

LOCAL MODELS IN THE RAMIFIED CASE. III UNITARY GROUPS

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Abstract We continue our study of the reduction of PEL Shimura varieties with parahoric level structure at primes p at which the group defining the Shimura variety ramifies. We describe ‘good’ p -adic integral models of these Shimura varieties and study their étale local structure. In the present paper we mainly concentrate on the case of unitary groups for a ramified quadratic extension. Some of our results are applications of the theory of twisted affine flag varieties that we developed in a previous paper.

Keywords: Shimura variety; unitary group; parahoric subgroup; local model

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Contents

1. Unitary Shimura varieties and moduli problems	515
1.1. Unitary Shimura varieties	515
1.2. Parahoric subgroups of the unitary similitude group	517
1.3. Reduction to level subgroups that are lattice chain stabilizers	520
1.4. Unitary moduli problems	523
1.5. Local models for GU	524
2. Affine Weyl groups and affine flag varieties: the μ -admissible set	526
2.1. Affine Weyl groups	526
2.2. The μ -admissible set	528
2.3. Calculation of the affine root system	529
2.4. The case of a unitary group	532
3. Affine flag varieties	534
3.1. Affine flag varieties and the coherence conjecture	534
3.2. Unitary affine flag varieties	536

- 3.3. Embedding of the special fibres of local models in partial affine flag varieties 537
- 3.4. Lifting of points in μ -strata to the generic fibre 538
- 4. The structure of local models 541
 - 4.1. Consequences of the coherence conjecture 541
 - 4.2. Vertex-wise admissibility 541
- 5. Special parahorics 542
 - 5.1. An affine chart 542
 - 5.2. The odd case 543
 - 5.3. The even case 544
 - 5.4. Symmetric pairs 545
 - 5.5. Generic smoothness 546
 - 5.6. Normality of some nilpotent orbits 548
- 6. The local models of Picard surfaces 549
- 7. Functor description: the spin condition 552
 - 7.1. Generalities on symmetric forms of even dimension 552
 - 7.2. The spin correction 555
- 8. Remarks on the case of (even) orthogonal groups 557
 - 8.1. The naive local models 558
 - 8.2. The corrected local models 559
 - 8.3. The split case: examples 560
- References 563

Introduction

One of the basic problems in the arithmetic theory of Shimura varieties is the construction of natural models over the ring of integers \mathcal{O}_E of the reflex field E . One would like to construct models which are flat and have mild singularities. Such models should be useful in various arithmetic applications. In particular, they should allow one to use the Lefschetz fixed point formula for the complex of nearby cycles to calculate the semisimple zeta function at a non-archimedean prime of E .

When the Shimura variety is the moduli space over $\text{Spec}(E)$ of abelian varieties with additional polarization, endomorphisms and level structure (a Shimura variety of PEL type), versions of the moduli problem sometimes make sense over $\text{Spec}(\mathcal{O}_E)$. In this case the corresponding moduli schemes define models over \mathcal{O}_E , and one may ask whether they satisfy the requirements spelled out above.

Let us fix a prime number p and consider a level structure of parahoric type in p . In this case one can expect that the problem of defining a natural integral model has a positive solution. When the parahoric is hyperspecial, and in most PEL cases, Kottwitz [19] has shown that the model defined by the natural extension of the moduli problem has good reduction. When the parahoric is defined in an elementary way as the stabilizer of a self-dual periodic lattice chain in a p -adic vector space, integral models of the above type were defined in [29]. When the group G only involves types A and C and splits over an unramified extension of \mathbf{Q}_p , all parahorics can be described in this way and

Görtz [10, 11] has shown in this case that the models defined in [29] are indeed flat with reduced special fibre and with irreducible components which are normal and with rational singularities. This follows earlier work by Deligne and Pappas [6], by Chai and Norman [4], and by de Jong [5]. In a few cases, these singularities can be resolved to construct semi-stable models or at least models with toroidal singularities. Here we mention work by Genestier [9], by Faltings [7, 8], and by Görtz [12]. When the group G is of type A or C and is split over \mathbf{Q}_p , Haines and Ngo [14] have calculated the semisimple trace of Frobenius on the sheaves of nearby cycles at any point in the reduction of the natural model at any prime ideal of \mathcal{O}_E of residue characteristic p . Indeed, they give a group-theoretical expression for this semisimple trace that had been conjectured earlier by Kottwitz.

In [23] the case of a unitary group that corresponds at p to a *ramified* quadratic extension of \mathbf{Q}_p was considered. The parahoric subgroup considered in [23] is the stabilizer of a self-dual lattice. In this case, it was shown in [23] that the models proposed in [29] are not flat in general. Furthermore, a closed subscheme of this model was defined and it was conjectured that this closed subscheme is flat and has other good properties. This conjecture is still open, although it is proved in [23] for signature type $(r, 1)$. In the intervening years it became clear that more generally, when the group G splits over a ramified extension of \mathbf{Q}_p , the models defined in [29], which were subsequently renamed *naive models*, are not the correct ones. Also, as was pointed out by Genestier to one of the authors several years ago, the naive models have pathological properties in the case of even orthogonal groups, even those that split over \mathbf{Q}_p .

There are two ways to overcome the shortcomings of the naive models. The first one is to force flatness by taking the flat closure of the generic fibre in the naive model, and to investigate the properties of the models obtained in this way. The second one is to strengthen the naive formulation of the moduli problem in order to obtain closed subschemes of the naive models and to show that these models have good properties, like flatness. In the case that the parahoric subgroup is the stabilizer of a self-dual periodic lattice chain, there is a well-known procedure [6, 29] to reduce these questions to problems of the corresponding *naive local models*. The advantage of this approach is that we are then dealing with varieties that can be defined in terms of linear algebra. Furthermore, the second approach sometimes leads in this way to very explicit problems on the structure of varieties given by matrix identities.

In [25] and [26] we considered groups G which after localization over \mathbf{Q}_p are of the form $\text{Res}_{F/\mathbf{Q}_p} G'$, where G' is the general linear group or the group of symplectic similitudes, and where F is a ramified extension of \mathbf{Q}_p . We gave three ways of defining good models in this case, which are all in the spirit of the first approach. The first one is in terms of a *splitting model* which in turn is defined in terms of the naive local model for the group G' . The second one is by flat closure. The third one is by flat closure for the maximal parahorics, and then by taking intersections inside the naive local model for G . It was shown in [26] that all three methods lead to the same models. The approach through a strengthening of the moduli problem turned out to be very complicated in the case of $G' = \text{GL}_n$. In the case $G' = \text{GSp}_{2n}$, we conjectured that the naive model

is flat. This conjecture is still open, but Görtz [13] has at least proved that the naive model is topologically flat in this case. Furthermore, we gave in [26] a partial calculation of the semisimple trace of Frobenius on the sheaves of nearby cycles for these cases, by reduction to the theorem of Haines and Ngo (here ‘partial’ means that we only compute these traces after a ramified base change). Note that in the cases considered in [26], all parahorics are stabilizers of self-dual periodic lattice chains.

In the present paper we deal with the other typical case of a group G which splits over a ramified extension of \mathbf{Q}_p , namely the group of unitary similitudes corresponding to a quadratic extension of \mathbf{Q} which is ramified at p . At the end we also comment on the case of an even orthogonal group since this case is closely intertwined with the case of the ramified unitary group when one approaches the construction problem in the second way.

To explain our results, we need to introduce some notation. We consider the group G of unitary similitudes for a hermitian vector space (W, ϕ) of dimension $n \geq 3$ over an imaginary quadratic field $K \subset \mathbf{C}$, and fix a conjugacy class of homomorphisms $h : \text{Res}_{\mathbf{C}/\mathbf{R}} G_m \rightarrow G_{\mathbf{R}}$ corresponding to a Shimura datum (G, X_h) of signature (r, s) with $s \leq r$. We assume that K/\mathbf{Q} is ramified over p and that $p \neq 2$. Let $F = K \otimes \mathbf{Q}_p$ and $V = W \otimes_{\mathbf{Q}} \mathbf{Q}_p$. We assume that the hermitian form on V is split, i.e. that there is a basis e_1, \dots, e_n such that

$$\phi(e_i, e_{n-j+1}) = \delta_{ij}, \quad \forall i, j = 1, \dots, n.$$

We fix a square root π of p . For $i = 0, \dots, n - 1$, set

$$A_i = \text{span}_{\mathcal{O}_F} \{ \pi^{-1} e_1, \dots, \pi^{-1} e_i, e_{i+1}, \dots, e_n \}.$$

We complete this into a self-dual periodic lattice chain by setting $A_{i+kn} = \pi^{-k} A_i$. Let $n = 2m + 1$ when n is odd and $n = 2m$ when n is even.

It turns out that when $n = 2m + 1$ is odd, the stabilizer of a partial self-dual periodic lattice chain is always a parahoric. The conjugacy classes of parahoric subgroups of $G(\mathbf{Q}_p)$ correspond in this way to non-empty subsets I of $\{0, \dots, m\}$ (stabilizer of the lattices A_j , where $j = \pm i + kn$ for some $i \in I$ and some $k \in \mathbf{Z}$). When $n = 2m$ is even, the situation is a little more complicated, since the stabilizer groups sometimes contain a parahoric subgroup with index 2. Also, the conjugacy classes of parahoric subgroups correspond in this case to subsets I of $\{0, \dots, m\}$ with the property that if $m - 1 \in I$ then also $m \in I$. Let us denote by M_I^{naive} the *naive local model* in the sense of [29] (see more on this below). This is a projective scheme over $\text{Spec}(\mathcal{O}_E)$. We also introduce the *local model*, the flat closure M_I^{loc} of the generic fibre in M_I^{naive} . In all cases we construct (naive respectively flat) models of the Shimura variety $\text{Sh}_C(G, X_h)$, where C is an open compact subgroup of the finite adèle group with parahoric p -component, which is étale locally around each point isomorphic to the naive local model respectively the local model for suitable I . To understand the structure of these models of the Shimura variety it therefore suffices to study their local versions.

The main tool towards this goal is, as already in Görtz’s paper [10], the embedding of the geometric special fibre of the naive local model in a partial affine flag variety over

the algebraic closure k of the residue field. In our case, we have to employ the affine flag varieties for non-split groups whose theory is developed in [27]. More precisely, let H be the quasi-split unitary group over $k((t))$ which corresponds to a ramified quadratic extension of $k((t))$ and let \mathcal{F} be the affine flag variety of H . We then obtain a closed embedding of the special fibre of M^{naive} in \mathcal{F} . Here M^{naive} denotes the naive local model for $I = \{0, \dots, m\}$. This embedding is equivariant for the action of the Iwahori subgroup, and hence its image is a union of Schubert varieties. These Schubert varieties are enumerated by certain elements of the affine Weyl group W_a of H . To describe this subset of W_a , recall that the Shimura datum (G, X_h) defines a minuscule coweight $\mu = \mu_{r,s}$ of H , and that to a coweight μ there is associated a finite subset $\text{Adm}(\mu)$ of W_a , the μ -admissible set [28]. One of the main results of the present paper is the following theorem.

Theorem 0.1. *The union of Schubert varieties over the μ -admissible set*

$$\mathcal{A}(\mu) = \bigcup_{w \in \text{Adm}(\mu)} S_w$$

is contained in \bar{M}^{loc} , the special fibre of the local model. If the coherence conjecture of [27] is true, then this containment is in fact an equality and \bar{M}^{loc} is reduced and all its irreducible components are normal and with rational singularities.

The coherence conjecture of [27] is an explicit formula for the dimension of the space of global sections on $\mathcal{A}(\mu)$ of the natural ample line bundle \mathcal{L} on \mathcal{F} . The conjecture is proved in the cases of GL_n and of GSp_{2n} , but is open in general for ramified unitary groups.

Something analogous is proved here also for local models M_I^{loc} for proper subsets I of $\{0, \dots, m\}$. In various special cases we can however dispense with the coherence conjecture of [27]. We prove the following result.

Theorem 0.2. *Let $I = \{0\}$ if n is odd, and $I = \{m\}$ if $n = 2m$ is even. The special fibre of the local model M_I^{loc} is irreducible and reduced and is normal, Frobenius split and with only rational singularities.*

We note that the parahoric subgroups corresponding to the subsets I in this theorem are *special* in the sense of Bruhat–Tits theory, but that when n is odd, there are special parahorics which are not conjugate to the parahoric subgroup for $I = \{0\}$.

For the case of Picard surfaces, we have a complete result for all parahorics.

Theorem 0.3. *Let $n = 3$.*

- (a) $M_{\{0\}}^{\text{naive}}$ is normal and Cohen–Macaulay. Furthermore, $M_{\{0\}}^{\text{naive}}$ is flat over $\text{Spec}(\mathcal{O}_E)$ and is smooth outside a single point of the special fibre. Blowing up this point yields a semi-stable model with special fibre consisting of two smooth surfaces meeting transversely along a smooth curve.
- (b) $M_{\{1\}}^{\text{loc}}$ is smooth over $\text{Spec}(\mathcal{O}_E)$, with special fibre isomorphic to \mathbf{P}^2 .

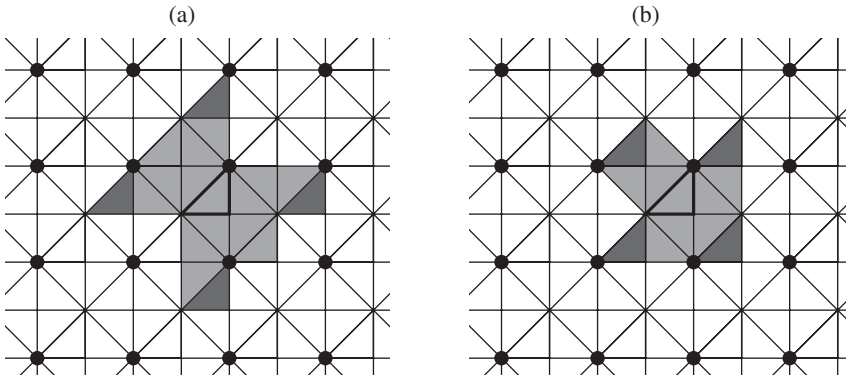


Figure 1. The admissible sets for (a) $U(2, 2)$ and (b) $U(3, 1)$.

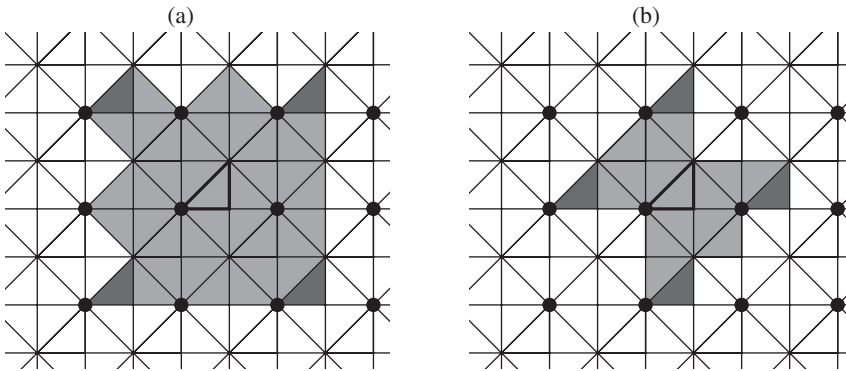


Figure 2. The admissible sets for (a) $U(3, 2)$ and (b) $U(4, 1)$.

- (c) $M_{\{0,1\}}^{\text{loc}}$ is normal and Cohen–Macaulay, with reduced special fibre. Its special fibre has two irreducible components which are normal and with only rational singularities. These two irreducible components meet along two smooth curves which intersect transversally in a single point.

It is an interesting question whether local models can be defined by first forming the flat closure for maximal parahorics, i.e. for subsets I consisting of a single element, and then taking intersections of their inverse images in the naive local model. This leads to the combinatorial question whether the μ -admissible set can be defined vertex-wise. Figures 1 and 2 show that this is indeed the case for the cases of relative rank 2. In fact, in these cases, the μ -admissible set coincides with the μ -permissible set [28, (3.9)], which is obviously defined vertex-wise. It would be interesting to know whether this is true even in higher rank cases.

Let us briefly explain these figures. The ambient tessellation by alcoves corresponds to the affine Weyl group associated to a finite root system Σ of type B_2 in the case of Figure 1, respectively of type C_2 in the case of Figure 2. The dots in these figures indicate the translation subgroup $Q(\Sigma^\vee)$ of W_a ; the vertices of the simplices correspond

to the elements in the coweight lattice $P(\Sigma^\vee)$. The base alcove is marked in bold face, and the extreme alcoves in $\text{Adm}(\mu)$ are shaded in darker grey. They are translates of the fundamental alcove under the four translation elements $\lambda_{r,s}$ in the Iwahori–Weyl group associated to the coweight $\mu = \mu_{r,s}$. The other elements of $\text{Adm}(\mu)$ are shaded in lighter grey.

In this paper we also pursue the approach of defining flat models through a strengthening of the moduli problem. Recall from [29] the definition of the naive local model M_I^{naive} . We change the notation slightly by now denoting by E the localization at p of the Shimura field, i.e. $E = \mathbf{Q}_p$ if $r = s$, and $E = F = K \otimes \mathbf{Q}_p$ if $r \neq s$. The naive local model represents a moduli problem over $\text{Spec}(\mathcal{O}_E)$. A point of M_I^{naive} with values in an \mathcal{O}_E -scheme S is given by a $\mathcal{O}_F \otimes \mathcal{O}_S$ -submodule

$$\mathcal{F}_j \subset \Lambda_j \otimes \mathcal{O}_S$$

for each $j \in \mathbf{Z}$ of the form $j = \pm i + kn$ for some $k \in \mathbf{Z}$ and some $i \in I$. The following conditions (a)–(d) are imposed.

- (a) As an \mathcal{O}_S -module, \mathcal{F}_j is locally on S a direct summand of rank n .
- (b) For each $j < j'$, there is a commutative diagram

$$\begin{array}{ccc} \Lambda_j \otimes \mathcal{O}_S & \longrightarrow & \Lambda_{j'} \otimes \mathcal{O}_S \\ \cup & & \cup \\ \mathcal{F}_j & \longrightarrow & \mathcal{F}_{j'} \end{array}$$

where the top horizontal map is induced by the inclusion $\Lambda_j \subset \Lambda_{j'}$, and for each j , the isomorphism $\pi : \Lambda_j \rightarrow \Lambda_{j-n}$ induces an isomorphism of \mathcal{F}_j with \mathcal{F}_{j-n} .

- (c) We have $\mathcal{F}_{-j} = \mathcal{F}_j^\perp$ where \mathcal{F}_j^\perp is the orthogonal complement of \mathcal{F}_j under the natural perfect pairing

$$(\Lambda_{-j} \otimes \mathcal{O}_S) \times (\Lambda_j \otimes \mathcal{O}_S) \rightarrow \mathcal{O}_S.$$

- (d) For each j , the characteristic polynomial equals

$$\det((T \cdot \text{id} - \pi) | \mathcal{F}_j) = (T - \pi)^s \cdot (T + \pi)^r \in \mathcal{O}_E[T].$$

Here π denotes a uniformizer of F with $\pi^2 = p$.

In [23] a closed subscheme of M_I^{naive} was defined by imposing an additional condition.

- (e) If $r \neq s$, we have

$$\wedge^{r+1}(\pi - \sqrt{p} | \mathcal{F}_j) = 0, \tag{0.1}$$

$$\wedge^{s+1}(\pi + \sqrt{p} | \mathcal{F}_j) = 0. \tag{0.2}$$

(Here $\sqrt{p} = 1 \otimes \pi \in \mathcal{O}_F \otimes \mathcal{O}_S$, while π stands for $\pi \otimes 1 \in \mathcal{O}_F \otimes \mathcal{O}_S$.)

Denote by M_I^\wedge the corresponding closed subscheme of M_I^{naive} . It has the same generic fibre. In [23] it is conjectured that M_I^\wedge is flat over $\text{Spec}(\mathcal{O}_E)$ when $I = \{0\}$, i.e. that condition (e) above cuts out the local model in $M_{\{0\}}^{\text{naive}}$. The conjecture for $I = \{0\}$ is then reduced to a question about schemes defined by explicit matrix identities. It turns out that something similar can be done for $I = \{m\}$ when $n = 2m$. Let us state here these questions.

Consider the scheme of matrices X in $M_{n \times n}$ over $\text{Spec}(k)$ described by

$$X^2 = 0, \quad {}^tX = HXH, \quad \text{char}_X(T) = T^n, \quad \wedge^{s+1}X = 0, \quad \wedge^{r+1}X = 0$$

if n is odd, respectively

$$X^2 = 0, \quad {}^tX = -JXJ, \quad \text{char}_X(T) = T^n \quad (\text{and } \wedge^{s+1}X = 0, \wedge^{r+1}X = 0 \text{ when } r \neq s)$$

if both n and s are even.

Is this scheme reduced (in which case it is normal, with rational singularities)?

Here $H = H_n$ denotes the antidiagonal unit matrix, and $J = J_{2m}$ the skew-symmetric matrix with square blocks 0_m on the diagonal and H_m , respectively $-H_m$, above the diagonal, respectively below the diagonal. Varieties given by similar matrix equations were also considered by Faltings [7]. However, it does not seem that his results can be used to answer the above question. Indeed, the schemes he considers are the special fibres of the (normalizations) of the flat closures of mixed characteristic schemes closely related to M_I^{naive} . We can only see, after the fact, that if the answer to our question is positive, then our schemes coincide with the ones considered in [7]. (Regardless, the results of [7] establish a connection between the singularities of the normalization of the local models and the complete symmetric varieties of de Concini–Procesi (see also [24]).)

We give examples which show that for more general I the condition (e) is not enough to define the local model. To treat a general index set I we propose in the present paper an additional condition.

- (f) (*Spin condition.*) The line $\wedge^n \mathcal{F}_j \subset \wedge^n (A_j \otimes \mathcal{O}_S)$ is contained in the subspace $(\wedge^n A_j \otimes \mathcal{O}_E)_\pm \otimes_{\mathcal{O}_E} \mathcal{O}_S$ with $\pm = (-1)^s$.

We refer to the body of the text for the definition of $(\wedge^n A_j \otimes \mathcal{O}_E)_\pm \otimes_{\mathcal{O}_E} \mathcal{O}_S$. We denote by M_I the closed subscheme of M_I^{naive} defined by the conditions (a)–(f). Then M_I has again the same generic fibre and we conjecture that M_I is flat over \mathcal{O}_E . Unfortunately, this seems very difficult to prove, although the computer evidence seems quite convincing. It is remarkable that, in contrast to the unramified cases of type A or C, one seems to need in these ramified cases and also for type D higher tensors to describe the local model.

We now give an overview of the paper. In § 1 we define the unitary Shimura varieties and reduce our structure problem to the case of local models. In § 2 we recall some facts about affine Weyl groups and about the μ -admissible set and make these explicit in the case of unitary groups. In § 3 we recall results from [27] on affine flag varieties, and state the

coherence conjecture. We also prove in this section that the points in Schubert varieties corresponding to elements in $\text{Adm}(\mu)$ can be lifted to the generic fibre of M^{naive} , i.e. the first statement of Theorem 0.1. In § 4 we give our general results. As pointed out above, they depend on some conjectures of a combinatorial nature. In § 5 we consider cases we can treat unconditionally. In particular, we relate the special fibres in the cases addressed in Theorem 0.2 to nilpotent orbits for the symmetric pairs $(\mathfrak{gl}(n), \mathfrak{o}(n))$ respectively $(\mathfrak{gl}(n), \mathfrak{sp}(n))$, and recover in this way some of Ohta's results about them [22], even in positive characteristics. We also treat the case of Picard surfaces in § 6. In § 7 we explain the spin condition and the evidence we have that it is sufficient to cut out the flat closure. In § 8 we give remarks on local models for even orthogonal groups and explain on examples how the spin condition also seems to eliminate the pathological points in the naive models for them found by Genestier. Actually, as this paper is being revised to its final version, we mention that Smithling has recently initiated a detailed study [32] of the even orthogonal local models introduced in § 8. Among other things, he shows that, in the split case, the spin condition is enough to cut out the flat local model topologically, i.e. up to nilpotents.

It should be pointed out that local models for ramified unitary groups are still quite mysterious. Many questions on them remain open. This also explains why in the present paper we have made abstract concepts as explicit as possible for the case at hand and why we have given many concrete examples. We hope that this will be useful for future work. One major open problem is that of calculating the semisimple trace of the Frobenius on the complex of nearby cycles. Besides the special case treated by Krämer [20] nothing is known.

The present paper should be viewed as a follow-up to [27]. The results in [27] on affine flag varieties for non-split groups over $k((t))$ are crucial to several of our results in this paper.

