a_1} \otimes \cdots \otimes V(i_m)_{a_m}$ as a simple subquotient (Lemma 5.13). We then prove that $\mathscr{C}_{\mathfrak{g},\alpha}$ is a block for any $\alpha \in \mathcal{W}$ (Theorem 5.14), which implies that the above decompositions are block decompositions of $\mathscr{C}_\mathfrak{g}$ and $\mathscr{C}^0_\mathfrak{g}$.

This paper is organized as follows. In § 2, we give the necessary background on quantum affine algebras, R-matrices, and the categories $\mathscr{C}_\mathfrak{g}$ and $\mathscr{C}^Q_\mathfrak{g}$. In § 3, we review the new invariants introduced in [KKOP20]. In § 4, we investigate properties of $\mathcal{W}$, Δ and $\mathsf{s}_{i,a}$ and state the main theorem for the root systems $(\mathcal{E}_0, \Delta_0)$ and $(\mathcal{E}, \Delta)$. In § 5, we prove the block decompositions of $\mathscr{C}_\mathfrak{g}$ and $\mathscr{C}^0_\mathfrak{g}$. Section 6 is devoted to a case-by-case proof of Theorem 4.6.

2. Preliminaries

CONVENTION.

(i) For a statement P, $\delta(P)$ is 1 or 0 according to whether P is true or not.
(ii) For an element a in a field $\mathbf{k}$ and $f(z) \in \mathbf{k}(z)$, we denote by $\mathrm{zero}_{z=a} f(z)$ the order of zero of $f(z)$ at $z = a$.

2.1 Quantum affine algebras

The quintuple $(\mathsf{A}, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$ is called an *affine Cartan datum* if it consists of the following components:

(i) an *affine Cartan matrix* $\mathsf{A} = (a_{ij})_{i,j \in I}$ with a finite index set I;
(ii) a free abelian group P of rank $|I| + 1$, called the *weight lattice*;
(iii) a set $\Pi = \{\alpha_i \in \mathsf{P} \mid i \in I\}$, whose elements are called *simple roots*;
(iv) the group $\mathsf{P}^\vee := \mathrm{Hom}_{\mathbb{Z}}(\mathsf{P}, \mathbb{Z})$, called the *coweight lattice*;
(v) a set $\Pi^\vee = \{h_i \mid i \in I\} \subset \mathsf{P}^\vee$, whose elements are *simple coroots*;

and if it satisfies the following properties:

(a) $\langle h_i, \alpha_j \rangle = a_{i,j}$ for any $i, j \in I$;
(b) for any $i \in I$, there exists $\Lambda_i \in \mathsf{P}$ such that $\langle h_j, \Lambda_i \rangle = \delta(i = j)$ for any $j \in I$;
(c) Π is linearly independent.

Let $\mathfrak{g}$ be the *affine Kac–Moody algebra* associated with $(\mathsf{A}, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$. We set $\mathsf{Q} := \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset \mathsf{P}$, which is called the *root lattice*, and $\mathsf{Q}^+ := \sum_{i \in I} \mathbb{Z}_{\geqslant 0} \alpha_i \subset \mathsf{Q}$. For $\beta = \sum_{i \in I} b_i \alpha_i \in \mathsf{Q}^+$, we write $|\beta| = \sum_{i \in I} b_i$. We denote by $\delta \in \mathsf{Q}$ the *imaginary root* and by $c \in \mathsf{Q}^\vee$ the *central element*. Note that the positive imaginary root Δ_+^{im} is equal to $\mathbb{Z}_{>0}\delta$ and the center of $\mathfrak{g}$ is generated by c. We write $\mathsf{P}_{\mathrm{cl}} := \mathsf{P}/(\mathsf{P} \cap \mathbb{Q}\delta)$, which is called the *classical weight lattice*, and take $\rho \in \mathsf{P}$ (respectively $\rho^\vee \in \mathsf{P}^\vee$) such that $\langle h_i, \rho \rangle = 1$ (respectively $\langle \rho^\vee, \alpha_i \rangle = 1$) for any $i \in I$. There exists a $\mathbb{Q}$-valued non-degenerate symmetric bilinear form $(\ ,\ )$ on P satisfying

$$\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{and} \quad \langle c, \lambda \rangle = (\delta, \lambda)$$

for any $i \in I$ and $\lambda \in \mathsf{P}$. We write $\mathsf{W} := \langle s_i \mid i \in I \rangle \subset \mathrm{Aut}(\mathsf{P})$ for the *Weyl group* of A, where $s_i(\lambda) := \lambda - \langle h_i, \lambda \rangle \alpha_i$ for $\lambda \in \mathsf{P}$. We will use the standard convention in [Kac90] of choosing $0 \in I$ except for type $A_{2n}^{(2)}$, in which case we take the longest simple root to be α_0, and for types $B_2^{(1)}$, $A_3^{(2)}$ and $E_k^{(1)}$ ($k = 6, 7, 8$), in which cases we take the following Dynkin diagrams.

$$(2.1)$$

Note that $B_2^{(1)}$ and $A_3^{(2)}$ in (2.1) are denoted by $C_2^{(1)}$ and $D_3^{(2)}$, respectively, in [Kac90].

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Let $\mathfrak{g}_0$ be the subalgebra of $\mathfrak{g}$ generated by the Chevalley generators e_i, f_i and h_i for $i \in I_0 := I \setminus \{0\}$, and let W_0 be the subgroup of W generated by s_i for $i \in I_0$. Note that $\mathfrak{g}_0$ is a finite-dimensional simple Lie algebra and W_0 contains the longest element w_0.

Let q be an indeterminate and $\mathbf{k}$ the algebraic closure of the subfield $\mathbb{C}(q)$ in the algebraically closed field $\hat{\mathbf{k}} := \bigcup_{m>0} \mathbb{C}((q^{1/m}))$. For $m, n \in \mathbb{Z}_{\geqslant 0}$ and $i \in I$, we define $q_i = q^{(\alpha_i, \alpha_i)/2}$ and

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[m-n]_i![n]_i!}.$$

Let d be the smallest positive integer such that $d((\alpha_i, \alpha_i)/2) \in \mathbb{Z}$ for all $i \in I$.

DEFINITION 2.1. The *quantum affine algebra* $U_q(\mathfrak{g})$ associated with an affine Cartan datum $(\mathsf{A}, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$ is the associative algebra over $\mathbf{k}$ with 1 generated by e_i, f_i ($i \in I$) and q^h ($h \in d^{-1}\mathsf{P}^\vee$) which satisfies the following relations:

  (i) $q^0 = 1$ and $q^h q^{h'} = q^{h+h'}$ for $h, h' \in d^{-1}\mathsf{P}^\vee$;
 (ii) $q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i$ and $q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i$ for $h \in d^{-1}\mathsf{P}^\vee$ and $i \in I$;
(iii) $e_i f_j - f_j e_i = \delta_{ij}((K_i - K_i^{-1})/(q_i - q_i^{-1}))$, where $K_i = q_i^{h_i}$;
 (iv) $\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} = 0$ for $i \neq j$;

here $e_i^{(k)} = e_i^k/[k]_i!$ and $f_i^{(k)} = f_i^k/[k]_i!$.

Let us denote by $U_q'(\mathfrak{g})$ the $\mathbf{k}$-subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i and $K_i^{\pm 1}$ ($i \in I$). The coproduct Δ of $U_q'(\mathfrak{g})$ is given by

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i,$$

and the bar involution$^-$of $U_q'(\mathfrak{g})$ is defined as

$$q^{1/m} \to q^{-1/m}, \quad e_i \mapsto e_i, \quad f_i \mapsto f_i, \quad K_i \mapsto K_i^{-1}.$$

Let $\mathscr{C}_\mathfrak{g}$ be the category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules, i.e. finite-dimensional modules M with a weight decomposition

$$M = \bigoplus_{\lambda \in \mathsf{P}_{\mathrm{cl}}} M_\lambda \quad \text{where } M_\lambda = \{u \in M \mid K_i u = q_i^{\langle h_i, \lambda \rangle} u\}.$$

Note that the trivial module $\mathbf{1}$ is contained in $\mathscr{C}_\mathfrak{g}$ and the tensor product $\otimes$ gives a monoidal category structure on $\mathscr{C}_\mathfrak{g}$. It is known that the Grothendieck ring $K(\mathscr{C}_\mathfrak{g})$ is a commutative ring [FR99]. A simple module L in $\mathscr{C}_\mathfrak{g}$ contains a non-zero vector $u \in L$ of weight $\lambda \in \mathsf{P}_{\mathrm{cl}}$ such that (i) $\langle h_i, \lambda \rangle \geqslant 0$ for all $i \in I_0$ and (ii) all the weights of L are contained in $\lambda - \sum_{i \in I_0} \mathbb{Z}_{\geqslant 0} \mathrm{cl}(\alpha_i)$, where $\mathrm{cl} \colon \mathsf{P} \to \mathsf{P}_{\mathrm{cl}}$ is the canonical projection. Such a λ is unique, and u is unique up to a constant multiple. We call λ the *dominant extremal weight* of L and u a *dominant extremal weight vector* of L.

Let $\mathsf{P}_{\mathrm{cl}}^0 := \{\lambda \in \mathsf{P}_{\mathrm{cl}} \mid \langle c, \lambda \rangle = 0\}$. For each $i \in I_0$ we set

$$\varpi_i := \gcd(\mathsf{c}_0, \mathsf{c}_i)^{-1} \mathrm{cl}(\mathsf{c}_0 \Lambda_i - \mathsf{c}_i \Lambda_0) \in \mathsf{P}_{\mathrm{cl}}^0,$$

where the central element c is equal to $\sum_{i \in I} \mathsf{c}_i h_i$. Note that $\mathsf{P}_{\mathrm{cl}}^0 = \bigoplus_{i \in I_0} \mathbb{Z} \varpi_i$. For any $i \in I_0$, there exists a unique simple module $V(\varpi_i)$ in $\mathscr{C}_\mathfrak{g}$ satisfying certain good conditions (see [Kas02, §5.2]), which is called the *ith fundamental representation*. Note that the dominant extremal weight of $V(\varpi_i)$ is ϖ_i.

For simple modules M and N in $\mathscr{C}_\mathfrak{g}$, we say that M and N *commute* or M commutes with N if $M \otimes N \simeq N \otimes M$. We say that M and N *strongly commute* or M *strongly commutes with*

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N if $M \otimes N$ is simple. Note that $M \otimes N$ is simple if and only if $N \otimes M$ is simple, since $K(\mathscr{C}_{\mathfrak{g}})$ is a commutative ring. It is clear that if simple modules M and N strongly commute, then they commute. We say that a simple module M is *real* if M strongly commutes with itself.

For an integrable $U'_q(\mathfrak{g})$-module M, we denote by M_z the *affinization* of M and by $z_M \colon M_z \to M_z$ the $U'_q(\mathfrak{g})$-module automorphism of weight δ. Note that $M_z \simeq \mathbf{k}[z^{\pm 1}] \otimes_{\mathbf{k}} M$ for an indeterminate z as a $\mathbf{k}$-vector space. For $x \in \mathbf{k}^{\times}$, we define

$$M_x := M_z/(z_M - x)M_z.$$

We call x a *spectral parameter*. The functor T_x defined by $T_x(M) = M_x$ is an endofunctor of $\mathscr{C}_{\mathfrak{g}}$ that commutes with tensor products (see [Kas02, §4.2] for details).

It is known that a fundamental representation is a *good module*, which is a simple $U'_q(\mathfrak{g})$-module with good properties including a *bar involution*, a crystal basis with *simple crystal graph*, and a *global basis* (see [Kas02] for the precise definition). We say that a $U'_q(\mathfrak{g})$-module M is *quasi-good* if

$$M \simeq V_c$$

for some good module V and $c \in \mathbf{k}^{\times}$. Note that any quasi-good module is a simple $U'_q(\mathfrak{g})$-module. Moreover the tensor product $M^{\otimes k} := \underbrace{M \otimes \cdots \otimes M}_{k \text{ times}}$ for a quasi-good module M and $k \in \mathbb{Z}_{\geqslant 1}$ is again quasi-good.

For a $U'_q(\mathfrak{g})$-module M, we denote by $\bar{M} = \{\bar{u} \mid u \in M\}$ the $U'_q(\mathfrak{g})$-module defined by $x\bar{u} := \overline{\bar{x}u}$ for $x \in U'_q(\mathfrak{g})$. Then we have

$$\overline{M_a} \simeq (\bar{M})_{\bar{a}} \quad \text{and} \quad \overline{M \otimes N} \simeq \bar{N} \otimes \bar{M} \quad \text{for any } M, N \in \mathscr{C}_{\mathfrak{g}} \text{ and } a \in \mathbf{k}^{\times}.$$

Note that $V(\varpi_i)$ is *bar-invariant*, i.e. $\overline{V(\varpi_i)} \simeq V(\varpi_i)$ (see [AK97, Appendix A]).

Let m_i be a positive integer such that

$$\mathsf{W}\pi_i \cap \left(\pi_i + \mathbb{Z}\delta \right) = \pi_i + \mathbb{Z}m_i\delta,$$

where π_i is an element of P such that $\mathrm{cl}(\pi_i) = \varpi_i$. Note that $m_i = (\alpha_i, \alpha_i)/2$ in the case where $\mathfrak{g}$ is the dual of an untwisted affine algebra, and $m_i = 1$ otherwise. Then for $x, y \in \mathbf{k}^{\times}$ we have (see [AK97, §1.3])

$$V(\varpi_i)_x \simeq V(\varpi_i)_y \quad \text{if and only if} \quad x^{m_i} = y^{m_i}. \tag{2.2}$$

We set

$$\sigma(\mathfrak{g}) := I_0 \times \mathbf{k}^{\times}/ \sim, \tag{2.3}$$

where the equivalence relation is given by

$$(i, x) \sim (j, y) \iff V(\varpi_i)_x \simeq V(\varpi_j)_y \iff i = j \text{ and } x^{m_i} = y^{m_j}. \tag{2.4}$$

We denote by $[(i, a)]$ the equivalence class of (i, a) in $\sigma(\mathfrak{g})$. When confusion is unlikely to arise, we simply write (i, a) for the equivalence class $[(i, a)]$.

The monoidal category $\mathscr{C}_{\mathfrak{g}}$ is rigid. For $M \in \mathscr{C}_{\mathfrak{g}}$, we denote by ${}^*M$ and M^* the right and left duals of M, respectively. We set

$$p^* := (-1)^{\langle \rho^{\vee}, \delta \rangle} q^{\langle c, \rho \rangle} \quad \text{and} \quad \tilde{p} := (p^*)^2 = q^{2\langle c, \rho \rangle}. \tag{2.5}$$

The integer $\langle \rho^\vee, \delta \rangle$ is called the *Coxeter number*, and $\langle c, \rho \rangle$ is called the *dual Coxeter number* (see [Kac90, Ch. 6]). For the reader's convenience we list p^* for all types in the following table.

Type of $\mathfrak{g}$	$A_n^{(1)}$	$B_n^{(1)}$	$C_n^{(1)}$	$D_n^{(1)}$	$A_{2n}^{(2)}$	$A_{2n-1}^{(2)}$	$D_{n+1}^{(2)}$
	$(n \geqslant 1)$	$(n \geqslant 2)$	$(n \geqslant 3)$	$(n \geqslant 4)$	$(n \geqslant 1)$	$(n \geqslant 2)$	$(n \geqslant 3)$
p^*	$(-q)^{n+1}$	q^{2n-1}	q^{n+1}	q^{2n-2}	$-q^{2n+1}$	$-q^{2n}$	$(-1)^{n+1}q^{2n}$
Type of $\mathfrak{g}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$	$F_4^{(1)}$	$G_2^{(1)}$	$E_6^{(2)}$	$D_4^{(3)}$
p^*	q^{12}	q^{18}	q^{30}	q^9	q^4	$-q^{12}$	q^6

(2.6)

Then for any $M \in \mathscr{C}_\mathfrak{g}$ we have

$$M^{**} \simeq M_{(\tilde{p})^{-1}} \quad \text{and} \quad {}^{**}M \simeq M_{\tilde{p}},$$

and for $i \in I_0$ and $x \in \mathbf{k}^\times$ we have

$$\left(V(\varpi_i)_x\right)^* \simeq V(\varpi_{i^*})_{(p^*)^{-1}x} \quad \text{and} \quad {}^*\left(V(\varpi_i)_x\right) \simeq V(\varpi_{i^*})_{p^*x}, \tag{2.7}$$

where $i^* \in I_0$ is defined by $\alpha_{i^*} = -w_0\,\alpha_i$ (see [AK97, Appendix A]). Note that the involution $i \mapsto i^*$ is the identity for all types except A_n, D_n and E_6, which are given as follows:

(a) (type A_n) $i^* = n + 1 - i$;

(b) (type D_n) $i^* = \begin{cases} n - (1 - \epsilon) & \text{if } n \text{ is odd and } i = n - \epsilon \ (\epsilon = 0, 1), \\ i & \text{otherwise}; \end{cases}$

(c) (type E_6) the map $i \mapsto i^*$ is determined by

$$i^* = \begin{cases} 6 & \text{if } i = 1, \\ i & \text{if } i = 2, 4, \\ 5 & \text{if } i = 3, \end{cases}$$

where the Dynkin diagram of type E_6 is given in (A.3) in Appendix A.

2.2 R-matrices

We recall the notion of R-matrices on $U'_q(\mathfrak{g})$-modules and their coefficients (see [Dri86], as well as [AK97, Appendices A and B] and [Kas02, §8], for details). Choose a basis $\{P_\nu\}_\nu$ of $U_q^+(\mathfrak{g})$ and a basis $\{Q_\nu\}_\nu$ of $U_q^-(\mathfrak{g})$ that are dual to each other with respect to a suitable coupling between $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$. For $U'_q(\mathfrak{g})$-modules M and N, we define

$$R_{M,N}^{\mathrm{univ}}(u \otimes v) := q^{(\mathrm{wt}(u),\mathrm{wt}(v))} \sum_\nu P_\nu v \otimes Q_\nu u \quad \text{for } u \in M \text{ and } v \in N,$$

so that $R_{M,N}^{\mathrm{univ}}$ gives a $U'_q(\mathfrak{g})$-linear homomorphism $M \otimes N \to N \otimes M$, called the *universal R-matrix*, provided that the infinite sum has a meaning. As $R_{M,N_z}^{\mathrm{univ}}$ converges in the z-adic topology for $M, N \in \mathscr{C}_\mathfrak{g}$, we have a morphism of $\mathbf{k}((z)) \otimes U'_q(\mathfrak{g})$-modules

$$R_{M,N_z}^{\mathrm{univ}} : \mathbf{k}((z)) \underset{\mathbf{k}[z^{\pm 1}]}{\otimes} (M \otimes N_z) \longrightarrow \mathbf{k}((z)) \underset{\mathbf{k}[z^{\pm 1}]}{\otimes} (N_z \otimes M).$$

Note that $R_{M,N_z}^{\mathrm{univ}}$ is an isomorphism.

Let M and N be non-zero modules in $\mathscr{C}_\mathfrak{g}$. The universal R-matrix $R_{M,N_z}^{\mathrm{univ}}$ is *rationally renormalizable* if there exists $f(z) \in \mathbf{k}((z))^\times$ such that

$$f(z)R_{M,N_z}^{\mathrm{univ}}\left(M \otimes N_z\right) \subset N_z \otimes M.$$

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In this case, we can choose $c_{M,N}(z) \in \mathbf{k}((z))^{\times}$ such that for any $x \in \mathbf{k}^{\times}$, the specialization of $R_{M,N_z}^{\mathrm{ren}} := c_{M,N}(z) R_{M,N_z}^{\mathrm{univ}} \colon M \otimes N_z \to N_z \otimes M$ at $z = x$,

$$R_{M,N_z}^{\mathrm{ren}}\big|_{z=x} \colon M \otimes N_x \to N_x \otimes M,$$

does not vanish. Note that R_{M,N_z}^{ren} and $c_{M,N}(z)$ are unique up to a multiple of $\mathbf{k}[z^{\pm 1}]^{\times} = \bigsqcup_{n \in \mathbb{Z}} \mathbf{k}^{\times} z^n$. We call $c_{M,N}(z)$ the *renormalizing coefficient*. We denote by $\mathbf{r}_{M,N}$ the specialization at $z = 1$,

$$\mathbf{r}_{M,N} := R_{M,N_z}^{\mathrm{ren}}\big|_{z=1} \colon M \otimes N \to N \otimes M, \tag{2.8}$$

and call it the *R-matrix*. The R-matrix $\mathbf{r}_{M,N}$ is well-defined up to a constant multiple whenever $R_{M,N_z}^{\mathrm{univ}}$ is rationally renormalizable. By the definition, $\mathbf{r}_{M,N}$ never vanishes.

Suppose that M and N are simple $U_q'(\mathfrak{g})$-modules in $\mathscr{C}_{\mathfrak{g}}$. Let u and v be dominant extremal weight vectors of M and N, respectively. Then there exists $a_{M,N}(z) \in \mathbf{k}[[z]]^{\times}$ such that

$$R_{M,N_z}^{\mathrm{univ}}(u \otimes v_z) = a_{M,N}(z)(v_z \otimes u).$$

Thus we have a unique $\mathbf{k}(z) \otimes U_q'(\mathfrak{g})$-module isomorphism

$$R_{M,N_z}^{\mathrm{norm}} := a_{M,N}(z)^{-1} R_{M,N_z}^{\mathrm{univ}}\big|_{\mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]}(M \otimes N_z)}$$

from $\mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (M \otimes N_z)$ to $\mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (N_z \otimes M)$, which satisfies

$$R_{M,N_z}^{\mathrm{norm}}(u \otimes v_z) = v_z \otimes u.$$

We call $a_{M,N}(z)$ the *universal coefficient* of M and N, and call $R_{M,N_z}^{\mathrm{norm}}$ the *normalized R-matrix*.

Let $d_{M,N}(z) \in \mathbf{k}[z]$ be a monic polynomial of the smallest degree such that the image of $d_{M,N}(z) R_{M,N_z}^{\mathrm{norm}}(M \otimes N_z)$ is contained in $N_z \otimes M$; we call it the *denominator of* $R_{M,N_z}^{\mathrm{norm}}$. Then we have

$$R_{M,N_z}^{\mathrm{ren}} = d_{M,N}(z) R_{M,N_z}^{\mathrm{norm}} \colon M \otimes N_z \longrightarrow N_z \otimes M \quad \text{up to a multiple of } \mathbf{k}[z^{\pm 1}]^{\times}.$$

Thus

$$R_{M,N_z}^{\mathrm{ren}} = a_{M,N}(z)^{-1} d_{M,N}(z) R_{M,N_z}^{\mathrm{univ}} \quad \text{and} \quad c_{M,N}(z) = \frac{d_{M,N}(z)}{a_{M,N}(z)}$$

up to a multiple of $\mathbf{k}[z^{\pm 1}]^{\times}$. In particular, $R_{M,N_z}^{\mathrm{univ}}$ is rationally renormalizable whenever M and N are simple.

