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WILD RAMIFICATION AND THE CHARACTERISTIC CYCLE OF AN ℓ-ADIC SHEAF

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Abstract We propose a geometric method to measure the wild ramification of a smooth étale sheaf along the boundary. Using the method, we study the graded quotients of the logarithmic ramification groups of a local field of characteristic $p > 0$ with arbitrary residue field. We also define the characteristic cycle of an ℓ-adic sheaf, satisfying certain conditions, as a cycle on the logarithmic cotangent bundle and prove that the intersection with the 0-section computes the characteristic class, and hence the Euler number.

Keywords: ℓ-adic sheaf; wild ramification; characteristic cycle

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Contents

Introduction

Let X be a separated scheme of finite type over a perfect field k of characteristic $p > 0$. We consider a smooth ℓ-adic étale sheaf $\mathcal{F}$ on a smooth dense open subscheme $U \subset X$

for a prime $\ell \neq p$. The ramification of $\mathcal{F}$ along the boundary $X \setminus U$ has been studied traditionally by using a finite étale covering of U trivializing $\mathcal{F}$ modulo ℓ. In this paper, we propose a new geometric method, inspired by the definition of the ramification groups [**1**, **2**, **4**].

The basic geometric construction used in this paper is the blowing-up at the ramification divisor embedded diagonally in the self log product. A precise definition will be given at the beginning of § 2.3. We will consider two types of blow-up. The preliminary one, called the log blow-up, is the blow-up $(X \times X)' \to X \times X$ at every $D_i \times D_i$ where D_i denotes an irreducible component of a divisor $D = X \setminus U$ with simple normal crossings in a smooth scheme X over k. The second one is the blow-up $(X \times X)^{(R)} \to (X \times X)'$ at $R = \sum_i r_i D_i$, with some rational multiplicities $r_i \geqslant 0$, embedded in the log diagonal $X \to (X \times X)'$. This construction globalizes that used in the definition of the ramification groups in [**1**] and [**2**] recalled in § 1.

Inspired by [**13**], we consider the ramification along the boundary of the smooth sheaf $\mathcal{H} = \mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, \mathrm{pr}_1^* \mathcal{F})$ on the dense open subscheme $U \times U \subset (X \times X)^{(R)}$. We introduce a measure of wild ramification by using the extension property of the identity regarded as a section of the restriction on the diagonal of the sheaf $\mathcal{H}$, in Definition 2.19.

Let $j^{(R)} : U \times U \to (X \times X)^{(R)}$ denote the open immersion. A key property of the sheaf $\mathcal{H}$ established in Propositions 2.25 and 2.26 is that the restriction of $j_*^{(R)} \mathcal{H}$ on the complement $(X \times X)^{(R)} \setminus U \times U$ admits a description by the Artin–Schreier sheaves defined by certain linear forms. This fact is derived from a groupoid structure of $(X \times X)^{(R)}$ inherited from the natural one on $X \times X$. We prove in Theorem 1.24 that this property at the generic point of an irreducible component implies the following properties of the ramification groups conjectured in [**4**, Conjecture 9.4]: the graded pieces of the ramification groups, known to be abelian, are killed by p and their character groups are described by differential forms.

The definition of the measure of the wild ramification in this paper is closely related to that of the characteristic class in [**3**]. In Definition 3.6, we propose a definition of the characteristic cycle of an ℓ-adic sheaf as a cycle of the logarithmic cotangent bundle, under the conditions (R) and (C) stated in § 3.2. Roughly speaking, the conditions mean that the ramification is controlled at the generic points of the irreducible components of the ramification divisor. Consequently, the characteristic cycle in this case does not have components supported on subvarieties of codimension at least two. We show that its intersection product with the 0-section computes the characteristic class, in Theorem 3.7. This is a generalization of Kato's formula in the rank one case [**15**].

One expects that the same construction works for $\mathcal{D}$-modules with irregular singularities. It should give another evidence for the analogy between the wild ramification of ℓ-adic sheaves and irregular singularities of $\mathcal{D}$-modules.