1. Unitary Shimura varieties and moduli problems

1.1. Unitary Shimura varieties

Let K be an imaginary quadratic field with an embedding $\epsilon : K \hookrightarrow \mathbf{C}$. Denote by \mathcal{O} the ring of integers of K . Denote by $a \mapsto \bar{a}$ the non-trivial automorphism of K . Let $W = K^n$ be an n -dimensional K -vector space, and suppose that $\phi : W \times W \rightarrow K$ is a non-degenerate hermitian form. We assume $n \geq 3$. Now set $W_{\mathbf{C}} = W \otimes_{K, \epsilon} \mathbf{C}$. Choosing a suitable isomorphism $W_{\mathbf{C}} \simeq \mathbf{C}^n$, we may write ϕ on $W_{\mathbf{C}}$ in a normal form $\phi(w_1, w_2) = {}^t \bar{w}_1 H w_2$, where

$$H = \text{diag}(-1, \dots, -1, 1, \dots, 1).$$

We denote by s (respectively r) the number of places, where -1 (respectively 1) appears in H . We will say that ϕ has signature (r, s) . By replacing ϕ by $-\phi$ if needed, we can make sure that $s \leq r$. We will assume that $s \leq r$ throughout the paper. Let $J : W_{\mathbf{C}} \rightarrow W_{\mathbf{C}}$ be the endomorphism given by the matrix $-\sqrt{-1}H$. We have $J^2 = -\text{id}$ and so the endomorphism J gives an \mathbf{R} -algebra homomorphism $h_0 : \mathbf{C} \rightarrow \text{End}_{\mathbf{R}}(W \otimes_{\mathbf{Q}} \mathbf{R})$

with $h_0(\sqrt{-1}) = J$ and hence a complex structure on $W \otimes_{\mathbf{Q}} \mathbf{R} = W_{\mathbf{C}}$. For this complex structure we have

$$\text{Tr}_{\mathbf{C}}(a; W \otimes_{\mathbf{Q}} \mathbf{R}) = s \cdot \epsilon(a) + r \cdot \bar{\epsilon}(a), \quad a \in K.$$

Denote by E the subfield of \mathbf{C} which is generated by the traces above (the ‘reflex field’). This is equal to \mathbf{Q} if $r = s$ and to K if $r \neq s$. The representation of K on $W \otimes_{\mathbf{Q}} \mathbf{R}$ with the above trace is defined over E , i.e. there is an n -dimensional E -vector space W_0 on which K acts such that

$$\text{Tr}_E(a; W_0) = s \cdot a + r \cdot \bar{a} \tag{1.1}$$

and such that $W_0 \otimes_E \mathbf{C}$ together with the above K -action is isomorphic to $W \otimes_{\mathbf{Q}} \mathbf{R}$ with the K -action induced by $\epsilon : K \hookrightarrow \mathbf{C}$ and the above complex structure.

Now let us fix a non-zero element $\alpha \in K$ with $\bar{\alpha} = -\alpha$. Set

$$\psi(x, y) = \text{Tr}_{K/\mathbf{Q}}(\alpha^{-1}\phi(x, y)), \tag{1.2}$$

which is a non-degenerate alternating form $W \otimes_{\mathbf{Q}} W \rightarrow \mathbf{Q}$. This satisfies

$$\psi(av, w) = \psi(v, \bar{a}w), \quad \text{for all } a \in K, v, w \in W. \tag{1.3}$$

By replacing α by $-\alpha$, we can make sure that the symmetric \mathbf{R} -bilinear form on $W_{\mathbf{C}}$ given by $\psi(x, Jy)$ for $x, y \in W_{\mathbf{C}}$ is positive definite.

Let $G = \text{GU}(\phi)$ be the unitary similitude group of the form ϕ :

$$G = \text{GU}(\phi) = \{g \in \text{GL}_K(W) \mid \phi(gv, gw) = c(g)\phi(v, w), c(g) \in \mathbf{G}_m\}.$$

Set

$$\text{GU}(r, s) := \{A \in \text{GL}_n(\mathbf{C}) \mid {}^t\bar{A}HA = c(A)H, c(A) \in \mathbf{R}^{\times}\}.$$

By the above discussion, the embedding $\epsilon : K \hookrightarrow \mathbf{C}$ induces an isomorphism $G(\mathbf{R}) \simeq \text{GU}(r, s)$. The group G is a reductive group over \mathbf{Q} which is also given by

$$G(\mathbf{Q}) = \{g \in \text{GL}_K(W) \mid \psi(gv, gw) = c(g)\psi(v, w), c(g) \in \mathbf{Q}^{\times}\}.$$

We define a homomorphism $h : \text{Res}_{\mathbf{C}/\mathbf{R}}\mathbf{G}_{m, \mathbf{C}} \rightarrow G_{\mathbf{R}}$ by restricting h_0 to \mathbf{C}^{\times} . Then $h(a)$ for $a \in \mathbf{R}^{\times}$ acts on $W \otimes_{\mathbf{Q}} \mathbf{R}$ by multiplication by a and $h(\sqrt{-1})$ acts as J . Consider $h_{\mathbf{C}}(z, 1) : \mathbf{C}^{\times} \rightarrow G(\mathbf{C}) \simeq \text{GL}_n(\mathbf{C}) \times \mathbf{C}^{\times}$. Up to conjugation, $h_{\mathbf{C}}(z, 1)$ is given by

$$\mu_{r,s}(z) = (\text{diag}(z^{(s)}, 1^{(r)}), z); \tag{1.4}$$

this is a cocharacter of G defined over the number field E . (Here and in the rest of the paper, we write $a^{(m)}$ to denote a list of m copies of a .)

Denote by $X_h = X_{r,s}$ the conjugation orbit of $h(i)$ under $G(\mathbf{R})$. The pair (G, h) gives rise to a Shimura variety $\text{Sh}(G, h)$ which is defined over the reflex field E . In particular, if

$$C = \prod_v C_v \subset G(\mathbf{A}_{\mathbf{Q}}^f), \quad \text{with } C_v \subset G(\mathbf{Q}_v),$$

is an open compact subgroup of the finite adelic points of G , we can consider $\text{Sh}_C(G, h)$; this is a quasi-projective variety over E whose set of complex points is identified with

$$\text{Sh}_C(G, h)(\mathbf{C}) = G(\mathbf{Q}) \backslash X_{r,s} \times G(\mathbf{A}_Q^f) / C.$$

Suppose now that p is an *odd* rational prime which ramifies in K . We denote by \mathcal{O}_{E_w} the ring of integers of the completion of E at the unique place above (p) . We are interested in the construction of models of $\text{Sh}(G, h)_C$ over \mathcal{O}_{E_w} and in the reductions of these models when the ‘level subgroup at p ’, $C_p \subset G(\mathbf{Q}_p)$, is a parahoric subgroup of $G(\mathbf{Q}_p)$. (We will also assume that the level ‘away from p ’, i.e. the subgroup $C^p = \prod_{v \neq p} C_v$, is sufficiently small, i.e. it is contained in the principal congruence subgroup for some $N \geq 3$ relatively prime to the discriminant of K .)

1.2. Parahoric subgroups of the unitary similitude group

It turns out that the parahoric subgroups are the neutral components of the subgroups of the unitary similitude group over the local field that stabilize certain sets of lattices. In this paragraph, we will explain this statement in a slightly more general context.

1.2.1. Let F_0 be a complete discretely valued field with ring of integers $\mathcal{O} = \mathcal{O}_{F_0}$ and perfect residue field k of characteristic not equal to 2 and uniformizer π_0 . Let F/F_0 be a ramified quadratic extension and let $\pi \in F$ be a uniformizer with $\pi^2 = \pi_0$, so that $\bar{\pi} = -\pi$. Set $\Gamma = \text{Gal}(F/F_0)$. Let V be an F -vector space of dimension $n \geq 3$ and let

$$\phi : V \times V \rightarrow F$$

be an F/F_0 -hermitian form. We assume that ϕ is split. This means that there exists a basis e_1, \dots, e_n of V such that

$$\phi(e_i, e_{n-j+1}) = \delta_{ij}, \quad \forall i, j = 1, \dots, n.$$

Set

$$\text{GU}(V, \phi) = \{g \in \text{GL}_F(V) \mid \phi(gx, gy) = c(g)\phi(x, y), \ c(g) \in F_0^\times\}.$$

We have an exact sequence of algebraic groups over F_0 :

$$1 \rightarrow \text{SU}(V, \phi) \rightarrow \text{GU}(V, \phi) \rightarrow D \rightarrow 1. \tag{1.5}$$

Here $\text{SU}(V, \phi)$ is also the derived group of $\text{GU}(V, \phi)$ and D is the torus $D = T/(T \cap \text{SU}(V, \phi))$ with T the standard (diagonal) maximal torus of $\text{GU}(V, \phi)$.

1.2.2. We have two associated F_0 -bilinear forms:

$$(x, y) = \frac{1}{2} \text{Tr}_{F/F_0}(\phi(x, y)), \quad \langle x, y \rangle = \frac{1}{2} \text{Tr}_{F/F_0}(\pi^{-1} \cdot \phi(x, y)).$$

The form (\cdot, \cdot) is symmetric while $\langle \cdot, \cdot \rangle$ is alternating. They satisfy the identities

$$(x, \pi y) = -(\pi x, y), \quad \langle x, \pi y \rangle = -\langle \pi x, y \rangle. \tag{1.6}$$

For any \mathcal{O}_F -lattice Λ in V we set,

$$\hat{\Lambda} = \{v \in V \mid \phi(v, \Lambda) \subset \mathcal{O}_F\} = \{v \in V \mid \langle v, \Lambda \rangle \subset \mathcal{O}\}.$$

Similarly, we set

$$\hat{\Lambda}^s = \{v \in V \mid (v, \Lambda) \subset \mathcal{O}\},$$

so that $\hat{\Lambda}^s = \pi^{-1} \cdot \hat{\Lambda}$. For $i = 0, \dots, n - 1$, set

$$A_i = \text{span}_{\mathcal{O}_F} \{\pi^{-1}e_1, \dots, \pi^{-1}e_i, e_{i+1}, \dots, e_n\}.$$

The lattice A_0 is self-dual for the alternating form $\langle \cdot, \cdot \rangle$.

1.2.3. We now distinguish two cases.

(a) $n = 2m + 1 \geq 3$ is odd. Then we have

$$T = \text{diag}(a_1, \dots, a_m, a, a\bar{a}\bar{a}_m^{-1}, \dots, a\bar{a}\bar{a}_1^{-1}),$$

$$T \cap \text{SU}(V, \phi) = \{\text{diag}(a_1, \dots, a_m, \bar{b}b^{-1}, \bar{a}_m^{-1}, \dots, \bar{a}_1^{-1}) \mid b = a_1 \cdots a_m\}.$$

We can see that

$$D = T / (T \cap \text{SU}(V, \phi)) \xrightarrow{\sim} \text{Res}_{F/F_0}(\mathbf{G}_m) \tag{1.7}$$

with the isomorphism given by sending an element of T as above to $a^{-1} \cdot (a_1\bar{a}_1^{-1}) \cdots (a_m\bar{a}_m^{-1})$.

Now let I be a non-empty subset of $\{0, \dots, m\}$ and consider the subgroup

$$P_I = \{g \in \text{GU}(V, \phi) \mid g \cdot A_i = A_i, \forall i \in I\}$$

of $\text{GU}(V, \phi)$ that preserves the lattice set $A_i, i \in I$. The following statement can be shown as in [27]. Notice that in this case $X_*(D)_\Gamma \simeq \mathbf{Z}$ and so the Kottwitz invariant of each element of P_I is trivial.

The subgroup P_I is a parahoric subgroup of $\text{GU}(V, \phi)$. Any parahoric subgroup of $\text{GU}(V, \phi)$ is conjugate to a subgroup P_I for a unique subset I . The sets $I = \{0\}$ and $I = \{m\}$ correspond to the special maximal parahoric subgroups.

In fact, $P_{\{0, \dots, m\}}$ is an Iwahori subgroup and its choice allows us to identify $\{0, \dots, m\}$ with the local Dynkin diagram $\Delta(\text{GU}(V, \phi))$; the index i corresponds to the vertex associated to $P_{\{i\}}$.

(b) $n = 2m \geq 4$. Then we have

$$T = \{\text{diag}(a_1, \dots, a_m, c\bar{a}_m^{-1}, \dots, c\bar{a}_1^{-1}) \mid c \in F_0^\times\}$$

$$T \cap \text{SU}(V, \phi) = \{\text{diag}(a_1, \dots, a_m, \bar{a}_m^{-1}, \dots, \bar{a}_1^{-1}) \mid a_1 \cdots a_m \in F_0^\times\}.$$

We can see that

$$D = T / (T \cap \text{SU}(V, \phi)) \xrightarrow{\sim} \mathbf{G}_m \times \ker(\text{Res}_{F/F_0}(\mathbf{G}_m) \xrightarrow{\text{Norm}} \mathbf{G}_m) \tag{1.8}$$

with the isomorphism given by sending an element of T as above to

$$(c, a_1 \cdots a_m \cdot \bar{a}_1^{-1} \cdots \bar{a}_m^{-1}).$$

(Here of course we use Hilbert’s Theorem 90.) We can now see that $X_*(D)_\Gamma \simeq \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ and that the Kottwitz invariant of an element $t \in T(F_0)$ as above is given by

$$\kappa(t) = (\text{val}_{F_0}(c), \text{val}_F(a_1 \cdots a_m) \bmod 2). \tag{1.9}$$

This shows that in this case the Kottwitz invariant $\kappa(g)$ of $g \in \text{GU}(V, \phi)(F_0)$ can be obtained as follows. Consider $d(g) := c(g)^{-m} \det_F(g) \in F^\times$; this has norm 1 and so we can write $d(g) = x \cdot \bar{x}^{-1}$. Then $\kappa(g) = (\text{val}_{F_0}(c(g)), \text{val}_F(x) \bmod 2)$. Notice that if g stabilizes a maximal ϕ -isotropic F -subspace $L \subset V$, then the second component of $\kappa(g)$ is the valuation modulo 2 of the determinant $\det(g \mid L)$. Also note that if g preserves a lattice, then the first component of $\kappa(g)$ is zero.

Now consider non-empty subsets $I \subset \{0, \dots, m\}$. As above, consider the subgroup

$$P_I = \{g \in \text{GU}(V, \phi) \mid gA_i = A_i, \forall i \in I\}$$

of $\text{GU}(V, \phi)$ that preserves the lattices $A_i, i \in I$. We also consider the kernel of the Kottwitz homomorphism, i.e.

$$P_I^0 = \{g \in P_I \mid \kappa(g) = 1\}.$$

We can see that if $m - 1 \in I$, then $P_I^0 = P_{I \cup \{m\}}^0$.

In fact, in this case the following statement holds:

The subgroup P_I^0 is a parahoric subgroup of $\text{GU}(V, \phi)$. Any parahoric subgroup of $\text{GU}(V, \phi)$ is conjugate to a subgroup P_I^0 for a unique subset I with the property that if $m - 1$ is in I , then m is also in I . For such a subset I , we have $P_I^0 = P_I$ exactly when I contains m . The set $I = \{m\}$ corresponds to a special maximal parahoric subgroup.

This follows from the results on parahoric subgroups of $\text{SU}(V, \phi)$ in [27]. To explain this, we also introduce

$$\Lambda_{m'} = \text{span}_{\mathcal{O}_F} \{\pi^{-1}e_1, \dots, \pi^{-1}e_{m-1}, e_m, \pi^{-1}e_{m+1}, e_{m+2}, \dots, e_n\}.$$

Then both Λ_m and $\Lambda_{m'}$ are self-dual for the symmetric form (\cdot, \cdot) . Now consider non-empty subsets J of $\{0, \dots, m - 2, m, m'\}$. Consider the subgroups

$$P_J = \{g \in \text{GU}(V, \phi) \mid gA_j = A_j, \forall j \in J\}, \quad P_J^0 = \{g \in P_J \mid \kappa(g) = 1\}.$$

Notice that if both m and m' are in J then $A_j, j \in J$, is not a lattice chain.

As in [27] we can see that P_J is parahoric exactly when J contains at least one of the two elements $\{m, m'\}$. Then $P_J^0 = P_J$. When J contains neither m nor m' , the kernel P_J^0 is a parahoric subgroup and $P_J/P_J^0 \simeq \mathbf{Z}/2\mathbf{Z}$. Now recall that [27] gives a description of the parahoric subgroups of $\text{SU}(V, \phi)$ via $P'_J = P_J \cap \text{SU}(V, \phi)$. In fact, $P_{\{0, \dots, m-2, m, m'\}}$ is

an Iwahori subgroup and its choice allows us to identify $\{0, \dots, m - 2, m, m'\}$ with the set of vertices of the local Dynkin diagram $\Delta := \Delta(\text{GU}(V, \phi)) = \Delta(\text{SU}(V, \phi))$; the index j corresponds to the vertex associated to the subgroup $P_{\{j\}}$.

For $J \subset \{0, \dots, m - 2, m, m'\}$ let J^* be the subset obtained by replacing m by m' and vice versa. Observe that if τ is the unitary automorphism defined by $e_m \mapsto e_{m+1}$, $e_{m+1} \mapsto e_m$, $e_i \mapsto e_i$, for $0 \leq i \leq m - 2$, then $\tau \cdot \Lambda_m = \Lambda_{m'}$, $\tau \cdot \Lambda_{m'} = \Lambda_m$, $\tau \cdot \Lambda_i = \Lambda_i$, for $0 \leq i \leq m - 2$. This shows that $\tau P_J \tau^{-1} = P_{J^*}$. By [33, 2.5] we can see that the action of $\text{GU}(V, \phi)$ on the local Dynkin diagram $\Delta = \{0, \dots, m - 2, m, m'\}$ factors through the second component of the Kottwitz homomorphism $\kappa : \text{GU}(V, \phi) \rightarrow \mathbf{Z}/2\mathbf{Z}$; now $\kappa(\tau) = -1$ and as we have seen τ fixes $0 \leq i \leq m - 2$ and exchanges m and m' . It follows that the conjugacy classes of parahoric subgroups of $\text{GU}(V, \phi)$ are parametrized by the orbits of $J \mapsto J^*$ on the set of non-empty subsets $J \subset \{0, \dots, m - 2, m, m'\}$. These are in turn parametrized by non-empty subsets $I \subset \{0, \dots, m - 1, m\}$ with the property that if $m - 1$ is in I , then m is also in I . The set $I = J^\# = (J^*)^\#$ that corresponds to $\{J, J^*\}$ is obtained by the following recipe. If both m and m' belong to J , put $J^\#$ to be the set which is obtained from J by replacing m' by $m - 1$. If $m' \notin J$, set $J^\# = J$. Finally, if $m' \in J$ but $m \notin J$, let $J^\# = J^*$. Observe now that for all J , we have $P_J = P_{J^\#}$ if $m \in J$ and $P_J = \tau P_{J^*} \tau^{-1} = \tau P_{J^\#} \tau^{-1}$ if $m' \in J$. Therefore, our statements on the parahoric subgroups of $\text{GU}(V, \phi)$ now follow from these observations and the results on parahoric subgroups of $\text{SU}(V, \phi)$ in § 4 of [27].

1.3. Reduction to level subgroups that are lattice chain stabilizers

We return to the set up of § 1.1, so $G = \text{GU}(W, \phi)$ is a unitary similitude group over \mathbf{Q} , and $X = X_{r,s}$. We will assume $s > 0$. (When $s = 0$ the corresponding Shimura varieties are zero dimensional.) Let p be an odd rational prime which ramifies in K . Assume that the form ϕ is split on $V = W \otimes_{\mathbf{Q}} \mathbf{Q}_p$ so that the set-up of § 1.2 applies. Let $C = C_p \cdot C^p$ with $C_p = P_I^0 \subset G(\mathbf{Q}_p)$ a parahoric subgroup as in § 1.2 and where $C^p \subset G(\mathbf{A}^{f,p})$ contains the principal congruence subgroup for some $N \geq 3$ relatively prime to the discriminant of K . Set $C'_p = P_I \subset G(\mathbf{Q}_p)$ for the corresponding stabilizer of the set of lattices; we have either $P_I^0 = P_I$ or $P_I/P_I^0 \simeq \mathbf{Z}/2\mathbf{Z}$. Set $C' = C'_p \cdot C^p$. Our first observation is that the Shimura varieties $\text{Sh}_C(G, X)$ and $\text{Sh}_{C'}(G, X)$ have isomorphic geometric connected components. This follows essentially from the fact that $C \cap G_{\text{der}}(\mathbf{A}^f) = C' \cap G_{\text{der}}(\mathbf{A}^f)$ (see below). Therefore, from the point of view of constructing reasonable integral models over \mathcal{O}_{E_w} , we may restrict our attention to $\text{Sh}_{C'}(G, X)$; since C'_p corresponds to a lattice set stabilizer, this Shimura variety is given by a simpler moduli problem.

Denote by

$$\nu : G \rightarrow D = G/G_{\text{der}}$$

the maximal torus quotient of G . We can see [19, § 7] that

$$D \simeq \begin{cases} \text{Res}_{K/\mathbf{Q}}(\mathbf{G}_m) & \text{if } n \text{ is odd,} \\ \mathbf{G}_m \times \ker(\text{Res}_{K/\mathbf{Q}}(\mathbf{G}_m) \rightarrow \mathbf{G}_m) & \text{if } n \text{ is even.} \end{cases} \tag{1.10}$$

Since G_{der} is simply connected, the set of connected components of the complex Shimura variety $\text{Sh}_C(G, X)(\mathbf{C})$ can be identified with the double coset

$$D(\mathbf{Q}) \backslash Y \times D(\mathbf{A}^f) / \nu(C),$$

where $Y = D(\mathbf{R}) / \text{Im}(Z(\mathbf{R}) \xrightarrow{\nu} D(\mathbf{R}))$, provided that $r > 0, s > 0$ (cf. [21, § 5, p. 311]). (Notice that $D(\mathbf{R}) = \mathbf{C}^\times$ if n is odd, respectively $D(\mathbf{R}) = \mathbf{R}^\times \times U_1$ if n is even, with U_1 the complex unit circle. The image of the real points $Z(\mathbf{R})$ of the centre Z of G is \mathbf{C}^\times , respectively $\mathbf{R}_{>0} \times U_1$. Therefore, $Y = \{1\}$ if n is odd, and $Y = \{\pm 1\}$ if n is even.) The action of the Galois group $\text{Gal}(\bar{E}/E)$ on the set of connected components factors through $\text{Gal}(E^{\text{ab}}/E)$ and is given as follows. Consider the composition $\nu \circ \mu_{r,s} : \mathbf{G}_{mE} \rightarrow G_E \rightarrow D_E$ which is defined over E , and set

$$\rho = \text{Norm}_{E/\mathbf{Q}} \circ \nu \circ \mu_{r,s} : \mathbf{G}_{mE} \rightarrow D_{\mathbf{Q}}.$$

If $\sigma \in \text{Gal}(E^{\text{ab}}/E)$ corresponds to the idele $x_\sigma \in \mathbf{A}_E^\times$ via Artin reciprocity, then

$$\sigma \cdot [y, d] = \rho(x_\sigma) \cdot [y, d], \tag{1.11}$$

with $\rho : \mathbf{A}_E^\times \rightarrow D(\mathbf{A}_{\mathbf{Q}})$ given as above. (Here we normalize the reciprocity isomorphism by asking that the local uniformizer corresponds to the inverse of the Frobenius.)

The conjugacy class of the cocharacter $\mu_{r,s}$ of G_E defines a conjugacy class of a local cocharacter $\mu_{r,s} : \mathbf{G}_{mE_w} \rightarrow G_{E_w}$. Suppose first that $r = s$. Then $E_w = \mathbf{Q}_p$ and we can assume that $\mu_{r,s}$ is given by

$$a \mapsto \text{diag}(a^{(s)}, 1^{(r)}) \in T(\mathbf{Q}_p) \subset G(\mathbf{Q}_p) \tag{1.12}$$

in the notation of § 1.2. If $r \neq s$ then $E_w = K_v$ and we can assume that $\mu_{r,s}$ is given by

$$a \mapsto (\text{diag}(a^{(s)}, 1^{(r)}), a) \in \text{GL}_n(K_v) \times K_v^\times \simeq G(K_v). \tag{1.13}$$

Note that the isomorphism $\text{GL}_n(K_v) \times K_v^\times \simeq G(K_v)$ takes the conjugation action on $G(K_v)$ to the involution $(A, c) \mapsto (\bar{c}(A^*)^{-1}, \bar{c})$ where $(A^*)^{-1}$ is the inverse of the hermitian adjoint. Therefore, we have

$$\text{Norm}_{E_w/\mathbf{Q}_p}(\mu_{r,s})(a) = \text{diag}((a\bar{a})^{(s)}, \bar{a}^{(r-s)}, 1^{(s)}) \in T(\mathbf{Q}_p) \subset G(\mathbf{Q}_p). \tag{1.14}$$

Under the identification of $D(\mathbf{Q}_p)$ as in § 1.2.3 this gives

$$\rho_{E_w}(a) = \bar{a}^{-1} \left(\frac{\bar{a}}{a} \right)^{m-s}, \quad \text{if } n \text{ is odd,} \tag{1.15}$$

$$\rho_{E_w}(a) = \left(a\bar{a}, \left(\frac{\bar{a}}{a} \right)^{m-s} \right), \quad \text{if } n \text{ is even and } r \neq s, \tag{1.16}$$

$$\rho_{E_w}(a) = (a, 1), \quad \text{if } n \text{ is even and } r = s. \tag{1.17}$$

Consider now the morphism between the corresponding Shimura varieties,

$$\pi_{C,C'} : \text{Sh}_C(G, X) \rightarrow \text{Sh}_{C'}(G, X).$$

Recall that if n is odd, we always have $C = C'$ and so we may restrict our attention to the case when $n = 2m$ is even.