The denominator formulas between fundamental representations are summarized for all types in Appendix A.

The next theorem follows from the results of [AK97, Cha10, Kas02, KKKO15]. In the theorem, (ii) follows essentially from [KKKO15, Corollary 3.16] together with properties of R-matrices (see also [KKOP20, Proposition 3.16 and Corollary 3.17]), and (i), (iii) and (iv) were conjectured in [AK97, §2] and proved in [AK97, §4] for affine types A and C, in [Kas02, §9] for general cases in terms of good modules, and in [Cha10, §§4 and 6] using the braid group actions.

THEOREM 2.2 [AK97, Cha10, Kas02, KKKO15].

(i) *For good modules M and N, the zeros of $d_{M,N}(z)$ belong to $\mathbb{C}[[q^{1/m}]]q^{1/m}$ for some $m \in \mathbb{Z}_{>0}$.*

(ii) *For simple modules M and N such that one of them is real, M_x and N_y strongly commute with each other if and only if $d_{M,N}(z) d_{N,M}(1/z)$ does not vanish at $z = y/x$.*

(iii) *Let M_k be a good module with a dominant extremal vector u_k of weight λ_k, and let $a_k \in \mathbf{k}^\times$ for $k = 1, \ldots, t$. Assume that a_j/a_i is not a zero of $d_{M_i, M_j}(z)$ for any $1 \leqslant i < j \leqslant t$. Then the following statements hold.*

   (a) *$(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ is generated by $u_1 \otimes \cdots \otimes u_t$.*
   (b) *The head of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ is simple.*
   (c) *Any non-zero submodule of $(M_t)_{a_t} \otimes \cdots \otimes (M_1)_{a_1}$ contains the vector $u_t \otimes \cdots \otimes u_1$.*
   (d) *The socle of $(M_t)_{a_t} \otimes \cdots \otimes (M_1)_{a_1}$ is simple.*
   (e) *Let $\mathbf{r}\colon (M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t} \to (M_t)_{a_t} \otimes \cdots \otimes (M_1)_{a_1}$ be the specialization of $\mathbf{r}_{M_1, \ldots, M_t} := \prod_{1 \leqslant j < k \leqslant t} \mathbf{r}_{M_j, M_k}$ at $z_k = a_k$; see (2.8). Then the image of $\mathbf{r}$ is simple and coincides with the head of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ and also with the socle of $(M_t)_{a_t} \otimes \cdots \otimes (M_1)_{a_1}$.*

(iv) *For any simple integrable $U_q'(\mathfrak{g})$-module M, there exists a finite sequence in $\sigma(\mathfrak{g})$ (see (2.3)) such that M has $\sum_{k=1}^t \varpi_{i_k}$ as a dominant extremal weight and is isomorphic to a simple subquotient of $V(\varpi_{i_1})_{a_1} \otimes \cdots V(\varpi_{i_t})_{a_t}$. Moreover, such a sequence $\big((i_1, a_1), \ldots, (i_t, a_t)\big)$ is unique up to a permutation.*
   We call $\sum_{k=1}^t (i_k, a_k) \in \mathbb{Z}^{\oplus \sigma(\mathfrak{g})}$ the affine highest weight of M.

2.3 Hernandez–Leclerc categories

Recall $\sigma(\mathfrak{g})$ in (2.3). For (i, x) and $(j, y) \in \sigma(\mathfrak{g})$, we put d arrows from (i, x) to (j, y), where d is the order of the zeros of $d_{V(\varpi_i), V(\varpi_j)}(z_{V(\varpi_j)}/z_{V(\varpi_i)})$ at $z_{V(\varpi_j)}/z_{V(\varpi_i)} = y/x$. Then $\sigma(\mathfrak{g})$ has a quiver structure. Note that (i, x) and (j, y) are linked in $\sigma(\mathfrak{g})$ if and only if the tensor product $V(\varpi_i)_x \otimes V(\varpi_j)_y$ is reducible [AK97, Corollary 2.4]. The denominator formulas are explicitly given in Appendix A.

We choose a connected component $\sigma_0(\mathfrak{g})$ of $\sigma(\mathfrak{g})$. Since a connected component of $\sigma(\mathfrak{g})$ is unique up to a spectral parameter shift, $\sigma_0(\mathfrak{g})$ is uniquely determined up to a quiver isomorphism. We set

$$q_s = q^{1/2} \quad \text{and} \quad q_t = q^{1/3}. \tag{2.9}$$

The *distance* $d(u, v)$ between two vertices u and v in a finite Dynkin diagram is the length of the path connecting them. For example, $d(1, 4) = 2$ in a Dynkin diagram of type D_4, and $d(1, 3) = 2$ in a Dynkin diagram of type F_4. We denote by $d_\circ(i, j)$ the distance between i and j in the Dynkin diagram of $\mathfrak{g}_0$. For the rest of this paper, we make the following choices of $\sigma_0(\mathfrak{g})$ (see table (2.6) for the range of n):

$$\sigma_0(X) := \{(i, (-q)^p) \in I_0 \times \mathbf{k}^\times \mid p \equiv_2 d_\circ(1, i)\} \quad (X = A_n^{(1)}, D_n^{(1)}, E_k^{(1)}(k = 6, 7, 8)),$$

$$\sigma_0(B_n^{(1)}) := \{(i, (-1)^{n+i} q_s q^m), (n, q^m) \mid 1 \leqslant i \leqslant n - 1,\, m \in \mathbb{Z}\},$$

$$\sigma_0(C_n^{(1)}) := \{(i, (-q_s)^p) \in I_0 \times \mathbf{k}^\times \mid p \equiv_2 d_\circ(1, i)\},$$

$$\sigma_0(F_4^{(1)}) := \{(i, (-1)^i q_s^{2p - \delta_{i,3}}) \in I_0 \times \mathbf{k}^\times \mid p \in \mathbb{Z}\},$$

$$\sigma_0(G_2^{(1)}) := \{(i, (-q_t)^p) \in I_0 \times \mathbf{k}^\times \mid p \equiv_2 d_\circ(2, i)\},$$

$$\sigma_0(A_{2n}^{(2)}) := \{(i, (-q)^p) \in I_0 \times \mathbf{k}^\times \mid p \in \mathbb{Z}\},$$

$$\sigma_0(A_{2n-1}^{(2)}) := \{(i, \pm(-q)^p), (n, (-q)^r) \mid 1 \leqslant i < n,\, p \equiv_2 i + 1,\, r \equiv_2 n + 1\},$$

$$\sigma_0(D_{n+1}^{(2)}) := \{(i, (\sqrt{-1}^{n+1-i})(-q)^p), (n, \pm(-q)^r) \mid 1 \leqslant i < n,\, p \equiv_2 i + 1,\, r \equiv_2 n + 1\},$$

$$\sigma_0(E_6^{(2)}) := \{(i, \pm q^r), (i', \sqrt{-1}(-q)^{r'}) \mid i \in \{1, 2\},\, i' \in \{3, 4\},\, r \equiv_2 i + 1,\, r' \equiv_2 i' + 1\},$$

$$\sigma_0(D_4^{(3)}) := \{(1, q^r), (1, \omega q^r), (1, \omega^2 q^r), (2, -q^{r+1}) \mid r \equiv_2 0\} \quad (\omega^2 + \omega + 1 = 0),$$

where $a \equiv_2 b$ means that $a \equiv b$ mod 2 (see [HL10, §3.7], [KKKO16, §4.1], [KO19, §6] and [OS19a, §6]). Note that in [OS19a, §6] the category $\mathscr{C}_{\mathfrak{g}}^Q$ and $\sigma_Q(\mathfrak{g})$ were dealt with only in exceptional cases, but it is easy to obtain $\sigma_0(\mathfrak{g})$ using $\sigma_Q(\mathfrak{g})$. We use the notation $B_2^{(1)}$ and $A_3^{(2)}$ instead of $C_2^{(1)}$ and $D_3^{(2)}$, respectively. Here we use the standard convention for Dynkin diagrams in [Kac90, Ch. 4] except for $A_{2n}^{(2)}$, $A_3^{(2)}$, $B_2^{(1)}$ and $E_k^{(1)}$ ($k = 6, 7, 8$), which are given in (2.1).

We define $\mathscr{C}_{\mathfrak{g}}^0$ to be the smallest full subcategory of $\mathscr{C}_{\mathfrak{g}}$ for which the following hold:

(a) $\mathscr{C}_{\mathfrak{g}}^0$ contains $V(\varpi_i)_x$ for all $(i, x) \in \sigma_0(\mathfrak{g})$;
(b) $\mathscr{C}_{\mathfrak{g}}^0$ is stable by taking subquotients, extensions and tensor products.

For symmetric affine types, this category was introduced in [HL10]. Note that every simple module in $\mathscr{C}_{\mathfrak{g}}$ is isomorphic to a tensor product of certain spectral parameter shifts of some simple modules in $\mathscr{C}_{\mathfrak{g}}^0$ (see [HL10, §3.7]).

2.4 The categories $\mathscr{C}_{\mathfrak{g}}^Q$

In this subsection, we recall very briefly a certain subcategory $\mathscr{C}_{\mathfrak{g}}^Q$ of $\mathscr{C}_{\mathfrak{g}}$ categorifying the coordinate ring $\mathbb{C}[N]$ of the maximal unipotent group N associated with a certain simple Lie algebra.

This subcategory $\mathscr{C}_{\mathfrak{g}}^Q$ was introduced in [HL15] for simply laced affine type ADE, in [KKKO16] for twisted affine types $A^{(2)}$ and $D^{(2)}$, in [KO19, OS19b] for untwisted affine types $B^{(1)}$ and $C^{(1)}$, and in [OS19a] for exceptional affine type. The quantum affine Schur–Weyl duality functor between the finite-dimensional module category of a quiver Hecke algebra and $\mathscr{C}_{\mathfrak{g}}^Q$ was also constructed in [KKK15] for untwisted affine types $A^{(1)}$ and $D^{(1)}$, in [KKKO16] for twisted affine types $A^{(2)}$ and $D^{(2)}$, in [KO19] for untwisted affine types $B^{(1)}$ and $C^{(1)}$, in [OS19a] for exceptional affine type, and in [Fuj20] for simply laced affine type ADE in a geometric manner.

We shall describe $\sigma_Q(\mathfrak{g})$ and $\mathscr{C}_{\mathfrak{g}}^Q$ by using Q-*data* [FO21]. A Q-datum generalizes a Dynkin quiver with a height function, which provides a uniform way of describing the Hernandez–Leclerc category $\mathscr{C}_{\mathfrak{g}}^Q$. Our brief explanation follows [FO21, §3] (see also [FHOO21, §4] and [KKOP21, §6]). Let $\mathfrak{g}$ be an affine Kac–Moody algebra and let $\mathfrak{g}_{\mathrm{fin}}$ be the simply laced finite-type Lie algebra corresponding to the affine type of $\mathfrak{g}$ in table (4.5). Let $\mathsf{I}_{\mathrm{fin}}$ be the index set of $\mathfrak{g}_{\mathrm{fin}}$ and let $\mathsf{D}_{\mathrm{fin}}$ be the Dynkin diagram for $\mathfrak{g}_{\mathrm{fin}}$.

We first assume that $\mathfrak{g}$ is of untwisted type. We define an Dynkin diagram automorphism ϱ of $\mathsf{D}_{\mathrm{fin}}$ as follows. For $\mathfrak{g} = A_n^{(1)}, D_n^{(1)}$ or $E_k^{(1)}$ type ($k = 6, 7, 8$) we set $\varrho := \mathrm{id}$, and for the remaining types ϱ is defined as follows (see [FO21, §3.1]).

$B_n^{(1)}$-type: $\left(\mathsf{D}_{\mathrm{fin}} : \underset{1}{\circ}\!\!-\!\!\underset{2}{\circ}\!\!-\!\! \cdots \cdots \underset{2n-2}{\circ}\!\!-\!\!\underset{2n-1}{\circ} \, , \, \varrho(k) = 2n - k \right) \Longrightarrow \mathsf{D}_{B_n} : \underset{1}{\circ}\!\!-\!\!\underset{2}{\circ}\!\!-\!\! \cdots \cdots \underset{n-1}{\circ}\!\!\Rightarrow\!\!\underset{n}{\circ},$

$C_n^{(1)}$-type: $\left(\mathsf{D}_{\mathrm{fin}} : \underset{1}{\circ}\!\!-\!\!\underset{2}{\circ}\!\! \cdots \cdots \underset{n-1}{\circ}\!\!\overset{\overset{\displaystyle n}{\circ}}{\underset{\underset{\displaystyle n+1}{\circ}}{<}} , \; \varrho(k) = \begin{cases} k & \text{if } k \leqslant n-1, \\ n+1 & \text{if } k = n, \\ n & \text{if } k = n+1 \end{cases} \right) \Longrightarrow \mathsf{D}_{C_n} : \underset{1}{\circ}\!\!-\!\!\underset{2}{\circ}\!\!-\!\! \cdots \cdots \underset{n-1}{\circ}\!\!\Leftarrow\!\!\underset{n}{\circ},$

$F_4^{(1)}$-type: $\left(\mathsf{D}_{\mathrm{fin}} : \begin{array}{c} \overset{\displaystyle 2}{\circ} \\ | \\ \underset{1}{\circ}\!-\!\underset{3}{\circ}\!-\!\underset{4}{\circ}\!-\!\underset{5}{\circ}\!-\!\underset{6}{\circ} \end{array} , \; \begin{cases} \varrho(1) = 6, \; \varrho(6) = 1, \\ \varrho(3) = 5, \; \varrho(5) = 3, \\ \varrho(4) = 4, \; \varrho(2) = 2 \end{cases} \right) \Longrightarrow \mathsf{D}_{F_4} : \underset{1}{\circ}\!-\!\underset{2}{\circ}\!\Rightarrow\!\underset{3}{\circ}\!-\!\underset{4}{\circ},$

$G_2^{(1)}$-type: $\left(\mathsf{D}_{\mathrm{fin}} : \underset{1}{\circ}\!-\!\underset{2}{\circ}\!\overset{\overset{\displaystyle \circ}{}}{\underset{\underset{\displaystyle 4}{\circ}}{<}} 3 , \; \begin{cases} \varrho(1) = 3, \; \varrho(3) = 4, \; \varrho(4) = 1, \\ \varrho(2) = 2 \end{cases} \right) \Longrightarrow \mathsf{D}_{G_2} : \underset{1}{\circ}\!\Rightarrow\!\underset{2}{\circ} \, .$

Let $I_0 = \{1, 2, \ldots, n\}$ be the index set of $\mathfrak{g}_0$. Note that $I_{\text{fin}} = I_0$ when $\mathfrak{g} = A_n^{(1)}, D_n^{(1)}, E_k^{(1)}$ ($k = 6, 7, 8$). Let $\text{ord}(\varrho)$ be the order of ϱ. For $i \in I_{\text{fin}}$, we denote by $\text{orb}(i)$ the orbit of i under the action ϱ and set $\mathsf{d}_i := |\text{orb}(i)|$. We identify the set of orbits of I_{fin} with I_0 by mapping $\text{orb}(i) \mapsto \min\{\text{orb}(i)\}$ for $\mathfrak{g} \neq F_4^{(1)}$ and mapping $\text{orb}(1) \mapsto 1$, $\text{orb}(3) \mapsto 2$, $\text{orb}(4) \mapsto 3$ and $\text{orb}(2) \mapsto 4$ for $\mathfrak{g} = F_4^{(1)}$. We write $\pi \colon I_{\text{fin}} \to I_0$ for the projection via this identification.

Definition 2.3 [FO21, Definition 3.5]. A function $\xi \colon I_{\text{fin}} \to \mathbb{Z}$ is called a *height function on* $(\mathsf{D}_{\text{fin}}, \varrho)$ if the following two conditions are satisfied.

(i) For any $i, j \in I_{\text{fin}}$ such that $\mathsf{d}(i, j) = 1$ and $\mathsf{d}_i = \mathsf{d}_j$, we have $|\xi_i - \xi_j| = \mathsf{d}_i$.
(ii) For any $i, j \in I_{\text{fin}}$ such that $\mathsf{d}(i, j) = 1$ and $1 = \mathsf{d}_i < \mathsf{d}_j = \text{ord}(\varrho)$, there exists a unique element $j^\circ \in \text{orb}(j)$ such that $|\xi_i - \xi_{j^\circ}| = 1$ and $\xi_{\varrho^k(j^\circ)} = \xi_{j^\circ} - 2k$ for any $0 \leqslant k < \text{ord}(\varrho)$.

Here $\mathsf{d}(i, j)$ denotes the distance between i and j in the Dynkin diagram D_{fin}. We call the triple $Q = (\mathsf{D}_{\text{fin}}, \varrho, \xi)$ a *Q-datum* for $\mathfrak{g}$.

For a Q-datum $Q = (\mathsf{D}_{\text{fin}}, \varrho, \xi)$ associated to $\mathfrak{g}$, let

$$\hat{I}_Q := \{(i, p) \in I_{\text{fin}} \times \mathbb{Z} \mid p - \xi_i \in 2\mathsf{d}_i\mathbb{Z}\}.$$

The *generalized ϱ-Coxeter element* $\tau_Q \in \mathsf{W}_{\text{fin}} \rtimes \text{Aut}(\mathsf{D}_{\text{fin}})$ associated with Q is defined in [FO21, Definition 3.33] and can be understood as a generalization of a Coxeter element. Here W_{fin} is the Weyl group of $\mathfrak{g}_{\text{fin}}$.

For $i \in I_0$, we denote by $o(i)$ the corresponding orbit of I_{fin}. For each $i \in I_0$, we denote by i° the unique vertex in the orbit $o(i)$ satisfying $\xi_{i^\circ} = \max\{\xi_j \mid j \in o(i)\}$. In this paper, we assume further that the height function ξ satisfies

$$\xi_{\varrho^k(i^\circ)} = \xi_{i^\circ} - 2k \quad \text{for each } i \in I_0 \text{ and } 0 \leqslant k < \mathsf{d}_i. \tag{2.10}$$

Let $\{i_1, i_2, \ldots, i_n\}$ be a total order of I_0 satisfying $\xi_{i_1^\circ} \geqslant \xi_{i_2^\circ} \geqslant \cdots \geqslant \xi_{i_n^\circ}$. Then we have

$$\tau_Q = s_{i_1^\circ} s_{i_2^\circ} \cdots s_{i_n^\circ} \varrho \in \mathsf{W}_{\text{fin}} \rtimes \text{Aut}(\mathsf{D}_{\text{fin}})$$

(see [FO21, §3.6] and also [FHOO21, Proposition 4.4] for more details).

Let Δ_Q^+ be the set of positive roots of $\mathfrak{g}_{\text{fin}}$, and let $\hat{\Phi} := \Delta_Q^+ \times \mathbb{Z}$. For each $i \in I_{\text{fin}}$ we define

$$\gamma_i^Q := (1 - \tau_Q^{\mathsf{d}_i})\Lambda_i \in \Delta_Q^+,$$

where Λ_i is the ith fundamental weight of $\mathfrak{g}_{\text{fin}}$. It is shown in [HL15, §2.2] and [FO21, Theorem 3.35] that there exists a unique bijection $\psi_Q \colon \hat{I}_Q \to \hat{\Phi}$ defined inductively as follows:

(i) $\psi_Q(i, \xi_i) = (\gamma_i^Q, 0)$;
(ii) if $\psi_Q(i, p) = (\beta, m)$, then define:
    (a) $\psi_Q(i, p \pm 2\mathsf{d}_i) = (\tau_Q^{\mp \mathsf{d}_i}(\beta), m)$ if $\tau_Q^{\mp \mathsf{d}_i}(\beta) \in \Delta_Q^+$;
    (b) $\psi_Q(i, p \pm 2\mathsf{d}_i) = (-\tau_Q^{\mp \mathsf{d}_i}(\beta), m \pm 1)$ if $\tau_Q^{\mp \mathsf{d}_i}(\beta) \in -\Delta_Q^+$.

Let $I_Q := \psi_Q^{-1}(\Delta_Q^+ \times \{0\}) \subset I_{\text{fin}} \times \mathbb{Z}$. Then one can describe

$$I_Q = \{(i, p) \in \hat{I}_Q \mid \xi_{i^*} - \text{ord}(\varrho)\mathsf{h}^\vee < p \leqslant \xi_i\},$$

where $\mathsf{h}^\vee$ is the dual Coxeter number of $\mathfrak{g}_0$ (see [FO21, Theorem 3.35] and also [FHOO21, Proposition 4.15]). We define

$$\sigma_Q(\mathfrak{g}) := \{\zeta(i, p) \mid (i, p) \in I_Q\},$$

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where we set $q_{\mathrm{sh}} := q^{1/\mathrm{ord}(\varrho)}$ and

$$\zeta(i,p) := \begin{cases} (\pi(i), (-q_{\mathrm{sh}})^p) & \text{if } \mathfrak{g} = A_n^{(1)},\ C_n^{(1)},\ D_n^{(1)},\ E_{6,7,8}^{(1)},\ G_2^{(1)}, \\ (\pi(i), (-1)^{i+n}(q_{\mathrm{sh}})^p) & \text{if } \mathfrak{g} = B_n^{(1)}, \\ (\pi(i), (-1)^{\pi(i)}(q_{\mathrm{sh}})^p) & \text{if } \mathfrak{g} = F_4^{(1)} \end{cases}$$

(see [FO21, §5.4]). We define

$$\phi_Q \colon \Delta_Q^+ \xrightarrow{\ \sim\ } \sigma_Q(\mathfrak{g}) \tag{2.11}$$

by $\phi_Q(\beta) := \zeta \circ \psi_Q^{-1}(\beta,0)$ for $\beta \in \Delta_Q^+$. The map ϕ_Q is bijective.