Notation

k denotes a perfect field of characteristic $p > 0$. A scheme over k is assumed to be separated of finite type over k. For a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank on a scheme X, $E = \boldsymbol{V}(\mathcal{E})$ denotes the contravariant vector bundle defined by the quasi-coherent

$\mathcal{O}_X$-algebra $S^\bullet\mathcal{E}$. Similarly, $\boldsymbol{P}(\mathcal{E})$ denotes the projective space bundle $\mathcal{P}roj\, S^\bullet\mathcal{E}$. The dual of $\mathcal{E}$ is denoted by $\mathcal{E}^\vee$. For a closed subscheme $X \subset Y$ defined by the ideal $\mathcal{I}_X \subset \mathcal{O}_Y$, the conormal sheaf $\mathcal{I}_X/\mathcal{I}_X^2$ is denoted by $\mathcal{N}_{X/Y}$.

ℓ denotes a prime number invertible in k and Λ denotes a finite local $\mathbb{Z}_\ell$-algebra.

1. Ramification groups

The theory of logarithmic ramification groups of a local field with imperfect residue field as developed in [**1**] and [**2**] relies on rigid geometry and on log geometry in an essential way. In §§ 1.1–1.3, we give some interpretations purely in terms of schemes, without using rigid geometry or log geometry. In § 1.3, we state the main result, Theorem 1.24, on the structure of the graded quotients. We prove it in § 1.4 by computing the nearby cycles.

In this section, K denotes a discrete valuation field, $\mathcal{O}_K$ denotes the valuation ring, and $\mathfrak{m}_K$ denotes the maximal ideal. The residue field $\mathcal{O}_K/\mathfrak{m}_K$ is denoted by F and $v_K : K \to \mathbb{Z} \cup \{\infty\}$ denotes the discrete valuation normalized by $v_K(\pi) = 1$ for a prime element π. We put $S = \operatorname{Spec} \mathcal{O}_K$. Throughout the section, a morphism of schemes over S will mean a morphism over S.

1.1. Basic constructions

Let A be a finite flat $\mathcal{O}_K$-algebra. We put $T = \operatorname{Spec} A$. We consider a closed immersion $T \to P$ to a smooth scheme P over S. Let $\mathcal{I}_T = \operatorname{Ker}(\mathcal{O}_P \to \mathcal{O}_T)$ be the ideal sheaf defining the closed subscheme T in P.

For a pair (m, n) of integers $m \geqslant 0$ and $n > 0$, let $Q = P_T^{[m/n]} \to P$ be the blow-up at the ideal $\mathcal{I}_T^n + \mathfrak{m}_K^m\mathcal{O}_P$ and $P_T^{(m/n)} \subset P_T^{[m/n]}$ be the complement of the support of $(\mathcal{I}_T^n\mathcal{O}_Q + \mathfrak{m}_K^m\mathcal{O}_Q)/\mathfrak{m}_K^m\mathcal{O}_Q$. The morphism $P_T^{(m/n)} \to P$ is affine and $P_T^{(m/n)}$ is defined by the quasi-coherent $\mathcal{O}_P$-subalgebra $\mathcal{O}_P[\mathfrak{m}_K^{-m}\mathcal{I}_T^n] \subset K \otimes \mathcal{O}_P$. The maps $P_T^{(m/n)} \to P_T^{[m/n]} \to P$ induce isomorphisms $P_{T,K}^{(m/n)} \to P_{T,K}^{[m/n]} \to P_K$ on the generic fibres. For $m = 0$, we have $P_T^{(0/n)} = P_T^{[0/n]} = P$. The immersion $T \to P$ is uniquely lifted to an immersion $T \to P^{(m/n)}$.

Let $d > 0$, $m' \geqslant 0$, $n' > 0$ be integers such that $m' \leqslant dm$ and $n' = dn$. Then the inclusion $(\mathfrak{m}_K^{-m}\mathcal{I}_T^n)^d \supset \mathfrak{m}_K^{-m'}\mathcal{I}_T^{n'}$ induces a canonical map $P_T^{(m/n)} \to P_T^{(m'/n')}$ that is an isomorphism on the generic fibres. If $(m', n') = (dm, dn)$, the canonical map $P_T^{(m/n)} \to P_T^{(m'/n')}$ is finite.