Proposition 1.1. *Assume $C \neq C'$. Then the morphism $\pi_{C,C'}$ is an étale $\mathbf{Z}/2\mathbf{Z}$ -cover which splits after base changing by an extension E'/E which is of degree 1 or 2 and is unramified over p . If $m - s$ is odd, then the cover is not trivial, $[E' : E] = 2$ and w remains prime in E' . If either $r = s$ or more generally $m - s$ is even, then either $E' = E$ or $[E' : E] = 2$ and w splits in E' .*

Proof. It is clear that $\pi_{C,C'}$ is an étale cover of degree 1 or 2. Let us consider the corresponding map between geometric connected components:

$$D(\mathbf{Q}) \setminus Y \times D(\mathbf{A}^f) / \nu(C) \rightarrow D(\mathbf{Q}) \setminus Y \times D(\mathbf{A}^f) / \nu(C'). \tag{1.18}$$

Obviously, this is a surjective group homomorphism with kernel either trivial or of order 2. We will show that the kernel is actually of order 2; this will imply that the cover $\pi_{C,C'}$ becomes trivial after base changing to the algebraic closure \bar{E} . Consider the element α in $Y \times D(\mathbf{A}^f)$ which is 1 at each place except at p where it is equal to

$$\alpha_p = (1, -1) \in \mathbf{Q}_p^\times \times \ker(\text{Norm}(K_v^\times \rightarrow \mathbf{Q}_p^\times)).$$

From §1.2, we can see that α_p is in $\nu(C'_p)$ but not in $\nu(C_p)$. We claim that α gives a non-trivial element of $D(\mathbf{Q}) \setminus Y \times D(\mathbf{A}^f) / \nu(C)$ whose image under (1.18) is trivial. Indeed, suppose there is $d \in D(\mathbf{Q})$ such that $d \cdot \alpha \in \nu(C)$. Since α is in $\nu(C')$ we obtain that d is in the intersection $D(\mathbf{Q}) \cap \nu(C')$; this is given by pairs of a unit of \mathbf{Q} and a unit of K which are both congruent to 1 modulo N . We can see that the only such pair is $(1, 1)$; this implies $\alpha \in \nu(C)$ which contradicts our choice. It remains to show that the cover becomes trivial after base changing by an extension E'/E as in the statement. By the formula for the Galois action on the set of connected components and the above we see that the cover $\pi_{C,C'} \otimes_E E_w$ is described via local class field theory by the map

$$\rho_{E_w} \text{ mod } \nu(C'_p) : E_w^\times \rightarrow D(\mathbf{Q}_p) / \nu(C_p) \rightarrow \mathbf{Z}/2\mathbf{Z}.$$

Here the last map is given by $(a, b/\bar{b}) \mapsto \text{val}(b) \text{ mod } 2$. The result now follows using (1.16), (1.17). □

Corollary 1.2. *Assume $C \neq C'$. Let $\mathcal{M}_{C'}$ be a model of $\text{Sh}_{C'}(G, X)$ over $\text{Spec}(\mathcal{O}_{E_w})$. Then there exists a unique model \mathcal{M}_C of $\text{Sh}_C(G, X)$ over $\text{Spec}(\mathcal{O}_{E_w})$, such that the following diagram is commutative*

$$\begin{array}{ccc} \text{Sh}_C(G, X) \otimes_E E_w & \hookrightarrow & \mathcal{M}_C \\ \downarrow & & \downarrow \\ \text{Sh}_{C'}(G, X) \otimes_E E_w & \hookrightarrow & \mathcal{M}_{C'} \end{array}$$

in which the vertical arrows are finite étale.

1.4. Unitary moduli problems

Here we follow [29] to define moduli schemes over \mathcal{O}_{E_w} whose generic fibre agrees with $\text{Sh}_{C'}(G, X) \otimes_E E_w$ when $C' = C'_p \cdot C^p$ with C'_p one of the lattice set stabilizer subgroups of § 1.2. More precisely, we consider non-empty subsets $I \subset \{0, \dots, m\}$ where $m = \lfloor n/2 \rfloor$ with the requirement that for $n = 2m$ even, if $m - 1$ is in I , then m is in I too. We will use the notation of § 1.2 with $F_0 = \mathbf{Q}_p$, $F = K_v$, $V = W \otimes_K K_v$. We can extend Λ_i , $i \in I$, given as in § 1.2.2, to a periodic self-dual lattice chain by first including the duals $\hat{\Lambda}_i^s = \Lambda_{n-i}$ for $i \neq 0$, and then all the π -multiples of our lattices. For $j \in \mathbf{Z}$ of the form $j = k \cdot n \pm i$ with $i \in I$ we put

$$\Lambda_j = \pi^{-k} \cdot \Lambda_i.$$

Then $\{\Lambda_j\}_j$ form a periodic lattice chain Λ_I (with $\pi \cdot \Lambda_j = \Lambda_{j-n}$) which satisfies $\hat{\Lambda}_j = \Lambda_{-j}$.

We set $C'_p = P_I$. We will use the construction of Chapter 6 of [29] applied to the current situation. In the notation of [29] we take $B = K$, $a \mapsto a^*$ is the conjugation of K/\mathbf{Q} , $V = K^n = \mathbf{Q}^{2n}$, $(x, y) = \psi(x, y)$. Then G is the unitary similitude group as before. We take the self-dual multichain \mathcal{L} of lattices to be Λ_j , $j = kn \pm i$ with $i \in I$. For simplicity, set $R = \mathcal{O}_{E_w}$.

We consider the moduli functor $\mathcal{A}_{C'}$ over $\text{Spec}(R)$ given in [29, Definition 6.9].

A point of $\mathcal{A}_{C'}$ with values in the $\text{Spec}(R)$ -scheme S is the isomorphism class of the following set of data $(A, \bar{\lambda}, \bar{\eta})$.

- (1) An \mathcal{L} -set of abelian schemes $A = \{A_j\}$, $j = kn \pm i$ with $i \in I$, over S (terminology of [29]). By definition, this amounts to the data of abelian schemes A_j over S up to prime-to- p -isogeny for each j , isogenies $A_j \rightarrow A_{j'}$ of height $\log_p(A_{j'}/A_j)$ for each pair $j \leq j'$, and periodicity isomorphisms $\theta^a : A_j \simeq A_{j-n \cdot \text{val}(a)}$ for each $a \in \mathcal{O}_K \otimes \mathbf{Z}_{(p)}$. These should satisfy the conditions of [29].
- (2) A \mathbf{Q} -homogeneous principal polarization $\bar{\lambda}$ of the \mathcal{L} -set A .
- (3) A C^p -level structure

$$\bar{\eta} : H_1(A, \mathbf{A}_f^p) = \left(\prod_{l \neq p} T_l(A_j) \right) \otimes \mathbf{Q} \simeq W \otimes \mathbf{A}_f^p \text{ mod } C^p$$

that respects the bilinear forms of both sides up to a constant in $(\mathbf{A}_f^p)^\times$ (see [29] for details).

The set A should satisfy the determinant condition (i) of [29].

Recall that we assume that C^p is sufficiently small, i.e. that it is contained in the principal congruence subgroup of level N for some $N \geq 3$ relatively prime to the discriminant of K . Then $\mathcal{A}_{C'}$ is representable by a quasi-projective scheme over $\text{Spec}(R)$ which we will also denote by $\mathcal{A}_{C'}$. Since the Hasse principle is satisfied for the unitary group, we can see as in [29] that there is a natural isomorphism

$$\mathcal{A}_{C'} \otimes_R E_w = \text{Sh}_{C'}(G, X) \otimes_E E_w. \tag{1.19}$$

1.5. Local models for GU

We fix non-negative integers r, s with $n = r + s$ and consider F/F_0 as in § 1.2. We set $E = F$ if $r \neq s$ and $E = F_0$ if $r = s$ (This is the reflex field of the local model we are about to define. Note that when we refer to local models E is always a local field.) As in § 1.4, we will consider non-empty subsets $I \subset \{0, \dots, m\}$ where $m = \lfloor n/2 \rfloor$ with the requirement that for n even, if $m - 1$ is in I , then m is in I too.

1.5.1. First consider $I = \{0, \dots, m\}$ which gives as above a complete lattice chain $\Lambda_j, j \in \mathbf{Z}$. We define as follows a functor M^{naive} on the category of \mathcal{O}_E -schemes. A point of M^{naive} with values in an \mathcal{O}_E -scheme S is given by a $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ -submodule

$$\mathcal{F}_j \subset \Lambda_j \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$$

for each $j \in \mathbf{Z}$. The following conditions are imposed.

- (a) As an \mathcal{O}_S -module, \mathcal{F}_j is locally on S a direct summand of rank n .
- (b) For each $j < j'$, there is a commutative diagram

$$\begin{array}{ccc} \Lambda_j \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S & \longrightarrow & \Lambda_{j'} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \\ \cup & & \cup \\ \mathcal{F}_j & \longrightarrow & \mathcal{F}_{j'} \end{array}$$

where the top horizontal map is induced by the inclusion $\Lambda_j \subset \Lambda_{j'}$, and for each j , the isomorphism $\pi : \Lambda_j \rightarrow \Lambda_{j-n}$ induces an isomorphism of \mathcal{F}_j with \mathcal{F}_{j-n} .

- (c) We have $\mathcal{F}_{-j} = \mathcal{F}_j^\perp$ where \mathcal{F}_j^\perp is the orthogonal complement of \mathcal{F}_j under the perfect pairing

$$(\Lambda_{-j} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \times (\Lambda_j \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \rightarrow \mathcal{O}_S$$

induced by $\langle \cdot, \cdot \rangle \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$.

Next note that \mathcal{F}_j is an $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ -module, hence \mathcal{O}_F and \mathcal{O}_E act on it. We require further that

- (d) for each j , the characteristic polynomial equals

$$\det((T \cdot \text{id} - \pi) | \mathcal{F}_j) = (T - \pi)^s \cdot (T + \pi)^r \in \mathcal{O}_S[T].$$

This concludes the definition of the functor M^{naive} , which is obviously representable by a projective scheme over $\text{Spec}(\mathcal{O}_E)$. We call M^{naive} the *naive* local model associated to the group $\text{GU}(V, \phi)$, the signature type (r, s) and for the complete lattice chain $\Lambda_j, j \in \mathbf{Z}$.

1.5.2. We can generalize this definition to incomplete self-dual periodic lattice chains, i.e. to all subsets I as above. For each such I , we obtain a functor M_I^{naive} by only giving the submodules \mathcal{F}_j for $j \in \mathbf{Z}$ of the form $j = k \cdot n \pm i$ with $i \in I$. Therefore, $M^{\text{naive}} = M_{\{0, \dots, m\}}^{\text{naive}}$. Denote by \mathcal{P}_I the (smooth) group scheme of automorphisms (up to similitude) of the polarized chain \mathcal{L} over \mathcal{O}_{F_0} ; then $\mathcal{P}_I(\mathcal{O}_{F_0}) = P_I$; the group scheme \mathcal{P}_I has $\text{GU}(V, \phi)$ as its generic fibre. Then $\mathcal{P}_I \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_E$ acts on M_I^{naive} . We have forgetful morphisms of projective schemes

$$M^{\text{naive}} \rightarrow M_I^{\text{naive}} \quad \text{and} \quad M_{I'}^{\text{naive}} \rightarrow M_I^{\text{naive}} \quad \text{for } I' \supset I. \tag{1.20}$$

1.5.3. Note that the map $\{\mathcal{F}_j\}_j \mapsto \ker(\pi - \sqrt{\pi_0} \mid \mathcal{F}_j)$ gives an isomorphism between $M_I^{\text{naive}} \otimes_{\mathcal{O}_E} F$ and the Grassmannian $\text{Gr}(s, n)_F$ of s -dimensional spaces in the n -dimensional space $V_0 = \ker(\pi - \sqrt{\pi_0} \mid V \otimes_{F_0} F)$. (Here, for simplicity, we set $\sqrt{\pi_0} = 1 \otimes \pi$, $\pi = \pi \otimes 1$.)

1.5.4. As is explained in [23, 26, 29], when $C'_p = P_I$, for $F_0 = \mathbf{Q}_p$ and $F = K_p$, the (naive) local model is connected to the moduli scheme via a diagram

$$\mathcal{A}_{C'} \xleftarrow{\pi} \tilde{\mathcal{A}}_{C'} \xrightarrow{\phi} M_I^{\text{naive}}, \tag{1.21}$$

where the morphism π is a $\mathcal{P}_I \otimes_{\mathbf{Z}_p} R$ -torsor and ϕ is a smooth morphism of relative dimension $\dim(G)$ which is $\mathcal{P}_I \otimes_{\mathbf{Z}_p} R$ -equivariant. Therefore, there is a relatively representable morphism of algebraic stacks

$$\mathcal{A}_{C'} \rightarrow [M_I^{\text{naive}} / (\mathcal{P}_I \otimes_{\mathbf{Z}_p} R)] \tag{1.22}$$

which is smooth of relative dimension $\dim(G)$. (See [23] and [26, § 15] for some more details.)

1.5.5. As was observed in [23], the schemes M_I^{naive} are almost never flat over R ; by the above, the same is true for $\mathcal{A}_{C'}$. In the present paper, we will examine the flat closure $\mathcal{A}_{C'}^{\text{flat}}$ of $\text{Sh}_{C'}(G, X)_{E_w} = \mathcal{A}_{C'} \otimes_R E_w$ in $\mathcal{A}_{C'}$. By the above, we can reduce certain local questions about $\mathcal{A}_{C'}^{\text{flat}}$ to similar questions about the flat closure of $M_I^{\text{naive}} \otimes_R E_w$ in M^{naive} . This last flat closure is, by definition, the *local model* M_I^{loc} . We can see that M_I^{loc} supports an action of $\mathcal{P}_I \otimes_{\mathbf{Z}_p} R$ and there is a relatively representable smooth morphism of relative dimension $\dim(G)$:

$$\mathcal{A}_{C'}^{\text{flat}} \rightarrow [M_I^{\text{loc}} / (\mathcal{P}_I \otimes_{\mathbf{Z}_p} R)]. \tag{1.23}$$

This of course implies that each closed point of $\mathcal{A}_{C'}^{\text{flat}}$ has an étale neighbourhood which is isomorphic to an étale neighbourhood of some corresponding point of M_I^{loc} .

Since the generic fibres $M_I^{\text{naive}} \otimes_R E_w$ are, for the various choices of I , all identical, we obtain for $I' \supset I$ commutative diagrams of projective morphisms, respectively closed embeddings,

$$\begin{array}{ccc} M_{I'}^{\text{loc}} & \hookrightarrow & M_{I'}^{\text{naive}} \\ \downarrow & & \downarrow \\ M_I^{\text{loc}} & \hookrightarrow & M_I^{\text{naive}} \end{array}$$

whose generic fibres are all the identity morphism. We can see that these correspond to similar diagrams between the schemes $\mathcal{A}_{C'}^{\text{flat}}$ and $\mathcal{A}_{C'}$ for the choices $C'_p = P_I$ or P_I .

1.5.6. Again let F/F_0 be as in §1.2. We can also define a subfunctor M_I^\wedge of M_I^{naive} by specifying that a point of M_I^\wedge with values in an \mathcal{O}_E -scheme S is given by an $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ -submodule $\mathcal{F}_j \subset \Lambda_j \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ for each $j = k \cdot n \pm i, i \in I$, that in addition to the conditions (a)–(d) above also satisfies the following condition.

(e) If $r \neq s$, for each $j = k \cdot n \pm i, i \in I$, we have

$$\wedge^{r+1}(\pi - \sqrt{\pi_0} \mid \mathcal{F}_j) = 0, \tag{1.24}$$

$$\wedge^{s+1}(\pi + \sqrt{\pi_0} \mid \mathcal{F}_j) = 0, \tag{1.25}$$

where $\pi = \pi \otimes 1, \sqrt{\pi_0} = 1 \otimes \pi \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_E$.

Denote also by M_I^\wedge the corresponding moduli scheme which is a closed subscheme of M_I^{naive} . Suppose that S is an E -scheme. Then conditions (d) and (e) are equivalent since the action of π on \mathcal{F}_j is semisimple. Therefore, the generic fibres $M_I^\wedge \otimes_{\mathcal{O}_E} E$ and $M_I^{\text{naive}} \otimes_{\mathcal{O}_E} E$ agree. It turns out that M_I^\wedge is in some cases flat; then $M_I^{\text{loc}} = M_I^\wedge$. This was shown when $I = \{0\}$ and $n = 3$ in [23]; when $I = \{0\}$ it is conjectured in [23] that $M_{\{0\}}^\wedge$ is flat. However, M_I^\wedge is not flat in general (see Remark 5.3).

In the sequel we will pursue two goals. The first goal is to understand the structure of M_I^{loc} . The second goal is to define M_I^{loc} as a closed subscheme of M_I^{naive} by imposing conditions similar to those defining M_I^\wedge .

2. Affine Weyl groups and affine flag varieties: the μ -admissible set

In this section, G will be a (connected) reductive group over a local field L with perfect residue field. We assume that G is residually split and hence quasi-split [33, 1.10].

2.1. Affine Weyl groups

We start by recalling some facts on affine Weyl groups [16, 28].

2.1.1. Let S be a maximal split torus in G and let T be its centralizer. Since G is quasi-split, T is a maximal torus in G . Let $N = N(T)$ be the normalizer of T ; denote by $T(L)_1$ the kernel of the Kottwitz homomorphism $\kappa_T : T(L) \rightarrow X_*(T)_I$; then $T(L)_1 = \mathcal{T}^0(\mathcal{O}_L)$ where \mathcal{T}^0 the connected Neron model of the torus T over \mathcal{O}_L . By definition, the *Iwahori–Weyl group associated to S* is the quotient group

$$\tilde{W} = N(L)/T(L)_1.$$

Since κ_T is surjective, the Iwahori–Weyl group \tilde{W} is an extension of the relative Weyl group $W_0 = N(L)/T(L)$ by $X_*(T)_I$:

$$0 \rightarrow X_*(T)_I \rightarrow \tilde{W} \rightarrow W_0 \rightarrow 1. \tag{2.1}$$

We have (see [16] and also [28, §2]).

Proposition 2.1. *Let B_0 be the Iwahori subgroup of $G(L)$ associated to an alcove contained in the apartment associated to the maximal split torus S . Then $G(L) = B_0 \cdot N(L) \cdot B_0$ and the map $B_0 \cdot n \cdot B_0 \mapsto n \in \tilde{W}$ induces a bijection*

$$B_0 \backslash G(L) / B_0 \xrightarrow{\sim} \tilde{W}.$$

If P is the parahoric subgroup of $G(L)$ associated to a facet contained in the apartment corresponding to S , then

$$P \backslash G(L) / P \xrightarrow{\sim} W^P \backslash \tilde{W} / W^P, \quad \text{where } W^P := (N(L) \cap P) / T(L)_1. \tag{2.2}$$

In fact, if P is the (special) parahoric subgroup $P_{\underline{x}}$ that corresponds to a special vertex \underline{x} in the apartment corresponding to S , then the subgroup $W^P \subset \tilde{W}$ maps isomorphically to W_0 under the quotient $\tilde{W} \rightarrow W_0$ and the exact sequence (2.1) represents the Iwahori–Weyl group as a semidirect product

$$\tilde{W} = W_0 \ltimes X_*(T)_I \tag{2.3}$$

(see [16]).

2.1.2. Now let $S_{\text{sc}}, T_{\text{sc}}$, respectively N_{sc} , be the inverse images of $S \cap G_{\text{der}}, T \cap G_{\text{der}}$, respectively $N \cap G_{\text{der}}$, in the simply connected covering G_{sc} of the derived group G_{der} . Then S_{sc} is a maximal split torus of G_{sc} and T_{sc} , respectively N_{sc} , is its centralizer, respectively normalizer. Hence

$$W_{\mathfrak{a}} := N_{\text{sc}}(L) / T_{\text{sc}}(L)_1$$

is the Iwahori–Weyl group of G_{sc} . This group is also called the *affine Weyl group associated to S* and is a Coxeter group. Indeed, we can recover $W_{\mathfrak{a}}$ in the following way. Let $N(L)_1$ be the intersection of $N(L)$ with the kernel $G(L)_1$ of the Kottwitz homomorphism $\kappa_G : G(L) \rightarrow \pi_1(G)_I$. Then one can see [16] that the natural homomorphism

$$W_{\mathfrak{a}} = N_{\text{sc}}(L) / T_{\text{sc}}(L)_1 \xrightarrow{\sim} N(L)_1 / T(L)_1 \tag{2.4}$$

is an isomorphism and that there is an exact sequence

$$1 \rightarrow W_{\mathfrak{a}} \rightarrow \tilde{W} \xrightarrow{\kappa_G} \pi_1(G)_I \rightarrow 1, \tag{2.5}$$

where $\pi_1(G)_I = X_*(T)_I / X_*(T_{\text{sc}})_I$. Now let B_0 be the Iwahori subgroup of $G(L)$ associated to an alcove C in the apartment corresponding to S and let \mathcal{S} be the set of reflections about the walls of C . Then by [3, 5.2.12] the quadruple $(G(L)_1, B_0, N(L)_1, \mathcal{S})$ is a double Tits system and $W_{\mathfrak{a}} = N(L)_1 / T(L)_1$ is the affine Weyl group of the affine root system $\Phi_{\mathfrak{a}}$ of S . The affine Weyl group $W_{\mathfrak{a}}$ acts simply transitively on the set of alcoves in the apartment of S . Since \tilde{W} acts transitively on the set of these chambers, \tilde{W} is the semi-direct product of $W_{\mathfrak{a}}$ with the normalizer Ω of the base alcove C , i.e. the subgroup of \tilde{W} which preserves the alcove,

$$\tilde{W} = W_{\mathfrak{a}} \rtimes \Omega. \tag{2.6}$$

We can identify Ω with $\pi_1(G)_I$.

Let us write $\mathcal{S} = \{s_i\}_{i \in I} \subset W_a$ for the finite set of reflections about the walls of C that generate the Coxeter group W_a . For each $w \in W_a$ its length $l(w)$ is the minimal number of factors in a product of the s_i representing w . Any such product realizing the minimum is called a reduced decomposition of w . We will denote by ‘ \leq ’ the corresponding Bruhat order. Recall its definition. Fix a reduced decomposition of $w \in W_a$. The elements $w' \leq w$ are obtained by replacing some factors in it by 1. (This set of such w ’s is independent of the choice of the reduced decomposition of w .) We extend the Bruhat order from W_a to \tilde{W} using the semi-direct product decomposition (2.6): for $w = w_1 \cdot \tau$, with $w_1 \in W_a$ and $\tau \in \pi_1(G)_I$, the elements smaller than w are the w' of the form $w' = w'_1 \cdot \tau$ with $w'_1 \leq w_1$ in W_a .

Let us denote by $\alpha_i \in \Phi_a$ the unique affine root with corresponding affine reflection equal to s_i (since the group is residually split $\frac{1}{2}\alpha_i \notin \Phi_a$ (cf. [33, 1.8])). We will denote by $\Delta = \Delta_G$ the (local) Dynkin diagram of the affine root system Φ_a (this can be obtained from the set $\{\alpha_i\}_{i \in I}$ (see [2, 1.4] and [33, 1.8])). For a subset $Y \subset \mathcal{S}$, we denote by $W_Y \subset W_a$ the subgroup generated by s_i with $i \in Y$; we set $P_Y = B_0 \cdot W_Y \cdot B_0$. By general properties of Tits systems these are subgroups of $G(L)_1 \subset G(L)$; by [3, 5.2.12 (i)] they are the parahoric subgroups of $G(L)$ that contain B . Using [2, 1.3.5] we see that we can identify P_Y with the parahoric subgroup P_{C_Y} , where C_Y is the facet consisting of $a \in \tilde{C}$ for which Y is exactly the set of reflections $s \in \mathcal{S}$ which fix a .

Finally, let us recall that there exists a reduced root system Σ such that the semi-direct product (2.3) (for G_{sc} instead of G) presents W_a as the affine Weyl group associated (in the sense of Bourbaki) to Σ :

$$W_a = W(\Sigma) \ltimes Q^\vee(\Sigma) \tag{2.7}$$

(cf. [2, 1.3.8] and [33, 1.7, 1.9]). In other words, we have identifications $W_0 \simeq W(\Sigma)$, $X_*(T_{sc})_I \simeq Q^\vee(\Sigma)$ compatible with the semidirect product decompositions (2.3) and (2.7).

2.2. The μ -admissible set

We next recall the definition of the μ -admissible set.