For the rest of this paper, we make the following choices of Q-data:

- for simply laced ADE type, $\mathrm{ord}(\varrho) = 1$ and the height function ξ is defined in Appendix A.1;

- for $\mathfrak{g} = B_n^{(1)}$, $\mathrm{ord}(\varrho) = 2$ and $Q =$;

- for $\mathfrak{g} = C_n^{(1)}$, $\mathrm{ord}(\varrho) = 2$ and $Q =$;

- for $\mathfrak{g} = F_4^{(1)}$, $\mathrm{ord}(\varrho) = 2$ and $Q =$;

- for $\mathfrak{g} = G_2^{(1)}$, $\mathrm{ord}(\varrho) = 3$ and $Q =$.

Here an underlined integer stands for the value of ξ_i at each vertex $i \in \mathsf{D}_{\mathrm{fin}}$ and an arrow $i \to j$ means that $\xi_i > \xi_j$ and $\mathrm{d}(i,j) = 1$ in the Dynkin diagram $\mathsf{D}_{\mathrm{fin}}$. Note that our choice of Q satisfies (2.10). Then τ_Q is given as follows:

- for simply laced ADE type, τ_Q is the same as τ in Appendix A.1;
- for $\mathfrak{g} = B_n^{(1)}, C_n^{(1)}$, $\tau_Q = s_1 s_2 \cdots s_n \varrho$;
- for $\mathfrak{g} = F_4^{(1)}$, $\tau_Q = s_1 s_2 s_3 s_4 \varrho$;
- for $\mathfrak{g} = G_2^{(1)}$, $\tau_Q = s_2 s_1 \varrho$.

In this case the set $\sigma_Q(\mathfrak{g})$ is contained in $\sigma_0(\mathfrak{g})$ in §2.3 and can be written explicitly as follows (where $a \leqslant_2 b$ means that $a \leqslant b$ and $a \equiv b \bmod 2$):

$$\sigma_Q(A_n^{(1)}) := \{(i, (-q)^k) \in \sigma_0(A_n^{(1)}) \mid i - 2n + 1 \leqslant_2 k \leqslant_2 -i + 1\},$$

$$\sigma_Q(B_n^{(1)}) := \{(i, (-1)^{n+i}q_s^k), (n, q^{k'}) \in \sigma_0(B_n^{(1)}) \mid i < n, -2n - 2i + 3 \leqslant_2 k \leqslant_2 2n - 2i - 1,$$
$$-2n + 2 \leqslant k' \leqslant 0\},$$

$$\sigma_Q(C_n^{(1)}) := \{(i, (-q_s)^k) \in \sigma_0(C_n^{(1)}) \mid -\mathrm{d}_\circ(1,i) - 2n \leqslant_2 k \leqslant_2 -\mathrm{d}_\circ(1,i)\},$$

$$\sigma_Q(D_n^{(1)}) := \{(i, (-q)^k) \in \sigma_0(D_n^{(1)}) \mid -\mathrm{d}_\circ(1,i) - 2n + 4 \leqslant_2 k \leqslant_2 -\mathrm{d}_\circ(1,i)\},$$

$$\sigma_Q(E_6^{(1)}) := \{(i, (-q)^k) \in \sigma_0(E_6^{(1)}) \mid \mathrm{d}_\circ(1,i) - 14 \leqslant_2 k \leqslant_2 -\mathrm{d}_\circ(1,i) + 2\delta_{i,2}\},$$

$$\sigma_Q(E_7^{(1)}) := \{(i, (-q)^k) \in \sigma_0(E_7^{(1)}) \mid -\mathrm{d}_\circ(1,i) - 16 + 2\delta_{i,2} \leqslant_2 k \leqslant_2 -\mathrm{d}_\circ(1,i) + 2\delta_{i,2}\},$$

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$$\sigma_Q(E_8^{(1)}) := \{(i, (-q)^k) \in \sigma_0(E_8^{(1)}) \mid -\mathrm{d}_\circ(1,i) - 28 + 2\delta_{i,2} \leqslant_2 k \leqslant_2 -\mathrm{d}_\circ(1,i) + 2\delta_{i,2}\},$$

$$\sigma_Q(F_4^{(1)}) := \left\{(i, (-1)^i q^k) \in \sigma_0(F_4^{(1)}) \;\middle|\; \mathrm{d}_\circ(i,3) - 10 + \frac{\delta_{i,3}}{2} \leqslant k \leqslant \mathrm{d}_\circ(i,3) - 2 + \frac{\delta_{i,3}}{2}\right\},$$

$$\sigma_Q(G_2^{(1)}) := \{(i, (-q_t)^k) \in \sigma_0(G_2^{(1)}) \mid -\mathrm{d}_\circ(2,i) - 10 \leqslant_2 k \leqslant_2 -\mathrm{d}_\circ(2,i)\},$$

where $\mathrm{d}_\circ(i,j)$ denotes the distance between i and j in the Dynkin diagram of $\mathfrak{g}_0$.

We now assume that $\mathfrak{g}$ is of twisted type. Then one can define

$$\sigma_Q(A_N^{(2)}) := \{(i, (-q)^k)^\star \mid (i, (-q)^k) \in \sigma_Q(A_N^{(1)})\} \quad (N = 2n-1 \text{ or } 2n),$$

$$\sigma_Q(D_{n+1}^{(2)}) := \{(i, (-q)^k)^\star \mid (i, (-q)^k) \in \sigma_Q(D_{n+1}^{(1)})\},$$

$$\sigma_Q(E_6^{(2)}) := \{(i, (-q)^k)^\star \mid (i, (-q)^k) \in \sigma_Q(E_6^{(1)})\},$$

$$\sigma_Q(D_4^{(3)}) := \{(i, (-q)^k)^\dagger \mid (i, (-q)^k) \in \sigma_Q(D_4^{(1)})\},$$

where for $(i,a) \in \sigma_0(\mathfrak{g}_N^{(1)})$ we set

$$(i,a)^\star = \begin{cases} (i,a) & \text{if } \mathfrak{g} = A_N^{(1)},\ i \leqslant \lfloor (N+1)/2 \rfloor \text{ or } \mathfrak{g} = E_6^{(1)},\ i = 1, \\ (N+1-i, (-1)^N a) & \text{if } \mathfrak{g} = A_N^{(1)},\ i > \lfloor (N+1)/2 \rfloor, \\ (i, \sqrt{-1}^{n+1-i} a) & \text{if } \mathfrak{g} = D_{n+1}^{(1)},\ i \leqslant n-1, \\ (n, (-1)^i a) & \text{if } \mathfrak{g} = D_{n+1}^{(1)},\ i \in \{n, n+1\}, \\ (2,a) & \text{if } \mathfrak{g} = E_6^{(1)},\ i = 3, \\ (2,-a) & \text{if } \mathfrak{g} = E_6^{(1)},\ i = 5, \\ (1,-a) & \text{if } \mathfrak{g} = E_6^{(1)},\ i = 6, \\ (3, \sqrt{-1}a) & \text{if } \mathfrak{g} = E_6^{(1)},\ i = 4, \\ (4, \sqrt{-1}a) & \text{if } \mathfrak{g} = E_6^{(1)},\ i = 2 \end{cases}$$

and

$$(i,a)^\dagger = \begin{cases} (2,a) & \text{if } i = 2, \\ (1, (\delta_{i,1} + \delta_{i,3}\omega + \delta_{i,4}\omega^2)a) & \text{if } i \neq 2 \end{cases}$$

(see [KKKO16, Proposition 4.3] and [OS19a, Proposition 6.5] for details of $\star$ and $\dagger$). The bijection $\phi_Q \colon \Delta_Q^+ \xrightarrow{\sim} \sigma_Q(\mathfrak{g})$ is defined by composing the bijection for untwisted type with the maps $\star$ and $\dagger$.

Comparing the above descriptions of $\sigma_Q(\mathfrak{g})$ with the descriptions of $\sigma_0(\mathfrak{g})$ given in §2.3, one can easily show that

$$\sigma_0(\mathfrak{g}) = \bigsqcup_{k \in \mathbb{Z}} \sigma_Q(\mathfrak{g})^{*k},$$

$$\sigma_Q(\mathfrak{g})^{*k} \cap \sigma_Q(\mathfrak{g})^{*k'} = \emptyset \quad \text{for } k, k' \in \mathbb{Z} \text{ with } k \neq k', \tag{2.12}$$

where $\sigma_Q(\mathfrak{g})^{*k} := \{(i^{*k}, (p^*)^k a) \mid (i,a) \in \sigma_Q(\mathfrak{g})\}$ with $i^{*k} = i$ if k is even and $i^{*k} = i^*$ if k is odd (see [FO21, Proposition 5.9]). Note that p^* is given in (2.5).

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Let $\mathscr{C}_{\mathfrak{g}}^Q$ be the smallest full subcategory of $\mathscr{C}_{\mathfrak{g}}^0$ with the following properties:

(a) $\mathscr{C}_{\mathfrak{g}}^Q$ contains $\mathbf{1}$ and $V(\varpi_i)_x$ for all $(i, x) \in \sigma_Q(\mathfrak{g})$;
(b) $\mathscr{C}_{\mathfrak{g}}^Q$ is stable by taking subquotients, extensions and tensor products.

It was shown in [HL15, Theorem 6.1], [KKKO16, Corollary 5.6], [KO19, Corollary 6.6] and [OS19a, §6] that the Grothendieck ring $K(\mathscr{C}_{\mathfrak{g}}^Q)$ of the monoidal category $\mathscr{C}_{\mathfrak{g}}^Q$ is isomorphic to the coordinate ring $\mathbb{C}[N]$ of the maximal unipotent group N associated with $\mathfrak{g}_{\mathrm{fin}}$. The set Δ_Q^+ has a convex order $\prec_Q$ arising from Q.

Let $\beta \in \Delta_Q^+$ and write $(i, a) = \phi_Q(\beta)$. Then set

$$V_Q(\beta) := V(\varpi_i)_a \in \mathscr{C}_{\mathfrak{g}}^Q.$$

Under the categorification, the modules $V_Q(\beta)$ correspond to the dual PBW vectors of $\mathbb{C}[N]$ with respect to the convex order $\prec_Q$ on Δ_Q^+.

The proposition below follows from [KKK15, §4.3], [KKKO16, Proposition 4.9 and Theorem 5.1], [KO19, §4.3] and [OS19a, §6].

PROPOSITION 2.4 [KKK15, KKKO16, KO19, OS19a]. *For a minimal pair (α, β) of a positive root $\gamma \in \Delta_Q^+$, $V_Q(\gamma)$ is isomorphic to the head of $V_Q(\alpha) \otimes V_Q(\beta)$. Here, (α, β) is called a minimal pair of γ if $\alpha \prec_Q \beta$, $\gamma = \alpha + \beta$ and there exists no pair (α', β') such that $\gamma = \alpha' + \beta'$ and $\alpha \prec_Q \alpha' \prec_Q \beta' \prec_Q \beta$.*

3. New invariants for pairs of modules

In this section, we recall several properties of the new invariants arising from R-matrices introduced in [KKOP20].

We set

$$\varphi(z) := \prod_{s=0}^{\infty} (1 - \tilde{p}^s z) = \sum_{n=0}^{\infty} \frac{(-1)^n \tilde{p}^{n(n-1)/2}}{\prod_{k=1}^{n}(1 - \tilde{p}^k)} z^n \in \mathbf{k}[[z]] \subset \hat{\mathbf{k}}[[z]],$$

where $\tilde{p}$ is given in (2.5). We consider the subgroup $\mathcal{G}$ of $\mathbf{k}((z))^\times$ given by

$$\mathcal{G} := \left\{ cz^m \prod_{a \in \mathbf{k}^\times} \varphi(az)^{\eta_a} \;\middle|\; \begin{array}{c} c \in \mathbf{k}^\times, \; m \in \mathbb{Z}, \\ \eta_a \in \mathbb{Z} \text{ vanishes except for finitely many } a \end{array} \right\}.$$

Note that if $R_{M,N_z}^{\mathrm{univ}}$ is rationally renormalizable for $M, N \in \mathscr{C}_{\mathfrak{g}}$, then the renormalizing coefficient $c_{M,N}(z)$ belongs to $\mathcal{G}$ (see [KKOP20, Proposition 3.2]). In particular, for simple modules M and N in $\mathscr{C}_{\mathfrak{g}}$, the universal coefficient $a_{M,N}(z)$ belongs to $\mathcal{G}$.

For a subset S of $\mathbb{Z}$, let $\tilde{p}^S := \{\tilde{p}^k \mid k \in S\}$. We define the group homomorphisms

$$\mathrm{Deg} \colon \mathcal{G} \to \mathbb{Z} \quad \text{and} \quad \mathrm{Deg}^\infty \colon \mathcal{G} \to \mathbb{Z}$$

by

$$\mathrm{Deg}(f(z)) = \sum_{a \in \tilde{p}^{\mathbb{Z}\leqslant 0}} \eta_a - \sum_{a \in \tilde{p}^{\mathbb{Z}>0}} \eta_a \quad \text{and} \quad \mathrm{Deg}^\infty(f(z)) = \sum_{a \in \tilde{p}^{\mathbb{Z}}} \eta_a$$

for $f(z) = cz^m \prod_{a \in \mathbf{k}^\times} \varphi(az)^{\eta_a} \in \mathcal{G}$.

LEMMA 3.1 [KKOP20, Lemma 3.4]. *Let $f(z) \in \mathcal{G}$.*

(i) *If $f(z) \in \mathbf{k}(z)^\times$, then we have $f(z) \in \mathcal{G}$,*

$$\mathrm{Deg}^\infty(f(z)) = 0 \quad \text{and} \quad \mathrm{Deg}(f(z)) = 2\,\mathrm{zero}_{z=1} f(z).$$

(ii) If $g(z), h(z) \in \mathcal{G}$ satisfy $g(z)/h(z) \in \mathbf{k}[z^{\pm 1}]$, then $\mathrm{Deg}(h(z)) \leqslant \mathrm{Deg}(g(z))$.

(iii) We have that $\mathrm{Deg}^\infty f(z) = -\mathrm{Deg}\big(f(\tilde{p}^n z)\big) = \mathrm{Deg}\big(f(\tilde{p}^{-n} z)\big)$ for $n \gg 0$.

(iv) If $\mathrm{Deg}^\infty\big(f(cz)\big) = 0$ for any $c \in \mathbf{k}^\times$, then $f(z) \in \mathbf{k}(z)^\times$.

The following invariants for a pair of modules M and N in $\mathscr{C}_\mathfrak{g}$ such that $R^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable were introduced in [KKOP20] by using the homomorphisms Deg and Deg^∞.

DEFINITION 3.2. For non-zero modules M and N in $\mathscr{C}_\mathfrak{g}$ such that $R^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable, we define the integers $\Lambda(M, N)$ and $\Lambda^\infty(M, N)$ by

$$\Lambda(M, N) := \mathrm{Deg}(c_{M,N}(z)),$$
$$\Lambda^\infty(M, N) := \mathrm{Deg}^\infty(c_{M,N}(z)).$$

We have $\Lambda(M, N) \equiv \Lambda^\infty(M, N) \bmod 2$.

PROPOSITION 3.3 [KKOP20, Lemma 3.7]. For any simple modules $M, N \in \mathscr{C}_\mathfrak{g}$ and $x \in \mathbf{k}^\times$, we have

$$\Lambda(M, N) = \Lambda(M_x, N_x) \quad \text{and} \quad \Lambda^\infty(M, N) = \Lambda^\infty(M_x, N_x).$$

PROPOSITION 3.4 [KKOP20, Lemmas 3.7 and 3.8 and Corollary 3.23]. Let M and N be simple modules in $\mathscr{C}_\mathfrak{g}$. Then the following hold:

(i) $\Lambda^\infty(M, N) = -\mathrm{Deg}^\infty(a_{M,N}(z))$;

(ii) $\Lambda^\infty(M, N) = \Lambda^\infty(N, M)$;

(iii) $\Lambda^\infty(M, N) = -\Lambda^\infty(M^*, N) = -\Lambda^\infty(M, {}^*N)$;

(iv) in particular, $\Lambda^\infty(M, N) = \Lambda^\infty(M^*, N^*) = \Lambda^\infty({}^*M, {}^*N)$.

PROPOSITION 3.5 [KKOP20, Lemma 3.7 and Proposition 3.18]. Let M and N be simple modules in $\mathscr{C}_\mathfrak{g}$. Then the following hold:

(i) $\Lambda(M, N) = \Lambda(N^*, M) = \Lambda(N, {}^*M)$;

(ii) in particular,

$$\Lambda(M, N) = \Lambda(M^*, N^*) = \Lambda({}^*M, {}^*N).$$

PROPOSITION 3.6 [KKOP20, Proposition 3.9]. Let M and N be modules in $\mathscr{C}_\mathfrak{g}$, and let M' and N' be non-zero subquotients of M and N, respectively. Assume that $R^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable. Then $R^{\mathrm{univ}}_{M',N'_z}$ is rationally renormalizable, and we have

$$\Lambda(M', N') \leqslant \Lambda(M, N) \quad \text{and} \quad \Lambda^\infty(M', N') = \Lambda^\infty(M, N).$$

PROPOSITION 3.7 [KKOP20, Proposition 3.11]. Let M, N and L be non-zero modules in $\mathscr{C}_\mathfrak{g}$, and let S be a non-zero subquotient of $M \otimes N$.

(i) Assume that $R^{\mathrm{univ}}_{M,L_z}$ and $R^{\mathrm{univ}}_{N,L_z}$ are rationally renormalizable. Then $R^{\mathrm{univ}}_{S,L_z}$ is rationally renormalizable, and we have

$$\Lambda(S, L) \leqslant \Lambda(M, L) + \Lambda(N, L) \quad \text{and} \quad \Lambda^\infty(S, L) = \Lambda^\infty(M, L) + \Lambda^\infty(N, L).$$

(ii) Assume that $R^{\mathrm{univ}}_{L,M_z}$ and $R^{\mathrm{univ}}_{L,N_z}$ are rationally renormalizable. Then $R^{\mathrm{univ}}_{L,S_z}$ is rationally renormalizable, and we have

$$\Lambda(L, S) \leqslant \Lambda(L, M) + \Lambda(L, N) \quad \text{and} \quad \Lambda^\infty(L, S) = \Lambda^\infty(L, M) + \Lambda^\infty(L, N).$$

COROLLARY 3.8 [KKOP20, Corollary 3.12]. Let M and N be simple modules in $\mathscr{C}_\mathfrak{g}$. Suppose that M (respectively N) is isomorphic to a subquotient of $V(\varpi_{i_1})_{a_1} \otimes V(\varpi_{i_2})_{a_2} \otimes \cdots \otimes V(\varpi_{i_k})_{a_k}$

(respectively $V(\varpi_{j_1})_{b_1} \otimes V(\varpi_{j_2})_{b_2} \otimes \cdots \otimes V(\varpi_{j_l})_{b_l}$). Then we have

$$\Lambda^\infty(M, N) = \sum_{1 \leqslant \nu \leqslant k, 1 \leqslant \mu \leqslant l} \Lambda^\infty(V(\varpi_{i_\nu})_{a_\nu}, V(\varpi_{j_\mu})_{b_\mu}).$$

For simple modules M and N in $\mathscr{C}_{\mathfrak{g}}$, we define $\mathfrak{d}(M, N)$ by

$$\mathfrak{d}(M, N) := \tfrac{1}{2}\big(\Lambda(M, N) + \Lambda(M^*, N)\big).$$

PROPOSITION 3.9 [KKOP20, Proposition 3.16 and Corollary 3.19]. *Let M and N be simple modules in $\mathscr{C}_{\mathfrak{g}}$. Then the following hold:*

(i) $\mathfrak{d}(M, N) = \operatorname{zero}_{z=1}\big(d_{M,N}(z)d_{N,M}(z^{-1})\big)$;
(ii) $\mathfrak{d}(M, N) = \tfrac{1}{2}\big(\Lambda(M, N) + \Lambda(N, M)\big)$;
(iii) *in particular, $\mathfrak{d}(M, N) = \mathfrak{d}(N, M)$.*

COROLLARY 3.10 [KKOP20, Corollaries 3.17 and 3.20]. *Let M and N be simple modules in $\mathscr{C}_{\mathfrak{g}}$.*

(i) *Suppose that one of M and N is real. Then M and N strongly commute if and only if $\mathfrak{d}(M, N) = 0$.*
(ii) *In particular, if M is real, then $\Lambda(M, M) = 0$.*

PROPOSITION 3.11 [KKOP20, Proposition 3.22]. *For simple modules M and N in $\mathscr{C}_{\mathfrak{g}}$, we have*

$$\Lambda(M, N) = \sum_{k \in \mathbb{Z}} (-1)^{k+\delta(k<0)} \mathfrak{d}(M, \mathscr{D}^k N),$$

$$\Lambda^\infty(M, N) = \sum_{k \in \mathbb{Z}} (-1)^k \mathfrak{d}(M, \mathscr{D}^k N),$$

where $\mathscr{D}^k N$ is defined as

$$\mathscr{D}^k N := \begin{cases} \underbrace{(\cdots(N^* \;)^* \cdots)^*}_{(-k)\text{ times}} & \text{if } k < 0, \\ \underbrace{{}^*(\cdots({}^*N)\cdots)}_{k\text{ times}} & \text{if } k \geqslant 0. \end{cases}$$

4. Root systems associated with $\mathscr{C}_{\mathfrak{g}}$

Let $\operatorname{Hom}(\sigma(\mathfrak{g}), \mathbb{Z})$ be the set of $\mathbb{Z}$-valued functions on $\sigma(\mathfrak{g})$. It is obvious that $\operatorname{Hom}(\sigma(\mathfrak{g}), \mathbb{Z})$ forms a torsion-free abelian group under addition. Let $M \in \mathscr{C}_{\mathfrak{g}}$ be a module such that $R^{\mathrm{univ}}_{M,V(\varpi_i)_z}$ is rationally renormalizable for any $i \in I_0$. Then we define $\mathsf{E}(M) \in \operatorname{Hom}(\sigma(\mathfrak{g}), \mathbb{Z})$ by

$$\mathsf{E}(M)(i, a) := \Lambda^\infty(M, V(\varpi_i)_a) \quad \text{for } (i, a) \in \sigma(\mathfrak{g}), \tag{4.1}$$

which is well-defined by (2.4).

