An index for betting with examples from games and sports

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1. *Introduction*

It is tempting to accept bets when the outcome has a positive expectation favouring the bettor. We examine situations where this is not enough of a criterion as a basis for betting. The argument we present shows that the probability of winning the bets in the long run has to be qualified in a certain way beyond positive expectation. We introduce an alternative index as a criterion to recommend to bettors.

2. *General set-up*

Some bets can have multiple possibilities. For instance, in many team sports, a game may end up with one team winning, drawing or losing the game.

In a general setting, we have a finite range, say of m values, for the net gain in a single bet, which we can list in increasing order as $x_1, x_2, x_3, \dots, x_m$, that can occur with probabilities $p_1, p_2, \dots, p_m$. When a bettor risks a unit (say a dollar), she gains x_i units with probability p_i.

In real life, some of the values for the net gain per monetary unit bet are negative, as no real-life bet can have all values in the range of outcomes being positive. In the following, we shall assume that at least one of the x_i is negative. Thus, without loss of generality, x_1 is negative.

We call the gain per unit of the nth bet the nth *outcome*. Accordingly, the nth outcome is a random variable X_n that assumes the values $x_1, \dots, x_m$, with probabilities $p_1, \dots, p_m$, and the outcomes in the different bets are independent. Such betting has average outcome $\sum_{i=1}^{m} x_i p_i$.

The bettor intends to participate in the bets indefinitely. Because of the possibility of loss, the cautious bettor does not risk all her money in one bet, even when the probability of gaining in an individual bet is favourable. We suppose that she risks a certain fixed proportion $b \in (0, 1)$ of her money at each bet, hoping to gain in the long run. Such a proportion must also be feasible. A natural constraint on b is discussed below. Even if the bettor loses in one individual bet after committing a feasible proportion b, she still has money to re-enter the betting game and a chance to recover.

If after n bets the bettor's money is less than her starting capital, we say that the bettor is in a losing position, and if the money in her account is more than her starting capital, we say that the bettor is in a winning position, otherwise we say she is even. Our interest is in the following question: *What is the probability that the bettor will be in a winning position after a large number of bets*?

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Let M_n be the bettor's money after n bets, where the starting capital is M_0. Recall that the bettor's strategy at the nth bet is to risk only bM_{n-1}, and for each dollar committed to the bet the reward is X_n. Then we have the recurrence relation

$$M_n = M_{n-1} + bM_{n-1}X_n = (1 + bX_n)M_{n-1}.$$

Let us write this as

$$M_n = Y_n M_{n-1}, \tag{1}$$

where Y_n is $1 + bX_n$. As X_j, for $j = 1, 2, \ldots$, are independent and identically distributed, so are Y_j, for $j = 1, 2, \ldots$.

One can then seek an optimal value of b that maximises the logarithm, thus maximising expected gains. The idea is classic in binary betting games; see [1]. The ensuing formula is called Kelly's criterion. We present extensions and ramifications.

As the bettor intends to bet repeatedly, her money should not drop to 0 or below. This imposes the constraint $1 + bX_n > 0$, for every possible realisation of X_n. This is automatically satisfied for the positive values in the range of X_n. But if one of the outcomes is negative, say $-u$, for some $u > 0$, then $1 - ub > 0$, or $b < 1/u$. If the bettor commits $1/u$ or more of her money in a single bet, her money can drop to 0 or less in a single bet, and she will not be able to continue betting, contrary to the intended strategy of perpetual betting. This being true for all possible negative values in the range of the outcome puts a restriction on b; we must have $b < -1/x_1$. It may happen that x_1 is fractional. In this case, the upper bound so imposed is higher than 1. Another natural bound on b is 1. Respecting all these constraints, b must fall in the feasible range

$$0 < b < \min\left\{\frac{1}{|x_1|}, 1\right\}.$$

Feasible b is assumed henceforth.

The recurrence (1) can be iterated to obtain

$$M_n = Y_n Y_{n-1} M_{n-2} = \ldots = Y_n Y_{n-1} \ldots Y_1 M_0.$$

It is natural to deal with logarithms to turn the product into a sum, a mathematically more tractable form in the calculus of random variables. In what follows, we use $\mathbb{P}$ for probability and $\mathbb{E}$ for expectation.