2.2.1. To $\mu \in X_*(T)$ we attach its image λ in the coinvariants $X_*(T)_I$. By (2.1) we can consider λ as an element in the Iwahori–Weyl group \tilde{W} of G . The *admissible subset of \tilde{W} associated to the coweight μ* is defined as

$$\text{Adm}(\mu) = \{w \in \tilde{W} \mid w \leq w_0(\lambda) \text{ for some } w_0 \in W_0\}. \tag{2.8}$$

Note that all elements of $\text{Adm}(\mu)$ have the same image in $\tilde{W}/W_a = \Omega$, namely the image of μ in $\pi_1(G)_I$. Furthermore, the set $\text{Adm}(\mu)$ only depends on the geometric conjugacy class of the one-parameter subgroup μ (cf. [28, § 3]).

2.2.2. In the present paper we find it more convenient to pass to the adjoint group. Let T_{ad} denote the image of T in the adjoint group G_{ad} . To μ we associate its image μ_{ad} in

$X_*(T_{\text{ad}})$ and its image λ_{ad} in $X_*(T_{\text{ad}})_I$. Now the set $\text{Adm}(\mu)$ is mapped bijectively to $\text{Adm}(\mu_{\text{ad}})$ under the homomorphism $\tilde{W} \rightarrow \tilde{W}_{\text{ad}}$, so that it suffices to consider the latter set, which we sometimes also denote by $\text{Adm}(\mu)$. We have a commutative diagram

$$\begin{array}{ccc}
 W_{\mathfrak{a}} & \xrightarrow{\sim} & W(\Sigma) \ltimes Q^{\vee}(\Sigma) \\
 \downarrow & & \downarrow \\
 \tilde{W}_{\text{ad}} & \longrightarrow & W(\Sigma) \ltimes P^{\vee}(\Sigma)
 \end{array} \tag{2.9}$$

Here all arrows are injective and $Q^{\vee}(\Sigma)$, respectively $P^{\vee}(\Sigma)$, is the group of coroots, respectively of coweights, of the finite root system Σ (cf. [28, § 3]). We have $\lambda_{\text{ad}} \in P^{\vee}(\Sigma)$ (cf. the proof of Lemma 3.1 in [28]). In the sequel we write W_0 for the finite Weyl group $W(\Sigma)$.

Denoting by τ_{ad} the common image of all elements of $\text{Adm}(\mu)$ or $\text{Adm}(\mu_{\text{ad}})$ in $\Omega_{\text{ad}} = \tilde{W}_{\text{ad}}/W_{\mathfrak{a}}$, we can define the subset $\text{Adm}(\mu)^{\circ}$ of $W_{\mathfrak{a}}$ by

$$\text{Adm}^{\circ}(\mu) = \{w \in W_{\mathfrak{a}} \mid w \cdot \tau_{\text{ad}} \in \text{Adm}(\mu_{\text{ad}})\}.$$

For a non-empty subset Y of the set of simple affine roots, let $Y^{\circ} \subset \mathbf{S}$ be the subset that corresponds to the set of simple reflections of the form $\{\tau_{\text{ad}} \cdot s_i \cdot \tau_{\text{ad}}^{-1} \mid i \in Y\}$ where s_i is the reflection corresponding to the simple affine root parametrized by $i \in Y$.

We may define the subset $\text{Adm}^Y(\mu)$ of \tilde{W}_{ad} by $W^Y \cdot \text{Adm}(\mu_{\text{ad}}) \cdot W^Y$ and the subset $\text{Adm}^Y(\mu)^{\circ}$ of $W_{\mathfrak{a}}$ by

$$\text{Adm}^Y(\mu)^{\circ} := W^Y \cdot \text{Adm}(\mu)^{\circ} \cdot W^{Y^{\circ}}.$$

Since $\tau_{\text{ad}} \cdot W^Y \cdot \tau_{\text{ad}}^{-1} = W^{Y^{\circ}}$ this is also equal to $(W^Y \cdot \text{Adm}(\mu_{\text{ad}}) \cdot W^Y) \cdot \tau_{\text{ad}}^{-1}$.

2.3. Calculation of the affine root system

The following recipe for obtaining the affine root system $\Phi_{\mathfrak{a}}$ was explained to us by Waldspurger. We may assume for this that the derived group G_{der} is simply connected. Let L'/L be a finite Galois extension which splits G . Since G is residually split, the extension L'/L is totally ramified. We extend the normalized valuation $\text{val} : L \rightarrow \mathbb{Z}$ to a valuation $\text{val} : L' \rightarrow (1/e)\mathbb{Z}$, where $e = |L' : L|$ is the ramification degree of L'/L . We identify $X_*(S)$ with $X_*(T)^I$ and $X^*(S)$ with $X^*(T)_I/\text{torsion}$, and denote by φ the natural map

$$\varphi : X^*(T) \rightarrow X^*(T)_I/\text{torsion} = X^*(S). \tag{2.10}$$

Let $\Phi^{\text{abs}} \subset X^*(T)$ be the set of (absolute) roots. The relative root system Φ is the image of Φ^{abs} under φ . Let $\Phi_{\mathfrak{a}}$ be the affine root system, which is a set of affine functions on $V = X_*(S) \otimes \mathbb{R}$ of the form $\beta + r$, for $\beta \in \Phi$ and $r \in \mathbb{R}$. There is a unique action of $N(L)$ on V by affine transformations such that the elements $t \in T(L)$ act by translations by $\text{ord}(t) \in V$. Here $\text{ord}(t)$ is uniquely defined by

$$\langle \text{ord}(t), \chi \rangle = -\text{val } \chi(t), \quad \chi \in X^*(T).$$

Let

$$l_\beta = |\varphi^{-1}(\beta) \cap \Phi^{\text{abs}}|, \quad \beta \in \Phi. \tag{2.11}$$

Let, for $\beta \in \Phi$,

$$R_\beta = \begin{cases} \frac{1}{l_\beta} \mathbb{Z} & \text{if } \frac{1}{2}\beta \notin \Phi, \\ \frac{1}{2l_\beta} + \frac{1}{l_\beta} \mathbb{Z} & \text{if } \frac{1}{2}\beta \in \Phi. \end{cases} \tag{2.12}$$

Proposition 2.2. *The affine root system Φ_a equals*

$$\Phi_a = \{\beta + r \mid \beta \in \Phi, r \in R_\beta\}.$$

Proof. For any $\alpha \in \Phi^{\text{abs}}$, we denote by U_α the corresponding root subgroup and by $\alpha^\vee \in X_*(T)$ the corresponding coroot. Similarly, for $\beta \in \Phi$ we have U_β and $\beta^\vee \in X_*(S)$ (in case $2\beta \in \Phi$, we have $U_{2\beta} \subset U_\beta$).

Let $\beta \in \Phi$. For $u \in U_\beta(L) \setminus \{1\}$, the intersection $U_{-\beta}(L) \cdot u \cdot U_{-\beta}(L) \cap N(L)$ consists of a single element, denoted by $n(u)$, with image in W_0 equal to the reflection s_β . The action of $n(u)$ on V is an affine reflection about a hyperplane parallel to $\text{Ker}(\beta)$. Let $r_\beta(u) \in \mathbb{R}$ be the real number such that this hyperplane is defined by the equation

$$\beta(v) + r_\beta(u) = 0, \quad v \in V.$$

We also set $r_\beta(1) = \infty$. For $r \in \mathbb{R}$, define

$$U_{\beta+r} = \{u \in U_\beta(L) \mid r_\beta(u) \geq r\}, \quad U_{\beta+r^+} = \{u \in U_\beta(L) \mid r_\beta(u) > r\}. \tag{2.13}$$

Then the affine roots in Φ_a are defined to be the affine functions $\beta + r$ such that

$$U_{\beta+r}/U_{\beta+r^+} \cdot U_{2\beta+2r} \neq \{1\} \tag{2.14}$$

(cf. [33, 1.6]). Here we set $U_{2\beta+2r} = \{1\}$, if $2\beta \notin \Phi$.

Let $\beta \in \Phi$. Assume first that $2\beta \notin \Phi$ and $\frac{1}{2}\beta \notin \Phi$, (i.e. β of type I). There exists a subextension $L \subset L_\beta \subset L'$ such that $l_\beta = |L_\beta : L|$ and such that the subgroup of $G(L)$ generated by $U_\beta(L)$ and $U_{-\beta}(L)$ is isomorphic to $SL_2(L_\beta)$ (here the fact that G_{der} is simply connected, enters). We choose this isomorphism such that $U_\beta(L)$ becomes identified with the upper unipotent matrices, and $U_{-\beta}(L)$ with the lower unipotent matrices and the intersection of $S(L)$ with this subgroup with the diagonal torus. For $b \in L_\beta \setminus \{0\}$ and $u = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, one calculates [3, p. 80]

$$n(u) = \begin{pmatrix} 1 & 0 \\ -b^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -b^{-1} & 1 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \cdot w, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The action of $n(u)$ on V is given by

$$v \mapsto s_\beta(v) - \text{val}(b) \cdot \beta^\vee(v). \tag{2.15}$$

Here s_β denotes the reflection about the hyperplane $\text{Ker } \beta$. It follows that $r_\beta(u) = \text{val}(b)$. Hence $\beta + r \in \Phi$ if and only if $r \in \text{val}(L_\beta)$. Since L_β/L is totally ramified, $\text{val}(L_\beta) = (1/l_\beta) \cdot \mathbb{Z}$. This proves the claim for affine functions of type $v \mapsto \beta(v) + r$, for β of type I.

Now let $\beta \in \Phi$ be of type II, i.e. such that $2\beta \in \Phi$ or $\frac{1}{2}\beta \in \Phi$. Then there is a tower of subextensions $L \subset L_\beta \subset L'_\beta \subset L'$ such that $|L'_\beta : L_\beta| = 2$ and such that

$$|L_\beta : L| = \begin{cases} \frac{1}{2}l_\beta & \text{if } 2\beta \in \Phi, \\ l_\beta & \text{if } \frac{1}{2}\beta \in \Phi. \end{cases}$$

The subgroup of $G(L)$ generated by $U_\beta(L)$ and $U_{\beta/2}(L)$ (here $U_{\beta/2}(L)$ is trivial by definition if $\frac{1}{2}\beta \notin \Phi$) and $U_{-\beta}(L)$ and $U_{-\beta/2}(L)$ is isomorphic to $\text{SU}_3(L_\beta)$, the special unitary group relative to the quadratic extension L'_β/L_β and the hermitian form with anti-diagonal unit matrix. We again choose this isomorphism in the evident way. Let β_0 , respectively β_0^\vee , the root, respectively coroot, of $\text{SU}_3(L_\beta)$, given by

$$\begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-1} \end{pmatrix} \mapsto t, \quad t \mapsto \begin{pmatrix} t^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-2} \end{pmatrix}.$$

The identification is chosen such that if $2\beta \in \Phi$, then U_β is identified with

$$U_{\beta_0} = \left\{ \begin{pmatrix} 1 & b & -c \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{pmatrix} \middle| c + \bar{c} = b\bar{b} \right\},$$

and if $\frac{1}{2}\beta \in \Phi$, then U_β is identified with

$$U_{2\beta_0} = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| c + \bar{c} = 0 \right\}.$$

Here $x \mapsto \bar{x}$ denotes the Galois automorphism of L'_β/L_β . For $u \in U_{\beta_0}$, written as above, one has [3, § 4.1]

$$n(u) = \begin{pmatrix} -c & 0 & 0 \\ 0 & \bar{c}c^{-1} & 0 \\ 0 & 0 & -\bar{c}^{-1} \end{pmatrix} w,$$

where

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Hence the action of $n(u)$ on V is given by

$$v \mapsto s_\beta(v) - \frac{1}{2} \text{val}(c) \cdot \beta_0^\vee. \tag{2.16}$$

It follows that

$$r_\beta(u) = \begin{cases} \frac{1}{2} \text{val}(c) & \text{if } 2\beta \in \Phi, \\ \text{val}(c) & \text{if } \frac{1}{2}\beta \in \Phi. \end{cases} \tag{2.17}$$

Hence the condition that $U_{\beta+r}/U_{\beta+r+} \cdot U_{2\beta+2r} \neq \{1\}$ is equivalent to

$$\begin{cases} \exists b \in L'_\beta \text{ with } r = \frac{1}{2} \max\{\text{val}(c) \mid c + \bar{c} = b\bar{b}\} & \text{if } 2\beta \in \Phi, \\ \exists c \in L'_\beta \text{ with } c + \bar{c} = 0, \text{ and } r = \text{val}(c) & \text{if } \frac{1}{2}\beta \in \Phi. \end{cases} \tag{2.18}$$

Since $p \neq 2$, and L'/L is totally ramified, this is equivalent to

$$\begin{cases} r \in \frac{1}{l_\beta} \mathbb{Z} & \text{if } 2\beta \in \Phi, \\ r \in \frac{1}{2l_\beta} + \frac{1}{l_\beta} \mathbb{Z} & \text{if } \frac{1}{2}\beta \in \Phi. \end{cases} \tag{2.19}$$

□

2.4. The case of a unitary group

We now take G to be the (quasi-split) unitary similitude group over $L = F_0$ corresponding to a tame ramified quadratic extension F/F_0 (as in § 1.2). Set $\Gamma = \text{Gal}(F/F_0) = \langle \tau \rangle$ and notice that we can take $L' = F$ in the notation of the previous paragraph. As in § 1.2.3 we denote by T the standard maximal torus of G and by S the maximal split subtorus of T . Our first purpose is to calculate explicitly the inclusion

$$W_a \subset \tilde{W}_{\text{ad}}.$$

Here $W_a \cong X_*(T_{\text{sc}})_\Gamma \rtimes W_0$ (as in (2.3), after the choice of a special vertex) is the Coxeter group generated by the reflections in the root hyperplanes in Φ_a . The group \tilde{W}_{ad} is obtained from W_a by the push-out via the inclusion $X_*(T_{\text{sc}})_\Gamma \hookrightarrow X_*(T_{\text{ad}})_\Gamma$.

Recall the definition of the cocharacter $\mu_{r,s} \in X_*(T)$ in § 1.1. Using $G(F) \simeq \text{GL}_n(F) \times F^\times$ we identify $X_*(T)$ with $\mathbb{Z}^n \times \mathbb{Z}$; then $\mu_{r,s} = ((1^{(s)}, 0^{(r-s)}), 1)$. Our second purpose is to determine the image $\lambda = \lambda_{r,s}$ of $\mu = \mu_{r,s} \in X_*(T)$ in $X_*(T_{\text{ad}})_\Gamma$ and the set

$$\text{Adm}(\mu) = \{w \in \tilde{W}_{\text{ad}} \mid w \leq w_0(\lambda), \text{ some } w_0 \in W_0\}.$$

Let $\text{Adm}_0(\mu)$ be the image of $W_0 \cdot \text{Adm}(\mu)$ in $W_0 \setminus \tilde{W}_{\text{ad}}/W_0$. Our third purpose is to determine $\text{Adm}_0(\mu)$.

2.4.1. *The case $A_{2m-1}^{(2)}$ (corresponding to $\text{GU}_{2m}(F/F_0)$) for $m \geq 2$*

Let $n = 2m$. In this case, $X_*(S)$ can be identified with the subgroup of $X_*(T) \simeq \mathbb{Z}^n \times \mathbb{Z}$ formed by the elements of the form $(x_1, \dots, x_m, -x_m, \dots, -x_1; y)$. The relative roots β are of the form $\pm x_i \pm x_j$ for $i \neq j$ and of the form $\pm 2x_i$, for $i, j \in \{1, \dots, m\}$. For the first kind of β we have $l_\beta = 2$, and for the second kind $l_\beta = 1$. Therefore, the affine root system Φ_a consists of the affine functions $\pm x_i \pm x_j + \frac{1}{2}\mathbb{Z}$ and $\pm 2x_i + \mathbb{Z}$. Hence the set

of root hyperplanes is the zero sets of the affine functions $\{\pm x_i \pm x_j + \frac{1}{2}\mathbf{Z}; \pm x_i + \frac{1}{2}\mathbf{Z}\}$. It follows that the root system Σ is of type B_m (i.e. Σ^\vee is of type C_m) and we have

$$W_a = Q^\vee \rtimes W_0 \subset \tilde{W}_{ad} = P^\vee \rtimes W_0,$$

with

$$\begin{aligned} W_0 &= S_m \rtimes \{\pm 1\}^m, \\ Q^\vee &= \left\{ x \in \mathbb{Z}^m \mid \sum(x) \equiv 0(2) \right\}, \\ P^\vee &= \mathbb{Z}^m. \end{aligned}$$

The map

$$X_*(T) \rightarrow X_*(T_{ad})_\Gamma \xrightarrow{\sim} \mathbb{Z}^m$$

is given by

$$(x_1, \dots, x_{2m}, y) \mapsto (x_1 - x_{2m}, x_2 - x_{2m-1}, \dots, x_m - x_{m+1}).$$

The image $\lambda = \lambda_s$ of $\mu_{r,s} = (1^{(s)}, 0^{(n-s)}, 1)$ in $X_*(T_{ad})_\Gamma = \mathbb{Z}^m$ is equal to $(1^{(s)}, 0^{(m-s)})$.

Let us identify $W_0 \setminus \tilde{W}_{ad}/W_0 = P^\vee/W_0$ with

$$P^\vee \cap \bar{C} = \{x \in \mathbb{Z}^m \mid x_1 \geq x_2 \geq \dots \geq x_m \geq 0\}.$$

The induced order from the Bruhat order is the dominance order. The positive coroots are (Bourbaki, table C_m)

$$e_i - e_j, e_i + e_j, 2e_i \ (i < j).$$

The only possibility of subtracting a positive coroot from λ_s and staying inside \bar{C} is by $e_{s-1} + e_s$. Hence we see inductively that

$$\text{Adm}_0(\mu_{r,s}) = \{\lambda = \lambda_s > \lambda_{s-2} > \dots\},$$

with the chain ending in λ_1 if s is odd and in λ_0 if s is even.

Note that λ_s is minuscule if and only if $s = 0$ or $s = 1$.

2.4.2. The case $A_{2m}^{(2)}$ (corresponding to $\text{GU}_{2m+1}(F/F_0)$) for $m \geq 1$

Set $n = 2m + 1$. In this case, $X_*(S)$ can be identified with the subgroup of $X_*(T) \simeq \mathbb{Z}^n \times \mathbb{Z}$ formed by the elements of the form $(x_1, \dots, x_m, 0, -x_m, \dots, -x_1; y)$. The relative roots β are of the form $\pm x_i \pm x_j$ for $i \neq j$ and of the form $\pm x_i$ and of the form $\pm 2x_i$, for $i, j \in \{1, \dots, m\}$. For the first and second kind of β we have $l_\beta = 2$, and for the third kind $l_\beta = 1$. Therefore, the affine root system Φ_a consists of the affine functions $\pm x_i \pm x_j + \frac{1}{2}\mathbf{Z}$ and $\pm x_i + \frac{1}{2}\mathbf{Z}$ and $\pm 2x_i + \frac{1}{2} + \mathbf{Z}$. Hence the set of root hyperplanes is the zero sets of the affine functions $\{\pm x_i \pm x_j + \frac{1}{2}\mathbf{Z}; \pm 2x_i + \frac{1}{2}\mathbf{Z}\}$. In this case, the root system Σ is of type C_m , and we obtain

$$W_a = Q^\vee \rtimes W_0 = \tilde{W}_{ad}.$$

Here

$$\begin{aligned} W_0 &= S_m \times \{\pm 1\}^m, \\ Q^\vee &= \mathbb{Z}^m \end{aligned}$$

(whereas $P^\vee = \mathbb{Z}^m + \mathbb{Z}(\frac{1}{2} \sum_{i=1}^m e_i)$). The map

$$X_*(T) \rightarrow X_*(T_{\text{ad}})_\Gamma \xrightarrow{\sim} \mathbb{Z}^m$$

is given by

$$(x_1, \dots, x_{2m+1}, y) \mapsto (x_1 - x_{2m+1}, x_2 - x_{2m}, \dots, x_m - x_{m+2}).$$

The image of $\mu_{r,s}$ is $\lambda_s = (1^{(s)}, 0^{(m-s)})$. Again we identify $W_0 \setminus W_a/W_0$ with

$$Q^\vee \cap \bar{C} = \{x \in \mathbb{Z}^m \mid x_1 \geq x_2 \geq \dots \geq x_m \geq 0\}.$$

In this case the positive coroots are

$$e_i - e_j, e_i + e_j, e_i \ (i < j),$$

and

$$\text{Adm}_0(\mu_{r,s}) = \{\lambda = \lambda_s > \lambda_{s-1} > \dots > \lambda_1 > \lambda_0 = 0\}.$$

Note that λ_s is minuscule if and only if $s = 0$.

3. Affine flag varieties

3.1. Affine flag varieties and the coherence conjecture

Let k be an algebraically closed field. We denote by $K = k((t))$ the field of Laurent power series with coefficients in k . Let G be a reductive algebraic group over K . We will assume for simplicity that the derived group of G is simply connected and absolutely simple, and also that G splits over a tamely ramified extension of K .

Let S be a maximal split torus in G . We fix an alcove C in the apartment corresponding to S , and denote by B the associated Iwahori subgroup of $G(K)$. Let \mathcal{S} be the set of reflections about the walls of C . The parahoric subgroups containing B correspond to the non-empty subsets Y of \mathcal{S} . More precisely, we associate to Y the unique parahoric subgroup containing B such that the associated subgroup $W^Y = W^{P^Y}$ in the sense of (2.1) is equal to the subgroup of W_a generated by the simple reflections for the simple affine roots α_i with $i \notin Y$. In particular, $P^{\mathcal{S}} = B$.

Let LG be the loop group associated to G (cf. [27]). This is the ind-group scheme over k which represents the functor $R \mapsto G(R((t)))$ on the category of k -algebras. To any parahoric subgroup P there is associated, by [3], a smooth affine group scheme with connected fibres over \mathcal{O}_K and with generic fibre equal to G . We will again denote this group scheme by P and let L^+P be the associated group scheme over k . Recall that L^+P represents the functor $R \mapsto P(R[[t]])$ on the category of k -algebras. Let $\mathcal{F}(G) = LG/L^+B$,

respectively $\mathcal{F}^Y(G) = L G/L^+ P^Y$ be the affine flag variety of G , respectively the partial affine flag variety of G corresponding to Y . These fpqc quotients are representable by ind-schemes.

By [27, § 10], there is a canonical isomorphism

$$\text{Pic}(\mathcal{F}^Y(G_{\text{der}})) \xrightarrow{\sim} \bigoplus_{i \in Y} \mathbb{Z} \cdot \epsilon_i, \tag{3.1}$$

given by sending a line bundle \mathcal{L} to the degrees of its restrictions to the projective lines \mathbb{P}_{α_i} corresponding to the simple affine roots α_i , for $i \in Y$. Put $\kappa(i) = 1$, unless the vector part of the simple affine root α_i is a multipliable root, in which case we set $\kappa(i) = 2$ (this last case only arises when G is of type $A_{2m}^{(2)}$). Let $\mathcal{L}'(Y)$ be the line bundle on $\mathcal{F}^Y(G_{\text{der}})$ which is sent to $\sum_i \kappa(i)\epsilon_i$ under (3.1). Then $\mathcal{L}'(Y)$ is ample, i.e. the restriction of $\mathcal{L}'(Y)$ to any Schubert variety S_w is ample, for any $w \in W^Y \setminus W_a/W^Y$.

Let μ be a cocharacter of G_{ad} . Recall from § 2.2.2 the subset $\text{Adm}^Y(\mu)^\circ$ of W_a . More precisely, in the notation of [27] we take $I = \mathbf{S} \setminus Y$ and $I^\circ = \tau_{\text{ad}}^{-1}(I) = \mathbf{S} \setminus Y^\circ$, with $Y^\circ = \tau_{\text{ad}}^{-1}(Y)$; then $\text{Adm}^Y(\mu)^\circ = \text{Adm}_I(\mu)^\circ$. We define the subset $\mathcal{A}^Y(\mu)^\circ$ of $\mathcal{F}^{Y^\circ}(G_{\text{der}})$ as a reduced union of L^+B -orbits

$$\mathcal{A}^Y(\mu)^\circ = \bigcup_{w \in \text{Adm}^Y(\mu)^\circ} L^+B \cdot n_w. \tag{3.2}$$

Note that the union of Schubert varieties in $\mathcal{F}^Y(G_{\text{ad}})$ defined by

$$\mathcal{A}^Y(\mu) = \bigcup_{w \in \text{Adm}^Y(\mu)} S_w$$

is the translation by τ of $\mathcal{A}^Y(\mu)^\circ$.