LEMMA 4.1. *Let M and N be simple modules in $\mathscr{C}_{\mathfrak{g}}$.*

(i) *We have $\mathsf{E}(M) = -\mathsf{E}(M^*) = -\mathsf{E}({}^*M)$.*
(ii) *Let $\{M_k\}_{1 \leqslant k \leqslant r}$ be a sequence of simple modules. Then for any non-zero subquotient S of $M_1 \otimes \cdots \otimes M_r$, we have*

$$\mathsf{E}(S) = \sum_{k=1}^r \mathsf{E}(M_k).$$

(iii) $\mathsf{E}(M) = \mathsf{E}(N)$ *if and only if $a_{M,V(\varpi_i)}(z)/a_{N,V(\varpi_i)}(z) \in \mathbf{k}(z)^\times$ for any $i \in I_0$.*

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Proof. Assertions (i) and (ii) easily follow from Propositions 3.4 and 3.7.

Let us show (iii). For $(i,a) \in \sigma(\mathfrak{g})$, the condition $\Lambda^\infty(M, V(\varpi_i)_a) = \Lambda^\infty(N, V(\varpi_i)_a)$ is equivalent to

$$\mathrm{Deg}^\infty(a_{M,V(\varpi_i)}(az)) = \mathrm{Deg}^\infty(a_{N,V(\varpi_i)}(az)).$$

Since $\mathrm{Deg}^\infty \colon \mathcal{G} \to \mathbb{Z}$ is a group homomorphism, it is equivalent to

$$\mathrm{Deg}^\infty\left(\frac{a_{M,V(\varpi_i)}(az)}{a_{N,V(\varpi_i)}(az)}\right) = 0 \quad \text{for any } a \in \mathbf{k}^\times.$$

Then (iii) follows from Lemma 3.1(iv). $\qquad\square$

For $(i,a) \in \sigma(\mathfrak{g})$, we set

$$\mathsf{s}_{i,a} := \mathsf{E}(V(\varpi_i)_a) \in \mathrm{Hom}(\sigma(\mathfrak{g}), \mathbb{Z})$$

and

$$\begin{aligned}
\mathcal{W} &:= \{\mathsf{E}(M) \mid M \text{ is simple in } \mathscr{C}_\mathfrak{g}\}, & \Delta &:= \{\mathsf{s}_{i,a} \mid (i,a) \in \sigma(\mathfrak{g})\} \subset \mathcal{W}, \\
\mathcal{W}_0 &:= \{\mathsf{E}(M) \mid M \text{ is simple in } \mathscr{C}_\mathfrak{g}^0\}, & \Delta_0 &:= \{\mathsf{s}_{i,a} \mid (i,a) \in \sigma_0(\mathfrak{g})\} \subset \mathcal{W}_0.
\end{aligned} \quad (4.2)$$

It is obvious that $\mathcal{W}_0 \subset \mathcal{W}$ and $\Delta_0 \subset \Delta$.

LEMMA 4.2.

(i) *We have $\mathcal{W} = \sum_{(i,a)\in\sigma(\mathfrak{g})} \mathbb{Z}\mathsf{s}_{i,a}$ and $\mathcal{W}_0 = \sum_{(i,a)\in\sigma_0(\mathfrak{g})} \mathbb{Z}\mathsf{s}_{i,a} = \sum_{(i,a)\in\sigma_Q(\mathfrak{g})} \mathbb{Z}\mathsf{s}_{i,a}$. In particular, $\mathcal{W}_0$ is a finitely generated free $\mathbb{Z}$-module.*

(ii) *There exists a unique symmetric bilinear form $(-,-)$ on $\mathcal{W}$ such that*

$$(\mathsf{E}(M), \mathsf{E}(N)) = -\Lambda^\infty(M, N)$$

for any simple modules $M, N \in \mathscr{C}_\mathfrak{g}$.

Proof. Assertion (i) follows from Theorem 2.2(iv), Lemma 4.1 and (2.12).

Let us show (ii). By Corollary 3.8, it reduces to the existence of a bilinear form $(-,-)$ on $\mathcal{W}$ such that

$$(\mathsf{s}_{i,a}, \mathsf{s}_{j,b}) = -\Lambda^\infty(V(\varpi_i)_a, V(\varpi_j)_b).$$

Therefore it is enough to show that for a sequence $\{(i_k, a_k)\}_{k=1,\dots,r}$ in $\sigma(\mathfrak{g})$ such that $\sum_{k=1}^r \mathsf{s}_{i_k,a_k} = 0$, we have $\sum_{k=1}^r \Lambda^\infty(V(\varpi_{i_k})_{a_k}, V(\varpi_j)_b) = 0$ for any $(j,b) \in \sigma(\mathfrak{g})$. Let us take a simple subquotient M of $V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_r})_{a_r}$. Then we have $\mathsf{E}(M) = \sum_{k=1}^r \mathsf{s}_{i_k,a_k} = 0$. Hence we obtain

$$\sum_{k=1}^r \Lambda^\infty(V(\varpi_{i_k})_{a_k}, V(\varpi_j)_b) = \Lambda^\infty(M, V(\varpi_j)_b) = -\mathsf{E}(M)(j,b) = 0. \qquad\square$$

LEMMA 4.3. *For $i \in I_0$ and $a \in \mathbf{k}^\times$, we have*

$$\mathfrak{d}(V(\varpi_i), \mathscr{D}^k V(\varpi_i)) = \delta(k = \pm 1) \quad \text{for } k \in \mathbb{Z}. \quad (4.3)$$

In particular,

$$(\mathsf{s}_{i,a}, \mathsf{s}_{i,a}) = -\Lambda^\infty(V(\varpi_i), V(\varpi_i)) = 2.$$

Proof. The statement $\Lambda^\infty(V(\varpi_i), V(\varpi_i)) = -2$ follows from (4.3) and Proposition 3.11.

Let us show (4.3). Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$, and write

$$d_{i,j}(z) := d_{V(\varpi_i),V(\varpi_j)}(z) \quad \text{for } i,j \in I.$$

The denominator formula for $d_{i,j}(z)$ is given in Appendix A. Using this formula, one can easily check that if ϵq^t ($|\epsilon| = 1$) is a zero of $d_{i,i}(z)$, then t should be between 1 and $h^\vee$. Combining this with Proposition 3.9, we obtain

$$\mathfrak{d}(V(\varpi_i), V(\varpi_j)_{(p^*)^k}) = \text{zero}_{z=1}\big(d_{i,j}((p^*)^k z)d_{j,i}((p^*)^{-k}z^{-1})\big) = 0 \quad \text{unless } k = \pm 1.$$

Now we shall show that $\mathfrak{d}(V(\varpi_i), \mathscr{D}^{\pm 1}V(\varpi_i)) = 1$.

Case of simply laced affine ADE type. In this case, the dual Coxeter number is equal to the Coxeter number. Then from the denominator formula in Appendix A it follows that

$$\mathfrak{d}(V(\varpi_i), \mathscr{D}^{\pm 1}V(\varpi_i)) = \tilde{c}_{i,i^*}(h^\vee - 1).$$

Since $\tilde{c}_{i,j}(k) = \tilde{c}_{j,i^*}(h^\vee - k)$ for $1 \leqslant k \leqslant h^\vee - 1$ (see [Fuj22, Lemma 3.7]) and $\tilde{c}_{i,i}(1) = 1$ by Proposition A.1, we have

$$\mathfrak{d}(V(\varpi_i), \mathscr{D}^{\pm 1}V(\varpi_i)) = \tilde{c}_{i,i}(1) = 1.$$

Other case. In this case, we know that $i^* = i$ for any $i \in I_0$. Thus we have

$$\mathfrak{d}(V(\varpi_i), \mathscr{D}^{\pm 1}V(\varpi_i)) = \mathfrak{d}(V(\varpi_i), V(\varpi_i)_{p^*}).$$

Using (2.6) and the denominator formula in Appendix A, one can compute directly that

$$\mathfrak{d}(V(\varpi_i), V(\varpi_i)_{p^*}) = 1. \qquad \square$$

For $t \in \mathbf{k}^\times$, $(i,a) \in \sigma(\mathfrak{g})$ and $f \in \text{Hom}(\sigma(\mathfrak{g}), \mathbb{Z})$, we define

$$\tau_t(i,a) := (i, ta) \quad \text{and} \quad (\tau_t f)(i,a) := f(i, t^{-1}a). \tag{4.4}$$

LEMMA 4.4.

(i) *For $(i,a) \in \sigma(\mathfrak{g})$, we have $\mathsf{s}_{i,a} = -\mathsf{s}_{i^*,ap^*} = -\mathsf{s}_{i^*,a(p^*)^{-1}}$.*
(ii) *For $t \in \mathbf{k}^\times$ and $(i,a) \in \sigma(\mathfrak{g})$, we have $\tau_t(\mathsf{s}_{i,a}) = \mathsf{s}_{i,ta}$.*

Proof. Assertion (i) follows from (2.7) and Lemma 4.1.
   (ii) For $(j,b) \in \sigma(\mathfrak{g})$, we have

$$(\tau_t(\mathsf{s}_{i,a}))(j,b) = (\mathsf{s}_{i,a})(j, t^{-1}b) = \Lambda^\infty(V(\varpi_i)_a, V(\varpi_j)_{t^{-1}b}) = \Lambda^\infty(V(\varpi_i)_{ta}, V(\varpi_j)_b)$$
$$= (\mathsf{s}_{i,ta})(j,b),$$

where the third equality follows from Proposition 3.3. Thus, we have the desired assertion. $\square$

For $t \in \mathbf{k}^\times$, $A \subset \sigma(\mathfrak{g})$ and $F \subset \text{Hom}(\sigma(\mathfrak{g}), \mathbb{Z})$, we set

$$A_t := \{\tau_t(a) \mid a \in A\} \quad \text{and} \quad F_t := \{\tau_t(f) \mid f \in F\}.$$

We write $\mathbf{k}_0$ for the stabilizer subgroup of $\sigma_0(\mathfrak{g})$ with respect to the action of $\mathbf{k}^\times$ on $\sigma(\mathfrak{g})$ through τ_t, i.e.

$$\mathbf{k}_0 := \{t \in \mathbf{k}^\times | (\sigma_0(\mathfrak{g}))_t = \sigma_0(\mathfrak{g})\}.$$

PROPOSITION 4.5. *The following hold:*

(i) $\sigma(\mathfrak{g}) = \bigsqcup_{a \in \mathbf{k}^\times/\mathbf{k}_0}(\sigma_0(\mathfrak{g}))_a$;
(ii) $\Delta = \bigsqcup_{a \in \mathbf{k}^\times/\mathbf{k}_0}(\Delta_0)_a$;
(iii) *for $k, k' \in \mathbf{k}^\times$ such that $k/k' \notin \mathbf{k}_0$, we have $\big((\mathcal{W}_0)_k, (\mathcal{W}_0)_{k'}\big) = 0$.*

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Proof. Assertion (i) follows from the fact that any connected component of $\sigma(\mathfrak{g})$ is a translation of $\sigma_0(\mathfrak{g})$.

(iii) It is enough to show that for $(i, a) \in (\sigma_0(\mathfrak{g}))_k$ and $(j, b) \in (\sigma_0(\mathfrak{g}))_{k'}$ we have $(\mathsf{s}_{i,a}, \mathsf{s}_{j,b}) = 0$. By the definition of $\sigma_0(\mathfrak{g})$, $V(\varpi_i)_a$ and $\mathscr{D}^m V(\varpi_j)_b$ strongly commute for any m, which tells us that

$$\Lambda^\infty(V(\varpi_i)_a, V(\varpi_j)_b) = 0$$

by Corollary 3.10 and Proposition 3.11.

(ii) It is enough to show that

$$\Delta_0 \cap (\Delta_0)_k = \emptyset \quad \text{for } k \in \mathbf{k}^\times / \mathbf{k}_0.$$

For $(i, a) \in \sigma_0(\mathfrak{g})$ and $(j, b) \in \sigma_0(\mathfrak{g})_k$, we have $(\mathsf{s}_{i,a}, \mathsf{s}_{i,a}) = 2$ by Lemma 4.3 and $(\mathsf{s}_{i,a}, \mathsf{s}_{j,b}) = 0$ by (iii). Thus we conclude that $\mathsf{s}_{i,a} \neq \mathsf{s}_{j,b}$. $\qquad\square$

We set

$$\mathcal{E} := \mathbb{R} \otimes_\mathbb{Z} \mathcal{W} \quad \text{and} \quad \mathcal{E}_0 := \mathbb{R} \otimes_\mathbb{Z} \mathcal{W}_0.$$

Then the pairing $(-, -)$ gives a symmetric bilinear form on $\mathcal{E}$. Theorem 4.6 below is the main theorem of this section; its proof is postponed until §6.

THEOREM 4.6.

(i) *The pair $(\mathcal{E}_0, \Delta_0)$ is an irreducible simply laced root system of type shown in the following table.*

Type of $\mathfrak{g}$	$A_n^{(1)}$	$B_n^{(1)}$	$C_n^{(1)}$	$D_n^{(1)}$	$A_{2n}^{(2)}$	$A_{2n-1}^{(2)}$	$D_{n+1}^{(2)}$
	$(n \geqslant 1)$	$(n \geqslant 2)$	$(n \geqslant 3)$	$(n \geqslant 4)$	$(n \geqslant 1)$	$(n \geqslant 2)$	$(n \geqslant 3)$
Type of $(\mathcal{E}_0, \Delta_0)$	A_n	A_{2n-1}	D_{n+1}	D_n	A_{2n}	A_{2n-1}	D_{n+1}
Type of $\mathfrak{g}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$	$F_4^{(1)}$	$G_2^{(1)}$	$E_6^{(2)}$	$D_4^{(3)}$
Type of $(\mathcal{E}_0, \Delta_0)$	E_6	E_7	E_8	E_6	D_4	E_6	D_4

$$(4.5)$$

(ii) *The bilinear form $(-, -)|_{\mathcal{W}_0}$ is positive definite. Moreover, it is Weyl group invariant, i.e. $s_\alpha(\Delta_0) \subset \Delta_0$ for any $\alpha \in \Delta_0$. Here $s_\alpha \in \mathrm{End}(\mathcal{E}_0)$ is the reflection defined by $s_\alpha(\lambda) = \lambda - (\alpha, \lambda)\alpha$.*

The next corollary follows from Proposition 4.5 and Theorem 4.6.

COROLLARY 4.7.

(i) *We have $\mathcal{W} = \bigoplus_{k \in \mathbf{k}^\times / \mathbf{k}_0} (\mathcal{W}_0)_k$.*

(ii) *As a root system, $((\mathcal{E}_0)_k, (\Delta_0)_k)$ is isomorphic to $(\mathcal{E}_0, \Delta_0)$ for $k \in \mathbf{k}^\times / \mathbf{k}_0$, and*

$$(\mathcal{E}, \Delta) = \bigsqcup_{k \in \mathbf{k}^\times / \mathbf{k}_0} ((\mathcal{E}_0)_k, (\Delta_0)_k).$$

Proof. We know already that $\mathcal{W} = \sum_{k \in \mathbf{k}^\times / \mathbf{k}_0} (\mathcal{W}_0)_k$. Since $(\mathcal{W}_0)_k$ and $(\mathcal{W}_0)_{k'}$ are orthogonal if $k/k' \notin \mathbf{k}_0$, the non-degeneracy of $(-, -)|_\mathcal{E}$ implies that $\mathcal{W} = \bigoplus_{k \in \mathbf{k}^\times / \mathbf{k}_0} (\mathcal{W}_0)_k$.

Assertion (ii) easily follows from (i) and Theorem 4.6. $\qquad\square$

The following corollary is an immediate consequence of Theorem 4.6.

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Corollary 4.8. *The following hold:*

(i) $(\lambda, \lambda) \in 2\mathbb{Z}_{>0}$ *for any* $\lambda \in \mathcal{W}_0 \setminus \{0\}$;
(ii) $\Delta_0 = \{\lambda \in \mathcal{W}_0 \mid (\lambda, \lambda) = 2\}$.

Hence the root system $(\mathcal{E}_0, \Delta_0)$ is completely determined by the pair $(\mathcal{W}_0, (-, -)|_{\mathcal{W}_0})$.

5. Block decomposition of $\mathscr{C}_\mathfrak{g}$

In this section, we give a block decomposition of $\mathscr{C}_\mathfrak{g}$ parameterized by $\mathcal{W}$.

5.1 Blocks

We recall the notion of blocks. Let $\mathcal{C}$ be an abelian category such that any object of $\mathcal{C}$ has finite length.

Definition 5.1. A *block* $\mathcal{B}$ of $\mathcal{C}$ is a full abelian subcategory with the following properties:

(i) there is a decomposition $\mathcal{C} = \mathcal{B} \oplus \mathcal{C}'$ for some full abelian subcategory $\mathcal{C}'$,
(ii) there is no non-trivial decomposition $\mathcal{B} = \mathcal{B}' \oplus \mathcal{B}''$ with full abelian subcategories $\mathcal{B}'$ and $\mathcal{B}''$.

The following lemma is obvious.

Lemma 5.2. *Let* $\mathcal{B}$ *be a full subcategory of* $\mathcal{C}$ *satisfying condition (i) in* Definition 5.1. *Then* $\mathcal{B}$ *has the following properties:*

(i) $\mathcal{B}$ *is stable by taking subquotients and extensions;*
(ii) *for simple objects* $S, S' \in \mathcal{C}$ *such that* $\mathrm{Ext}^1_\mathcal{C}(S, S') \not\simeq 0$, *if one of them belongs to* $\mathcal{B}$ *then so does the other.*

Lemma 5.3. *Let* $X, X' \in \mathcal{C}$. *Suppose that* $\mathrm{Ext}^1_\mathcal{C}(S, S') = 0$ *for any simple subquotients* S *and* S' *of* X *and* X', *respectively. Then we have* $\mathrm{Ext}^1_\mathcal{C}(X, X') = 0$.

Proof. Let ℓ and ℓ' be the lengths of X and X', respectively. We use induction on $\ell + \ell'$. If X and X' are simple, then the claimed result is clear by the assumption.

Suppose that X' is not simple. Then there exists an exact sequence $0 \to M \to X' \to N \to 0$ with a simple M. It in turn gives the exact sequence

$$\mathrm{Ext}^1_\mathcal{C}(X, M) \to \mathrm{Ext}^1_\mathcal{C}(X, X') \to \mathrm{Ext}^1_\mathcal{C}(X, N).$$

By the induction hypothesis we have $\mathrm{Ext}^1_\mathcal{C}(X, M) = \mathrm{Ext}^1_\mathcal{C}(X, N) = 0$, which tells us that $\mathrm{Ext}^1_\mathcal{C}(X, X') = 0$.

The case where X is not simple can be proved in the same manner. $\square$

Lemma 5.4. *Let* $\mathfrak{c}$ *be the set of isomorphism classes of simple objects of* $\mathcal{C}$, *and let* $\mathfrak{c} = \bigsqcup_{a \in A} \mathfrak{c}_a$ *be a partition of* $\mathfrak{c}$. *We assume that*

> *for* $a, a' \in A$ *such that* $a \neq a'$ *and a simple object* S *(respectively* S') *belonging to* $\mathfrak{c}_a$ *(respectively* $\mathfrak{c}_{a'}$), *one has* $\mathrm{Ext}^1_\mathcal{C}(S, S') = 0$.

For $a \in A$, *let* $\mathcal{C}_a$ *be the full subcategory of* $\mathcal{C}$ *consisting of objects* X *such that any simple subquotient of* X *belongs to* $\mathfrak{c}_a$. *Then* $\mathcal{C} = \bigoplus_{a \in A} \mathcal{C}_a$.

Proof. It is enough to show that any object X of $\mathcal{C}$ has a decomposition $X \simeq \bigoplus_{a \in A} X_a$ with $X_a \in \mathcal{C}_a$. In order to prove this, we shall argue by induction on the length of X. We may assume

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that X is non-zero. Let us take a subobject Y of X such that X/Y is simple. Then the induction hypothesis implies that $Y = \bigoplus_{a \in A} Y_a$ with $Y_a \in \mathcal{C}_a$.

Take $a_0 \in A$ such that X/Y belongs to $\mathfrak{c}_{a_0}$. Then define $Z \in \mathcal{C}$ by the exact sequence

$$0 \to \bigoplus_{a \neq a_0} Y_a \to X \to Z \to 0. \tag{5.1}$$

Since we have an exact sequence $0 \to Y_{a_0} \to Z \to X/Y \to 0$, Z belongs to $\mathcal{C}_{a_0}$. Then Lemma 5.3 tells us that $\mathrm{Ext}^1(Z, \bigoplus_{a \neq a_0} Y_a) = 0$. Hence the exact sequence (5.1) splits, i.e. $X \simeq Z \oplus \bigoplus_{a \neq a_0} Y_a$. □

Let $\approx$ be the equivalence relation on the set of isomorphism classes of simple objects of $\mathcal{C}$ generated by the relation $\approx'$ defined as follows: for simple objects $S, S' \in \mathcal{C}$,

$$[S] \approx' [S'] \quad \text{if and only if} \quad \mathrm{Ext}^1_{\mathcal{C}}(S, S') \neq 0.$$

THEOREM 5.5. *Let A be the set of $\approx$-equivalence classes. For $a \in A$, let $\mathcal{C}_a$ be the full subcategory of $\mathcal{C}$ consisting of objects X such that any simple subquotient of X belongs to a. Then $\mathcal{C}_a$ is a block, and the category $\mathcal{C}$ has a decomposition $\mathcal{C} = \bigoplus_{a \in A} \mathcal{C}_a$. Moreover, any block of $\mathcal{C}$ is equal to $\mathcal{C}_a$ for some a.*

Proof. Lemma 5.4 implies the decomposition

$$\mathcal{C} = \bigoplus_{a \in A} \mathcal{C}_a.$$

Moreover, since a is a $\approx$-equivalence class, there is no non-trivial decomposition of $\mathcal{C}_a$ for any $a \in A$. □

The next corollary follows directly from Theorem 5.5.

COROLLARY 5.6. *Let X be an indecomposable object of $\mathcal{C}$. Then X belongs to some block. In particular, all the simple subquotients of X belong to the same block.*

5.2 Direct decomposition of $\mathscr{C}_{\mathfrak{g}}$

In this subsection, we shall prove that $\mathscr{C}_{\mathfrak{g}}$ has a decomposition parameterized by elements of $\mathcal{W}$.

LEMMA 5.7. *For modules $M, N \in \mathscr{C}_{\mathfrak{g}}$, there exists an isomorphism*

$$\Psi \colon \mathbf{k}[z^{\pm 1}] \otimes \mathrm{Hom}_{U_q'(\mathfrak{g})}(N, \mathbf{1}) \otimes \mathrm{Hom}_{U_q'(\mathfrak{g})}(\mathbf{1}, M) \xrightarrow{\sim} \mathrm{Hom}_{U_q'(\mathfrak{g})}(N, M_z) \tag{5.2}$$

defined by $\Psi(a(z) \otimes f \otimes g) = a(z)(g \circ f)$ *for* $a(z) \in \mathbf{k}[z^{\pm 1}]$, $f \in \mathrm{Hom}_{U_q'(\mathfrak{g})}(N, \mathbf{1})$ *and* $g \in \mathrm{Hom}_{U_q'(\mathfrak{g})}(\mathbf{1}, M)$.