The probability of being in a winning position after n bets is

$$\begin{aligned}\mathbb{P}(M_n > M_0) &= \mathbb{P}(Y_n Y_{n-1} \ldots Y_1 M_0 > M_0) \\ &= \mathbb{P}(\ln Y_n + \ln Y_{n-1} + \ldots + \ln Y_1 > 0) \qquad (2) \\ &= \mathbb{P}\left(\left(\sum_{i=1}^{n} \ln Y_i\right) - \beta n > -\beta n\right) \\ &= \mathbb{P}\left(\frac{\left(\sum_{i=1}^{n} \ln Y_i\right) - \beta n}{\sigma\sqrt{n}} > -\frac{\beta}{\sigma}\sqrt{n}\right),\end{aligned}$$

where β and $\sigma^2 > 0$ are the mean and variance of $\ln Y_1$. In the sequel we shall use the notation

$$S_n = \ln Y_1 + \ldots + \ln Y_n. \tag{3}$$

3. *The betting index*

For a given distribution of X_1, both β and σ^2 are functions of b. We shall call $\beta = \beta(b) = \mathbb{E}\left[\ln Y_1\right] = \mathbb{E}\left[\ln\left(1 + bX_1\right)\right]$ the *betting index*; we write the function β with its argument when advantageous to clarity.

Let Z be the normally distributed random variable with mean 0 and variance 1. According to the central limit theorem, for large n the random variable

$$S_n^* = \frac{S_n - \beta n}{\sigma\sqrt{n}} \tag{4}$$

has an approximate standard normal distribution (and in the limit, as $n \to \infty$, becomes distributed like Z).

Theorem 1: After n bets, the probability of being in a winning position converges as follows:

$$\mathbb{P}\left(M_n > M_0\right) \to \begin{cases} 0, & \text{if } \beta < 0; \\ \frac{1}{2}, & \text{if } \beta = 0; \\ 1, & \text{if } \beta > 0. \end{cases}$$

Proof: We give an argument along the lines of [2]. As already seen in (2) and the definitions (3) and (4), the probability $\mathbb{P}\left(M_n > M_0\right)$ of being in a winning position is the same as $\mathbb{P}\left(S_n > 0\right) = \mathbb{P}\left(S_n^* > -\sigma^{-1}\beta\sqrt{n}\right)$.

If $\beta = 0$, this probability becomes $\mathbb{P}\left(S_n^* > 0\right)$. According to the central limit theorem, this probability converges to $\frac{1}{2}$.

Consider next positive β. Let a be any arbitrary negative real number. As n can be arbitrarily large, $-\sigma^{-1}\beta\sqrt{n} < a$, for n large enough. For large n, the event $S_n^* \leqslant -\sigma^{-1}\beta\sqrt{n}$ implies $S_n^* < a$. It then follows that, for large n,

$$\mathbb{P}\left(S_n \leqslant 0\right) = \mathbb{P}\left(S_n^* \leqslant -\sigma^{-1}\beta\sqrt{n}\right) \leqslant \mathbb{P}\left(S_n^* < a\right).$$

According to the central limit theorem, as $n \to \infty$, we have

$$\limsup_{n\to\infty} \mathbb{P}\left(S_n \leqslant 0\right) \leqslant \limsup_{n\to\infty} \mathbb{P}\left(S_n^* < a\right) = F_Z(a),$$

where F_Z is the cumulative distribution function of Z. Taking the limit as $a \to -\infty$, since the left-hand side is free of a, its limit (as $a \to -\infty$) is itself, while the right-hand side is diminished to 0. Thus we get

$$\limsup_{n\to\infty} \mathbb{P}\left(S_n \leqslant 0\right) \leqslant 0.$$

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We must also have

$$0 \leqslant \liminf_{n\to\infty} \mathbb{P}\left(S_n \leqslant 0\right) \leqslant \limsup_{n\to\infty} \mathbb{P}\left(S_n \leqslant 0\right) \leqslant 0.$$

Hence $\lim_{n\to\infty} \mathbb{P}\left(S_n \leqslant 0\right)$ exists and is equal to 0, or equivalently

$$\lim_{n\to\infty} \mathbb{P}\left(M_n > M_0\right) = \lim_{n\to\infty} \mathbb{P}\left(S_n > 0\right) = 1.$$

The argument for $\beta < 0$ is similar and we omit it.