We recall the *coherence conjecture* from [27, § 10]. It concerns the dimensions $h_Y^{(\mu)} = \dim H^0(\mathcal{A}^Y(\mu)^\circ, \mathcal{L}'(Y)^{\otimes k})$, as Y ranges over the non-empty subsets of \mathbf{S} . For any minuscule coweight μ of G_{ad} we introduce the polynomial

$$h^{(\mu)}(k) = \dim H^0(X(\mu), \mathcal{L}(\mu)^{\otimes ek}). \tag{3.3}$$

Here $e = [K' : K]$ is the degree of the splitting field K' of G , and $X(\mu) = G_{K'}/P(\mu)$ is the homogeneous projective variety associated to μ , and $\mathcal{L}(\mu)$ is the ample generator of the Picard group of $X(\mu)$. If $\mu = \mu_1 + \dots + \mu_r$ is a sum of minuscule coweights, we set $h^{(\mu)} = h^{(\mu_1)} \dots h^{(\mu_r)}$. The conjecture then states that

$$h_Y^{(\mu)}(k) = h^{(\mu)}(|Y| \cdot k), \tag{3.4}$$

provided that μ is a sum of minuscule coweights for G_{ad} . In [27] this conjecture is proved for $G = \text{GL}_n$ and for $G = \text{GSp}_{2n}$. In the present paper we need the conjecture for $G = \text{GU}(V, \phi)$, in which case it is an open question.

3.2. Unitary affine flag varieties

Assume $\text{char}(k) \neq 2$. We consider the unitary similitude group described in § 1.2 in the special case when $F_0 = K = k((t))$ with uniformizer $\pi_0 = t$, and $F = K' = k((u))$ with uniformizer u with $u^2 = t$. We also have the hermitian vector space (V, ϕ) of dimension $n \geq 3$ over K' , which we assume split, with distinguished basis e_1, \dots, e_n . In this case, we will denote the standard lattice chain by $\lambda_0, \dots, \lambda_{n-1}$. As in § 1.4, we can complete this into a periodic self-dual lattice chain λ in $V = K'^n$. We denote by $G = \text{GU}(V, \phi)$ the group of unitary similitudes, which is an algebraic group over K , of the type considered in § 3.1. The simultaneous stabilizer of all the lattices in the lattice chain λ is an Iwahori subgroup B as before. As in § 1.2 we set $m = \lfloor n/2 \rfloor$.

Let $I \subset \{0, \dots, m\}$ be a non-empty subset. We also require as usual for $n = 2m$ that if $m - 1 \in I$, then also $m \in I$ (cf. § 1.2). Let us write $I = \{i_0 < i_1 < \dots < i_k\}$. We consider the part of the ‘standard’ lattice chain

$$\lambda_{i_0} \subset \lambda_{i_1} \subset \dots \subset \lambda_{i_k} \subset u^{-1}\lambda_{i_0}.$$

We can complete this to a periodic self-dual lattice chain $\lambda_I = \{\lambda_j\}_j$ in $V = K'^n$ as in § 1.4. (Recall that here j is of the form $k \cdot n \pm i$, $i \in I$, $k \in \mathbf{Z}$, and $\lambda_{k \cdot n - i} = u^{-k} \cdot \hat{\lambda}_i$, $\lambda_{k \cdot n + i} = u^{-k} \cdot \lambda_i$.)

Let us now consider the functor \mathcal{F}_I which to a k -algebra R associates the set of pairs of $R[[u]]$ -lattice chains

$$L_{i_0} \subset L_{i_1} \subset \dots \subset L_{i_k} \subset u^{-1}L_{i_0}$$

in $V \hat{\otimes}_{K'} R((u)) = R((u))^n$ together with an element $\alpha \in R((t))^\times / R[[t]]^\times$ which satisfy the following conditions.

- (a) For any $q \in \{0, \dots, k\}$, we have

$$L_{i_q} \subset u^{-1}\alpha^{-1}\hat{L}_{i_q} \subset u^{-1}L_{i_q}.$$

- (b) The quotients $L_{i_{q+1}}/L_{i_q}$, $u^{-1}\alpha^{-1}\hat{L}_{i_q}/L_{i_q}$, $u^{-1}L_{i_0}/L_{i_{q+1}}$ are projective R -modules of rank equal to the rank of the corresponding quotients for the standard chain (when $q = k$, these conditions have to be interpreted in the obvious way).

Here, we may think of α as corresponding to a ‘similitude factor’. The ind-group scheme $\text{L}G$ over k acts naturally on \mathcal{F}_I by $g \cdot (\{L_i\}, \alpha) \mapsto (\{g \cdot L_i\}, c(g)^{-1} \cdot \alpha)$. By following the arguments of [27, § 4] (which deals with the case of the unitary group, i.e. when $\alpha = 1$), we see that there is an $\text{L}G$ -equivariant isomorphism $\text{L}G/\text{L}^+P_I \simeq \mathcal{F}_I$ of sheaves for the fpqc topology. Note, however, that if $n = 2m$, \mathcal{F}_I is not always a partial flag variety associated to G , since the stabilizer group P_I is not always a parahoric subgroup. In fact, this happens if and only if $m \notin I$ (cf. § 1.2). In this case the parahoric subgroup P_I^0 has index 2 in P_I and the partial flag variety $\text{L}G/\text{L}^+P_I^0$ is isomorphic to the disjoint sum of two copies of \mathcal{F}_I .

Recall from § 1.2.3 that we identified the local Dynkin diagram of $\text{SU}(V, \phi)$ with $\{0, \dots, m\}$ when $n = 2m + 1$ is odd, and with $\{0, \dots, m - 2, m, m'\}$ when $n = 2m$

is even. Using the general notation for affine flag varieties introduced in §3.1, we have $LG/L^+P_I^0 = \mathcal{F}^Y$. Here $Y = I$ when n is odd or n is even and $m - 1 \notin I$. If n is even and $m - 1 \in I$, then $Y = (I \setminus \{m - 1\}) \cup \{m'\}$.

3.3. Embedding of the special fibres of local models in partial affine flag varieties

We now return to the set-up of §1.5. In particular, we fix integers r, s with $r + s = n$ and denote by E the reflex field for $(F/F_0, r, s)$. Let k denote an algebraic closure of the residue field of \mathcal{O}_E . Similarly to [27] we construct a natural embedding of the geometric special fibres $\bar{M}_I^{\text{naive}} = M_I^{\text{naive}} \otimes_R k$ into a partial affine flag variety associated to the unitary similitude group G . For this we fix identifications compatible with the actions of π , respectively u ,

$$\Lambda_j \otimes_{\mathcal{O}_{F_0}} k = \lambda_j \otimes_{k[[t]]} k,$$

which sends the natural bases of each side to one another. We therefore also obtain a $k[[u]]/(u^2)$ -module chain isomorphism

$$\Lambda_\bullet \otimes_{\mathcal{O}_{F_0}} k \simeq \lambda_\bullet \otimes_{k[[t]]} k,$$

which is in fact an isomorphism of polarized periodic module chains [29]. Let R be a k -algebra. For an R -valued point of M^{naive} we have

$$\mathcal{F}_j \subset \Lambda_j \otimes_{\mathcal{O}_{F_0}} R = (\lambda_j \otimes_{k[[t]]} k) \otimes_k R.$$

Let $L_j := L_{\mathcal{F}_j} \subset \lambda_j \otimes_{k[[t]]} R[[t]]$ be the inverse image of \mathcal{F}_j under the canonical projection

$$\lambda_j \otimes_{k[[t]]} R[[t]] \rightarrow \lambda_j \otimes_{k[[t]]} R.$$

Also set $\alpha = -t^{-1}$. Then for this choice of α ,

$$L_0 \subset L_1 \subset \dots \subset L_m \subset u^{-1}L_0 = L_n$$

is a lattice chain in $R((u))^n$ which satisfies conditions (a) and (b) in §3.2. This gives a well-defined point in $\mathcal{F}(R) = (LG/L^+B)(R)$. We obtain in this way a morphism

$$\iota : M^{\text{naive}} \otimes_{\mathcal{O}_E} k \rightarrow \mathcal{F},$$

which is a closed immersion (of ind-schemes).

In the case of incomplete lattice chains, one can proceed in a similar way and obtain an embedding

$$\iota_I : M_I^{\text{naive}} \otimes_{\mathcal{O}_E} k \hookrightarrow \mathcal{F}_I.$$

For $I' \supset I$, the following diagram is commutative:

$$\begin{CD} M_{I'}^{\text{naive}} \otimes_{\mathcal{O}_E} k @>>> \mathcal{F}_{I'} \\ @VVV @VVV \\ M_I^{\text{naive}} \otimes_{\mathcal{O}_E} k @>>> \mathcal{F}_I \end{CD}$$

The horizontal morphisms are equivariant for the actions of L^+P_I , respectively L^+P_I , in the sense of [26, § 6].

Note that in [27], we have set $L_j := u^{-1}L_{\mathcal{F}_j}$ to produce an embedding of $M_I^{\text{naive}} \otimes_{\mathcal{O}_E} k$ into $\text{LU}_n/L^+P_I(\text{U}_n)$, where $P_I(\text{U}_n)$ is the stabilizer of our lattice chain in the unitary group. Observe that the similitude of scalar multiplication by u^{-1} is our choice of $\alpha = -t^{-1}$ here. Hence, we obtain a commutative diagram

$$\begin{array}{ccc}
 M_I^{\text{naive}} \otimes_{\mathcal{O}_E} k & \xrightarrow{\mathcal{F}_j \mapsto u^{-1} \cdot L_{\mathcal{F}_j}} & \text{LU}_n/L^+P_I(\text{U}_n) \\
 & \searrow i_I & \downarrow u \cdot \text{id} \\
 & & \mathcal{F}_I = \text{LG}/L^+P_I
 \end{array}$$

Let $\mu = \mu_{r,s}$ denote the cocharacter of G given under the isomorphism $G \otimes_K K' \cong \text{GL}_n \times \mathbb{G}_m$ by

$$\mu_{r,s}(z) = (\text{diag}(z^{(s)}, 1^{(r)}), z). \tag{3.5}$$

Let $\mathcal{A}(\mu) = \bigcup_{w \in \text{Adm}(\mu)} S_w$. This is a closed reduced subset of the full flag variety $\mathcal{F} = \text{LG}/L^+B$. Similarly, for $I \subset \{0, \dots, m\}$, we denote by $\mathcal{A}^I(\mu)$ the image of $\mathcal{A}(\mu)$ in \mathcal{F}_I .

Proposition 3.1. *$\mathcal{A}^I(\mu)$ is contained in the image of $M_I^{\text{loc}} \otimes_{\mathcal{O}_E} k$ under ι_I .*

It obviously suffices to prove this in the case of the full flag variety, i.e. for $I = \{0, \dots, m\}$. This is done in the next subsection.

3.4. Lifting of points in μ -strata to the generic fibre

Recall that we are treating the Iwahori case of signature (r, s) with $s \leq r$. Recall our notation of the standard lattices Λ_j ; these come with distinguished \mathcal{O}_F -generators. For example, for Λ_0 these are e_1, \dots, e_n . For Λ_m , if $n = 2m$ is even, these are $f_1 := -\pi^{-1}e_1, \dots, f_m := -\pi^{-1}e_m, f_{m+1} := e_{m+1}, \dots, f_n := e_n$. We have

$$\langle \pi e_i, e_{n+1-j} \rangle = \delta_{i,j}$$

and (for $n = 2m$)

$$(f_i, \pi f_{n+1-j}) = \pm \delta_{i,j}$$

where $\langle \cdot, \cdot \rangle$, respectively (\cdot, \cdot) , is the alternating, respectively symmetric, form associated to the hermitian form ϕ , as in § 1.2.2. In particular, Λ_0 , respectively Λ_m is self-dual for $\langle \cdot, \cdot \rangle$, respectively (\cdot, \cdot) . In general, if

$$\Lambda = \text{span}_{\mathcal{O}_F} \{ \pi^{-1}e_1, \dots, \pi^{-1}e_i, e_{i+1}, \dots, e_n \},$$

we will set

$$f_1^\Lambda := -\pi^{-1}e_1, \quad \dots, \quad f_i^\Lambda := -\pi^{-1}e_i, \quad f_{i+1}^\Lambda := e_{i+1}, \quad \dots, \quad f_n^\Lambda := e_n.$$

If T is a subset of $\{1, \dots, n\} = [1, n]$, we will set $T^* = \{n + 1 - a \mid a \in T\}$. We denote by S a subset of $[1, n]$ of cardinality s ; then $R = R_S$ will denote the complement of S^* .

Note that R has cardinality $r = n - s$ and that if $S = [1, s]$, then $R = [1, r]$. We will consider subsets S of cardinality s for which $S \cap S^* = \emptyset$, i.e. $S \subset R_S$.

Set

$$\tilde{\pi} = \pi \otimes 1 - 1 \otimes \pi \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F.$$

We will write this as $\tilde{\pi} = \pi - \sqrt{\pi_0}$. For such a subset S we will consider

$$\mathcal{F}_S^A = \langle f_{S^*}^A, \pi f_{S^*}^A, \tilde{\pi} f_{R \setminus S}^A \rangle \subset \Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F,$$

where $f_T^A = \{f_t^A\}_{t \in T}$ and where $\langle \cdot, \cdot \rangle$ denotes the \mathcal{O}_{F_0} -module generated by these vectors (not to be confused with the notation for the alternating form). Notice that $R \setminus S = [1, n] \setminus (S \cup S^*)$. We claim this corresponds to an \mathcal{O}_F -point of the naive local model for signature (r, s) and the complete lattice chain. Note the following identity:

$$\pi \cdot (\tilde{\pi}v) = -\sqrt{\pi_0} \cdot (\tilde{\pi}v),$$

so that $\tilde{\pi}v$ is an eigenvector for the action of π with eigenvalue $-\sqrt{\pi_0} = -1 \otimes \pi$. This shows that \mathcal{F}_S^A is stable for the action of $\pi = \pi \otimes 1$ and the characteristic polynomial is

$$\det(T \cdot I - \pi \mid \mathcal{F}_S^A) = (T^2 - \pi_0)^s (T - \sqrt{\pi_0})^{r-s} = (T + \sqrt{\pi_0})^r (T - \sqrt{\pi_0})^s.$$

Now we explain why \mathcal{F}_S^A gives an \mathcal{O}_F -point of M^{naive} . It is clear that the sequence respects the inclusions conditions of § 1.5.1 (b). It remains to check the isotropy condition (c). Since this is a closed condition it is enough to check it over F . Since $\Lambda \otimes_{\mathcal{O}_{F_0}} F = \Lambda_0 \otimes_{\mathcal{O}_{F_0}} F$ it is enough to check that

$$\langle \mathcal{F}_S^{A_0}, \mathcal{F}_S^{A_0} \rangle = 0$$

for all such S . Recall that $f_j^{A_0} = e_j$ for all j . Since $S \cap S^* = \emptyset$, we have $\langle e_i, \pi e_j \rangle = 0$ if $i, j \in S^*$. We also have

$$\langle \tilde{\pi}v, \tilde{\pi}w \rangle = \langle \pi v - \sqrt{\pi_0}v, \pi w - \sqrt{\pi_0}w \rangle = 0.$$

Also $\langle e_i, e_j \rangle = 0$ for all i, j and $\langle e_i, \tilde{\pi}e_j \rangle = 0$ if $i \in S^*, j \in R \setminus S$. Similarly $\langle \pi e_i, \tilde{\pi}e_j \rangle = 0$ for $i \in S^*, j \in R \setminus S$.

This shows that \mathcal{F}_S^A gives an \mathcal{O}_F -point of M^{naive} . Denote by

$$\bar{\mathcal{F}}_S^A \subset \Lambda \otimes_{\mathcal{O}_{F_0}} (\mathcal{O}_F / (\pi) \otimes_{\mathcal{O}_F / (\pi)} k) = \Lambda \otimes_{\mathcal{O}_{F_0}} k$$

the reduction of the subspaces \mathcal{F}_S^A modulo the maximal ideal of \mathcal{O}_F . Denote by L_{S_\bullet} the lattice chain that corresponds to the subspaces $\bar{\mathcal{F}}_S^A$ as in § 3.2. Then

$$u^2 \lambda_\bullet \subset L_{S_\bullet} \subset \lambda_\bullet.$$

(We also think of L_{S_\bullet} as giving a point of the affine flag variety $\mathcal{F} = \text{LG}/L^+B$ for the unitary similitude group G .)

Recall from §§ 2.4.1 and 2.4.2 that $W_0 = S_m \rtimes \{\pm 1\}^m$, no matter whether n is even or odd. For each $w \in W_0$, and each subset $S \subset [1, n]$ of cardinality $s \leq m$ with $S \cap S^* = \emptyset$, $w \cdot S$ is of this type again and this action is transitive.

Lemma 3.2. *The relative position $\text{inv}(\lambda_\bullet, L_{S\bullet})$ in the Iwahori–Weyl group \tilde{W} of G is $w_S(\lambda)$. Here λ is as in § 2.2.2 the image of $\mu \in X_*(T)$ in the coinvariants $X_*(T)_I$ and $w_S \in W_0$ has the property that $w_S \cdot [1, s] = S$.*

Proof. Here we may think of the Iwahori–Weyl group \tilde{W} of the unitary similitude group $G \subset \text{Res}_{F/F_0}(\text{GL}_n \times \mathbf{G}_m)$ as a subgroup of the Iwahori–Weyl group for $\text{GL}_n \times \mathbf{G}_m$. This follows easily when we represent these groups as quotients of normalizer groups; on the translation part this induces the injection given by the norm

$$X_*(T)_I \rightarrow X_*(T) = \mathbf{Z}^n \times \mathbf{Z}. \tag{3.6}$$

(Note here that $X_*(T)_I$ is torsion free.) Under this map the element λ is sent to

$$\text{Norm}(\mu_{r,s}) = ((2^s, 1^{r-s}, 0^s), 2).$$

Set

$$\tau_{r,s} = \text{diag}(-t^{(s)}, u^{(r-s)}, 1^{(s)})$$

for a corresponding torus element in the unitary similitude group.

Set $\Lambda = \Lambda_0$ if n is odd and $\Lambda = \Lambda_m$ if $n = 2m$ is even (this amounts to choosing a special vertex for G). Similarly, set λ for λ_0 or λ_m , and L_S for the lattices corresponding to $\tilde{\mathcal{F}}_S^{\Lambda_0}$ or $\tilde{\mathcal{F}}_S^{\Lambda_m}$ respectively. Then

$$L_{[1,s]\bullet} = \tau_{r,s} \cdot \lambda_\bullet.$$

This shows that the claim is true for $S = [1, s]$ and $w_{[1,s]}$ the identity; the same is true for any other choice of $w_{[1,s]}$ since each such choice centralizes $\tau_{r,s}$. Now write $\lambda_j = t_j \cdot \lambda$ where t_j is a (diagonal) translation element for GL_n . Then we also have $(L_S)_j = t_j \cdot L_S$ for any j . Now we think of elements of W_0 as permutation matrices in GL_n in the standard basis f_i^Λ of Λ . We can see that the permutation w_S takes the $k[[u]]$ -basis

$$\{u^2 f_1^\Lambda, \dots, u^2 f_s^\Lambda, u f_{s+1}^\Lambda, \dots, u f_r^\Lambda, f_{r+1}^\Lambda, \dots, f_n^\Lambda\}$$

of $L_{[1,s]}$ to the basis

$$\{f_{S^*}^\Lambda, u f_{R-S}^\Lambda, u^2 f_S^\Lambda\}$$

of L_S . This shows that $L_S = w_S \cdot L_{[1,s]}$ which combined with the above gives

$$(L_S)_j = t_j w_S L_{[1,s]} = t_j w_S \tau_{r,s} \lambda = t_j w_S \tau_{r,s} w_S^{-1} \lambda = t_j w_S \tau_{r,s} w_S^{-1} t_j^{-1} \lambda_j. \tag{3.7}$$

This amounts to $(L_S)_\bullet = w_S \tau_{r,s} w_S^{-1} \Lambda_\bullet$ which proves the claim. □

The claim shows that the \mathcal{O}_F -points of M^{loc} that we constructed above (one for each subset S) reduce to points lying in the Schubert varieties that correspond to the extreme elements of the set $\text{Adm}(\mu)$; all these extreme elements are then obtained by this construction. Since $\text{Adm}(\mu)$ is closed under the Bruhat order, the set $\mathcal{A}(\mu)$ is closed in \mathcal{F} , and is contained in the closed subset $\iota(M^{\text{loc}} \otimes_{\mathcal{O}_E} k)$. This proves Proposition 3.1.

4. The structure of local models

4.1. Consequences of the coherence conjecture

It is shown in [27, § 11] that as a consequence of Proposition 3.1, we have the following statement.

Theorem 4.1. *Assume the validity of the coherence conjecture for the pair $(G, \mu) = (\text{GU}(V, \phi), \mu_{r,s})$. Then the geometric special fibre of M_I^{loc} is reduced and is isomorphic to $\mathcal{A}^I(\mu)$. Its irreducible components are normal and with rational singularities.*

4.2. Vertex-wise admissibility

We conjecture that admissibility for the pair $(G, \mu) = (\text{GU}(V, \phi), \mu_{r,s})$ has a vertex by vertex characterization, in the following sense.

Conjecture 4.2.

$$\text{Adm}^I(\mu) = \bigcap_{i \in I} \pi_{I, \{i\}}^{-1}(\text{Adm}^{\{i\}}(\mu))$$

(intersection in $W^I \backslash \tilde{W} / W^I$).

Note that in general, the Bruhat order between two given elements in an affine Weyl group cannot be tested vertex by vertex (cf. [15]), which makes this a somewhat daring conjecture.

We also have a conjectural description of the extreme admissible elements. Recall that we are assuming $s \leq r$. By the previous section, all the extreme elements of the set $\text{Adm}^I(\mu_{r,s})$ correspond to sequences of subspaces \mathcal{F}_i for which the rank of $(\pi|_{\mathcal{F}_i})$ is maximal, i.e. equal to s , for all $i \in I$.

Conjecture 4.3. *The converse is true: given a sequence of subspaces \mathcal{F}_i (in the special fibre of the naive local model M_I^{naive}) for which the rank of $(\pi|_{\mathcal{F}_i})$ is equal to s for all $i \in I$, then the corresponding stratum is admissible and extreme, i.e. of the form $w_0(\lambda)$ for some $w_0 \in W_0$ (or rather its image in $W^I \backslash \tilde{W} / W^I$).*

Remark 4.4. For $\text{Res}_{F/F_0} \text{GL}_n$ (here again F/F_0 is a ramified quadratic extension) this statement is true; this follows from [25]. In fact, in this case, an element of the naive local model lies in a Schubert cell for an element of $\text{Adm}^I(\mu)$ if and only if the rank of π on each subspace is less than or equal to s and the extreme elements are the ones for which the rank is s . This stronger statement is false in the case of a ramified unitary group. For example, let $n = 2m$ be even and let $I = \{m\}$. Then the subspace $\mathcal{F} = \pi A_m / \pi_0 A_m$ of $A_m / \pi_0 A_m$ has rank of $\pi|_{\mathcal{F}}$ equal to zero, but does not belong to $\mathcal{A}^I(\mu)$ if s is odd (cf. § 2.4.1).

In the global context of § 1.4, Görtz and Wedhorn ask whether the points in the special fibre of $\mathcal{A}_{C'}$, for which the stronger statement fails, are all supersingular.

Proposition 4.5. *Assume the validity of Conjecture 4.2. Then*

$$\mathcal{A}^I(\mu) = \bigcap_{i \in I} \pi_{I, \{i\}}^{-1}(\mathcal{A}^{\{i\}}(\mu))$$

(scheme-theoretic intersection inside $\mathcal{F}_I = \mathrm{LG}/L^+P_I$). Assuming in addition the coherence conjecture (for the pair $(G, \mu) = (\mathrm{GU}(V, \phi), \mu_{r,s})$,

$$M_I^{\mathrm{loc}} = \bigcap_{i \in I} \pi_{I, \{i\}}^{-1}(M_{\{i\}}^{\mathrm{loc}})$$

(scheme-theoretic intersection inside M_I^{naive}).

Proof. In either case, it is clear that the left-hand side is contained in the right-hand side. It suffices to prove the second identity for the special fibres, in which case it follows from the first identity and Theorem 4.1, since we are assuming the coherence conjecture. The first identity on the level of sets is an immediate consequence of Conjecture 4.2. Since by [27, Theorem 8.4] the Schubert varieties in \mathcal{F} are Frobenius split, the standard argument allows us to conclude that the scheme-theoretic intersection involved is reduced. \square

5. Special parahorics

In this section we consider special parahorics in the sense of Bruhat–Tits [33], corresponding to the cases $I = \{0\}$ when n is odd and $I = \{m\}$ when $n = 2m$ is even. One further case of a special parahoric occurs for $n = 2m + 1$ odd and $I = \{m\}$. This case is considered in a recent preprint by Arzdorf [1].

In the beginning of this section, we give affine charts for the local models M_I^{naive} and M_I^\wedge . For simplicity, we will omit the subscript $\{0\}$ or $\{m\}$ and write simply M^{naive} , M^\wedge , etc. As we will see, the reduced special fibres of these charts are the closures of certain nilpotent orbits for the ‘classical symmetric pairs’ $(\mathfrak{gl}(n), \mathfrak{o}(n))$, respectively $(\mathfrak{gl}(n), \mathfrak{sp}(n))$.

5.1. An affine chart

We distinguish two cases.