Proof. Note that $\mathbf{k}[z^{\pm 1}] \otimes \mathrm{Hom}_{U_q'(\mathfrak{g})}(\mathbf{1}, M) \xrightarrow{\sim} \mathrm{Hom}_{U_q'(\mathfrak{g})}(\mathbf{1}, M_z)$. There is a quotient N' of N which is a direct sum of copies of $\mathbf{1}$ and $\mathrm{Hom}(N', \mathbf{1}) \xrightarrow{\sim} \mathrm{Hom}(N, \mathbf{1})$. Since (5.2) for N' is obviously an isomorphism, Ψ is injective.

To prove that Ψ is surjective, we shall decompose a given non-zero $f \colon N \to M_z$ into $N \to \mathbf{1}^{\oplus \ell} \to M_z$ for some $\ell \in \mathbb{Z}_{>0}$. Here $\mathbf{1}^{\oplus \ell}$ is the direct sum of ℓ copies of the trivial module $\mathbf{1}$. Without loss of generality, we may assume that f is injective. We set $\mathrm{wt}(N) := \{\lambda \in \mathsf{P}_{\mathrm{cl}} \mid N_\lambda \neq 0\}$.

If $\mathrm{wt}(N) = \{0\}$, then N should be isomorphic to $\mathbf{1}^{\oplus \ell}$ for some $\ell \in \mathbb{Z}_{>0}$, which is the desired result.

Now suppose that $\mathrm{wt}(N) \neq \{0\}$. We choose a non-zero weight $\lambda \in \mathrm{wt}(N)$.

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Note that the $U_q'(\mathfrak{g})$-module structure on M_z extends to a $U_q(\mathfrak{g})$-module structure and we have a weight decomposition $M_z = \bigoplus_{\mu \in \mathsf{P}}(M_z)_\mu$. Then

$$f(N_\lambda) \subset \bigoplus_{\mu \in \mathsf{P},\, \mathrm{cl}(\mu)=\lambda} (M_z)_\mu,$$

where $\mathrm{cl}\colon \mathsf{P} \to \mathsf{P}_{\mathrm{cl}}$ is the classical projection. There exist $w \in \mathsf{W}$ and a non-zero integer n such that $w(\mu) = \mu + n\delta$ for any $\mu \in \mathrm{cl}^{-1}(\lambda)$. We now consider the braid group action T_w defined by w on an integral module (see [Lus90, Sai94]). Then the $\mathbf{k}$-linear automorphism T_w sends $(M_z)_\mu$ to $(M_z)_{w\mu}$. The space $f(N_\lambda)$ is invariant under the automorphism T_w, but any non-zero finite-dimensional subspace of $\bigoplus_{\mu \in \mathsf{P},\, \mathrm{cl}(\mu)=\lambda}(M_z)_\mu$ cannot be invariant under T_w. This is a contradiction. $\square$

Pʀᴏᴘᴏꜱɪᴛɪᴏɴ 5.8. *For modules $M, N \in \mathscr{C}_{\mathfrak{g}}$ and a simple module $L \in \mathscr{C}_{\mathfrak{g}}$, we have the isomorphisms*

$$\mathbf{k}[z^{\pm 1}] \otimes \mathrm{Hom}_{U_q'(\mathfrak{g})}(M, N) \xrightarrow{\;\sim\;} \mathrm{Hom}_{\mathbf{k}[z^{\pm 1}] \otimes U_q'(\mathfrak{g})}(M \otimes L_z, N \otimes L_z).$$

Proof. By Lemma 5.7, we obtain that

$$\mathrm{Hom}_{\mathbf{k}[z^{\pm 1}] \otimes U_q'(\mathfrak{g})}(M \otimes L_z, N \otimes L_z) \simeq \mathrm{Hom}_{U_q'(\mathfrak{g})}(N^* \otimes M, (L \otimes L^*)_z)$$

$$\simeq \mathbf{k}[z^{\pm 1}] \otimes \mathrm{Hom}_{U_q'(\mathfrak{g})}(N^* \otimes M, \mathbf{1}) \otimes \mathrm{Hom}_{U_q'(\mathfrak{g})}(\mathbf{1}, L \otimes L^*)$$

$$\simeq \mathbf{k}[z^{\pm 1}] \otimes \mathrm{Hom}_{U_q'(\mathfrak{g})}(M, N). \qquad \square$$

Lᴇᴍᴍᴀ 5.9. *Let M and N be simple modules in $\mathscr{C}_{\mathfrak{g}}$. If*

$$\frac{c_{M,L}(z)}{c_{N,L}(z)} \notin \mathbf{k}(z) \quad \text{for some simple module } L \in \mathscr{C}_{\mathfrak{g}},$$

then we have

$$\mathrm{Ext}^1_{U_q'(\mathfrak{g})}(M, N) = 0.$$

Proof. Let $L \in \mathscr{C}_{\mathfrak{g}}$ be a simple module such that $c_{M,L}(z)/c_{N,L}(z) \notin \mathbf{k}(z)$.

We shall prove that any exact sequence

$$0 \to N \to X \to M \to 0$$

splits. We set $\hat{L}_z := \mathbf{k}((z)) \otimes_{\mathbf{k}[z^{\pm 1}]} L_z$, where L_z is the affinization of L. Then the following diagram commutes.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N \otimes \hat{L}_z & \longrightarrow & X \otimes \hat{L}_z & \longrightarrow & M \otimes \hat{L}_z & \longrightarrow & 0 \\
& & {\scriptstyle R^{\mathrm{univ}}_{N,\hat{L}_z}} \downarrow \wr & & {\scriptstyle R^{\mathrm{univ}}_{X,\hat{L}_z}} \downarrow \wr & & {\scriptstyle R^{\mathrm{univ}}_{M,\hat{L}_z}} \downarrow \wr & & \\
0 & \longrightarrow & \hat{L}_z \otimes N & \longrightarrow & \hat{L}_z \otimes X & \longrightarrow & \hat{L}_z \otimes M & \longrightarrow & 0
\end{array}
$$

We set

$$f(z) := \frac{c_{M,L}(z)}{c_{N,L}(z)} \notin \mathbf{k}(z) \quad \text{and} \quad R := c_{M,L}(z) R^{\mathrm{univ}}_{X,\hat{L}_z} \colon X \otimes \hat{L}_z \to \hat{L}_z \otimes X.$$

It follows from

$$c_{M,L}(z) R^{\mathrm{univ}}_{M,\hat{L}_z}(M \otimes L_z) \subset L_z \otimes M \quad \text{and} \quad c_{N,L}(z) R^{\mathrm{univ}}_{N,\hat{L}_z}(N \otimes L_z) \subset L_z \otimes N$$

that

$$R(X \otimes L_z) \subset L_z \otimes X + \hat{L}_z \otimes N \quad \text{and} \quad R(N \otimes L_z) \subset f(z)(L_z \otimes N).$$

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Therefore R induces the $\mathbf{k}[z^{\pm 1}] \otimes U_q'(\mathfrak{g})$-linear homomorphism

$$\mathcal{R} \colon M \otimes L_z \simeq \frac{X \otimes L_z}{N \otimes L_z} \longrightarrow \frac{\mathbf{k}(z) \otimes L_z \otimes X + \hat{L}_z \otimes N}{\mathbf{k}(z) \otimes L_z \otimes X + f(z)\mathbf{k}(z) \otimes L_z \otimes N}.$$

We set $\mathcal{P} := \mathbf{k}((z))/(\mathbf{k}(z) + f(z)\mathbf{k}(z))$. Since

$$\frac{\mathbf{k}(z) \otimes L_z \otimes X + \hat{L}_z \otimes N}{\mathbf{k}(z) \otimes L_z \otimes X + f(z)\mathbf{k}(z) \otimes L_z \otimes N} \simeq \frac{\hat{L}_z \otimes N}{\mathbf{k}(z) \otimes L_z \otimes N + f(z)\mathbf{k}(z) \otimes L_z \otimes N}$$
$$\simeq \mathcal{P} \otimes_{\mathbf{k}[z^{\pm 1}]} L_z \otimes N,$$

we have the homomorphism of $\mathbf{k}[z^{\pm 1}] \otimes U_q'(\mathfrak{g})$-modules

$$\mathcal{R} \colon M \otimes L_z \longrightarrow \mathcal{P} \otimes_{\mathbf{k}[z^{\pm 1}]} L_z \otimes N.$$

Let us show that $\mathcal{R}$ vanishes.

Assume that $\mathcal{R} \neq 0$. Then

$$\mathrm{Hom}_{\mathbf{k}[z^{\pm 1}] \otimes U_q'(\mathfrak{g})}(M \otimes L_z, \mathcal{P} \otimes_{\mathbf{k}[z^{\pm 1}]} L_z \otimes N) \simeq \mathcal{P} \otimes_{\mathbf{k}[z^{\pm 1}]} \mathrm{Hom}_{\mathbf{k}[z^{\pm 1}] \otimes U_q'(\mathfrak{g})}(M \otimes L_z, L_z \otimes N)$$

implies that $\mathrm{Hom}_{\mathbf{k}[z^{\pm 1}] \otimes U_q'(\mathfrak{g})}(M \otimes L_z, L_z \otimes N) \not\simeq 0$.

Since $\mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (M \otimes L_z)$ and $\mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (L_z \otimes N)$ are simple $\mathbf{k}(z) \otimes U_q'(\mathfrak{g})$-modules, they are isomorphic. Since $\mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (L_z \otimes N)$ and $\mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (N \otimes L_z)$ are isomorphic, we conclude that $\mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (M \otimes L_z) \simeq \mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (N \otimes L_z)$. On the other hand, Proposition 5.8 implies that

$$\mathbf{k}(z) \otimes \mathrm{Hom}_{U_q'(\mathfrak{g})}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{k}(z) \otimes U_q'(\mathfrak{g})}(\mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} M \otimes L_z, \mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} N \otimes L_z).$$

Hence $\mathrm{Hom}_{U_q'(\mathfrak{g})}(M, N) \neq 0$, and we obtain that M and N are isomorphic, which is a contradiction. Therefore $\mathcal{R} = 0$, which means that

$$R\big(\mathbf{k}(z) \otimes (X \otimes L_z)\big) \subset \mathbf{k}(z) \otimes L_z \otimes X + f(z)\mathbf{k}(z) \otimes L_z \otimes N.$$

Let us consider the composition

$$\Phi \colon K := R\big(\mathbf{k}(z) \otimes (X \otimes L_z)\big) \cap \big(\mathbf{k}(z) \otimes L_z \otimes X\big) \rightarrowtail \mathbf{k}(z) \otimes L_z \otimes X \twoheadrightarrow \mathbf{k}(z) \otimes L_z \otimes M.$$

We have

$$R\big(\mathbf{k}(z) \otimes (X \otimes L_z)\big) \cap \hat{L}_z \otimes N = R\big(\mathbf{k}(z) \otimes (N \otimes L_z)\big) = f(z)\mathbf{k}(z) \otimes L_z \otimes N.$$

Hence $\ker(\Phi) = K \cap \big(\mathbf{k}(z) \otimes L_z \otimes N\big) = \big(f(z)\mathbf{k}(z) \otimes L_z \otimes N\big) \cap \big(\mathbf{k}(z) \otimes L_z \otimes N\big)$ vanishes, which means that Φ is a monomorphism.

Since $\mathbf{k}(z) \otimes L_z \otimes M$ and $\mathbf{k}(z) \otimes L_z \otimes N$ are simple $\mathbf{k}(z) \otimes U_q'(\mathfrak{g})$-modules, $\mathbf{k}(z) \otimes L_z \otimes X$ has length 2. Similarly, $R\big(\mathbf{k}(z) \otimes (X \otimes L_z)\big)$ also has length 2. On the other hand, $\mathbf{k}(z) \otimes L_z \otimes X + f(z)\mathbf{k}(z) \otimes L_z \otimes N$ has length no more than 3, which implies that K does not vanish. Hence Φ is an isomorphism. Thus we conclude that the homomorphism

$$\mathrm{Hom}(\mathbf{k}(z) \otimes L_z \otimes M, \mathbf{k}(z) \otimes L_z \otimes X) \rightarrow \mathrm{Hom}(\mathbf{k}(z) \otimes L_z \otimes M, \mathbf{k}(z) \otimes L_z \otimes M)$$
$$= \mathbf{k}(z) \, \mathrm{id}_{\mathbf{k}(z) \otimes L_z \otimes M}$$

is surjective. Then Proposition 5.8 implies that this homomorphism is isomorphic to

$$\mathbf{k}(z) \otimes \mathrm{Hom}(M, X) \twoheadrightarrow \mathbf{k}(z) \otimes \mathrm{Hom}(M, M).$$

Thus we conclude that $\mathrm{Hom}(M, X) \rightarrow \mathrm{Hom}(M, M)$ is surjective, that is,

$$0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0$$

splits. $\qquad\square$

For $\alpha \in \mathcal{W}$, let $\mathscr{C}_{\mathfrak{g},\alpha}$ be the full subcategory of $\mathscr{C}_{\mathfrak{g}}$ consisting of objects X such that $\mathsf{E}(S) = \alpha$ for any simple subquotient S of X.

Theorem 5.10. *There exist the decompositions*

$$\mathscr{C}_{\mathfrak{g}} = \bigoplus_{\alpha \in \mathcal{W}} \mathscr{C}_{\mathfrak{g},\alpha} \quad and \quad \mathscr{C}_{\mathfrak{g}}^0 = \bigoplus_{\alpha \in \mathcal{W}_0} \mathscr{C}_{\mathfrak{g},\alpha}.$$

Proof. Let $\alpha, \beta \in \mathcal{W}$ with $\alpha \neq \beta$. For simple modules $M \in \mathscr{C}_{\mathfrak{g},\alpha}$ and $N \in \mathscr{C}_{\mathfrak{g},\beta}$, Lemma 4.1(iii) says that $a_{M,V(\varpi_i)}(z)/a_{N,V(\varpi_i)}(z) \notin \mathbf{k}(z)$ for some $i \in I_0$. Hence Lemma 5.9 implies that $\mathrm{Ext}^1_{U_q'(\mathfrak{g})}(M, N) = 0$. The desired result then follows from Lemma 5.4. $\square$

5.3 The block $\mathscr{C}_{\mathfrak{g},\alpha}$

Recall the automorphism τ_t on $\sigma(\mathfrak{g})$ defined in (4.4). For $(i,a) \in \sigma(\mathfrak{g})$ we write

$$V(i,a) := V(\varpi_i)_a.$$

Note that $V(\tau_t \alpha) = V(\alpha)_t$ for $\alpha \in \sigma(\mathfrak{g})$ and $t \in \mathbf{k}^\times$. For $\alpha \in \sigma(\mathfrak{g})$, we define $\alpha^* \in \sigma(\mathfrak{g})$ by

$$V(\alpha^*) \simeq V(\alpha)^*.$$

Thus we have

$$\alpha^{**} = \tau_{\tilde{p}^{-1}}(\alpha) \quad \text{for } \alpha \in \sigma(\mathfrak{g}).$$

Lemma 5.11. *Let $\alpha_1, \ldots, \alpha_k \in \sigma(\mathfrak{g})$ for $k \in \mathbb{Z}_{>0}$. Then all the simple subquotients of $V(\alpha_1) \otimes V(\alpha_2) \otimes \cdots \otimes V(\alpha_k)$ are contained in the same block of $\mathscr{C}_{\mathfrak{g}}$.*

Proof. There exists a permutation $\sigma \in \mathfrak{S}_k$ such that the tensor product $V(\alpha_{\sigma(1)}) \otimes V(\alpha_{\sigma(2)}) \otimes \cdots \otimes V(\alpha_{\sigma(k)})$ has a simple head by Theorem 2.2, and hence it is indecomposable. Thus, all the simple subquotients of $V(\alpha_{\sigma(1)}) \otimes V(\alpha_{\sigma(2)}) \otimes \cdots \otimes V(\alpha_{\sigma(k)})$ are contained in the same block by Corollary 5.6. Since any simple subquotient of $V(\alpha_1) \otimes V(\alpha_2) \otimes \cdots \otimes V(\alpha_k)$ is isomorphic to some simple subquotient of $V(\alpha_{\sigma(1)}) \otimes V(\alpha_{\sigma(2)}) \otimes \cdots \otimes V(\alpha_{\sigma(k)})$, we obtain the desired result. $\square$

We set

$$\mathcal{P} := \bigoplus_{\alpha \in \sigma(\mathfrak{g})} \mathbb{Z}\mathsf{e}_\alpha, \quad \mathcal{P}_0 := \bigoplus_{\alpha \in \sigma_0(\mathfrak{g})} \mathbb{Z}\mathsf{e}_\alpha$$

and

$$\mathcal{P}^+ := \sum_{\alpha \in \sigma(\mathfrak{g})} \mathbb{Z}_{\geqslant 0}\mathsf{e}_\alpha \subset \mathcal{P},$$

where e_α is a symbol. Define a group homomorphism

$$p \colon \mathcal{P} \twoheadrightarrow \mathcal{W}, \quad \mathsf{e}_{(i,a)} \mapsto \mathsf{s}_{i,a},$$

and set

$$p_0 := p|_{\mathcal{P}_0} \colon \mathcal{P}_0 \twoheadrightarrow \mathcal{W}_0.$$

By Proposition 4.5, we have

$$\mathcal{P} = \bigoplus_{k \in \mathbf{k}^\times/\mathbf{k}_0} (\mathcal{P}_0)_k. \tag{5.3}$$

Let $\mathcal{Q}_0$ be the subgroup of $\mathcal{P}_0$ generated by elements of the form $\sum_{k=1}^m \mathsf{e}_{\alpha_k}$ $(\alpha_k \in \sigma_0(\mathfrak{g}))$ such that the trivial module $\mathbf{1}$ appears in $V(\alpha_1) \otimes V(\alpha_2) \otimes \cdots \otimes V(\alpha_m)$ as a simple subquotient. We then have $p_0(\mathcal{Q}_0) = 0$.

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We set

$$\mathcal{Q} := \bigoplus_{k \in \mathbf{k}^\times / \mathbf{k}_0} (\mathcal{Q}_0)_k. \tag{5.4}$$

Recall $\phi_Q \colon \Delta_Q^+ \xrightarrow{\sim} \sigma_Q(\mathfrak{g})$ in (2.11). Let $\Pi_Q \subset \Delta_Q^+$ be the set of simple roots of the positive root system Δ_Q^+ and Q_Q the corresponding root lattice. Hence we have $\Pi_Q \subset \Delta_Q^+ \subset \mathsf{Q}_Q$.

In the proof of the following lemma, we do not use Theorem 4.6.

LEMMA 5.12. *For $\alpha \in \sigma_0(\mathfrak{g})$, denote by $\bar{\mathsf{e}}_\alpha \in \mathcal{P}_0/\mathcal{Q}_0$ the image of e_α under the projection $\mathcal{P}_0 \to \mathcal{P}_0/\mathcal{Q}_0$.*

  (i) *The map $\Delta_Q^+ \ni \alpha \mapsto \bar{\mathsf{e}}_{\phi_Q(\alpha)} \in \mathcal{P}_0/\mathcal{Q}_0$ extends to an additive map $\psi_Q' \colon \mathsf{Q}_Q \to \mathcal{P}_0/\mathcal{Q}_0$.*
 (ii) *We have that ψ_Q' is surjective, i.e.*

$$\mathcal{P}_0/\mathcal{Q}_0 = \sum_{\beta \in \Pi_Q} \mathbb{Z}\bar{\mathsf{e}}_{\phi_Q(\beta)}.$$

(iii) *Let $\psi_Q \colon \mathsf{Q}_Q \to \mathcal{W}_0$ be the composition $\mathsf{Q}_Q \xrightarrow{\psi_Q'} \mathcal{P}_0/\mathcal{Q}_0 \longrightarrow \mathcal{W}_0$. Then*

$$\psi_Q(\beta) = \mathsf{E}\big(V_Q(\beta)\big).$$

(iv) *We have that ψ_Q is surjective, i.e. $\mathcal{W}_0 = \sum_{\alpha \in \phi_Q(\Pi_Q)} \mathbb{Z}p_0(\mathsf{e}_\alpha)$.*

Proof. (i) The map $\Pi_Q \ni \alpha \mapsto \bar{\mathsf{e}}_{\phi_Q(\alpha)} \in \mathcal{P}_0/\mathcal{Q}_0$ extends to a linear map $\psi_Q' \colon \mathsf{Q}_Q \to \mathcal{P}_0/\mathcal{Q}_0$. It is enough to show that $\bar{\mathsf{e}}_{\phi_Q(\gamma)} = \psi_Q'(\gamma)$ for any $\gamma \in \Delta_Q^+$. Let us show this by induction on the length of γ. If γ is not a simple root, take a minimal pair (β, β') of γ (see Proposition 2.4). Since $V_Q(\gamma)$ appears as a composition factor of $V_Q(\beta) \otimes V_Q(\beta')$ by Proposition 2.4, we have

$$\bar{\mathsf{e}}_{\phi_Q(\gamma)} = \bar{\mathsf{e}}_{\phi_Q(\beta)} + \bar{\mathsf{e}}_{\phi_Q(\beta')} = \psi_Q'(\beta) + \psi_Q'(\beta') = \psi_Q'(\gamma).$$

Assertion (ii) follows from (i), and (iii) follows from (ii) and a surjective map $\mathcal{P}_0/\mathcal{Q}_0 \twoheadrightarrow \mathcal{W}_0$. $\square$

In the proof of the following lemma, we use the fact that the rank of $\mathcal{W}_0$ is at least the rank of Δ_Q^+ (stated in Theorem 4.6, whose proof is postponed to §6; see (6.3)).

LEMMA 5.13. *We have the isomorphisms*

$$\mathcal{P}_0/\mathcal{Q}_0 \xrightarrow{\sim} \mathcal{W}_0 \quad and \quad \mathcal{P}/\mathcal{Q} \xrightarrow{\sim} \mathcal{W}.$$

Proof. The second isomorphism easily follows from the first isomorphism together with (5.3) and (5.4). So we need only show that $\mathcal{P}_0/\mathcal{Q}_0 \twoheadrightarrow \mathcal{W}_0$ is an isomorphism.