The events $\mathscr{E}_n = \left\{Z > -\frac{\beta}{\sigma}\sqrt{n}\right\}_{n=1}^{\infty}$ are monotone increasing (decreasing) if β is positive (nonpositive).* By the continuity theorem [3], we obtain

$$\lim_{n\to\infty} \mathbb{P}\left(M_n > M_0\right) = \mathbb{P}\left(Z > -\frac{\beta}{\sigma}\lim_{n\to\infty}\sqrt{n}\right).$$

Note that

$$\frac{\beta}{\sigma}\lim_{n\to\infty}\sqrt{n} \to \begin{cases} -\infty, & \text{if } \beta < 0; \\ 0, & \text{if } \beta = 0; \\ +\infty, & \text{if } \beta > 0. \end{cases}$$

Hence we have

$$\mathbb{P}\left(M_n > M_0\right) \to \begin{cases} \mathbb{P}\left(Z > \infty\right) = 0, & \text{if } \beta < 0; \\ \mathbb{P}\left(Z > 0\right) = \frac{1}{2}, & \text{if } \beta = 0; \\ \mathbb{P}\left(Z > -\infty\right) = 1, & \text{if } \beta > 0. \end{cases}$$

The interpretation of Theorem 1 is that in the long run a winning position is assured if the betting index is positive, whereas the bettor is even if that index is 0, and loss is assured if that index is negative. Such a betting game is recommended only if the betting index $\beta = \mathbb{E}\left[\ln\left(1 + bX_1\right)\right]$ is positive. It is not enough for X_j, for $j = 1, 2, \ldots,$ to have a positive expectation to gain in the long run. We illustrate this point with a few examples.

4. *American roulette*

Suppose a player bets half her money $\left(b = \frac{1}{2}\right)$ on the event that the colour black is the outcome of spinning a roulette wheel. There are 38 numbers on the wheel, of which 18 are black, 18 are red and two are green. If the player wins the game, for each dollar she bets, she receives a dollar from the house (with probability $18/38 \approx 0.47$); otherwise for each dollar she bets she gives the house a dollar (with probability $20/38 \approx 0.53$).

* By monotone increasing we mean $\mathscr{E}_n \subseteq \mathscr{E}_{n+1}$, and by monotone decreasing we mean $\mathscr{E}_n \supseteq \mathscr{E}_{n+1}$.

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In this example, X_1 has the distribution

$$X_1 = \begin{cases} +1, & \text{with probability } \frac{18}{38}; \\ -1, & \text{with probability } \frac{20}{38}. \end{cases}$$

Here $x_1 = -1$ and the feasible range of b is $0 < b < 1$; the player can bet any proper proportion of her money with no real restriction. With $b = \frac{1}{2}$, we have

$$\beta = \mathbb{E}\left[\ln(1 + bX_1)\right] = \frac{18}{38}\ln\left(\frac{3}{2}\right) + \frac{20}{38}\ln\left(\frac{1}{2}\right) = -0.1727518860\ldots < 0.$$

As the betting index is negative, a gambler playing this game will almost certainly lose in the long run.

Some players mix their bets by choosing more than one possible outcome. For example, a gambler may bet on black and on the number 12 (which is red). If the ball on the roulette wheel ends up on the number 12, the player gets \$36.00 for every dollar she bets; if the outcome is any of the black numbers, the player gets \$1.00 for every dollar she bets. A player may bet half her money each time, and her strategy is to split the chips she is placing as a bet evenly between black and 12. For example, if her capital is \$100.00, she bets \$50.00 of this amount, with \$25.00 on black and \$25.00 on the number 12. Here we have

$$X_1 = \left\{ \begin{array}{ll} +18 - \frac{1}{2}, & \text{with probability } \frac{1}{38}; \\ \frac{1}{2} - \frac{1}{2}, & \text{with probability } \frac{18}{38}; \\ -1, & \text{with probability } \frac{19}{38}. \end{array} \right\}$$

The corresponding betting index is

$$\begin{aligned} \beta &= \mathbb{E}\left[\ln\left(1 + bX_1\right)\right] \\ &= \frac{1}{38}\ln\left(1 + \frac{35}{4}\right) + \frac{18}{38}\ln\left(1 + \tfrac{1}{4}(1 - 1)\right) + \frac{19}{38}\ln\left(1 - \frac{1}{2}\right) \\ &= -0.2866455039\ldots \\ &< 0. \end{aligned}$$

Following this strategy, ultimate loss is almost guaranteed.