(A) Assume that $n = 2m + 1$ is odd; then we take $I = \{0\}$. Recall that $\Lambda := \Lambda_0$ is equipped with the perfect alternating form $\langle \cdot, \cdot \rangle$ which satisfies $\langle \pi e_i, e_{n+1-j} \rangle = \delta_{ij}$. The corresponding \mathcal{O}_{F_0} -basis is

$$\{e_1, \dots, e_n, \pi e_1, \dots, \pi e_n\}$$

and the matrix of $\langle \cdot, \cdot \rangle$ in this basis is

$$J_{2n} = \begin{pmatrix} 0 & -H_n \\ H_n & 0 \end{pmatrix},$$

where H_n is the unit antidiagonal matrix (of size n). Let $\mathcal{L} = \langle e_1, \dots, e_n \rangle$ be the standard Lagrangian \mathcal{O}_{F_0} -direct summand of Λ , and denote by Q the Siegel parabolic subgroup of $\mathrm{Sp}(\Lambda, \langle \cdot, \cdot \rangle)$ that preserves \mathcal{L} .

(B) Assume that $n = 2m$ is even; then we take $I = \{m\}$. Recall that $\Lambda := \Lambda_m$ is equipped with the perfect symmetric form (\cdot, \cdot) . Set $f_i = -\pi^{-1}e_i$, for $1 \leq i \leq m$, $f_i = e_i$, for $m + 1 \leq i \leq n$. Consider the \mathcal{O}_{F_0} -basis of Λ_m given by

$$\{f_1, \dots, f_n, -\pi f_1, \dots, -\pi f_m, \pi f_{m+1}, \dots, \pi f_n\}.$$

The matrix of (\cdot, \cdot) in this basis is H_{2n} . Here we let Q be the parabolic subgroup of $O(\Lambda_m, (\cdot, \cdot))$ that preserves the totally isotropic \mathcal{O}_{F_0} -direct summand $\mathcal{L} = \langle f_1, \dots, f_n \rangle$.

In each case, the Lie algebra $\text{Lie}(Q)$ consists of the \mathcal{O}_{F_0} -endomorphisms A of Λ that satisfy $A(\mathcal{L}) \subset \mathcal{L}$ and

$$\langle Av, w \rangle + \langle v, Aw \rangle = 0, \quad \text{respectively } (Av, w) + (v, Aw) = 0. \tag{5.1}$$

For simplicity, denote by X the restriction of A to \mathcal{L} . Now let $N_{r,s}$ be the scheme of endomorphisms A in the Lie algebra $\text{Lie}(Q) \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_E$ that satisfy the equations

$$A^2 = \pi_0 \cdot I_{2n}$$

and

$$\text{char}_X(T) = (T - \sqrt{\pi_0})^s (T + \sqrt{\pi_0})^r.$$

Also, denote by $N_{r,s}^\wedge$ the closed subscheme of $N_{r,s}$ given by A which when $r \neq s$ satisfy in addition:

$$\wedge^{r+1}(X - \sqrt{\pi_0} \cdot I) = 0, \quad \wedge^{s+1}(X + \sqrt{\pi_0} \cdot I) = 0. \tag{5.2}$$

As in [25, (1.3)] we see that there is a diagram

$$M^{\text{naive}} \xleftarrow{\pi} \tilde{M}^{\text{naive}} \xrightarrow{\phi} N_{r,s}, \tag{5.3}$$

where π is a Q -torsor, and where ϕ is a smooth morphism. Here

$$\tilde{M}^{\text{naive}}(S) = \{(\mathcal{F} \subset \Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S, \alpha)\},$$

where \mathcal{F} gives a point of $M^{\text{naive}}(S)$ and α is a choice of basis v_1, \dots, v_{2n} of $\Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ with matrix J_{2n} , respectively H_{2n} , for the bilinear form on $\Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ and such that the subspace \mathcal{F} is generated by v_1, \dots, v_n . Here $\phi((\mathcal{F}, \alpha))$ is given by the endomorphism $\alpha^{-1} \cdot \pi \cdot \alpha$ (which we can express by a matrix A using $\{v_i\}$). Similarly, by restriction along the closed immersions $M^\wedge \subset M^{\text{naive}}$, $N_{r,s}^\wedge \subset N_{r,s}$ we also obtain

$$M^\wedge \xleftarrow{\pi^\wedge} \tilde{M}^\wedge \xrightarrow{\phi^\wedge} N_{r,s}^\wedge, \tag{5.4}$$

with π^\wedge a Q -torsor and ϕ^\wedge smooth again. Notice that the morphisms ϕ, ϕ^\wedge are not surjective; for example, the closed fibre of their image lies in the open subset where the rank of A is exactly n .

5.2. The odd case

Assume we are in case (A). For each signature (r, s) , we can consider the point of $M^\wedge(k) \subset M^{\text{naive}}(k)$ given by

$$\mathcal{F}_0 = \pi \cdot (\Lambda_0 \otimes_{\mathcal{O}_{F_0}} k) \subset \Lambda_0 \otimes_{\mathcal{O}_{F_0}} k.$$

To obtain an affine chart of the naive local model around this point consider the subspaces \mathcal{F} which are of the form

$$\mathcal{F} = \{\pi \cdot v + X \cdot v \mid v \in \mathcal{O}_S^n\},$$

with v a column vector in $\mathcal{O}_S^n = \mathcal{O}_S \cdot e_1 \oplus \cdots \oplus \mathcal{O}_S \cdot e_n$. Then we can see as in [23, p. 596] that the corresponding open subscheme $U_{r,s}$ of the local model M^{naive} is the scheme of $(n \times n)$ -matrices X which satisfy

$$X^2 = \pi_0 \cdot I, \quad {}^tX = HXH, \quad \text{char}_X(T) = (T - \sqrt{\pi_0})^s(T + \sqrt{\pi_0})^r. \tag{5.5}$$

Here $H = H_n$. Similarly, the corresponding open subscheme $U_{r,s}^\wedge$ of the wedge local model M^\wedge is the scheme of $(n \times n)$ -matrices X which satisfy the equations (5.5) above together with the equations of minors (5.2).

To connect this with the previous picture, notice that over $U_{r,s}$ we can split the Q -torsor π by picking the symplectic basis $\{v_i\}$ of $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ to be $\pi e_1 + X e_1, \dots, \pi e_n + X e_n, -e_1, \dots, -e_n$. Then $U_{r,s}$ can be identified with a subscheme of the scheme of matrices A of $N_{r,s}$ of the form

$$A = \begin{pmatrix} X & -I \\ 0 & -X \end{pmatrix}.$$

Similarly for $U_{r,s}^\wedge$.

5.3. The even case

Assume now that we are in case (B). We can see as above that an open affine chart $U_{r,s}$ of M^{naive} around the point $\mathcal{F}_0 = \pi(\Lambda_m \otimes_{\mathcal{O}_{F_0}} k) \subset \Lambda_m \otimes_{\mathcal{O}_{F_0}} k$ is given by the scheme of $n \times n$ -matrices X which satisfy

$$X^2 = \pi_0 \cdot I, \quad {}^tX = -JXJ, \quad \text{char}_X(T) = (T - \sqrt{\pi_0})^s(T + \sqrt{\pi_0})^r. \tag{5.6}$$

Here $J = J_n$. Similarly, the corresponding open subscheme $U_{r,s}^\wedge$ of M^\wedge is the scheme of X which satisfy the equations (5.6) above together with the equations of minors (5.2).

Suppose $s \neq 0$. It is then also useful to consider affine charts ${}_1U_{r,s}, {}_1U_{r,s}^\wedge$ of $M^{\text{naive}}, M^\wedge$ around the point

$$\mathcal{F}_1 = \langle f_1, \pi f_1, \pi f_2, \dots, \pi f_{n-1} \rangle \subset \Lambda_m \otimes_{\mathcal{O}_{F_0}} k$$

of $M^\wedge(k) \subset M^{\text{naive}}(k)$ (especially when s is odd). Set

$$\begin{aligned} A' &= \text{span}_{\mathcal{O}_{F_0}} \{f_1, \pi f_1, \pi f_2, \dots, \pi f_{n-1}\}, \\ A'' &= \text{span}_{\mathcal{O}_{F_0}} \{f_n, \pi f_n, f_2, \dots, f_{n-1}\} \end{aligned}$$

so that $\Lambda_m = A' \oplus A''$. Notice that the matrix of the form (\cdot, \cdot) in the basis

$$\{f_1, \pi f_1, \pi f_2, \dots, \pi f_{n-1}, f_n, \pi f_n, f_2, \dots, f_{n-1}\} \tag{5.7}$$

is

$$\begin{pmatrix} 0 & S \\ {}^tS & 0 \end{pmatrix},$$

where S is the skew matrix of size n :

$$S = \begin{pmatrix} {}^tJ_2 & 0 \\ 0 & J_{n-2} \end{pmatrix}.$$

Now we can find an affine chart ${}_1U_{r,s}$ around the point \mathcal{F}_1 by looking at graphs of linear maps $f : A' \rightarrow A''$ so that

$$\mathcal{F} = \{v + f(v) \mid v \in A' \otimes \mathcal{O}_S\}.$$

Write the map f by an $n \times n$ -matrix

$$X = \begin{pmatrix} T & B \\ C & Y \end{pmatrix}$$

by using the basis (5.7) above. Here T is a 2×2 -matrix and Y an $(n - 2) \times (n - 2)$ -matrix; the matrices B and C are of sizes $2 \times (n - 2)$ and $(n - 2) \times 2$ respectively. We can see that the condition that \mathcal{F} is isotropic is given by

$$S {}^tX = XS. \tag{5.8}$$

This translates to $J_2 \cdot {}^tT = T \cdot J_2$, $J_{n-2} \cdot {}^tY = Y \cdot J_{n-2}$ and $C \cdot {}^tJ_2 = J_{n-2} \cdot {}^tB$. The first condition implies that T is diagonal and scalar, $T = \text{diag}(x, x)$; the last condition shows that C is determined by B . The condition that \mathcal{F} is π -stable translates to

$$Y^2 = \pi_0 \cdot I_{n-2}, \quad B_1 = B_2 \cdot Y,$$

where B_i is the i th row of B ; therefore B_1 is determined by B_2 . We can see that the action of π on \mathcal{F} is now given by the block sum of the matrices $\begin{pmatrix} 0 & \pi_0 \\ 1 & 0 \end{pmatrix}$ and Y . This allows us to translate the characteristic polynomial and exterior power conditions to conditions about the matrix Y . We obtain

$$\text{char}_Y(T) = (T - \sqrt{\pi_0})^{s-1} (T + \sqrt{\pi_0})^{r-1}, \tag{5.9}$$

and for $M_{r,s}^\wedge$

$$\wedge^r(Y - \sqrt{\pi_0} \cdot I_{n-2}) = 0, \quad \wedge^s(Y + \sqrt{\pi_0} \cdot I_{n-2}) = 0. \tag{5.10}$$

We conclude that in this case, the chart ${}_1U_{r,s}$ is the product of the affine scheme $U_{r-1,s-1}$ with an affine space of dimension $n - 1$ over \mathcal{O}_E (which corresponds to the free coordinates x , $B_2 = (b_{21}, \dots, b_{2n-2})$). In particular, we can see that if s is 1, then ${}_1U_{r,s}$ is smooth.

5.4. Symmetric pairs

Let \mathfrak{g} be a reductive Lie algebra over a field k of odd characteristic. Let θ be a non-trivial involution of \mathfrak{g} . Let $\mathfrak{g} = \mathfrak{f} + \mathfrak{p}$ be the Cartan decomposition so that $\mathfrak{f} = \{X \in \mathfrak{g} : \theta(X) = X\}$, $\mathfrak{p} = \{X \in \mathfrak{g} : \theta(X) = -X\}$. Then $(\mathfrak{g}, \mathfrak{f})$ form a ‘symmetric pair’; we

will call \mathfrak{p} the associated vector space (we can think of it as an infinitesimal version of a symmetric space). Now let G be the adjoint group of \mathfrak{g} and let $H = G^\theta$ be the subgroup of elements of G fixed by the involution. The group H acts on \mathfrak{p} (via the adjoint action). The orbits of H on \mathfrak{p} , and in particular the orbits of elements of \mathfrak{p} which are nilpotent in \mathfrak{g} , have been studied by several people starting with Kostant and Rallis (see, for example, [17, 18, 22]). When $k = \mathbf{R}$, \mathfrak{g} is semisimple and θ is a Cartan involution, then $H(\mathbf{R})$ is compact and one has the so-called Kostant–Sekiguchi correspondence [31]: this is a bijective correspondence between the nilpotent $H_{\mathbf{C}}$ orbits of $\mathfrak{p}_{\mathbf{C}}$ and the nilpotent orbits of $H(\mathbf{R})$ on the real Lie algebra $\mathfrak{g}(\mathbf{R})$. (This plays no role in what follows.) Here we consider two ‘classical cases’.

Let V be a vector space over k of dimension n with a non-degenerate bilinear form $h : V \times V \rightarrow k$ which is either symmetric or alternating. If $X \in \text{End}(V)$ the adjoint X^* is defined as usual by $h(Xv, w) = h(v, X^*w)$. Then $\theta(X) = -X^*$ gives an involution of the Lie algebra $\text{End}(V) \simeq \mathfrak{gl}(n)$.

(A) The form is symmetric given by the antidiagonal matrix $H = H_n$. Then $(\mathfrak{g}, \mathfrak{f}) = (\mathfrak{gl}(n), \mathfrak{o}(n))$; the vector space \mathfrak{p} is $\{X \in \text{Mat}_{n \times n}(k) \mid {}^tX = HXH\}$ and the orbits are for the action of the (split) orthogonal group. The nilpotent orbits in \mathfrak{p} are parametrized by partitions $P(n)$ of n . For $\lambda \in P(n)$, the corresponding orbit is $\mathfrak{p}_\lambda = \mathfrak{p} \cap \mathcal{O}_\lambda$ with \mathcal{O}_λ the GL_n -orbit of a nilpotent matrix with Jordan blocks given by λ .

(B) The form is alternating given by the skew-symmetric matrix $J = J_{2m}$ where $n = 2m$. Then $(\mathfrak{g}, \mathfrak{f}) = (\mathfrak{gl}(n), \mathfrak{sp}(n))$; the vector space \mathfrak{p} is $\{X \in \text{Mat}_{n \times n}(k) \mid {}^tX = -JXJ\}$. The orbits are for the action of the symplectic group and are parametrized by partitions of m ; if $\lambda = (a_1, a_2, \dots, a_s)$ is a partition of m then $\mathfrak{p}_\lambda = \mathfrak{p} \cap \mathcal{O}_{\lambda^{(2)}}$ with $\lambda^{(2)} = (a_1, a_1, a_2, a_2, \dots, a_s, a_s)$ a partition of $n = 2m$.

(Note that the parametrization of the nilpotent orbits \mathfrak{p}_λ stated above is shown by Sekiguchi [30] (see also [22]) when $k = \mathbf{C}$. This result extends to any k of odd characteristic.)

5.5. Generic smoothness

Here we show that our results so far allow us to deduce that the special fibre \bar{M}^{loc} is irreducible and generically smooth in these special cases (when $n = 2m + 1$ and $I = \{0\}$, or $n = 2m$ and $I = \{m\}$).

By § 3.3, \bar{M}^{loc} is connected and projective and is a union of left orbits for P_I in the affine Grassmannian for GU_n . We can see that there is a unique closed P_I -orbit in \bar{M}^{loc} . This closed orbit has to be contained in $\mathcal{A}^I(\mu)$. From the description of $\text{Adm}_0(\mu_{r,s})$ in §§ 2.4.1 and 2.4.2 we see that the closed orbit is the point $\{\mathcal{F}_0\}$ if $n = 2m + 1$, or if $n = 2m$ and s is even. If $n = 2m$ and s is odd, then the closed orbit is the orbit of $\mathcal{F}_1 = \text{span}_{\mathcal{O}_F} \{f_1, \pi f_1, \pi f_2, \dots, \pi f_{n-1}\}$. By the above, we can now obtain a description of an affine chart of M^{loc} that contains a point from the closed orbit. In each case, such a chart is given by the flat closure $U_{r,s}^{\text{flat}}$ of $U_{r,s}^\wedge$, respectively ${}_1U_{r,s}^{\text{flat}}$ of ${}_1U_{r,s}^\wedge$. (This is the same as the flat closure of $U_{r,s}$, respectively ${}_1U_{r,s}$.)

Now consider the flat closure $V_{r,s}^{\text{flat}}$ of the scheme $V_{r,s}$ of $n \times n$ matrices X over \mathcal{O}_E that satisfy

$$X^2 = \pi_0 \cdot I, \quad \text{char}_X(T) = (T - \sqrt{\pi_0})^r (T + \sqrt{\pi_0})^s$$

$$\wedge^{s+1}(X - \sqrt{\pi_0} \cdot I) = 0, \quad \wedge^{r+1}(X + \sqrt{\pi_0} \cdot I) = 0, \quad \text{when } r \neq s.$$

By [25], we see that the scheme $V_{r,s}^{\text{flat}}$ has relative dimension $2rs$. Its special fibre is reduced, irreducible and is the union of the nilpotent orbits \mathcal{O}_ρ for GL_n that correspond to partitions ρ of n with $\rho \leq (2^s, 1^r) = (r, s)^\vee$ (the partition dual to (r, s)). Its smooth locus $V_{r,s}^{\text{sm}}$ is the complement of the closure $\bar{\mathcal{O}}_{(2^{s-1}, 1^{r+1})}$. The involution σ given by $X \mapsto H^t X H$ when n is odd, respectively $X \mapsto -J^t X J$ when n is even, acts on $V_{r,s}$, $V_{r,s}^{\text{flat}}$ and on $\mathcal{O}_{(2^s, 1^r)}$.

Suppose first that either n is odd, or that both n and s are even. Then, by definition, $U_{r,s}^\wedge$ is the fixed point scheme $(V_{r,s})^\sigma$; over the generic fibre E this scheme has dimension rs . In this case, by Proposition 1 of [22], we can see that there is at least one σ -fixed point on $\mathcal{O}_{(2^s, 1^r)}$; since the action is tame, the fixed point scheme $(\mathcal{O}_{(2^s, 1^r)})^\sigma$ is smooth. In fact, it follows from Proposition 5 in [18] and its proof (they deal with the case $k = \mathbf{C}$ but their proof is valid for every algebraically closed field of odd characteristic) that each irreducible component of $(\mathcal{O}_{(2^s, 1^r)})^\sigma$ has dimension half of the dimension of the corresponding nilpotent orbit $\mathcal{O}_{(2^s, 1^r)}$. Hence, this dimension is $\dim((\mathcal{O}_{(2^s, 1^r)})^\sigma) = rs$. We can now deduce that $(V_{r,s}^{\text{sm}})^\sigma \subset U_{r,s}^\wedge$ is smooth over \mathcal{O}_E of relative dimension rs and provides an open subscheme of $U_{r,s}^{\text{flat}}$ with (smooth) special fibre $(\mathcal{O}_{(2^s, 1^r)})^\sigma$. In fact, by Proposition 1 of [22], $(\mathcal{O}_{(2^s, 1^r)})^\sigma$ is a single G -orbit. Here $G = G_n = \text{Sp}_n$ in the case when n is even, and hence $(\mathcal{O}_{(2^s, 1^r)})^\sigma$ is irreducible in this case. In the case when n is odd, $G_n = O_n$ is disconnected. However, the action of G_n extends to an action of the group \tilde{G}_n of orthogonal similitudes, with the centre of \tilde{G}_n acting trivially. Since \tilde{G}_n is connected (n is odd), we again conclude that $(\mathcal{O}_{(2^s, 1^r)})^\sigma$ is irreducible. In the notation introduced in the previous paragraph, we may now identify $(\mathcal{O}_{(2^s, 1^r)})^\sigma$ with $\mathfrak{p}_{(2^s, 1^r)}$. It follows that the inclusion $\bar{U}_{r,s}^{\text{flat}} \subset \bar{U}_{r,s}^\wedge = \overline{\mathfrak{p}_{(2^s, 1^r)}}$ is an equality on points. It now follows that the special fibre of $U_{r,s}^{\text{flat}}$ is irreducible and generically smooth. We also deduce that \bar{M}^{loc} is irreducible.

Suppose now that n is even and s is odd. (Note that in this case, $(\mathcal{O}_{(2^r, 1^s)})^\sigma = \emptyset$.) Then we can reduce to the smaller case where $n = r + s$ is replaced by $n - 2$ partitioned by $(r - 1, s - 1)$; indeed, by our work in the last part of §5.3 and the above, we can see that the special fibre of the flat closure ${}_1U_{r,s}^{\text{flat}}$ is generically the product of an affine space with $(\mathcal{O}_{(2^{s-1}, 1^{r-1})})^\sigma$ and hence it is generically smooth and irreducible.

As a corollary of the main result of [27] we can now show the following result.

Theorem 5.1. *Let $I = \{0\}$ if n is odd, and $I = \{m\}$ if $n = 2m$ is even. The special fibre of the local model M_I^{loc} is irreducible and reduced and is normal, Frobenius split and with only rational singularities.*

Proof. By the previous considerations the underlying reduced scheme $(\bar{M}_I^{\text{loc}})_{\text{red}}$ is a Schubert variety equal to $\mathcal{A}^I(\mu)$. By the main result of [27] we deduce that $(\bar{M}_I^{\text{loc}})_{\text{red}}$ is normal, Frobenius split and has only rational singularities. On the other hand, we

saw that the special fibre \bar{M}_I^{loc} contains an open dense subset which is reduced. By Hironaka’s Lemma (EGA IV.5.12.8) we deduce that \bar{M}_I^{loc} is reduced, which proves the claim (see also [27, Remark 11.4]). \square

Conjecture 5.2. *Let n be odd, or both n and s be even. Then the scheme of matrices $U_{r,s}^\wedge$ is flat over $\text{Spec } \mathcal{O}_E$. Equivalently, consider the space of matrices X in M_n over $\text{Spec } k$ with*

$$X^2 = 0, \quad {}^tX = HXH, \quad \text{char}_X(T) = T^n, \quad \wedge^{s+1}X = 0, \quad \wedge^{r+1}X = 0,$$

if n is odd, respectively

$$X^2 = 0, \quad {}^tX = -JXJ, \quad \text{char}_X(T) = T^n$$

and

$$\wedge^{s+1}X = 0, \quad \wedge^{r+1}X = 0, \quad \text{when } r \neq s,$$

if n and s are even.

Then this scheme is reduced (in which case it is normal, with rational singularities).

Remark 5.3.

- (a) Conjecture 5.2 implies that, under the assumptions n is odd, or both $n = 2m$ and s even, $M_{\{0\}}^\wedge = M_{\{0\}}^{\text{loc}}$, respectively $M_{\{m\}}^\wedge = M_{\{m\}}^{\text{loc}}$.
- (b) Suppose that n is even and s is odd. Recall that we denote by $U_{r,s}^\wedge$ the affine chart around the point \mathcal{F}_0 of $M^\wedge(k) = M_{\{m\}}^\wedge(k)$. By the above, we have $U_{r,s}^\wedge = (V_{r,s}^\wedge)^\sigma$. However, we can see that in this case the generic fibre $U_{r,s}^\wedge \otimes_{\mathcal{O}_E} E$ is empty. (Indeed, when π_0 is invertible, if ${}^tXJ = JX$ and $X^2 = \pi_0$, then the matrix $(\sqrt{\pi_0})^{-1}X$ belongs to the symplectic group and hence has determinant 1. Therefore, both eigenvalues $\sqrt{\pi_0}, -\sqrt{\pi_0}$ of X have to appear with even multiplicity.) In fact, in this case $(\mathcal{O}_{(2^s, 1^r)})^\sigma$ is empty and an argument as above shows that $\dim(U_{r,s}^\wedge) < rs$. It follows that the point \mathcal{F}_0 does not lift to characteristic zero. Hence $M_{\{m\}}^\wedge$ and also $M_{\{m\}}^{\text{naive}}$ are not flat over \mathcal{O}_E .

5.6. Normality of some nilpotent orbits

In the proof of the previous theorem we used some facts about nilpotent orbits for symmetric pairs. Conversely, we can use the results on local models (which ultimately rely on structure theorems for affine flag varieties) to deduce results on nilpotent orbits.

Theorem 5.4.

- (a) *Suppose that n is odd. Then the Zariski closure of the nilpotent orbit $\mathfrak{p}_{(2^s, 1^r)}$ for the symmetric pair $(\mathfrak{gl}(n), \mathfrak{o}(n))$ is normal.*
- (b) *Suppose that n and s are both even. Then the Zariski closure of the nilpotent orbit $\mathfrak{p}_{(2^s, 1^r)}$ for the symmetric pair $(\mathfrak{gl}(n), \mathfrak{sp}(n))$ is normal.*

(When $k = \mathbf{C}$ this result is a very special case of the results of Ohta [22]; his methods are particular to characteristic 0. Note that Ohta shows that the Zariski closure of any nilpotent orbit for the pair $(\mathfrak{gl}(n), \mathfrak{sp}(n))$ is normal, but exhibits examples for pairs $(\mathfrak{gl}(n), \mathfrak{o}(n))$ of non-normal orbit closures.)