Let r be the rank of Δ_Q^+. By (6.3), the rank of $\mathcal{W}_0$ is at least r. Let us consider a surjective homomorphism

$$\mathcal{P}_0/\mathcal{Q}_0 \twoheadrightarrow \mathcal{W}_0. \tag{5.5}$$

By Lemma 5.12, $\mathcal{P}_0/\mathcal{Q}_0$ is generated by r elements. Hence (5.5) is an isomorphism. $\square$

For $\lambda = \sum_{t=1}^k \mathsf{e}_{\alpha_t} \in \mathcal{P}^+$, we set

$$\bar{V}(\lambda) := [V(\alpha_1) \otimes V(\alpha_2) \otimes \cdots \otimes V(\alpha_k)] \in K(\mathscr{C}_\mathfrak{g}).$$

Note that for $\lambda, \mu \in \mathcal{P}^+$, if $\mathbf{1}$ appears in $\bar{V}(\lambda)$ and $\bar{V}(\mu)$, then $\mathbf{1}$ also appears in $\bar{V}(\lambda) \otimes \bar{V}(\mu)$. Hence any element of $\mathcal{Q}$ can be written as $\lambda - \mu$ with $\lambda, \mu \in \mathcal{P}^+$ such that $\mathbf{1}$ appears in both $\bar{V}(\lambda)$ and $\bar{V}(\mu)$.

THEOREM 5.14. *For any $\alpha \in \mathcal{W}$, the subcategory $\mathscr{C}_{\mathfrak{g},\alpha}$ is a block of $\mathscr{C}_\mathfrak{g}$.*

Proof. Let $\alpha \in \mathcal{W}$, and let S and S' be simple modules in $\mathscr{C}_{\mathfrak{g},\alpha}$. We shall show that S and S' belong to the same block.

Thanks to Theorem 2.2(iv), there exist $\lambda, \lambda' \in \mathcal{P}^+$ such that S appears in $\bar{V}(\lambda)$ and S' appears in $\bar{V}(\lambda')$. By Lemma 5.13, we have $\lambda - \lambda' \in \ker p = \mathcal{Q}$. Then there exist $\mu, \mu' \in \mathcal{P}^+$ that satisfy the following:

- $\lambda - \lambda' = \mu' - \mu$;
- $\mathbf{1}$ appears in $\bar{V}(\mu)$ and $\bar{V}(\mu')$.

Thus the following hold:

(a) $\lambda + \mu = \lambda' + \mu'$, i.e. $\bar{V}(\lambda + \mu) = \bar{V}(\lambda' + \mu')$;
(b) S appears in $\bar{V}(\lambda) \otimes \bar{V}(\mu) = \bar{V}(\lambda + \mu)$;
(c) S' appears in $\bar{V}(\lambda') \otimes \bar{V}(\mu') = \bar{V}(\lambda' + \mu')$.

This tells us that S and S' belong to the same block by Lemma 5.11. $\qquad\square$

Combining Theorem 5.10 with Theorem 5.14, we have the following block decomposition.

COROLLARY 5.15. *There exist the block decompositions*
$$\mathscr{C}_{\mathfrak{g}} = \bigoplus_{\beta \in \mathcal{W}} \mathscr{C}_{\mathfrak{g},\beta} \quad and \quad \mathscr{C}_{\mathfrak{g}}^0 = \bigoplus_{\beta \in \mathcal{W}_0} \mathscr{C}_{\mathfrak{g},\beta}.$$

Remark 5.16. Lemma 5.13 gives a group presentation of $\mathcal{W}$ which parameterizes the block decomposition of $\mathscr{C}_{\mathfrak{g}}$. When $\mathfrak{g}$ is of untwisted type, the block decomposition of $\mathscr{C}_{\mathfrak{g}}$ was given in [CM05] and [EM03]. Considering [CM05] and [EM03] in our setting, their results give another group presentation of $\mathcal{W}$. Let us explain more precisely what this means in our setting.

Suppose that $\mathfrak{g}$ is of untwisted type. We define
$$\mathcal{P}_S := \bigoplus_{(i,a) \in \sigma(\mathfrak{g}), \, i \in S} \mathbb{Z}\mathsf{e}_{(i,a)},$$
where
$$S = \begin{cases} \{1\} & \text{if } \mathfrak{g} \text{ is of type } A_n^{(1)}, C_n^{(1)} \text{ or } E_6^{(1)}, \\ \{n\} & \text{if } \mathfrak{g} \text{ is of type } B_n^{(1)} \text{ or } D_n^{(1)} \text{ } (n \text{ odd}), \\ \{n-1, n\} & \text{if } \mathfrak{g} \text{ is of type } D_n^{(1)} \text{ } (n \text{ even}), \end{cases}$$
and S is $\{2\}$, $\{4\}$, $\{7\}$ or $\{8\}$ if $\mathfrak{g}$ is of type $G_2^{(1)}$, $F_4^{(1)}$, $E_7^{(1)}$ or $E_8^{(1)}$, respectively.

One can show that $p(\mathcal{P}_S) = \mathcal{W}$. Thus we have the surjective homomorphism
$$p_S := p|_{\mathcal{P}_S} \colon \mathcal{P}_S \twoheadrightarrow \mathcal{W}.$$

Then the results in [CM05, Proposition 4.1 and Appendix A] and [EM03, Lemma 4.6 and §6] explain that the kernel $\ker(p_S)$ is generated by the subset G described as follows:

(a) if $\mathfrak{g}$ is of type $A_n^{(1)}$, then $G = \{\sum_{k=0}^{n} \mathsf{e}_{(1,tq^{2k})} \mid t \in \mathbf{k}^{\times}\}$;
(b) if $\mathfrak{g}$ is of type $B_n^{(1)}$, then $G = \{\mathsf{e}_{(n,t)} + \mathsf{e}_{(n,tq^{2n-1})} \mid t \in \mathbf{k}^{\times}\}$;
(c) if $\mathfrak{g}$ is of type $C_n^{(1)}$, then $G = \{\mathsf{e}_{(1,t)} + \mathsf{e}_{(1,tq^{n+1})} \mid t \in \mathbf{k}^{\times}\}$;
(d) if $\mathfrak{g}$ is of type $D_n^{(1)}$ and n is odd, then $G = \{\mathsf{e}_{n,t} + \mathsf{e}_{n,tq^2} + \mathsf{e}_{n,tq^{2n-2}} + \mathsf{e}_{n,tq^{2n}} \mid t \in \mathbf{k}^{\times}\}$;
(e) if $\mathfrak{g}$ is of type $D_n^{(1)}$ and n is even, then $G = \{\mathsf{e}_{(n-1,t)} + \mathsf{e}_{(n-1,tq^2)} + \mathsf{e}_{(n,tq^{2n-2})} + \mathsf{e}_{(n,tq^{2n})}, \mathsf{e}_{(n-1,t)} + \mathsf{e}_{(n-1,tq^{2n-2})}, \mathsf{e}_{(n,t)} + \mathsf{e}_{(n,tq^{2n-2})} \mid t \in \mathbf{k}^{\times}\}$;

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(f) if $\mathfrak{g}$ is of type $E_6^{(1)}$, then $G = \{\mathsf{e}_{(1,t)} + \mathsf{e}_{(1,tq^8)} + \mathsf{e}_{(1,tq^{16})}, \mathsf{e}_{(1,t)} + \mathsf{e}_{(1,tq^2)} + \mathsf{e}_{(1,tq^4)} + \mathsf{e}_{(1,tq^{12})} + \mathsf{e}_{(1,tq^{14})} + \mathsf{e}_{(1,tq^{16})} \mid t \in \mathbf{k}^\times\}$;

(g) if $\mathfrak{g}$ is of type $E_7^{(1)}$, then $G = \{\mathsf{e}_{(7,t)} + \mathsf{e}_{(7,tq^{18})}, \mathsf{e}_{(7,t)} + \mathsf{e}_{(7,tq^2)} + \mathsf{e}_{(7,tq^{12})} + \mathsf{e}_{(7,tq^{14})} + \mathsf{e}_{(7,tq^{24})} + \mathsf{e}_{(7,tq^{26})} \mid t \in \mathbf{k}^\times\}$;

(h) if $\mathfrak{g}$ is of type $E_8^{(1)}$, then $G = \{\mathsf{e}_{(8,t)} + \mathsf{e}_{(8,tq^{30})}, \mathsf{e}_{(8,t)} + \mathsf{e}_{(8,tq^{20})} + \mathsf{e}_{(8,tq^{40})}, \ \mathsf{e}_{(8,t)} + \mathsf{e}_{(8,tq^{12})} + \mathsf{e}_{(8,tq^{24})} + \mathsf{e}_{(8,tq^{36})} + \mathsf{e}_{(8,tq^{48})} \mid t \in \mathbf{k}^\times\}$;

(i) if $\mathfrak{g}$ is of type $F_4^{(1)}$, then $G = \{\mathsf{e}_{(4,t)} + \mathsf{e}_{(4,tq^9)}, \mathsf{e}_{(4,t)} + \mathsf{e}_{(4,tq^6)} + \mathsf{e}_{(4,tq^{12})} \mid t \in \mathbf{k}^\times\}$;

(j) if $\mathfrak{g}$ is of type $G_2^{(1)}$, then $G = \{\mathsf{e}_{(2,t)} + \mathsf{e}_{(2,tq^4)}, \mathsf{e}_{(2,t)} + \mathsf{e}_{(2,t(-q_t)^8)} + \mathsf{e}_{(2,t(-q_t)^{16})} \mid t \in \mathbf{k}^\times\}$.

We remark that there are typos in the descriptions for types E_8 and F_4 in [CM05, Appendix A].

6. Proof of Theorem 4.6

6.1 Strategy of the proof

We now start to prove Theorem 4.6. We shall use the same notation as in §§ 2.3 and 2.4. Recall the explicit descriptions for $\sigma_0(\mathfrak{g})$ and $\sigma_Q(\mathfrak{g})$. Let $\Pi_Q = \{\alpha_i\}_{i\in\mathrm{I}_\mathrm{fin}}$ be the set of simple roots of Δ_Q^+, and let Q_Q be the root lattice of $\mathfrak{g}_\mathrm{fin}$. Hence

$$\Pi_Q \subset \Delta_Q^+ \subset \mathsf{Q}_Q.$$

Then, by Lemma 5.12, we have

$$\mathcal{W}_0 = \sum_{i\in\mathrm{I}_\mathrm{fin}} \mathbb{Z}\mathsf{s}_{\phi_Q(\alpha_i)}, \tag{6.1}$$

where $\phi_Q \colon \Delta_Q^+ \xrightarrow{\sim} \sigma_Q(\mathfrak{g})$ is the bijection given in (2.11).

Let $\mathsf{M}_Q := (\mathsf{m}_{i,j}^Q)_{i,j\in\mathrm{I}_\mathrm{fin}}$ be the square matrix given by

$$\mathsf{m}_{i,j}^Q := \big(\mathsf{s}_{\phi_Q(\alpha_i)}, \mathsf{s}_{\phi_Q(\alpha_j)}\big).$$

Thanks to Lemma 4.3, we know that

$$\mathsf{m}_{i,i}^Q = 2 \quad \text{for any } i \in \mathrm{I}_\mathrm{fin}.$$

To prove Theorem 4.6, it suffices to show that M_Q is the Cartan matrix of the finite simple Lie algebra $\mathfrak{g}_\mathrm{fin}$, i.e.

$$\big(\mathsf{s}_{\phi_Q(\alpha_i)}, \mathsf{s}_{\phi_Q(\alpha_j)}\big) = (\alpha_i, \alpha_j). \tag{6.2}$$

Indeed, (6.2) implies the following lemma, and Theorem 4.6 is its immediate consequence.

LEMMA 6.1. *Assume* (6.2). *Then the map* $\Delta_Q^+ \ni \beta \mapsto \mathsf{E}\big(V_Q(\beta)\big) \in \Delta_0 \subset \mathcal{W}_0$ *extends uniquely to an additive isomorphism*

$$\psi_Q \colon \mathsf{Q}_Q \xrightarrow{\sim} \mathcal{W}_0.$$

Moreover, it preserves the inner products of Q_Q *and* $\mathcal{W}_0$.

Proof. Since the Cartan matrix is a symmetric positive-definite matrix, $\{\mathsf{s}_{\phi_Q(\alpha_i)}\}_{i\in\mathrm{I}_\mathrm{fin}}$ is linearly independent. Hence we obtain that

$$\text{the rank of } \mathcal{W}_0 \text{ is at least the rank } r \text{ of } \mathfrak{g}_\mathrm{fin}. \tag{6.3}$$

On the other hand, Lemma 5.12 implies that $\psi_Q \colon \mathsf{Q}_Q \to \mathcal{W}_0$ is surjective. Hence ψ_Q is an isomorphism. Moreover, (6.2) shows that ψ_Q preserves the inner products of Q_Q and $\mathcal{W}_0$. The other assertions then easily follow. $\square$

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6.2 Calculation of the inner products

In this subsection, we give a type-by-type proof of (6.2).

Lᴇᴍᴍᴀ 6.2. *Suppose that* $\mathfrak{g}$ *is of affine ADE type. Let* $i, j \in I_0$.

(i) *For* $t \in \mathbb{Z}$, *we have*

$$\mathfrak{d}(V(\varpi_i), V(\varpi_j)_{(-q)^t}) = \delta(2 \leqslant |t| \leqslant h)\, \tilde{c}_{i,j}(|t| - 1),$$

where h *is the Coxeter number of* $\mathfrak{g}$ *and* $\tilde{c}_{i,j}(k)$ *is the integer defined in* (A.1) *in* Appendix A.

(ii) *If* $0 < t < 2h$, *then we have*

$$\Lambda^\infty(V(\varpi_i), V(\varpi_j)_{(-q)^t}) = \tilde{c}_{i,j}(t - 1) - \tilde{c}_{i,j}(t + 1)$$

and $\Lambda^\infty(V(\varpi_i), V(\varpi_j)) = -2\delta_{i,j}$.

Proof. (i) For $i, j \in I$, we write $d_{i,j}(z) := d_{V(\varpi_i), V(\varpi_j)}(z)$. Combining Proposition 3.9 with the denominator formula

$$d_{i,j}(z) = \prod_{k=1}^{h-1}(z - (-q)^{k+1})^{\tilde{c}_{i,j}(k)}$$

given in (A.2), we compute

$$\mathfrak{d}(V(\varpi_i), V(\varpi_j)_{(-q)^t}) = \delta(2 \leqslant t \leqslant h)\, \tilde{c}_{i,j}(t-1) + \delta(2 \leqslant -t \leqslant h)\, \tilde{c}_{i,j}(-t-1)$$

$$= \delta(2 \leqslant |t| \leqslant h)\, \tilde{c}_{i,j}(|t| - 1).$$

(ii) For $a \in \mathbb{Z}$, let $[a] := \prod_{n=0}^\infty (1 - (-q)^a \tilde{p}^n z)$. Combining the equation (A.13) in [AK97] with the denominator formula (A.2), we have

$$a_{i,j}((-q)^t z) = \prod_{1 \leqslant k \leqslant h-1} \frac{([h+k+1+t]^{\tilde{c}_{j,i^*}(k)})([h-k-1+t]^{\tilde{c}_{j,i^*}(k)})}{([k+1+t]^{\tilde{c}_{i,j}(k)})([2h-k-1+t]^{\tilde{c}_{i,j}(k)})}$$

$$= \prod_{1 \leqslant k \leqslant h-1} \frac{([h+k+1+t]^{-\tilde{c}_{i,j}(h+k)})([h-k-1+t]^{-\tilde{c}_{i,j}(h+k)})}{([k+1+t]^{\tilde{c}_{i,j}(k)})([2h-k-1+t]^{\tilde{c}_{i,j}(k)})}$$

$$= \prod_{1 \leqslant k \leqslant 2h-1} \frac{1}{([k+1+t]^{\tilde{c}_{i,j}(k)})([2h-k-1+t]^{\tilde{c}_{i,j}(k)})}$$

for any $t \in \mathbb{Z}$, up to a constant multiple. For the second equality, we used

$$\tilde{c}_{i,j}(h+k) = -\tilde{c}_{i,j}(h-k) = -\tilde{c}_{j^*,i}(k) \quad \text{for } 1 \leqslant k \leqslant h-1,$$

which comes from [Fuj22, Lemma 3.7 (4) and (5)]. Hence we have

$$\Lambda^\infty(V(\varpi_i), V(\varpi_j)_{(-q)^t}) = -\mathrm{Deg}^\infty(a_{i,j}((-q)^t z))$$

$$= \sum_{1 \leqslant k \leqslant 2h-1} \big(\tilde{c}_{i,j}(k)(\delta(k+1+t \equiv 0 \bmod 2h) + \delta(2h-k-1+t \equiv 0 \bmod 2h)) \big)$$

$$= \tilde{c}_{ij}(2h-t-1) + \tilde{c}_{ij}(t-1)$$

$$= -\tilde{c}_{ij}(t+1) + \tilde{c}_{ij}(t-1)$$

for $1 \leqslant t \leqslant 2h - 1$. If $t = 0$, then we have

$$\Lambda^\infty(V(\varpi_i), V(\varpi_j)) = 2\tilde{c}_{i,j}(2h-1) = -2\tilde{c}_{i,j}(1) = -2\delta_{i,j},$$

as desired. $\qquad\square$

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Type $A_n^{(1)}$. If $n = 1$, then it is obvious that M_Q is a Cartan matrix, so we may assume that $n \geqslant 2$. Recall the explicit description of $\sigma_Q(\mathfrak{g})$ for type $A_n^{(1)}$. Note that the Dynkin quiver corresponding to $\sigma_Q(\mathfrak{g})$ is given in (A.3). In this case, $h = n + 1$ and

$$\phi_Q(\alpha_i) = (1, (-q)^{2-2i}) \in \sigma_Q(\mathfrak{g}) \quad \text{for } i \in \mathrm{I}_{\mathrm{fin}} = \{1, \dots, n\}$$

by [KKK15, Lemma 3.2.3]. For example, if it is of type $A_4^{(1)}$, then elements $(i, (-q)^k)$ of $\sigma_Q(\mathfrak{g})$ with the values of ϕ_Q^{-1} can be drawn as follows.

$i \backslash k$	-6	-5	-4	-3	-2	-1	0
1	(0001)		(0010)		(0100)		(1000)
2		(0011)		(0110)		(1100)	
3			(0111)		(1110)		
4				(1111)			

Here $(a_1, a_2, a_3, a_4) := \sum_{k=1}^{4} a_k \alpha_k \in \Delta_Q^+$ is placed at the position $\phi_Q(a_1, a_2, a_3, a_4)$, and the underlined ones are simple roots. Using the formula given in Appendix A.1, one can compute that $\tilde{c}_{1,1}(2k) = 0$ and

$$\tilde{c}_{1,1}(2k+1) = (\tau^k \alpha_1, \varpi_1) = (\alpha_{k+1}, \varpi_1) = \delta_{k,0} \quad \text{for } 0 \leqslant k < n.$$

Lemma 6.2 implies that

$$\Lambda^\infty(V(\varpi_1), V(\varpi_1)_{(-q)^{2k}}) = \delta_{k,1} \quad \text{for } k \in \mathbb{Z} \text{ with } 1 \leqslant k \leqslant n - 1.$$

Therefore, for $i > j$ we have

$$\mathsf{m}_{i,j}^Q = -\Lambda^\infty(V(\varpi_1), V(\varpi_1)_{(-q)^{2(i-j)}}) = -\delta_{i-j,1},$$

which tells us that M_Q is a Cartan matrix of type A_n.

Type $B_n^{(1)}$. Recall the explicit description of $\sigma_Q(\mathfrak{g})$ for type $B_n^{(1)}$ ($n \geqslant 2$), which can be obtained from [KO19]. Note that the Dynkin diagram of $B_2^{(1)}$ is given in (2.1). In this case $\mathfrak{g}_{\mathrm{fin}}$ is of type A_{2n-1}, and for $i \in \mathrm{I}_{\mathrm{fin}} = \{1, \dots, 2n-1\}$ we have

$$\phi_Q(\alpha_i) = \begin{cases} (1, (-1)^{n+1} q_s^{2n+1-4i}) & \text{if } 1 \leqslant i \leqslant n-1, \\ (n, q^{-2n+2}) & \text{if } i = n, \\ (n, q^{-2n+3}) & \text{if } i = n+1, \\ (1, (-1)^{n+1} q_s^{-6n+4i-1}) & \text{if } n+2 \leqslant i \leqslant 2n-1. \end{cases}$$

For example, if it is of type $B_3^{(1)}$, then elements of $\sigma_Q(\mathfrak{g})$ with the values of ϕ_Q^{-1} can be drawn as follows.

$i \backslash k$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	
1				(00111)		(11110)		(01000)		(00001)		(10000)	$: (-1)^{i+3} q_s^k$
2		(00110)		(01110)		(01111)		(11111)		(11000)			
3	(00100)		(00010)		(01100)		(00011)		(11100)				$: q_s^k$

Here we set $(a_1, a_2, a_3, a_4, a_5) := \sum_{k=1}^{5} a_k \alpha_k \in \Delta_Q^+$, and the underlined ones are simple roots. Combining Propositions 3.11 and 3.9 with the denominator formula given in Appendix A, we

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compute that $\mathfrak{d}(V_Q(\alpha_i), \mathscr{D}^k V_Q(\alpha_j)) = 0$ for $i \neq j$ and $k \neq 0$ and that

$$\Lambda^\infty(V(\varpi_1), V(\varpi_1)_{q^k}) = \mathfrak{d}(V(\varpi_1), V(\varpi_1)_{q^k})$$
$$= \delta_{k,2} \quad \text{for } k = 1, 2, \ldots, 2n-4,$$
$$\Lambda^\infty(V(\varpi_n), V(\varpi_1)_{(-1)^{n+1}q_s^t}) = \mathfrak{d}(V(\varpi_n), V(\varpi_1)_{(-1)^{n+1}q_s^t})$$
$$= \delta_{t,2n+1} \quad \text{for } t = 2n-1, 2n+1, \ldots, 6n-7,$$
$$\Lambda^\infty(V(\varpi_n), V(\varpi_n)_q) = 1.$$

Therefore, for $i > j$ we obtain

$$\mathsf{m}_{i,j}^Q = -\delta_{i-j,1},$$

which tells us that M_Q is a Cartan matrix of type A_{2n-1}.