For a rational number $r = m/n \geqslant 0$, let $P_T^{(r)}$ be the normalization of $P_T^{(m/n)}$. For $r > 0$, let $\widehat{P_T^{(r)}}$ be the formal completion of $P_T^{(r)}$ along the closed fibre $P_{T,F}^{(r)}$. For $r' \leqslant r$, the canonical maps $P_T^{(r)} \to P_T^{(r')}$ of schemes and

$$\widehat{P_T^{(r)}} \to \widehat{P_T^{(r')}}$$

of affine formal schemes are induced.

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We compare the construction above to those in [**1**] and [**2**].

Example 1.1. Assume K is complete.

(1) Let $Z = (z_1, \dots, z_n)$ be a system of generators of a finite flat $\mathcal{O}_K$-algebra A and consider the closed immersion $T = \operatorname{Spec} A \to P = \operatorname{Spec} \mathcal{O}_K[X_1, \dots, X_n]$ defined by Z. Then, the affinoid variety X_Z^a in [**1**, 3.1] is defined by the formal $\mathcal{O}_K$-scheme $\widehat{P_T^{(r)}}$ for $a = r$.

(2) Let $T \to P$ be a closed immersion of a finite flat $\mathcal{O}_K$-scheme T to a smooth scheme P and let Spf $\boldsymbol{A}$ be the formal completion $\hat{P}|_T$ of P along the closed subscheme T. Then the affinoid variety $X^j(\boldsymbol{A} \to A)$ in [**2**, Definition 1.5] is defined by the formal $\mathcal{O}_K$-scheme $\widehat{P_T^{(r)}}$ for $j = r$.

Lemma 1.2. *Let T be a finite flat scheme over S and $T \to P$ and $T \to Q$ be closed immersions to smooth schemes over S. Let $P \to Q$ be a smooth morphism over S such that the diagram*

$$\begin{array}{ccc} T & \longrightarrow & P \\ & \searrow & \downarrow \\ & & Q \end{array}$$

is commutative. Then, for a positive integer $r > 0$, the map $P \to Q$ induces a smooth map $P_T^{(r)} \to Q_T^{(r)}$ and an isomorphism

$$P_{T,F}^{(r)} \to Q_{T,F}^{(r)} \times_{T_F} \boldsymbol{V}(\mathfrak{m}_K^{-r} \otimes_{\mathcal{O}_K} \Omega^1_{P/Q} \otimes_{\mathcal{O}_P} \mathcal{O}_{T_F}).$$

Proof. It suffices to show the assertions with (r) replaced by $(r/1)$. We show the map $P_T^{(r/1)} \to Q_T^{(r/1)}$ is smooth. Let $t \in T$ be a closed point and d be the relative dimension of $P \to Q$ at the image of t. The section defined by $T \to P$ of the smooth morphism $P \times_Q T \to T$ is a regular immersion of codimension d. By choosing a minimal set of generators of the ideal and by lifting them, we find a neighbourhood $V \subset P$ of the image of t and an étale morphism $V \to \boldsymbol{A}^d_Q = Q[X_1, \dots, X_d]$ inducing an open immersion $T \cap V \to T \subset T \times_Q \boldsymbol{A}^d_Q$ to the 0-section. Then, $P_T^{(r/1)} \times_P V$ is isomorphic to $V \times_{\boldsymbol{A}^d_Q} Q_T^{(r/1)}[X_1/\pi^r, \dots, X_d/\pi^r]$ and is smooth over $Q_T^{(r/1)}$.