Proof. By Theorem 5.1 the special fibre \bar{M}^{loc} is reduced and normal. By our discussion above, the Zariski closure of the nilpotent orbit $\mathfrak{p}_{(2^s, 1^r)}$ can be identified with an open affine subscheme of \bar{M}^{loc} . The result follows. \square

6. The local models of Picard surfaces

In this case ($G = \text{GU}(2, 1)$), there are three conjugacy classes of parahoric subgroups. In what follows we will prove Theorem 0.3 of the introduction.

(A) Let $I = \{0\}$. Then, by [23, Theorem 4.5, Remark 4.15], we have the following statement.

Proposition 6.1. *$M_{\{0\}}^{\text{naive}}$ is normal and Cohen–Macaulay. Furthermore, $M_{\{0\}}^{\text{naive}}$ is flat over $\text{Spec}(\mathcal{O}_F)$ and is smooth outside the special point \mathcal{F}_0 of the special fibre. Blowing up this special point yields a semi-stable model with special fibre consisting of two smooth surfaces meeting transversely along a smooth curve.*

We note that in [20] it is shown that the blow-up scheme represents a moduli problem analogous to the Demazure resolution of a Schubert variety in the Grassmannian.

(B) Let $I = \{1\}$. In this case we have the following statement.

Proposition 6.2. *$M_{\{1\}}^{\text{loc}}$ is smooth over $\text{Spec}(\mathcal{O}_F)$.*

Proof. The dual \hat{A}_1^s of A_1 with respect to the symmetric bilinear form (\cdot, \cdot) is A_2 . More precisely, the matrix of this bilinear form with respect to the \mathcal{O}_{F_0} -basis $e_3, \pi^{-1}e_1, e_2, \pi e_3, e_1, \pi e_2$ (in this order) is equal to

$$D = \begin{pmatrix} K & H \\ {}^tH & -\pi_0 K \end{pmatrix},$$

where

$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We describe an open neighbourhood of the point \mathcal{F}_0 in the special fibre. As in §5.2, we find that it can be given as a subscheme of the space of matrices A of the form

$$A = \begin{pmatrix} X \\ I \end{pmatrix},$$

where I is the unit matrix of size 3, and where X is a square matrix of size 3 with indeterminates as entries. The following conditions are imposed (the first condition corresponds

to the isotropy condition imposed on $\mathcal{F} = \text{Image}(A)$, the second one to the π -invariance of \mathcal{F}):

- (i) ${}^tA \cdot D \cdot A = 0$,
- (ii) $X^2 = \pi_0 \cdot I$.

Now condition (i) comes to

$${}^tXKX + ({}^tHX + {}^tXH) - \pi_0 \cdot K = 0. \tag{6.1}$$

Write X in the form

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix},$$

where X_1 is a square matrix of size 2 and X_4 a square matrix of size 1. Also let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

be the left upper corner of H . Let us write

$$X_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad X_3 = (x, y).$$

Then Equation (6.1) becomes the matrix equation

$$\begin{pmatrix} {}^tX_3X_3 & {}^tX_3X_4 \\ X_4X_3 & X_4^2 \end{pmatrix} + \begin{pmatrix} {}^tJX_1 + {}^tX_1J & {}^tJX_2 \\ {}^tX_2J & -\pi_0 \end{pmatrix} = 0, \tag{6.2}$$

where tX_3X_3 means

$$\begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}.$$

The condition (ii) becomes

$$X^2 = \begin{pmatrix} X_1^2 + X_2X_3 & X_1X_2 + X_2X_4 \\ X_3X_1 + X_4X_3 & X_3X_2 + X_4^2 \end{pmatrix} = \pi_0 \cdot I. \tag{6.3}$$

From the left lower corner in (6.2) we get ${}^tX_2 = -X_4 \cdot (y, -x)$. From the left upper corner we obtain the identity

$$-\begin{pmatrix} -2c & a - d \\ a - d & 2b \end{pmatrix} = \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}. \tag{6.4}$$

On the other hand, for the characteristic polynomial we are imposing the condition

$$\text{char}_X(T) = (T + \sqrt{\pi_0})^2(T - \sqrt{\pi_0}).$$

In particular $\text{tr}(X) = -\sqrt{\pi_0}$, i.e. $a + d + X_4 = -\sqrt{\pi_0}$. Taken together with (6.4), we see that X is uniquely determined by (x, y) and X_4 with $X_4^2 = \pi_0$. In the generic fibre of our scheme of matrices X we see that we have $X_4 = \sqrt{\pi_0}$. (Use (6.3), (6.4) and the characteristic polynomial condition.) Therefore, the identity $X_4 = \sqrt{\pi_0}$ persists in the flat closure. We conclude that the intersection of the flat closure $M_{\{1\}}^{\text{loc}}$ with an open neighbourhood of \mathcal{F}_0 can be identified with a closed subscheme of the affine space in (x, y) over $\text{Spec}(\mathcal{O}_F)$. It therefore has to coincide with this affine space. Hence $M_{\{1\}}^{\text{loc}}$ is smooth in a neighbourhood of \mathcal{F}_0 . But then the singular locus is empty since otherwise, as a closed subset in the special fibre invariant under the parahoric subgroup, it would have to contain \mathcal{F}_0 . \square

Remarks 6.3.

- (i) One can see that $M_{\{1\}}^{\text{naive}}$ and even $M_{\{1\}}^\wedge$ are not flat; computer calculations indicate that they have non-reduced special fibres (cf. Remark 7.4 (iv)).
- (ii) The special fibre \bar{M} of $M_{\{1\}}^{\text{loc}}$ is isomorphic to \mathbf{P}^2 . To see this, let $\bar{V} = V/R$, where $V = \Lambda_1 \otimes_{\mathcal{O}_{F_0}} k$, and where $R = (\pi e_2)$ is the one-dimensional radical of the symmetric bilinear form on V . Then \bar{V} is a non-degenerate quadratic space of dimension 5, and the image of $\pi\Lambda$ in \bar{V} is a two-dimensional isotropic subspace \bar{V}_2 . Now any \mathcal{F} corresponding to a point in \bar{M} contains R , and hence defines a two-dimensional isotropic subspace $\bar{\mathcal{F}}$ in \bar{V} . The fact that \mathcal{F} is π -stable implies that the intersection $\bar{\mathcal{F}} \cap \bar{V}_2$ is non-trivial. We see that \bar{M} is contained in the Schubert variety of isotropic planes in \bar{V} which have a non-trivial intersection with the fixed isotropic plane \bar{V}_2 . However, the Grassmannian of isotropic planes in \bar{V} is isomorphic to the Grassmannian of (isotropic) lines in a four-dimensional symplectic space $(W, \langle \cdot, \cdot \rangle)$. Indeed, \bar{V} may be identified with a natural subspace of $\wedge^2 W$, and a line l in W is mapped to $l \wedge l^\perp$. If L is mapped to \bar{V}_2 under this map, then the Schubert variety in question is identified with the set of lines in W which are contained in the three-dimensional space L^\perp . Hence this Schubert variety is isomorphic to \mathbf{P}^2 . It follows that \bar{M}_{red} is contained in \mathbf{P}^2 , and therefore by Proposition 6.2, $M_{\{1\}}^{\text{loc}} \otimes_{\mathcal{O}_F} k$ is isomorphic to \mathbf{P}^2 .

(C) Let $I = \{0, 1\}$. In this case we have the following statement.

Proposition 6.4. *The local model $M_{\{0,1\}}^{\text{loc}}$ is normal and Cohen–Macaulay and has reduced special fibre. It has two irreducible components which are normal and with only rational singularities. These two irreducible components meet along two smooth curves which intersect transversally in a single point.*

Proof. For the special fibre $\bar{M}_{\{0,1\}}^{\text{loc}}$ there is a chain of closed immersions, where $\mu = \mu(2,1)$,

$$\mathcal{A}^{\{0,1\}}(\mu) \subset \bar{M}_{\{0,1\}}^{\text{loc}} \subset \pi_{\{0\}}^{-1}(\bar{M}_{\{0\}}^{\text{loc}}) \cap \pi_{\{1\}}^{-1}(\bar{M}_{\{1\}}^{\text{loc}}) = \pi_{\{0\}}^{-1}(\mathcal{A}^{\{0\}}(\mu)) \cap \pi_{\{1\}}^{-1}(\mathcal{A}^{\{1\}}(\mu)). \tag{6.5}$$

For the μ -admissible set we have

$$\text{Adm}^{\{0,1\}}(\mu) = \pi_{\{0\}}^{-1}(\text{Adm}^{\{0\}}(\mu)) \cap \pi_{\{1\}}^{-1}(\text{Adm}^{\{1\}}(\mu)).$$

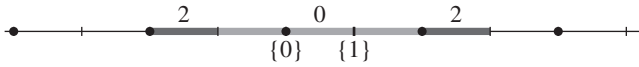


Figure 3. The admissible set for $U(2, 1)$.

This is shown by Figure 3. Indeed, in the picture, the extreme simplices of $\text{Adm}^{\{0,1\}}(\mu)$ (with respect to the Bruhat order) are translates of the base simplex by the translation elements $\{\pm 2\}$ which have length 2, and the other simplices in $\text{Adm}^{\{0,1\}}(\mu)$ correspond to the elements smaller than either of these two elements in the affine Weyl group. On the other hand, $\pi_{\{0\}}^{-1}(\text{Adm}^{\{0\}}(\mu))$ is the set of simplices Δ such that the distance from the vertex labelled $\{0\}$ in the picture to the vertex of the same type in Δ is less than or equal to 2, and $\pi_{\{1\}}^{-1}(\text{Adm}^{\{1\}}(\mu))$ is the set of simplices Δ such that the distance from the vertex labelled $\{1\}$ in the picture to the vertex of the same type in Δ is less than or equal to 2. It is visibly true that the set of simplices Δ satisfying both conditions is the set of simplices marked in the picture, i.e. is equal to $\text{Adm}^{\{0,1\}}(\mu)$.

Hence the above inclusions are bijections on the sets of closed points. By Frobenius splitness of Schubert varieties in the flag variety of the unitary group proved in [27], the intersection on the right-hand side of (6.5) is reduced. Hence all inclusions in (6.5) are equalities. It follows that $\bar{M}_{\{0,1\}}^{\text{loc}}$ is reduced and that all its irreducible components are Schubert varieties, hence by [27] are normal and with rational singularities. From Figure 3 we deduce that there are two irreducible components corresponding to the extreme elements of $\text{Adm}(\mu)$. They intersect in the two Schubert varieties corresponding to the elements of length one in $\text{Adm}(\mu)$ and these are normal curves which intersect in a single reduced point corresponding to the element of length zero in $\text{Adm}(\mu)$. We still need to prove that $\bar{M}_{\{0,1\}}^{\text{loc}}$ is Cohen–Macaulay. However, both irreducible components X and Y are Cohen–Macaulay, and the intersection $X \cap Y$ is Cohen–Macaulay and of codimension 1 in both X and Y . Hence $X \cup Y$ is Cohen–Macaulay (cf. [10, Lemma 4.22]). \square

7. Functor description: the spin condition

7.1. Generalities on symmetric forms of even dimension

Let F be a field of characteristic not equal to 2. Fix a separable closure F^{sep} of F . Let V be a vector space over F of even dimension $2n$. We will denote F -duals by * , so that $V^* := \text{Hom}_F(V, F)$ is the F -dual of V . There is a perfect, F -bilinear pairing

$$\wedge^n V \times \wedge^n V \rightarrow \wedge^{2n} V \tag{7.1}$$

defined by

$$(v_1 \wedge \cdots \wedge v_n, v'_1 \wedge \cdots \wedge v'_n) \mapsto v_1 \wedge \cdots \wedge v_n \wedge v'_1 \wedge \cdots \wedge v'_n.$$

This gives an isomorphism

$$c : \wedge^{2n} V \otimes_F (\wedge^n V)^* \rightarrow \wedge^n V \tag{7.2}$$

with inverse given by

$$v'_1 \wedge \cdots \wedge v'_n \mapsto (v_1 \wedge \cdots \wedge v_n \mapsto v_1 \wedge \cdots \wedge v_n \wedge v'_1 \wedge \cdots \wedge v'_n).$$

7.1.1. Now suppose that V supports a perfect symmetric F -bilinear form

$$h : V \times V \rightarrow F.$$

The form h induces an F -linear isomorphism $b = b_h : V \rightarrow V^*$. For simplicity, let us set \mathcal{D} for the determinant F -line $\wedge^{2n} V$. Then b induces an isomorphism ('the discriminant')

$$d : \mathcal{D}^{\otimes 2} \xrightarrow{\sim} F.$$

Notice that if $e = e_1 \wedge \cdots \wedge e_{2n} \in \mathcal{D}$ with $\{e_1, \dots, e_{2n}\}$ an F -basis of V then

$$d(e \otimes e) = D := (-1)^n \det((h(e_i, e_j))_{1 \leq i, j \leq 2n}) \tag{7.3}$$

is the discriminant of the form h in the basis $\{e_i\}_i$.

Define an F -algebra structure on $A := F \oplus \mathcal{D}$ by setting

$$(a, e) \cdot (a', e') = (aa' + d(e \otimes e'), ae' + a'e).$$

We call A the discriminant algebra of h . A choice of a non-zero element e in the F -line \mathcal{D} gives an isomorphism

$$F[x]/(x^2 - d(e \otimes e)) \xrightarrow{\sim} A, \tag{7.4}$$

and A is either isomorphic to $F \times F$, respectively a quadratic extension of F , if the discriminant is a square, respectively not a square, in F .

7.1.2. We will consider the composition

$$a := c \cdot (\text{id} \otimes_F \wedge^n b) : \mathcal{D} \otimes_F \wedge^n V \rightarrow \wedge^n V, \tag{7.5}$$

where $\wedge^n b : \wedge^n V \rightarrow \wedge^n(V^*) = (\wedge^n V)^*$ is given by $b = b_h$. A choice of a non-zero element $e \in \mathcal{D}$ provides us with an F -linear map:

$$a_e = a(e \otimes -) : \wedge^n V \rightarrow \wedge^n V. \tag{7.6}$$

For $\lambda \in F^\times$, we have $a_{\lambda \cdot e} = \lambda a_e$.

Proposition 7.1. *The map $a : \mathcal{D} \otimes_F \wedge^n V \rightarrow \wedge^n V$ provides an $A = F \oplus \mathcal{D}$ -module structure which extends the F -vector space structure on $\wedge^n V$.*

Proof. In view of the above, it is enough to show that if $e = e_1 \wedge \cdots \wedge e_{2n}$, with $\{e_1, \dots, e_{2n}\}$ an orthogonal F -basis of V , then

$$a_e^2 = D \cdot \text{id}_{\wedge^n V},$$

with

$$D = (-1)^n \prod_{i=1}^{2n} h(e_i, e_i).$$

For a subset $S = \{i_1, \dots, i_n\} \subset \{1, \dots, 2n\}$ with $i_1 < \dots < i_n$ set

$$e_S = e_{i_1} \wedge \dots \wedge e_{i_n} \in \wedge^n V, \quad D_S = h(e_{i_1}, e_{i_1}) \dots h(e_{i_n}, e_{i_n}) \in F^\times.$$

Denote by $h^{\wedge n}$ the symmetric form on $\wedge^n V$ given by the map $\wedge^n b : \wedge^n V \rightarrow (\wedge^n V)^* = \wedge^n V^*$. If $S \neq T$, then $h^{\wedge n}(e_S, e_T) = 0$. If $S = T$, then $h^{\wedge n}(e_S, e_S) = D_S$. Therefore, as S runs over all subsets of $\{1, \dots, 2n\}$ of order n , the elements e_S form an orthogonal basis of $\wedge^n V$ for the form $h^{\wedge n}$ and the map $\wedge^n b$ is given by

$$e_S = e_{i_1} \wedge \dots \wedge e_{i_n} \mapsto D_S \cdot e_S^* = e_{i_1}^* \wedge \dots \wedge e_{i_n}^*.$$

On the other hand, the map c sends e_S^* to $\eta_S \cdot e_{S^c}$ where $S^c = \{1, \dots, 2n\} \setminus S$ is the complement of S and $\eta_S = \pm 1$ is such that

$$e_S \wedge e_{S^c} = \eta_S \cdot e.$$

Note that $\eta_S \cdot \eta_{S^c} = (-1)^n$ (indeed, the form (7.1) is symmetric when n is even, and alternating when n is odd). We conclude that the map a_e sends e_S to $D_S \eta_S \cdot e_{S^c}$. Hence, the square a_e^2 maps e_S to $D_S D_{S^c} \eta_S \eta_{S^c} \cdot e_S$ and the result follows. □

Now suppose that R is an A -algebra. Then the base change $\wedge^n V \otimes_F R = \wedge^n (V \otimes_F R)$ supports an $A \otimes_F R$ -module structure where the first factor acts via the A -module structure of Proposition 7.1. Hence, it splits into a direct sum of two R -modules

$$\wedge^n V \otimes_F R = (\wedge^n V \otimes_F R)_+ \oplus (\wedge^n V \otimes_F R)_-, \tag{7.7}$$

where the $+$, respectively $-$, part is where $a \otimes 1$ acts as $1 \otimes a$, respectively as $1 \otimes \tau(a)$, and $\tau : A \rightarrow A$ is the non-trivial involution over F .

7.1.3. Suppose now that the form h is split, i.e. the vector space V has a basis $\{e_1, \dots, e_{2n}\}$ with $h(e_i, e_{2n+1-j}) = \delta_{ij}$. In this case, $D = 1$, $A = F \times F$, and Proposition 7.1 implies that for $e = e_1 \wedge \dots \wedge e_{2n}$, we have

$$a_e^2 = \text{id}_{\wedge^n V}. \tag{7.8}$$

The element e gives the structure of an A -algebra to F by sending e to 1 and we have

$$\wedge^n V = (\wedge^n V)_+ \oplus (\wedge^n V)_-,$$

where $(\wedge^n V)_+$, respectively $(\wedge^n V)_-$, is the eigenspace of $\wedge^n V$ where a_e has eigenvalue $+1$, respectively -1 . The subspaces $(\wedge^n V)_+$, $(\wedge^n V)_-$ depend on the choice of (split) basis but the two element set $\{(\wedge^n V)_+, (\wedge^n V)_-\}$ does not. We can see that the action of the special orthogonal group $\text{SO}((V, h))$ on $\wedge^n V$ preserves the subspaces $(\wedge^n V)_+$, $(\wedge^n V)_-$. The element of determinant -1 in $O(V, h)$ permutes the two subspaces.

In this case, we can easily obtain an explicit description of the subspaces $(\wedge^n V)_\pm$, as follows. For each subset S of $\{1, \dots, 2n\}$ of order n we set

$$2n + 1 - S := \{t \mid t = 2n + 1 - s, s \in S\}.$$

Define a permutation

$$\sigma_S : \{1, \dots, 2n\} \rightarrow \{1, \dots, 2n\}$$

by sending $\{1, \dots, n\}$ to the elements of $2n + 1 - S$ in *decreasing* order and sending the remaining $\{n + 1, \dots, 2n\}$ to the elements of the complement $(2n + 1 - S)^c$ in *increasing* order.

Lemma 7.2. *The subspace $(\wedge^n V)_\pm$ of $\wedge^n V$ is generated by the elements*

$$e_S \pm \text{sign}(\sigma_S)e_{(2n+1-S)^c}$$

for S running over all subsets of $\{1, \dots, 2n\}$ of order n .

Proof. A straightforward calculation from the definitions gives that

$$a_e(e_S) = \text{sign}(\sigma_S)e_{(2n+1-S)^c}.$$

Hence, since $a_e^2 = \text{id}_{\wedge^n V}$, we have $\text{sign}(\sigma_S) = \text{sign}(\sigma_{(2n+1-S)^c})$. The result now follows. □

7.1.4. We continue to assume that the form h on V is split with a basis $\{e_1, \dots, e_{2n}\}$ such that $h(e_i, e_{2n+1-j}) = \delta_{ij}$. If W is a totally isotropic subspace of V of dimension n , then the line $\wedge^n W$ is either contained in $(\wedge^n V)_+$ or in $(\wedge^n V)_-$. From the above we can see that $e_1 \wedge \dots \wedge e_n \subset (\wedge^n V)_+$ while $e_1 \wedge \dots \wedge e_{n-1} \wedge e_{n+1} \subset (\wedge^n V)_-$. Now any basis of an n -dimensional isotropic subspace of V can be completed to form a split basis of V . Hence, using induction, we can conclude that if W, W' are arbitrary isotropic n -dimensional subspaces of V then $\wedge^n W$ and $\wedge^n W'$ are contained in the same eigenspace $(\wedge^n V)_\pm$ if and only if

$$\dim(W \cap W') \equiv n \pmod{2}. \tag{7.9}$$

7.2. The spin correction

We now return to the notation of §1.2. In particular, F/F_0 is a tamely ramified quadratic extension with automorphism $a \mapsto \bar{a}$, and $\phi : V \times V \rightarrow F$ is a perfect hermitian form on the F -vector space V of dimension $n \geq 3$. Suppose $\{e_1, \dots, e_n\}$ is a basis of V with $\phi(e_i, e_{n+1-j}) = \delta_{ij}$. We will apply the constructions of §7.1 to V considered as a $2n$ -dimensional F_0 -vector space with the symmetric form

$$h(v, w) = \frac{1}{2} \text{Tr}_{F/F_0}(\phi(v, w)).$$

Write $m = [n/2]$ and consider the F_0 -basis

$$\{-\pi^{-1}e_1, \dots, -\pi^{-1}e_m, e_{m+1}, \dots, e_n, e_1, \dots, e_m, \pi e_{m+1}, \dots, \pi e_n\}$$

of V . The \mathcal{O}_{F_0} -lattice spanned by this basis is invariant for the action of $\mathcal{O}_F = \mathcal{O}_{F_0}[\pi]$ and is equal to Λ_m .

There are two cases.

- (I) $n = 2m + 1$ is odd. Then the discriminant D of h in this basis is equal to π_0 and the discriminant algebra A is isomorphic to F . Hence, the form h is not split over F_0 . However, after base changing to F we can replace in this list the vectors $e_{m+1}, \pi e_{m+1}$ by

$$e_{m+1} - \frac{\pi e_{m+1}}{\sqrt{\pi_0}}, \quad \frac{1}{2} \left(e_{m+1} + \frac{\pi e_{m+1}}{\sqrt{\pi_0}} \right).$$

The resulting new (ordered) basis splits the form $h \otimes_{F_0} F$. We can consider the corresponding eigenspaces $(\wedge^n V \otimes_{F_0} F)_{\pm}$ of $\wedge^n V \otimes_{F_0} F$.

- (II) $n = 2m$ is even. Then the above basis splits the form h and we can consider the eigenspaces $(\wedge^n V)_{\pm}$ of $\wedge^n V$.

Now consider a choice of (r, s) with $n = r + s$. Recall our definition of the reflex field E . We have $E = F_0$, if $r = s$, and $E = F$ if $r \neq s$.

Note that we can always make sense of the eigenspaces $(\wedge^n V \otimes_{F_0} E)_{\pm}$ of $\wedge^n V \otimes_{F_0} E$. For example, if $r = s$, then n is even and

$$(\wedge^n V \otimes_{F_0} E)_{\pm} = (\wedge^n V)_{\pm}.$$

If Λ is one of the \mathcal{O}_F -lattices Λ_j of V defined in §1.2, we set

$$(\wedge^n \Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_E)_{\pm} := (\wedge^n \Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_E) \cap (\wedge^n V \otimes_{F_0} E)_{\pm}. \tag{7.10}$$

7.2.1. Now let I be a subset of $\{0, \dots, m\}$ as in §1.4 and recall the definition of the ‘naive’ unitary local models M_I^{naive} of §1.5 corresponding to I and (r, s) . Recall also the definition of the closed subscheme M_I^{\wedge} of M_I^{naive} defined by requiring the additional exterior power condition (e) of (1.5.6).

We define a subfunctor M_I of M_I^{naive} by specifying that a point of M_I with values in an \mathcal{O}_E -scheme S is given by an $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ -submodule $\mathcal{F}_j \subset \Lambda_j \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ for each $j = k \cdot n \pm i, i \in I$, that in addition to the conditions (a)–(e) also satisfies the following condition.

- (f) (*Spin condition.*) For each $j = k \cdot n \pm i, i \in I$, the line $\wedge^n \mathcal{F}_j \subset \wedge^n (\Lambda_j \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S)$ is contained in the subspace $(\wedge^n \Lambda_j \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_E)_{\pm} \otimes_{\mathcal{O}_E} \mathcal{O}_S$ with $\pm = (-1)^s$.

It follows from the definition that M_I is represented by a closed subscheme of M_I^{naive} which is contained in M_I^{\wedge} ; we also denote this subscheme by M_I .