Type $C_n^{(1)}$. Recall the explicit description of $\sigma_Q(\mathfrak{g})$ for type $C_n^{(1)}$ ($n \geqslant 3$), which can be obtained from [KO19]. In this case $\mathfrak{g}_{\mathrm{fin}}$ is of type D_{n+1}, and for $1 \leqslant i \leqslant n+1$ we have

$$\phi_Q(\alpha_i) = \begin{cases} (1, (-q_s)^{2-2i}) & \text{if } 1 \leqslant i \leqslant n, \\ (n, (-q_s)^{-3n+1}) & \text{if } i = n+1. \end{cases}$$

For example, if it is of type $C_4^{(1)}$, then elements of $\sigma_Q(\mathfrak{g})$ with the values of ϕ_Q^{-1} can be drawn as follows.

$i\backslash k$	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	
1				$\binom{1}{1110}$		$\underline{\binom{0}{0001}}$		$\binom{0}{0010}$		$\binom{0}{0100}$		$\underline{\binom{0}{1000}}$	$: (-q_s)^k$
2			$\binom{1}{0110}$		$\binom{1}{1111}$		$\binom{0}{0011}$		$\binom{0}{0110}$		$\binom{0}{1100}$		
3		$\binom{1}{0010}$		$\binom{1}{0111}$		$\binom{1}{1121}$		$\binom{0}{0111}$		$\binom{0}{1110}$			
4	$\underline{\binom{1}{0000}}$		$\binom{1}{0011}$		$\binom{1}{0121}$		$\binom{1}{1221}$		$\binom{0}{1111}$				

Here we set $\binom{a_5}{a_1 a_2 a_3 a_4} := \sum_{k=1}^5 a_k \alpha_k \in \Delta_Q^+$, and the underlined ones are simple roots. Combining Propositions 3.11 and 3.9 with the denominator formula given in Appendix A, we compute that $\mathfrak{d}(V_Q(\alpha_i), \mathscr{D}^k V_Q(\alpha_j)) = 0$ for $i \neq j$ and $k \neq 0$ and that

$$\Lambda^\infty(V(\varpi_1), V(\varpi_1)_{(-q_s)^k}) = \mathfrak{d}(V(\varpi_1), V(\varpi_1)_{(-q_s)^k})$$
$$= \delta_{k,2} \quad \text{for } k = 2, 4, \ldots, 2n-2,$$
$$\Lambda^\infty(V(\varpi_n), V(\varpi_1)_{(-q_s)^t}) = \mathfrak{d}(V(\varpi_n), V(\varpi_1)_{(-q_s)^t})$$
$$= \delta_{t,n+3} \quad \text{for } t = n+1, n+3, \ldots, 3n-1.$$

Therefore, for $i > j$ we have

$$\mathsf{m}_{i,j}^Q = \begin{cases} -1 & \text{if } (i \leqslant n \text{ and } i-j = 1) \text{ or } (i,j) = (n+1, n-1), \\ 0 & \text{otherwise}, \end{cases}$$

which says that M_Q is a Cartan matrix of type D_{n+1}.

Type $D_n^{(1)}$. Recall the explicit description of $\sigma_Q(\mathfrak{g})$ for type $D_n^{(1)}$ ($n \geqslant 4$). Note that the Dynkin quiver corresponding to $\sigma(\mathfrak{g})_Q$ is given in (A.3). In this case $h = 2n-2$, and for $1 \leqslant i \leqslant n$ we

have

$$\phi_Q(\alpha_i) = \begin{cases} (1, (-q)^{-2(i-1)}) & \text{if } i \leqslant n-2, \\ (n-1, (-q)^{-3n+6}) & \text{if } (i=n-1 \text{ and } n \text{ is even}) \text{ or } (i=n \text{ and } n \text{ is odd}), \\ (n, (-q)^{-3n+6}) & \text{if } (i=n \text{ and } n \text{ is even}) \text{ or } (i=n-1 \text{ and } n \text{ is odd}) \end{cases}$$

by [KKK15, Lemma 3.2.3]. For example, if it is of type $D_5^{(1)}$, then elements $(i, (-q)^k)$ of $\sigma_Q(\mathfrak{g})$ with the values of ϕ_Q^{-1} can be drawn as follows.

$i \backslash k$	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
1				$\binom{1}{1111}$		$\binom{0}{0010}$		$\binom{0}{0100}$		$\binom{0}{1000}$
2			$\binom{1}{0111}$		$\binom{1}{1121}$		$\binom{0}{0110}$		$\binom{0}{1100}$	
3		$\binom{1}{0011}$		$\binom{1}{0121}$		$\binom{1}{1221}$		$\binom{0}{1110}$		
4	$\binom{1}{0000}$		$\binom{0}{0011}$		$\binom{1}{0110}$		$\binom{0}{1111}$			
5	$\binom{0}{0001}$		$\binom{1}{0010}$		$\binom{0}{0111}$		$\binom{1}{1110}$			

Here we set $\binom{a_5}{a_1 a_2 a_3 a_4} := \sum_{k=1}^{5} a_k \alpha_k \in \Delta_Q^+$, and the underlined ones are simple roots. Using the formula given in Appendix A.1, one can compute that for $1 \leqslant k < h$,

$$\tilde{c}_{1,1}(k) = \delta_{k,1} + \delta_{k,2n-3}, \quad \tilde{c}_{n,1}(k) = \tilde{c}_{n-1,1}(k) = \delta_{k,n-1},$$
$$\tilde{c}_{n,n}(k) = \tilde{c}_{n-1,n-1}(k) = \delta(k \equiv 1 \mod 4),$$
$$\tilde{c}_{n,n-1}(k) = \tilde{c}_{n-1,n}(k) = \delta(k \equiv 3 \mod 4).$$

Combining this with Lemma 6.2, we compute that

$$\Lambda^\infty(V(\varpi_1), V(\varpi_1)_{(-q)^k}) = \delta_{k,2} \quad \text{for } 2 \leqslant k \leqslant h-4,$$
$$\Lambda^\infty(V(\varpi_n), V(\varpi_1)_{(-q)^k}) = \delta_{k,n} \quad \text{for } n \leqslant k \leqslant 3n-6,$$
$$\Lambda^\infty(V(\varpi_n), V(\varpi_{n-1})) = 0.$$

Therefore, for $i > j$ we have

$$\mathsf{m}_{i,j}^Q = \begin{cases} -1 & \text{if } (i \leqslant n-1 \text{ and } i-j=1) \text{ or } (i,j) = (n, n-2), \\ 0 & \text{otherwise}, \end{cases}$$

which says that M_Q is a Cartan matrix of type D_n.

Type $A_{2n}^{(2)}$. Recall the explicit description of $\sigma_Q(\mathfrak{g})$ for type $A_{2n}^{(2)}$ $(n \geqslant 1)$, which can be obtained from [KKKO16]. In this case $\mathfrak{g}_{\text{fin}}$ is of type A_{2n}, and for $1 \leqslant i \leqslant 2n$ we have

$$\phi_Q(\alpha_i) = (1, (-q)^{2-2i}).$$

For example, if it is of type $A_4^{(2)}$, then elements of $\sigma_Q(\mathfrak{g})$ with the values of ϕ_Q^{-1} can be drawn as follows.

$i\backslash k$	-6	-5	-4	-3	-2	-1	0
1	(0001)		(0010)		(0100)		(1000)
2		(0011)		(0110)		(1100)	
2			(0111)		(1110)		
1				(1111)			

Here $(a_1, a_2, a_3, a_4) := \sum_{k=1}^4 a_k \alpha_k \in \Delta_Q^+$, and the underlined ones are simple roots. It follows from Propositions 3.11 and 3.9 and the denominator formula in Appendix A that $\mathfrak{d}(V_Q(\alpha_i), \mathscr{D}^k V_Q(\alpha_j)) = 0$ for $i \neq j$ and $k \neq 0$ and that

$$\Lambda^\infty(V(\varpi_1), V(\varpi_1)_{(-q)^k}) = \mathfrak{d}(V(\varpi_1), V(\varpi_1)_{(-q)^k})$$
$$= \delta_{k,2} \quad \text{for } k = 2, 4, \ldots, 4n-2.$$

Therefore, for $i > j$ we have

$$\mathsf{m}_{i,j}^Q = -\Lambda^\infty(V(\varpi_1), V(\varpi_1)_{(-q)^{2(i-j)}}) = -\delta_{i-j,1},$$

which tells us that M_Q is a Cartan matrix of type A_{2n}.

Type $A_{2n-1}^{(2)}$. Recall the explicit description of $\sigma_Q(\mathfrak{g})$ for type $A_{2n-1}^{(2)}$ $(n \geq 2)$, which can be obtained from [KKKO16]. In this case $\mathfrak{g}_{\text{fin}}$ is of type A_{2n-1}, and for $1 \leq i \leq 2n-1$ we have

$$\phi_Q(\alpha_i) = (1, (-q)^{2-2i}).$$

For example, if it is of type $A_5^{(2)}$, then elements of $\sigma_Q(\mathfrak{g})$ with the values of ϕ_Q^{-1} can be drawn as follows.

$i\backslash k$	-8	-7	-6	-5	-4	-3	-2	-1	0	
1	(00001)		(00010)		(00100)		(01000)		(10000)	$: (-q)^k$
2		(00011)		(00110)		(01100)		(11000)		
3			(00011)		(01110)		(11100)			
2					(01111)		(11110)			$: -(-q)^k$
1					(11111)					

Here $(a_1, a_2, a_3, a_4, a_5) := \sum_{k=1}^5 a_k \alpha_k \in \Delta_Q^+$, and the underlined ones are simple roots. Note that $V(\varpi_n)_a \simeq V(\varpi_n)_{-a}$. It follows from Propositions 3.11 and 3.9 and the denominator formula in Appendix A that $\mathfrak{d}(V_Q(\alpha_i), \mathscr{D}^k V_Q(\alpha_j)) = 0$ for $i \neq j$ and $k \neq 0$ and that

$$\Lambda^\infty(V(\varpi_1), V(\varpi_1)_{(-q)^k}) = \mathfrak{d}(V(\varpi_1), V(\varpi_1)_{(-q)^k})$$
$$= \delta_{k,2} \quad \text{for } k = 2, 4, \ldots, 4n-4.$$

Thus we obtain

$$\mathsf{m}_{i,j}^Q = -\Lambda^\infty(V(\varpi_1), V(\varpi_1)_{(-q)^{2(i-j)}}) = -\delta_{i-j,1}, \quad \text{for } i > j,$$

which implies that M_Q is a Cartan matrix of type A_{2n-1}.

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Type $D_{n+1}^{(2)}$. Recall the explicit description of $\sigma_Q(\mathfrak{g})$ for type $D_{n+1}^{(2)}$ ($n \geqslant 3$), which can be obtained from [KKKO16]. In this case $\mathfrak{g}_{\mathrm{fin}}$ is of type D_{n+1}, and for $1 \leqslant i \leqslant n+1$ we have

$$\phi_Q(\alpha_i) = \begin{cases} (1, (\sqrt{-1})^n(-q)^{-2(i-1)}) & \text{if } i \leqslant n-1, \\ (n, (-1)^i(-q)^{-3n+3}) & \text{if } i = n, n+1. \end{cases}$$

For example, if it is of type $D_5^{(2)}$, then elements of $\sigma_Q(\mathfrak{g})$ with the values of ϕ_Q^{-1} can be drawn as follows.

$i \backslash k$	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	
1				$\binom{1}{1111}$		$\underline{\binom{0}{0010}}$		$\binom{0}{0100}$		$\binom{0}{1000}$	$: (-q)^k$
2			$\binom{1}{0111}$		$\binom{1}{1121}$		$\binom{0}{0110}$		$\binom{0}{1100}$		$: -\sqrt{-1}(-q)^k$
3		$\binom{1}{0011}$		$\binom{1}{0121}$		$\binom{1}{1221}$		$\binom{0}{1110}$			$: -(-q)^k$
4	$\underline{\binom{1}{0000}}$		$\binom{0}{0011}$		$\binom{1}{0110}$		$\binom{0}{1111}$				$: (-q)^k$
4	$\underline{\binom{0}{0001}}$		$\binom{1}{0010}$		$\binom{0}{0111}$		$\binom{1}{1110}$				$: -(-q)^k$

Here we set $\binom{a_5}{a_1a_2a_3a_4} := \sum_{k=1}^{5} a_k\alpha_k \in \Delta_Q^+$, and the underlined ones are simple roots. Note that $V(\varpi_i)_a \simeq V(\varpi_i)_{-a}$ for $i < n$. It follows from Propositions 3.11 and 3.9 and the denominator formula in Appendix A that $\mathfrak{d}(V_Q(\alpha_i), \mathscr{D}^k V_Q(\alpha_j)) = 0$ for $i \neq j$ and $k \neq 0$ and that

$$\Lambda^\infty(V(\varpi_1), V(\varpi_1)_{(-q)^k}) = \mathfrak{d}(V(\varpi_1), V(\varpi_1)_{(-q)^k})$$
$$= \delta_{k,2} \quad \text{for } k = 2, 4, \ldots, 2n-4,$$
$$\Lambda^\infty(V(\varpi_n), V(\varpi_1)_{\pm\sqrt{-1}^n(-q)^k}) = \mathfrak{d}(V(\varpi_n), V(\varpi_1)_{\pm\sqrt{-1}^n(-q)^k})$$
$$= \delta_{k,n+1} \quad \text{for } k = n+1, n+3, \ldots, 3n-3,$$
$$\Lambda^\infty(V(\varpi_n), V(\varpi_n)_{-1}) = 0,$$

which give the values of $\mathsf{m}_{i,j}^Q$. Thus, one can check that the matrix M_Q is a Cartan matrix of type D_{n+1}.

Type $E_6^{(1)}$. Recall the explicit description of $\sigma_Q(\mathfrak{g})$ for type $E_6^{(1)}$. The Dynkin quiver corresponding to $\sigma(\mathfrak{g})_Q$ is given in (A.3). In this case, $h = 12$ and elements $(i, (-q)^k)$ of $\sigma_Q(\mathfrak{g})$ with the values of ϕ_Q^{-1} can be drawn as follows.

$i \backslash k$	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
1	$\underline{\binom{000}{001}}$		$\underline{\binom{000}{010}}$		$\underline{\binom{000}{100}}$		$\binom{011}{111}$		$\binom{101}{110}$		$\binom{010}{100}$		$\binom{001}{000}$		$\binom{100}{000}$
3		$\binom{000}{011}$		$\binom{000}{110}$		$\binom{011}{211}$		$\binom{112}{221}$		$\binom{111}{210}$		$\binom{011}{100}$		$\binom{101}{000}$	
4			$\binom{000}{111}$		$\binom{011}{221}$		$\binom{112}{321}$		$\binom{122}{321}$		$\binom{112}{210}$		$\binom{111}{100}$		
2				$\binom{010}{111}$		$\binom{001}{110}$		$\binom{111}{211}$		$\binom{011}{110}$		$\binom{101}{100}$		$\underline{\binom{010}{000}}$	
5				$\binom{001}{111}$		$\binom{111}{221}$		$\binom{011}{210}$		$\binom{112}{211}$		$\binom{111}{110}$			
6					$\binom{101}{111}$		$\binom{010}{110}$		$\binom{001}{100}$		$\binom{111}{111}$				

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Here we set $\binom{a_1a_2a_3}{a_4a_5a_6} := \sum_{i=1}^{6} a_i\alpha_i \in \Delta_Q^+$, and the underlined ones are simple roots. Using the formula given in Appendix A.1, one can compute that for $1 \leqslant k < h$,

$$\tilde{c}_{1,1}(k) = \delta_{k,1} + \delta_{k,7}, \quad \tilde{c}_{1,2}(k) = \delta_{k,4} + \delta_{k,8}.$$

By Lemma 6.2, we compute

$$\Lambda^\infty(V(\varpi_1), V(\varpi_1)_{(-q)^k}) = \delta_{k,2} + \delta_{k,8} \quad \text{for } k = 2,4,8,10,12,14,$$
$$\Lambda^\infty(V(\varpi_1), V(\varpi_2)_{(-q)^k}) = \delta_{k,9} \quad \text{for } k = -1,1,9,11,13,$$

which give the values of $\mathsf{m}_{i,j}^Q$. Therefore, one can check that the matrix M_Q is a Cartan matrix of type E_6.

Type $E_7^{(1)}$. Recall the explicit description of $\sigma_Q(\mathfrak{g})$ for type $E_7^{(1)}$. The Dynkin quiver corresponding to $\sigma(\mathfrak{g})_Q$ is given in (A.3). In this case, $h = 18$ and elements $(i, (-q)^k)$ of $\sigma_Q(\mathfrak{g})$ with the values of ϕ_Q^{-1} can be drawn as follows.

$i\backslash k$	−21	−20	−19	−18	−17	−16	−15	−14	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0
1						$\binom{1011}{111}$		$\binom{0101}{110}$		$\binom{0011}{100}$		$\binom{1112}{111}$		$\binom{0111}{110}$		$\binom{1011}{100}$		$\binom{0101}{000}$		$\underline{\binom{0010}{000}}$		$\underline{\binom{1000}{000}}$
3					$\binom{0011}{111}$		$\binom{1112}{221}$		$\binom{0112}{210}$		$\binom{1123}{211}$		$\binom{1223}{221}$		$\binom{1122}{210}$		$\binom{1112}{100}$		$\binom{0111}{000}$		$\binom{1010}{000}$	
4				$\binom{0001}{111}$		$\binom{0112}{221}$		$\binom{1123}{321}$		$\binom{1224}{321}$		$\binom{1234}{321}$		$\binom{2234}{321}$		$\binom{1223}{210}$		$\binom{1122}{100}$		$\binom{1111}{000}$		
2					$\binom{0101}{111}$		$\binom{0011}{110}$		$\binom{1112}{211}$		$\binom{0112}{110}$		$\binom{1122}{211}$		$\binom{1112}{110}$		$\binom{0111}{100}$		$\binom{1011}{000}$		$\underline{\binom{0100}{000}}$	
5			$\binom{0000}{111}$		$\binom{0001}{110}$		$\binom{0112}{211}$		$\binom{1123}{221}$		$\binom{1223}{321}$		$\binom{1123}{210}$		$\binom{1223}{221}$		$\binom{1122}{110}$		$\binom{1111}{100}$			
6		$\binom{0000}{011}$		$\binom{0000}{110}$		$\binom{0001}{100}$		$\binom{0112}{111}$		$\binom{1122}{221}$		$\binom{1112}{210}$		$\binom{0112}{100}$		$\binom{1122}{111}$		$\binom{1111}{110}$				
7	$\underline{\binom{0000}{001}}$		$\underline{\binom{0000}{010}}$		$\underline{\binom{0000}{100}}$		$\underline{\binom{0001}{000}}$		$\binom{0111}{111}$		$\binom{1011}{110}$		$\binom{0101}{100}$		$\binom{0011}{000}$		$\binom{1111}{111}$					

Here we set $\binom{a_1a_2a_3a_4}{a_5a_6a_7} := \sum_{i=1}^{7} a_i\alpha_i \in \Delta_Q^+$, and the underlined ones are simple roots. Using the formula given in Appendix A.1, one can compute that for $1 \leqslant k < h$,

$$\tilde{c}_{1,1}(k) = \delta_{k,1} + \delta_{k,7} + \delta_{k,11} + \delta_{k,17}, \quad \tilde{c}_{1,2}(k) = \delta_{k,4} + \delta_{k,8} + \delta_{k,10} + \delta_{k,14},$$
$$\tilde{c}_{7,1}(k) = \delta_{k,6} + \delta_{k,12}, \quad \tilde{c}_{7,2}(k) = \delta_{k,5} + \delta_{k,9} + \delta_{k,13}, \quad \tilde{c}_{7,7}(k) = \delta_{k,1} + \delta_{k,9} + \delta_{k,17}.$$

By Lemma 6.2, we compute

$$\Lambda^\infty(V(\varpi_1), V(\varpi_1)_{(-q)^2}) = 1, \quad \Lambda^\infty(V(\varpi_1), V(\varpi_2)_{(-q)}) = \Lambda^\infty(V(\varpi_2), V(\varpi_1)_{(-q)}) = 0,$$
$$\Lambda^\infty(V(\varpi_7), V(\varpi_1)_{(-q)^k}) = \delta_{k,13} \quad \text{for } k = 13,15,17,19,21,$$
$$\Lambda^\infty(V(\varpi_7), V(\varpi_2)_{(-q)^k}) = \delta_{k,14} \quad \text{for } k = 14,16,18,20,$$
$$\Lambda^\infty(V(\varpi_7), V(\varpi_7)_{(-q)^k}) = \delta_{k,2} \quad \text{for } k = 2,4,6,$$

which give the values of $\mathsf{m}_{i,j}^Q$. Therefore, one can check that the matrix M_Q is a Cartan matrix of type E_7.

Type $E_8^{(1)}$. Recall the explicit description of $\sigma_Q(\mathfrak{g})$ for type $E_8^{(1)}$. The Dynkin quiver corresponding to $\sigma(\mathfrak{g})_Q$ is given in (A.3). In this case, $h = 30$ and elements $(i, (-q)^k)$ of $\sigma_Q(\mathfrak{g})$ with

the values of ϕ_Q^{-1} can be drawn as follows.

Here we set $\begin{pmatrix} a_1 a_2 \\ a_3 a_4 \\ a_5 a_6 \\ a_7 a_8 \end{pmatrix} := \sum_{i=1}^{8} a_i \alpha_i \in \Delta_Q^+$, and the underlined ones are simple roots. Using the formula given in Appendix A.1, one can compute that for $1 \leqslant k < h$,

$$\tilde{c}_{1,1}(k) = \delta(k = 1, 7, 11, 13, 17, 19, 23, 29),$$
$$\tilde{c}_{1,2}(k) = \delta(k = 4, 8, 10, 12, 14, 16, 18, 20, 22, 26),$$
$$\tilde{c}_{8,1}(k) = \delta(k = 7, 13, 17, 23),$$
$$\tilde{c}_{8,2}(k) = \delta(k = 6, 10, 14, 16, 20, 24),$$
$$\tilde{c}_{8,8}(k) = \delta(k = 1, 11, 19, 29).$$

By Lemma 6.2, we compute

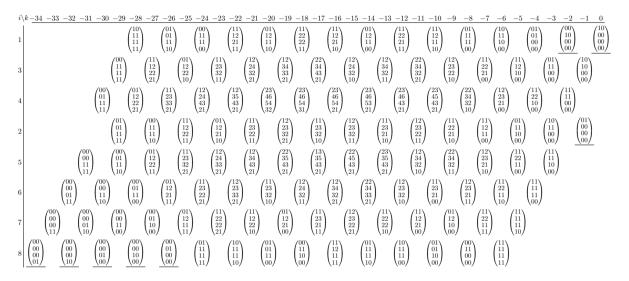$$\Lambda^\infty(V(\varpi_1), V(\varpi_1)_{(-q)^2}) = 1, \quad \Lambda^\infty(V(\varpi_1), V(\varpi_2)_{(-q)}) = \Lambda^\infty(V(\varpi_2), V(\varpi_1)_{(-q)}) = 0,$$
$$\Lambda^\infty(V(\varpi_8), V(\varpi_1)_{(-q)^k}) = \delta_{k,24} \quad \text{for } k = 24, 26, 28, 30, 32, 34,$$
$$\Lambda^\infty(V(\varpi_8), V(\varpi_2)_{(-q)^k}) = \delta_{k,25} \quad \text{for } k = 25, 27, 29, 31, 33,$$
$$\Lambda^\infty(V(\varpi_8), V(\varpi_8)_{(-q)^k}) = \delta_{k,2} \quad \text{for } k = 2, 4, 6, 8,$$

which give the values of $\mathsf{m}_{i,j}^Q$. Therefore, one can check that the matrix M_Q is a Cartan matrix of type E_8.