5. *Binary games*

In the previous example, it may appear obvious that ultimate loss is inevitable since the probability of winning an individual game is less than 0.5, with a net negative average outcome.

However, in certain binary games the probability of winning an individual game can be above 0.5, and yet eventual loss is certain. Consider a general binary game, in which the outcome X_j of the jth bet is +1 with probability p, or −1 with probability $1 - p$. As in the previous example,

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there is no restriction on the proportion of the money bet; she can bet any proportion $b \in (0, 1)$. If the gambler bets half her money $\left(b = \frac{1}{2}\right)$ at each game (a case discussed in [4]), we have

$$\beta = \mathbb{E}\left[\ln\left(1 + \tfrac{1}{2}X_1\right)\right] = p \ln\left(\frac{3}{2}\right) + (1 - p) \ln\left(\frac{1}{2}\right) = p \ln 3 - \ln 2.$$

Here $\dfrac{\ln 2}{\ln 3} = 0.6309297534\ldots$ is a critical probability, below which β is negative, at which β is zero, and above which β is positive. According to Theorem 1, the probability of ultimately being in a winning position converges in the following fashion:

$$\mathbb{P}\left(M_n > M_0\right) \to \begin{cases} 0, & \text{if } p < \frac{\ln 2}{\ln 3}; \\ \frac{1}{2}, & \text{if } p = \frac{\ln 2}{\ln 3}; \\ 1, & \text{if } p > \frac{\ln 2}{\ln 3}. \end{cases}$$

The point in this discussion is that p can be well above 0.5, and the gambler has a positive expectation for the gain in a bet (one individual bet), yet over an extended number of bets she is losing. Any value of p above 0.5 and below $\dfrac{\ln 2}{\ln 3}$ meets this description. For example, with $p = 0.6$, the gambler will ultimately be in a losing position, while betting half her money at each game.

For general p and b, the value of β associated with the binary game is

$$\beta(b) = p \ln(1 + b) + (1 - p) \ln(1 - b).$$

For any fixed p this function is concave (as it has a negative second derivative), has value 0 at $b = 0$, and achieves its maximum at $b = 2p - 1$ (which may be negative and hence not feasible). For all $p \leqslant 0.5$, the function $\beta(b)$ stays negative throughout the entire interval $0 < b < 1$. No matter what feasible b is chosen, it is a losing game. However, for $p > 0.5$, the betting index is positive on some interval contained in $(0, 1)$, and reaches its maximum at the feasible value $b = 2p - 1$ in the interval $(0, 1)$.

In this example, when the expected outcome is positive, the betting index might still be negative for some range of b. That is, positive expectation of gain in one game is not a sufficient condition for long-term winning. One has to have the right choice of b to have high chances of winning.

6. *Sports Betting*

We use an example pertaining to the sports betting market, an increasingly popular field of gambling. We focus primarily on *spread betting*, which is the most common form. For every game in the National Football League (NFL), oddsmakers set a *point spread* for the result of the

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game. This spread is meant to represent the expected result of the game. For instance, if Dallas Cowboys are playing Washington, a spread of Dallas +7.5 or Washington −7.5 means that the oddsmakers believe that Washington is expected to win by 7.5 points, and vice versa. In order to win this bet, the team you place your money on must cover the points. That is, if you bet on Washington, they must win by 8 or more points in order for you to win the bet. This spread is set in such a way that picking either side should, in theory, be a 50-50 result or a fair "coin flip". However, what is unique to sports betting is the *vigorish* added for each bet. Vigorish is the additional 'tax' placed on sports bets; typically, if the bet is won, you get only $\frac{10}{11}$ dollars (about 91 cents) back for every 1 dollar you place.

Some bettors continue betting on games of this nature in succession for an entire season, and even continue through future seasons. While the teams change and the associated spread points vary from game to game, the bettor is essentially entering a series of bets which correspond to flipping fair coins. In this example, the gain per dollar in the nth bet is X_n, which corresponds to the distribution

$$X_1 = \begin{cases} +\frac{10}{11}, & \text{with probability } \frac{1}{2}; \\ -1, & \text{with probability } \frac{1}{2}. \end{cases}$$

It is fair to assume that the X_n are independent.