7.2.2. We claim that the generic fibres of M_I and M_I^{naive} are equal. Suppose that S is an E -scheme. Then, we can see that condition (f) is open and closed on S . Recall that the generic fibre $M_I^{\text{naive}} \otimes_{\mathcal{O}_E} F$ is isomorphic to the Grassmannian $\text{Gr}(r, n)_F$. Hence, $M_I^{\text{naive}} \otimes_{\mathcal{O}_E} E$ is connected and so it is enough to show that (f) is satisfied at some point of the generic fibre. It is enough to check this for one choice of signature. Indeed, we can easily find subspaces $\mathcal{F}_j, \mathcal{F}'_j$ for signatures (r, s) and $(r - 1, s + 1)$ whose intersection has

dimension $n - 1$; using (7.9) we can inductively reduce to the case of a single signature choice. Let us take this to be $(m + 1, m)$ if $n = 2m + 1$ and (m, m) if $n = 2m$. The subspaces given by

$$\mathcal{F}^{\text{odd}} = \langle \pi^{-1}e_1, \dots, \pi^{-1}e_m, e_1, \dots, e_m, \pi e_{m+1} - \sqrt{\pi_0}e_{m+1} \rangle$$

and

$$\mathcal{F}^{\text{even}} = \langle \pi^{-1}e_1, \dots, \pi^{-1}e_m, e_1, \dots, e_m \rangle,$$

respectively, give points of $M_I^{\text{naive}} \otimes_{\mathcal{O}_E} E$ for these signatures. We can now see, by calculating the dimension of the intersection of \mathcal{F}^{odd} , respectively $\mathcal{F}^{\text{even}}$, with the standard isotropic subspace spanned by the first n -basis vectors of our chosen split basis of $V \otimes_{F_0} F$, that these satisfy condition (f). To recap, we have closed immersions

$$M_I^{\text{loc}} \subset M_I \subset M_I^\wedge \subset M_I^{\text{naive}} \tag{7.11}$$

which are isomorphisms on the generic fibre.

Conjecture 7.3. *The scheme M_I is flat over \mathcal{O}_E . Equivalently, we have $M_I^{\text{loc}} = M_I$.*

Remarks 7.4.

- (i) The definition of M_I and the above conjecture are also motivated by a similar construction that we have found to be effective in the case of even orthogonal groups (see §8).
- (ii) This conjecture is supported by some computational evidence that we obtained with the help of the computer software Macaulay 2. In particular, we verified the conjecture for the local models of unitary groups in three and four variables when $F_0 = \mathbf{F}_p((t))$, $F = \mathbf{F}_p((u))$ with $u^2 = t$, for various (small) primes $p > 2$.
- (iii) Recall from Remark 5.3 that the moduli scheme M_I^\wedge , in which we omit condition (f), is not flat over \mathcal{O}_E in general. Indeed, for $n = 2m$, $I = \{m\}$ and signature (r, s) with s odd the subspace $\pi(\Lambda_m \otimes_{\mathcal{O}_{F_0}} k) \subset \Lambda_m \otimes_{\mathcal{O}_{F_0}} k$ gives a point of the special fibre of $M_{\{m\}}^\wedge$ which does not lift to the generic fibre. We can see that this point does not satisfy the spin condition (f).
- (iv) Another interesting example is provided by the case $n = 3$, $(r, s) = (2, 1)$. Then we can see that both $M_{\{1\}}^\wedge$ and $M_{\{0,1\}}^\wedge$ are not flat while our computer calculations suggest that $M_{\{1\}}$ and $M_{\{0,1\}}$ are flat.

8. Remarks on the case of (even) orthogonal groups

Here we will discuss a particular type of orthogonal local model that controls the singularities of certain PEL Shimura varieties; we will concentrate on the algebraic definition of these models and leave the connection to group theory, loop groups and affine flag

varieties for another occasion.* The fact that the orthogonal group is not connected introduces additional complications. Also for the connection to the reduction of Shimura varieties we will refer to [29].

We continue with the notation of § 7.1. In addition, assume that F is local with ring of integers \mathcal{O}_F , uniformizer π and residue field k of characteristic not equal to 2. If Λ is an \mathcal{O}_F -lattice in the F -vector space V , we denote by $\hat{\Lambda}$ its dual lattice, i.e. the image of $\Lambda^* := \text{Hom}_{\mathcal{O}_F}(\Lambda, \mathcal{O}_F)$ under the map

$$\Lambda^* \rightarrow \Lambda^* \otimes_{\mathcal{O}_F} F = V^* \xrightarrow{b^{-1}} V.$$

The restriction of the form h gives a perfect \mathcal{O}_F -bilinear pairing

$$h_\Lambda : \Lambda \times \hat{\Lambda} \rightarrow \mathcal{O}_F.$$

In what follows we will adhere to the terminology of [29] (note, however, that in their set-up $F = \mathbf{Q}_p$). Let $\mathcal{L} = \{\Lambda\}$ be a self-dual periodic chain of \mathcal{O}_F -lattices in V . Recall that a lattice chain in V is by definition a (non-empty) collection of lattices such that if Λ, Λ' are in the chain then either $\Lambda \subset \Lambda'$ or $\Lambda' \subset \Lambda$. A lattice chain defines a category with objects the lattices and morphisms given by inclusions. ‘Periodic’ means that $\Lambda \in \mathcal{L}$ implies that $a\Lambda \in \mathcal{L}$ for every $a \in F^\times$; ‘self-dual’ means that $\Lambda \in \mathcal{L}$ implies that $\hat{\Lambda} \in \mathcal{L}$.

8.1. The naive local models

Let us formulate a moduli problem M^{naive} on the category of \mathcal{O}_F -schemes. A point of M^{naive} with values in an \mathcal{O}_F -scheme S is given by a functor from the category given by the lattice chain \mathcal{L} to the category of \mathcal{O}_S -modules

$$\Lambda \mapsto \mathcal{F}_\Lambda \tag{8.1}$$

together with a morphism of functors given by injections

$$j_\Lambda : \mathcal{F}_\Lambda \rightarrow \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_S \tag{8.2}$$

such that

- (a) via j_Λ , \mathcal{F}_Λ identifies with an \mathcal{O}_S -locally direct summand of rank n of $\Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_S$;
- (b) if $\Lambda' = \pi\Lambda$, the isomorphism $\Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_S \rightarrow \pi\Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_S = \Lambda' \otimes_{\mathcal{O}_F} \mathcal{O}_S$ given by multiplication by the uniformizer π induces an isomorphism between \mathcal{F}_Λ and $\mathcal{F}_{\Lambda'}$;
- (c) we have $\mathcal{F}_{\hat{\Lambda}} = \mathcal{F}_\Lambda^\perp$, where $\mathcal{F}_\Lambda^\perp$ is the orthogonal complement of $\mathcal{F}_\Lambda \subset \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_S$ under the perfect pairing

$$(\hat{\Lambda} \otimes_{\mathcal{O}_F} \mathcal{O}_S) \times (\Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_S) \rightarrow \mathcal{O}_S \tag{8.3}$$

given by $h_\Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_S$.

* This connection has recently been examined in detail by Smithling [32].

We can see that the functor M^{naive} is represented by a projective scheme over $\text{Spec } \mathcal{O}_F$ which we will also denote by M^{naive} ; this is the naive local model defined in [29]. If S is an F -scheme, then the conditions above imply that $W := \mathcal{F}_A \subset V \otimes_F \mathcal{O}_S$ is independent of the choice of $A \in \mathcal{L}$. We can see that the generic fibre M_F^{naive} of the naive local model is the orthogonal Grassmannian of isotropic subspaces W of V of dimension n .

8.2. The corrected local models

It was observed by Genestier (personal correspondence) that the naive local models are not flat in general. In this paragraph, we suggest a correction in their definition that should address this problem. We continue with the notation of the previous paragraph.

Recall $A = F \oplus \mathcal{D}$ is the discriminant algebra of h . By (7.7) we have

$$\wedge^n V \otimes_F A = (\wedge^n V \otimes_F A)_+ \oplus (\wedge^n V \otimes_F A)_-,$$

where the $+$, respectively $-$, part is where $a \otimes 1$ acts as $1 \otimes a$, respectively as $1 \otimes \tau(a)$, and $\tau : A \rightarrow A$ is the non-trivial involution over F .

Now let \mathcal{O}_A denote the integral closure of \mathcal{O}_F in A . For any \mathcal{O}_F -lattice Λ as above consider the \mathcal{O}_A -lattice $\Lambda_{\mathcal{O}_A} := \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_A$. We have $\wedge^n(\Lambda_{\mathcal{O}_A}) \subset \wedge^n(V \otimes_F A)$. Let us set

$$(\wedge^n \Lambda_{\mathcal{O}_A})_{\pm} = \wedge^n(\Lambda_{\mathcal{O}_A}) \cap (\wedge^n V \otimes_F A)_{\pm}.$$

This is an \mathcal{O}_A -lattice in $(\wedge^n V \otimes_F A)_{\pm}$.

Denote by M the closed subscheme of the base change $M^{\text{naive}} \otimes_{\mathcal{O}_F} \mathcal{O}_A$ whose S -points for an \mathcal{O}_A -scheme S are given by S -points $\mathcal{F}_A \subset \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_S$ of $M^{\text{naive}} \otimes_{\mathcal{O}_F} \mathcal{O}_A$ which satisfy $\wedge^n_{\mathcal{O}_S}(\mathcal{F}_A) \subset (\wedge^n \Lambda_{\mathcal{O}_A})_+ \otimes_{\mathcal{O}_A} \mathcal{O}_S$ for all A . (Here we identify $(\wedge^n \Lambda_{\mathcal{O}_A})_{\pm} \otimes_{\mathcal{O}_A} \mathcal{O}_S$ with a subsheaf of $\wedge^n_{\mathcal{O}_S}(\Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_S)$.) The corrected local model is by definition M regarded as a scheme over $\text{Spec}(\mathcal{O}_F)$. By its construction M comes with a morphism

$$M \rightarrow \text{Spec}(\mathcal{O}_A) \rightarrow \text{Spec}(\mathcal{O}_F). \tag{8.4}$$

There is a natural morphism of \mathcal{O}_F -schemes $q : M \rightarrow M^{\text{naive}}$. We will see below that q induces an isomorphism on the generic fibres. However, q is not a closed immersion in general.

Conjecture 8.1. *The scheme M is flat over $\text{Spec } \mathcal{O}_F$.*

In addition to the examples of the following paragraphs, there is a fair amount of computational evidence for this (when $n \leq 8$, $F = \mathbf{F}_p((t))$, with p a small prime).

8.2.1. To understand this construction a little better, let us consider the situation over F , i.e. the generic fibres. Suppose that W is an S -valued point of M_F^{naive} with S an F -scheme (i.e. a totally isotropic subbundle of $V \otimes_F \mathcal{O}_S$); the short exact sequence

$$0 \rightarrow W \rightarrow V \otimes_F \mathcal{O}_S \rightarrow (V \otimes_F \mathcal{O}_S)/W \xrightarrow{b^{-1}} W^* \rightarrow 0$$

induces an isomorphism $\mathcal{O}_S \simeq \wedge^n W \otimes \wedge^n(W^*) \simeq \det(V) \otimes_F \mathcal{O}_S \simeq \mathcal{D} \otimes_F \mathcal{O}_S$. We can see that this in turns equips \mathcal{O}_S with an A -algebra structure. We deduce that M_F^{naive} has

a natural structure of an A -scheme. This structure has the property that if W gives an S -valued point of M_F^{naive} , then the homomorphism $A \otimes_F \wedge^n W \rightarrow \wedge^n W$ induced by $S \rightarrow \text{Spec}(A)$ is obtained by restricting the homomorphism $A \otimes_F \wedge^n V \rightarrow \wedge^n V$ of Proposition 7.1. Therefore, we have

$$M_F^{\text{naive}} = M_F.$$

Now denote by F^{sep} a separable closure of F ; we fix an F -homomorphism $A \rightarrow F^{\text{sep}}$. The Plücker map $W \mapsto \wedge^n W$ gives an embedding in projective space

$$M_{F^{\text{sep}}} \rightarrow \mathbf{P}(\wedge^n V_{F^{\text{sep}}}).$$

We can deduce that the image of $M_{F^{\text{sep}}}$ under the Plücker embedding

$$M_{F^{\text{sep}}} \rightarrow \mathbf{P}(\wedge^n V_{F^{\text{sep}}}) = \mathbf{P}((\wedge^n V_{F^{\text{sep}}})_+ \oplus (\wedge^n V_{F^{\text{sep}}})_-)$$

lies in the (disjoint) union of the two linear subspaces $\mathbf{P}((\wedge^n V_{F^{\text{sep}}})_+)$ and $\mathbf{P}((\wedge^n V_{F^{\text{sep}}})_-)$ where the decomposition is obtained as above. Since the form h splits over F^{sep} we can choose a basis for $V_{F^{\text{sep}}}$ as in § 7.1.3. Since $\text{SO}(V_{F^{\text{sep}}}, h_{F^{\text{sep}}})$ is connected, from § 7.1.3 and the above, we conclude that $M_{F^{\text{sep}}}$ has exactly two connected components each isomorphic to the special orthogonal Grassmannian $\text{SO}(V_{F^{\text{sep}}}, h_{F^{\text{sep}}})/P$; they are separated by asking that the top exterior power of the isotropic subspace is contained in $(\wedge^n V_{F^{\text{sep}}})_+$, or $(\wedge^n V_{F^{\text{sep}}})_-$ respectively. In fact, it is well known that both of these components are of dimension $\binom{n}{2}$.

8.3. The split case: examples

Assume that the form h is split and choose a basis $\{e_1, \dots, e_{2n}\}$ as in § 7.1.3. Set $\Lambda_0 = \langle e_1, \dots, e_{2n} \rangle$ and let $\Pi : V \rightarrow V$ be the linear map defined by $\Pi(e_1) = \pi e_{2n}$, $\Pi(e_i) = e_{i-1}$, $1 < i \leq 2n$. We have $\Pi^{2n} = \pi$. Now let us set $\Lambda_{-i} = \Pi^i \Lambda_0$; we can see that $\hat{\Lambda}_i = \Lambda_{-i}$ so that the form induces perfect \mathcal{O}_F -bilinear pairings

$$\Lambda_{-i} \times \Lambda_i \rightarrow \mathcal{O}_F. \tag{8.5}$$

Now if $I \subset \{0, \dots, n\}$ is a non-empty subset,

$$\mathcal{L}_I = \{\pi^k \Lambda_{\pm i} \mid k \in \mathbf{Z}, i \in I\}$$

is a self-dual periodic lattice chain in V . We will denote by M_I^{naive} the ‘naive’ local model associated to V , h and the lattice chain \mathcal{L}_I . As it was observed by Genestier (personal correspondence), when $\{0, n\} \subset I$ the scheme $M_I^{\text{naive}} \rightarrow \text{Spec } \mathcal{O}_F$ is not flat; he pointed out that the problem is created by the existence of two connected components in the (isomorphic) generic fibres $(M_{\{i\}}^{\text{naive}})_F \simeq \text{OGr}(n, 2n)_F$, $i \in I$. Roughly speaking, the closures of several pairs of these components appear in the special fibre of the scheme

$$M_I^{\text{naive}} \subset \prod_{j \in I \cup -I} \text{Gr}(n, 2n)_{\mathcal{O}_F}.$$

(See the examples below.) His observation motivated our work in these sections.

Example 8.2. Let us consider the case of the orthogonal group O_2 over \mathbf{Q}_p , i.e. take $n = 1$, $F = \mathbf{Q}_p$, $\pi = p$, and $I = \{0, 1\}$. In this case we have $\Lambda_0 = \langle e_1, e_2 \rangle$, $\Lambda_{-1} = \Pi \Lambda_0 = \langle e_1, pe_2 \rangle$. For a \mathbf{Z}_p -scheme S the S -points of the naive local model M^{naive} are given by diagrams

$$\begin{array}{ccccc}
 \Lambda_{-1} \otimes_{\mathbf{Z}_p} \mathcal{O}_S & \xrightarrow{\phi} & \Lambda_0 \otimes_{\mathbf{Z}_p} \mathcal{O}_S & \xrightarrow{\psi} & \Lambda_{-1} \otimes_{\mathbf{Z}_p} \mathcal{O}_S \\
 \cup & & \cup & & \cup \\
 \mathcal{F}_{-1} & \longrightarrow & \mathcal{F}_0 & \longrightarrow & \mathcal{F}_{-1}
 \end{array} \tag{8.6}$$

such that $\mathcal{F}_0, \mathcal{F}_{-1}$ are locally direct summands of rank 1, \mathcal{F}_0 is isotropic for h , and \mathcal{F}_{-1} is isotropic for the perfect form on $\Lambda_{-1} \otimes_{\mathbf{Z}_p} \mathcal{O}_S$ given by $p^{-1}h$. In these diagrams, the map ϕ is given by base changing the inclusion $\Lambda_{-1} \subset \Lambda_0$ while the map ψ is given by base changing the map $\Lambda_0 \rightarrow \Lambda_{-1}$ given by multiplication by p .

Setting

$$\mathcal{F}_0 = \langle ae_1 + be_2 \rangle, \quad \mathcal{F}_{-1} = \langle ce_1 + dpe_2 \rangle,$$

we see that the orthogonality conditions give the (homogeneous) equations:

$$ab = 0, \quad cd = 0.$$

If p is invertible in \mathcal{O}_S , the submodule \mathcal{F}_{-1} is determined by \mathcal{F}_0 . We can see that there are two possibilities for \mathcal{F}_0 : either $\mathcal{F}_0 = \langle e_1 \rangle$, or $\mathcal{F}_0 = \langle e_2 \rangle$ and so the generic fibre $M^{\text{naive}}_{\mathbf{Q}_p}$ consists of a disjoint union of two copies of $\text{Spec } \mathbf{Q}_p$. Let us now consider the special fibre, i.e. assume that $p = 0$ in \mathcal{O}_S . Taking into account that $\phi(\mathcal{F}_{-1}) = \langle ce_1 \rangle \subset \mathcal{F}_0 = \langle ae_1 + be_2 \rangle$, $\psi(\mathcal{F}_0) = \langle bpe_2 \rangle \subset \mathcal{F}_{-1} = \langle ce_1 + dpe_2 \rangle$, we see that there are three possibilities:

$$\mathcal{F}_{-1} = \langle e_1 \rangle, \quad \mathcal{F}_0 = \langle e_1 \rangle$$

or

$$\mathcal{F}_{-1} = \langle pe_2 \rangle, \quad \mathcal{F}_0 = \langle e_2 \rangle$$

or

$$\mathcal{F}_{-1} = \langle pe_2 \rangle, \quad \mathcal{F}_0 = \langle e_1 \rangle.$$

Therefore, the special fibre consists of three copies of $\text{Spec } \mathbf{F}_p$ and so M^{naive}_I is not flat over $\text{Spec } \mathbf{Z}_p$. The point corresponding to the third possibility above does not lift to the generic fibre. Notice that in this case

$$(\wedge^n V)_+ = \langle e_1 \rangle, \quad (\wedge^n V)_- = \langle e_2 \rangle.$$

We can now readily see that $M = \text{Spec } \mathbf{Z}_p \sqcup \text{Spec } \mathbf{Z}_p$ which is of course flat.

Example 8.3. In this example, we take $n = 2$ (which corresponds to O_4), $F = \mathbf{Q}_p$, $\pi = p$, and $I = \{1\}$. We have

$$(\wedge^2 V)_+ = \langle e_1 \wedge e_2, e_3 \wedge e_4, e_1 \wedge e_4 + e_2 \wedge e_3 \rangle.$$

The lattices are

$$\begin{aligned} \Lambda_{-1} &= \Pi \Lambda_0 = \langle e_1, e_2, e_3, pe_4 \rangle, \\ \Lambda_1 &= \Pi^{-1} \Lambda_0 = \langle e_1/p, e_2, e_3, e_4 \rangle, \\ p^{-1} \Lambda_{-1} &= p^{-1} \Pi \Lambda_0 = \langle e_1/p, e_2/p, e_3/p, e_4 \rangle \end{aligned}$$

and the naive local model is given by diagrams

$$\begin{array}{ccccc} \langle e_1, e_2, e_3, pe_4 \rangle_S & \xrightarrow{\phi} & \langle e_1/p, e_2, e_3, e_4 \rangle_S & \xrightarrow{\psi} & \langle e_1/p, e_2/p, e_3/p, e_4 \rangle_S \\ \cup & & \cup & & \cup \\ \mathcal{F}_{-1} & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_{-1}^{(1/p)} \end{array}$$

with \mathcal{F}_i of rank 2 such that $\mathcal{F}_1 = (\mathcal{F}_{-1})^\perp$.

In this case

$$\begin{aligned} (\wedge^2 \Lambda_{-1})_+ &= \langle e_1 \wedge e_2, e_3 \wedge pe_4, e_1 \wedge pe_4 + p \cdot (e_2 \wedge e_3) \rangle, \\ (\wedge^2 \Lambda_1)_+ &= \langle e_1/p \wedge e_2, e_3 \wedge pe_4, p \cdot (e_1/p \wedge e_4) + e_2 \wedge e_3 \rangle. \end{aligned}$$

Our additional spin conditions amount to

$$\wedge^2 \mathcal{F}_{-1} \subset \langle e_1 \wedge e_2, e_3 \wedge pe_4, e_1 \wedge pe_4 + p \cdot (e_2 \wedge e_3) \rangle_S, \tag{8.7}$$

$$\wedge^2 \mathcal{F}_1 \subset \langle e_1/p \wedge e_2, e_3 \wedge e_4, p \cdot (e_1/p \wedge e_4) + e_2 \wedge e_3 \rangle_S. \tag{8.8}$$

We will examine the special fibre; thus we assume $p = 0$ in \mathcal{O}_S . If $(\mathcal{F}_{-1}, \mathcal{F}_1)$ corresponds to an $\bar{\mathbf{F}}_p$ -valued point of M , we can see that either e_2 or e_3 is in \mathcal{F}_1 . When $e_2 \in \mathcal{F}_1$, then $\mathcal{F}_1 = \langle e_2, ae_1/p + be_3 \rangle$, $\mathcal{F}_{-1} = \langle e_1, -ae_2 + bpe_4 \rangle$, where $(a; b) \in \mathbf{P}^1$. If $e_3 \in \mathcal{F}_1$, then $\mathcal{F}_1 = \langle e_3, ce_2 + de_4 \rangle$, $\mathcal{F}_{-1} = \langle pe_4, -ce_1 + de_3 \rangle$, with $(c; d) \in \mathbf{P}^1$. Those two projective lines intersect at the point $\mathcal{F}_1 = \langle e_2, e_3 \rangle$ (at which $\mathcal{F}_{-1} = \langle e_1, pe_4 \rangle$). This describes the (reduced) special fibre of the corrected local model. It is not hard to see that both irreducible components lift to characteristic zero. For example, the point $\mathcal{F}_1 = \langle e_3, e_4 \rangle$, $\mathcal{F}_{-1} = \langle e_3, pe_4 \rangle$ is a smooth point on the second component that lifts; similarly the point $\mathcal{F}_1 = \langle e_1/p, e_2 \rangle$, $\mathcal{F}_{-1} = \langle e_1, e_2 \rangle$ is a smooth point of the first component that lifts.

Now on to a ‘scheme-theoretic’ calculation. We will calculate an affine chart of the special fibre at the singular point $\mathcal{F}_{-1} = \langle e_1, pe_4 \rangle$, $\mathcal{F}_1 = \langle e_2, e_3 \rangle$. Set

$$\mathcal{F}_{-1} = \langle e_1 + x_2e_2 + x_3e_3, pe_4 + y_2e_2 + y_3e_3 \rangle.$$

Then we can see that

$$\mathcal{F}_1 = (\mathcal{F}_{-1})^\perp = \left\langle e_2 - y_3 \frac{e_1}{p} - x_3e_4, e_3 - y_2 \frac{e_1}{p} - x_2e_4 \right\rangle.$$

The condition $\phi(\mathcal{F}_{-1}) \subset \mathcal{F}_1$ translates to

$$x_2x_3 = y_2y_3 = x_2y_3 + x_3y_2 = 0, \tag{8.9}$$

while $\psi(\mathcal{F}_1) \subset \mathcal{F}_{-1}^{(1/p)}$ translates to

$$x_2y_2 = x_3y_3 = x_2y_3 + x_3y_2 = 0. \quad (8.10)$$

The spin conditions (8.7), (8.8) give

$$y_3 = x_2 = 0, \quad x_2y_3 - y_2x_3 = 0.$$

These equations together with (8.9), (8.10) amount to

$$x_2 = y_3 = 0, \quad y_2x_3 = 0.$$

This together with the ‘set-theoretic’ arguments above imply that the special fibre is reduced and is the union of two \mathbf{P}^1 s intersecting transversely at a point. Since, both of these \mathbf{P}^1 s lift, we also conclude that M_I is flat over \mathbf{Z}_p .

Notice that if we omit the spin conditions, we obtain

$$x_2x_3 = y_2y_3 = x_2y_2 = x_3y_3 = x_2y_3 + x_3y_2 = 0.$$

These are then the equations for an affine chart of the special fibre of the naive local model; this scheme has four irreducible components and is non-reduced at the origin. We can see that M_I^{naive} cannot be flat over \mathbf{Z}_p .

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