Type $F_4^{(1)}$. Recall the explicit description of $\sigma_Q(\mathfrak{g})$ for type $F_4^{(1)}$, which can be obtained from [OS19a]. In this case $\mathfrak{g}_{\text{fin}}$ is of type E_6, and elements of $\sigma_Q(\mathfrak{g})$ with the values of ϕ_Q^{-1} are drawn as follows.

Here we set $\left(\begin{smallmatrix} a_1 a_2 a_3 \\ a_4 a_5 a_6 \end{smallmatrix}\right) := \sum_{i=1}^{6} a_i \alpha_i \in \Delta_Q^+$, and the underlined ones are simple roots. It follows from Propositions 3.11 and 3.9 and the denominator formula in Appendix A that $\mathfrak{d}(V_Q(\alpha_i), \mathscr{D}^k V_Q(\alpha_j)) = 0$ for $i \neq j$ and $k \neq 0$ and that

$$\Lambda^\infty(V(\varpi_1), V(\varpi_1)_{q_s^k}) = \mathfrak{d}(V(\varpi_1), V(\varpi_1)_{q_s^k}) = \delta_{k,4} \quad \text{for } k = 2, 4,$$

$$\Lambda^\infty(V(\varpi_3), V(\varpi_1)_{q_s^k}) = \mathfrak{d}(V(\varpi_3), V(\varpi_1)_{q_s^k}) = \delta_{k,15} \quad \text{for } k = 15, 17, 19,$$

$$\Lambda^\infty(V(\varpi_4), V(\varpi_1)_{-q_s^k}) = \mathfrak{d}(V(\varpi_4), V(\varpi_1)_{-q_s^k})$$
$$= \delta_{k,14} \quad \text{for } k = -2, 0, 2, 12, 14, 16,$$

$$\Lambda^\infty(V(\varpi_3), V(\varpi_4)_{-q_s^k}) = \mathfrak{d}(V(\varpi_3), V(\varpi_4)_{-q_s^k}) = 1 \quad \text{for } k = 3, 17,$$

$$\Lambda^\infty(V(\varpi_4), V(\varpi_4)_{q_s^{14}}) = \mathfrak{d}(V(\varpi_4), V(\varpi_4)_{q_s^{14}}) = 0,$$

which give the values of $\mathsf{m}_{i,j}^Q$. Thus, one can check that the matrix M_Q is a Cartan matrix of type E_6.

Type $G_2^{(1)}$. Recall the explicit description of $\sigma_Q(\mathfrak{g})$ for type $G_2^{(1)}$, which can be obtained from [OS19a]. In this case $\mathfrak{g}_{\mathrm{fin}}$ is of type D_4, and elements of $\sigma_Q(\mathfrak{g})$ with the values of ϕ_Q^{-1} can be drawn as follows.

$i\backslash k$	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	
1	$\left(\begin{smallmatrix}1\\000\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}1\\011\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}1\\111\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}1\\121\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}0\\111\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}0\\110\end{smallmatrix}\right)$		$: (-q_t)^k$
2		$\left(\begin{smallmatrix}1\\010\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}0\\001\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}1\\110\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}0\\011\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}0\\100\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}0\\010\end{smallmatrix}\right)$	

Here we set $\left(\begin{smallmatrix} a_4 \\ a_1 a_2 a_3 \end{smallmatrix}\right) := \sum_{k=1}^{4} a_k \alpha_k \in \Delta_Q^+$, and the underlined ones are simple roots. It follows from Propositions 3.11 and 3.9 and the denominator formula in Appendix A that $\mathfrak{d}(V_Q(\alpha_i), \mathscr{D}^k V_Q(\alpha_j)) = 0$ for $i \neq j$ and $k \neq 0$ and that

$$\Lambda^\infty(V(\varpi_1), V(\varpi_2)_{(-q_t)^k}) = \mathfrak{d}(V(\varpi_1), V(\varpi_2)_{(-q_t)^k}) = \delta_{k,11} \quad \text{for } k = 3, 9, 11,$$

$$\Lambda^\infty(V(\varpi_2), V(\varpi_2)_{(-q_t)^k}) = \mathfrak{d}(V(\varpi_2), V(\varpi_2)_{(-q_t)^k}) = \delta_{k,2} + \delta_{k,8} \quad \text{for } k = 2, 6, 8,$$

which give the values of $\mathsf{m}_{i,j}^Q$. Thus, one can check that the matrix M_Q is a Cartan matrix of type D_4.

Type $E_6^{(2)}$. Recall the explicit description of $\sigma_Q(\mathfrak{g})$ for type $E_6^{(2)}$, which can be obtained from [OS19a]. In this case $\mathfrak{g}_{\mathrm{fin}}$ is of type E_6, and elements of $\sigma_Q(\mathfrak{g})$ with the values of ϕ_Q^{-1} can be drawn as follows.

$i\backslash k$	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	
1	$\left(\begin{smallmatrix}000\\001\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}000\\010\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}000\\100\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}011\\111\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}101\\110\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}010\\100\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}001\\000\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}100\\000\end{smallmatrix}\right)$	$: (-q)^k$
2		$\left(\begin{smallmatrix}000\\011\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}000\\110\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}011\\211\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}112\\221\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}111\\210\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}011\\100\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}101\\000\end{smallmatrix}\right)$		
3			$\left(\begin{smallmatrix}000\\111\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}011\\221\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}112\\321\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}122\\321\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}112\\210\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}111\\100\end{smallmatrix}\right)$			$: \sqrt{-1}(-q)^k$
4				$\left(\begin{smallmatrix}010\\111\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}001\\110\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}111\\211\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}011\\110\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}101\\100\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}010\\000\end{smallmatrix}\right)$		
2				$\left(\begin{smallmatrix}001\\111\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}111\\221\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}011\\210\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}112\\211\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}111\\110\end{smallmatrix}\right)$				$: -(-q)^k$
1					$\left(\begin{smallmatrix}101\\111\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}010\\110\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}001\\100\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}111\\111\end{smallmatrix}\right)$					

Here we set $\left(\begin{smallmatrix} a_1 a_2 a_3 \\ a_4 a_5 a_6 \end{smallmatrix}\right) := \sum_{i=1}^{6} a_i \alpha_i \in \Delta_Q^+$, and the underlined ones are simple roots. Note that $V(\varpi_i)_a \simeq V(\varpi_i)_{-a}$ for $i = 3, 4$. It follows from Propositions 3.11 and 3.9 and the denominator formula in Appendix A that $\mathfrak{d}(V_Q(\alpha_i), \mathscr{D}^k V_Q(\alpha_j)) = 0$ for $i \neq j$ and $k \neq 0$ and that

$$\Lambda^\infty(V(\varpi_1), V(\varpi_1)_{q^k}) = \mathfrak{d}(V(\varpi_1), V(\varpi_1)_{q^k}) = \delta_{k,2} + \delta_{k,8} \quad \text{for } k = 2, 4, 8, 10, 12, 14,$$

$$\Lambda^\infty(V(\varpi_1), V(\varpi_4)_{\sqrt{-1}q^k}) = \mathfrak{d}(V(\varpi_1), V(\varpi_4)_{\sqrt{-1}q^k})$$
$$= \delta_{k,9} \quad \text{for } k = -1, 0, 1, 9, 11, 13,$$

which give the values of $\mathsf{m}_{i,j}^Q$. Thus, one can check that the matrix M_Q is a Cartan matrix of type E_6.

Type $D_4^{(3)}$. Recall the explicit description of $\sigma_Q(\mathfrak{g})$ for type $D_4^{(3)}$, which can be obtained from [OS19a]. In this case $\mathfrak{g}_{\text{fin}}$ is of type D_4, and elements of $\sigma_Q(\mathfrak{g})$ with the values of ϕ_Q^{-1} can be drawn as follows.

$i \backslash k$	-6	-5	-4	-3	-2	-1	0	
1			$\left(\begin{smallmatrix}1\\111\end{smallmatrix}\right)$		$\underline{\left(\begin{smallmatrix}0\\010\end{smallmatrix}\right)}$		$\underline{\left(\begin{smallmatrix}0\\100\end{smallmatrix}\right)}$	$: q^k$
2		$\left(\begin{smallmatrix}1\\011\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}1\\121\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}0\\110\end{smallmatrix}\right)$		$: -q^k$
1	$\underline{\left(\begin{smallmatrix}0\\001\end{smallmatrix}\right)}$		$\left(\begin{smallmatrix}1\\010\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}0\\111\end{smallmatrix}\right)$			$: \omega q^k$
1	$\underline{\left(\begin{smallmatrix}1\\000\end{smallmatrix}\right)}$		$\left(\begin{smallmatrix}0\\011\end{smallmatrix}\right)$		$\left(\begin{smallmatrix}1\\110\end{smallmatrix}\right)$			$: \omega^2 q^k$

Here we set $\left(\begin{smallmatrix} a_4 \\ a_1 a_2 a_3 \end{smallmatrix}\right) := \sum_{k=1}^{4} a_k \alpha_k \in \Delta_Q^+$, and the underlined ones are simple roots. Note that $V(\varpi_2)_a \simeq V(\varpi_2)_{\omega^t a}$ for $t = 1, 2$. It follows from Propositions 3.11 and 3.9 and the denominator formula in Appendix A that $\mathfrak{d}(V_Q(\alpha_i), \mathscr{D}^k V_Q(\alpha_j)) = 0$ for $i \neq j$ and $k \neq 0$ and that

$$\Lambda^\infty(V(\varpi_1), V(\varpi_1)_{\omega^t q^k}) = \begin{cases} 1 & \text{if } (t,k) = (0,2), (1,4), (2,4), \\ 0 & \text{if } (t,k) = (1,0), (2,0), (1,6), (2,6), \end{cases}$$

which give the values of $\mathsf{m}_{i,j}^Q$. Thus, one can check that the matrix M_Q is a Cartan matrix of type D_4.

Appendix A. Denominator formulas

The denominator formulas were studied and computed in [AK97, DO94, Fuj22, KKK15, Oh15, OS19a]. In this appendix we present the denominator formulas for all types.

Let $q_s, q_t \in \mathbf{k}^\times$ be such that $q = q_s^2 = q_t^3$, and let $\omega \in \mathbf{k}$ be such that $\omega^2 + \omega + 1 = 0$. For $i, j \in I$, we set

$$d_{i,j}(z) := d_{V(\varpi_i), V(\varpi_j)}(z).$$

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A.1 Simply laced affine ADE types

Suppose that the Cartan matrix $\mathsf{C} = (c_{i,j})_{i,j \in I_0}$ is of type A_n, D_n or E_k ($k = 6, 7, 8$). The quantum Cartan matrix $\mathsf{C}(z) = (c_{i,j}(z))_{i,j \in I_0}$ is defined by

$$c_{i,j}(z) := \delta(i = j)(z + z^{-1}) + \delta(i \neq j)c_{i,j}.$$

We denote by $\tilde{\mathsf{C}}(z) = (\tilde{c}_{i,j}(z))_{i,j \in I_0}$ the inverse of $\mathsf{C}(z)$, and write

$$\tilde{c}_{i,j}(z) = \sum_{k \in \mathbb{Z}_{\geqslant 0}} \tilde{c}_{i,j}(k) z^k \quad \text{for } i, j \in I_0. \tag{A.1}$$

Then the following beautiful formula is given in [Fuj22, Theorem 2.10]:

$$d_{i,j}(z) = \prod_{k=1}^{h-1} (z - (-q)^{k+1})^{\tilde{c}_{i,j}(k)}, \tag{A.2}$$

where h is the Coxeter number. Note that the dual Coxeter number is equal to the Coxeter number in this case.

Let $\mathfrak{g}_0$ be a simple Lie algebra of type ADE with index set I_0, and let Q be a Dynkin quiver of $\mathfrak{g}$. Let $\xi \colon I_0 \to \mathbb{Z}$ be a height function such that $\xi_j = \xi_i - 1$ for $i \to j$ in Q. Choose a total order $>$ on I such that $i > j$ for $\xi_i > \xi_j$ and write $I_0 = \{i_1 > i_2 > \cdots > i_n\}$. We set $\tau := s_{i_1} \cdots s_{i_n}$, which is a Coxeter element. For $i \in I_0$ we set $\gamma_i := \sum_{j \in B(i)} \alpha_j$, where $B(i)$ is the subset of I_0 consisting of all elements j such that there is a path from j to i in Q. Then we have the following.

PROPOSITION A.1 [HL15, Proposition 2.1]. *For $i, j \in I$ and $k \in \mathbb{Z}_{>0}$, we have*

$$\tilde{c}_{i,j}(k) = \begin{cases} (\tau^{(k+\xi_i-\xi_j-1)/2}(\gamma_i), \varpi_j) & \text{if } k + \xi_i - \xi_j - 1 \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

In this paper, we make the following choice of Dynkin quivers:

$$\tag{A.3}$$

In this case we have the following data, which allow us to compute $\tilde{c}_{i,j}(k)$ explicitly.

(a) (*Type A_n*) $\tau = s_1 s_2 \cdots s_n$, $\xi_i = 1 - i$ and $\gamma_i = \sum_{j=1}^{i} \alpha_j$.

(b) (*Type D_n*) $\tau = s_1 s_2 \cdots s_{n-1} s_n$ and

$$\xi_i = \begin{cases} 1 - i & \text{if } i < n - 1, \\ -n + 2 & \text{if } i = n - 1, n, \end{cases} \qquad \gamma_i = \begin{cases} \displaystyle\sum_{j=1}^{i} \alpha_j & \text{if } i < n, \\ \displaystyle\sum_{j=1}^{n-2} \alpha_j + \alpha_n & \text{if } i = n. \end{cases}$$

(c) (*Type E_n, $n = 6, 7, 8$*) $\tau = s_1 s_2 \ldots s_n$, $\xi_1 = 0$, $\xi_2 = -1$ and $\xi_k = 2 - k$ for $k = 3, 4, \ldots, n$, and $\gamma_1 = \alpha_1$, $\gamma_2 = \alpha_2$, $\gamma_3 = \alpha_1 + \alpha_3$ and $\gamma_t = \sum_{k=1}^{t} \alpha_k$ for $t = 4, \ldots, n$.

Indeed, in the figures of §6.2, the root γ_i is the rightmost one in the row labeled by i, and τ corresponds to horizontal translation by -2. Hence one can read such values of $\tilde{c}_{i,j}(k)$ easily from the figures.

A.2 Other classical affine types

The denominator formulas for other classical affine types can be found in [AK97, Appendix C.4] for type $C_n^{(1)}$ and in [Oh15, Appendix] for types $B_n^{(1)}$, $D_{n+1}^{(2)}$ and $A_N^{(2)}$ ($N = 2n, 2n - 1$).

(i) Type $B_n^{(1)}$ ($n \geqslant 2$):
   (a) $d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} \left(z - (-q)^{|k-l|+2s}\right)\left(z + (-q)^{2n-k-l-1+2s}\right)$ for $1 \leqslant k, l \leqslant n - 1$;
   (b) $d_{k,n}(z) = \prod_{s=1}^{k} \left(z - (-1)^{n+k} q_s^{2n-2k-1+4s}\right)$ for $1 \leqslant k \leqslant n - 1$;
   (c) $d_{n,n}(z) = \prod_{s=1}^{n} \left(z - (q_s)^{4s-2}\right)$.

(ii) Type $C_n^{(1)}$ ($n \geqslant 2$):
   (a) $d_{k,l}(z) = \prod_{s=1}^{\min(k,l,n-k,n-l)} \left(z - (-q_s)^{|k-l|+2s}\right) \prod_{s=1}^{\min(k,l)} \left(z - (-q_s)^{2n+2-k-l+2s}\right)$ for $1 \leqslant k, l \leqslant n$.

(iii) Type $A_{2n-1}^{(2)}$ ($n \geqslant 2$):
   (a) $d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} \left(z - (-q)^{|k-l|+2s}\right)\left(z + (-q)^{2n-k-l+2s}\right)$ for $1 \leqslant k, l \leqslant n$.

(iv) Type $A_{2n}^{(2)}$ ($n \geqslant 1$):
   (a) $d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} \left(z - (-q)^{|k-l|+2s}\right)\left(z - (-q)^{2n+1-k-l+2s}\right)$ for $1 \leqslant k, l \leqslant n$.

(v) Type $D_{n+1}^{(2)}$ ($n \geqslant 3$):
   (a) $d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} \left(z^2 - (-q^2)^{|k-l|+2s}\right)\left(z^2 - (-q^2)^{2n-k-l+2s}\right)$ for $1 \leqslant k, l \leqslant n - 1$;
   (b) $d_{k,n}(z) = \prod_{s=1}^{k} \left(z^2 + (-q^2)^{n-k+2s}\right)$ for $1 \leqslant k \leqslant n - 1$;
   (c) $d_{n,n}(z) = \prod_{s=1}^{n} \left(z + (-q^2)^s\right)$ for $k = l = n$.

A.3 Other exceptional affine types

The denominator formulas for exceptional affine type can be found in [OS19a, §§4 and 7].

(i) Type $G_2^{(1)}$:
   (a) $d_{1,1}(z) = (z - q_t^6)(z - q_t^8)(z - q_t^{10})(z - q_t^{12})$;
   (b) $d_{1,2}(z) = (z + q_t^7)(z + q_t^{11})$;
   (c) $d_{2,2}(z) = (z - q_t^2)(z - q_t^8)(z - q_t^{12})$.

(ii) Type $F_4^{(1)}$:
   (a) $d_{1,1}(z) = (z - q_s^4)(z - q_s^{10})(z - q_s^{12})(z - q_s^{18})$;
   (b) $d_{1,2}(z) = (z + q_s^6)(z + q_s^8)(z + q_s^{10})(z + q_s^{12})(z + q_s^{14})(z + q_s^{16})$;
   (c) $d_{1,3}(z) = (z - q_s^7)(z - q_s^9)(z - q_s^{13})(z - q_s^{15})$;
   (d) $d_{1,4}(z) = (z + q_s^8)(z + q_s^{14})$;
   (e) $d_{2,2}(z) = (z - q_s^4)(z - q_s^6)(z - q_s^8)^2(z - q_s^{10})^2(z - q_s^{12})^2(z - q_s^{14})^2(z - q_s^{16})(z - q_s^{18})$;
   (f) $d_{2,3}(z) = (z + q_s^5)(z + q_s^7)(z + q_s^9)(z + q_s^{11})^2(z + q_s^{13})(z + q_s^{15})(z + q_s^{17})$;
   (g) $d_{2,4}(z) = (z - q_s^6)(z - q_s^{10})(z - q_s^{12})(z - q_s^{16})$;
   (h) $d_{3,3}(z) = (z - q_s^2)(z - q_s^6)(z - q_s^8)(z - q_s^{10})(z - q_s^{12})^2(z - q_s^{16})(z - q_s^{18})$;
   (i) $d_{3,4}(z) = (z + q_s^3)(z + q_s^7)(z + q_s^{11})(z + q_s^{13})(z + q_s^{17})$;
   (j) $d_{4,4}(z) = (z - q_s^2)(z - q_s^8)(z - q_s^{12})(z - q_s^{18})$.

(iii) Type $D_4^{(3)}$:

    (a) $d_{1,1}(z) = (z - q^2)(z - q^6)(z - \omega q^4)(z - \omega^2 q^4)$;

    (b) $d_{1,2}(z) = (z^3 + q^9)(z^3 + q^{15})$;

    (c) $d_{2,2}(z) = (z^3 - q^6)(z^3 - q^{12})^2(z^3 - q^{18})$.

(iv) Type $E_6^{(2)}$:

    (a) $d_{1,1}(z) = (z - q^2)(z + q^6)(z - q^8)(z + q^{12})$;

    (b) $d_{1,2}(z) = (z + q^3)(z - q^5)(z - q^7)(z + q^7)(z + q^9)(z - q^{11})$;

    (c) $d_{1,3}(z) = (z^2 + q^8)(z^2 + q^{12})(z^2 + q^{16})(z^2 + q^{20})$;

    (d) $d_{1,4}(z) = (z^2 + q^{10})(z^2 + q^{18})$;

    (e) $d_{2,2}(z) = (z - q^2)(z - q^4)(z - q^6)(z - q^8)^2(z - q^{10})(z + q^4)(z + q^6)^2(z + q^8)$
       $(z + q^{10})(z + q^{12})$;

    (f) $d_{2,3}(z) = (z^2 + q^6)(z^2 + q^{10})^2(z^2 + q^{14})^2(z^2 + q^{18})^2(z^2 + q^{22})$;

    (g) $d_{2,4}(z) = (z^2 + q^8)(z^2 + q^{12})(z^2 + q^{16})(z^2 + q^{20})$;

    (h) $d_{3,3}(z) = (z^2 - q^4)(z^2 - q^8)^2(z^2 - q^{12})^3(z^2 - q^{16})^3(z^2 - q^{20})^2(z^2 - q^{24})$;

    (i) $d_{3,4}(z) = (z^2 - q^6)(z^2 - q^{10})(z^2 - q^{14})^2(z^2 - q^{18})^2(z^2 - q^{22})$;

    (j) $d_{4,4}(z) = (z^2 - q^4)(z^2 - q^{12})(z^2 - q^{16})(z^2 - q^{24})$.

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Simply laced root systems arising from quantum affine algebras

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