For all b, we have

$$\beta(b) = \mathbb{E}\left[\ln\left(1 + bX_1\right)\right] = \frac{1}{2}\ln\left(1 + \frac{10}{11}b\right) + \frac{1}{2}\ln(1 - b).$$

Note that $\beta = \beta(b)$ has a negative derivative and $\beta(0) = 0$. So, for $\beta \in (0, 1)$, the function $\beta(b)$ is negative, yielding a negative betting index. This would mean that if the spread is the best possible prediction of the game, placing bets would result in a negative betting index, regardless of the proportion of bankroll b you place.

This is a betting game that *in theory* should not be played. However, the betting market, unlike all casino games such as roulette, is not completely efficient. For instance, a handicapper or statistical modeller also attempts to predict the result of that same game, and her analysis of the chances may arrive at different odds, such as a 60 percent chance of Washington winning by more than 7.5. If such a modeller's prediction remains consistently better than the odds announced by the NFL, her true betting index is

$$\beta(b) = \mathbb{E}\left[\ln\left(1 + bX_1\right)\right] = \frac{6}{10}\ln\left(1 + \frac{10}{11}b\right) + \frac{4}{10}\ln(1 - b).$$

For some b, this index is positive. For instance $\beta(0.16) = 0.01173957012\ldots$. Hence, if the handicapper can consistently predict some results more accurately than the oddsmakers, there is reason to believe that she can see long-term profit in the market.

7. *Optimising bets*

Given that a betting game has good prospects, the bettor would be interested in good advice on how to approach the betting scheme. For instance, a bettor on binary games may be interested in a good choice of $b \in (0, 1)$ to apportion money for the bets.

If, for the given p, there is no range of b for which the betting index is above 0, the bettor would be advised not to bet at all. Consider again the binary game discussed in Section 5. The left plot in Figure 1 shows the betting index for this game, for $p = 0.4$, for all possible b. For such low p, the index is always negative.

On the other hand, suppose for the given p there is a range of b for which the betting index is above 0. We know that S_n is stochastically increasing in b. In this case, the bettor is advised to bet the largest possible b proportion of her money on each bet, while the index remains positive. This way the bettor is assured of winning. For the binary game discussed, the right plot in Figure 1 shows the betting index for $p = 0.6$, for all possible b. For such high p, the index is positive only for $b < 0.38939$. The bettor is advised to bet an amount around 0.38 of her money. The probability of ultimately winning is 1.

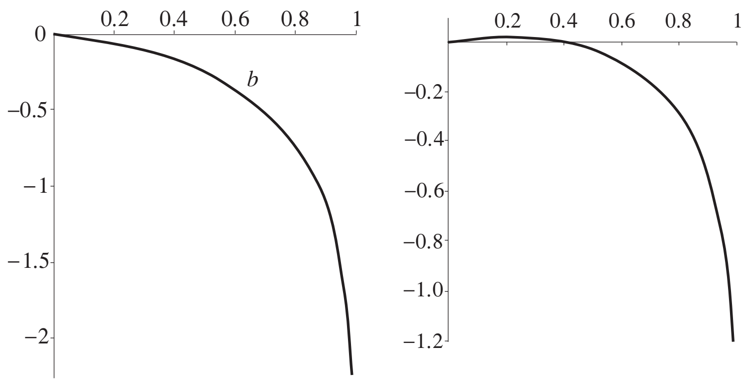

FIGURE 1: Plot of the betting index against the proportion bet, when $p = 0.4$ (left) and when $p = 0.6$ (right)

8. *Conclusion*

We presented a betting index, β, that guides a practitioner to a prudent strategy. Almost surely, in the long run, if the index is negative the bettor will lose, if the index is positive the bettor will gain, and if the index is $\frac{1}{2}$ the bettor will come out even. We corroborated the formal approach with examples from binary games, such as roulette, and the moneyline betting in the National Football League. The practitioner might be looking for advice on the proportion of her capital to bet. We said a word about the best proportion to bet, which may often be 0, that is, not to bet at all when the games are not designed to favour bettors.

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