Since the map $Q_{T,F}^{(r/1)} \to Q_{T,F}$ factors through the closed immersion $T_F \to Q_{T,F}$, the isomorphism

$$P_T^{(r/1)} \times_P V \to V \times_{\boldsymbol{A}^d_Q} Q_T^{(r/1)}[X_1/\pi^r, \dots, X_d/\pi^r]$$

above induces an open immersion $P_{T,F}^{(r/1)} \times_P V \to Q_{T,F}^{(r/1)}[X_1/\pi^r, \dots, X_d/\pi^r]$. Since $Q_{T,F}^{(r/1)}[X_1/\pi^r, \dots, X_d/\pi^r]$ is canonically identified with

$$Q_{T,F}^{(r/1)} \times_{T_F} \boldsymbol{V}(\mathfrak{m}_K^{-r} \otimes_{\mathcal{O}_K} \Omega^1_{P/Q} \otimes_{\mathcal{O}_P} \mathcal{O}_{T_F}),$$

the assertion follows. □

Let $\bar{T}$ denote the normalization of T. For positive integers $m, n > 0$ and for $r = m/n$, the immersion $T \to P$ induces an immersion $T \to P_T^{(m/n)}$ and hence a finite map $\bar{T} \to P_T^{(r)}$. The latter further induces a map on the formal completions.

Lemma 1.3. *Let $T = \operatorname{Spec} A$ be a finite flat scheme over S and $T \to P$ be a closed immersion to a smooth scheme over S. Assume that T_K is isomorphic to the disjoint union of finitely many copies of* $\operatorname{Spec} K$. *Then there exists an integer $r > 0$ such that the map $\bar{T} \to P_T^{(r)}$ is a closed immersion.*

Proof. By the assumption on T_K, the semi-local ring A is the product of finitely many local rings and the normalization of A is generated over $\mathcal{O}_K$ by the idempotents in $A \otimes_{\mathcal{O}_K} K$. Hence, we may assume $P = \operatorname{Spec} R$ is affine and hence $P_T^{(r)} = \operatorname{Spec} R^{(r)}$ is also affine. It is sufficient to show that, for every idempotent $e \in A \otimes_{\mathcal{O}_K} K$, there exists an integer $r > 0$ such that e is in the image of $R^{(r)} \to A \otimes_{\mathcal{O}_K} K$. Take a non-zero element $a \in \mathfrak{m}_K$ such that $ae \in A$. We show that $r = 2v_K(a)$ satisfies the condition.

Take a lifting $f \in R$ of $ae \in A$. Then $g = f^2 - af$ is in the kernel $I = \operatorname{Ker}(R \to A)$. Since g/a^2 is in $R^{(r/1)}$, the solution $f/a \in R^{(r/1)} \otimes_{\mathcal{O}_K} K$ of the equation $X^2 - X = g/a^2$ lies in $R^{(r)}$. □

We study the relation of the basic construction with a base change of discrete valuation rings. Let $T \to P$ be a closed immersion of a finite scheme to a smooth scheme over $S = \operatorname{Spec} \mathcal{O}_K$ as above. Let $S' = \operatorname{Spec} \mathcal{O}_{K'} \to S$ be a surjection of spectra of discrete valuation rings of ramification index e. Then, the base change $T' = T \times_S S' \to P' = P \times_S S'$ is a closed immersion of a finite flat scheme to a smooth scheme over S'. For integers $m, n > 0$, the induced map $P'^{[em/n]}_{T'} \to P_T^{[m/n]} \times_S S'$ is an isomorphism. Hence, for $r = m/n$, the scheme $P'^{(er)}_{T'}$ is the normalization of $P_T^{(r)} \times_S S'$ and the formal scheme $\widehat{P'^{(er)}_{T'}}$ is the normalization of $\widehat{P_T^{(r)}} \times_{\hat{S}} \hat{S}'$. Note that we need not assume that the fraction field extension nor the residue field extension is finite.

We prepare some facts on the properties (S_k) and (R_k) of locally noetherian schemes [**11**, Chapter IV, §§ 5.7, 5.8].

Lemma 1.4. *Let $f : X \to S$ be a flat scheme of finite type over a regular noetherian scheme S. For a point $s \in S$, we put $c(s) = \dim \mathcal{O}_{S,s}$. Let $k \geqslant 0$ be an integer.*

(1) *The following conditions are equivalent.*

(a) *For every point $s \in S$, the fibre X_s satisfies the condition* $(\mathrm{S}_{k-c(s)})$.

(b) *X satisfies the condition* (S_k).

(2) *Condition (a) implies condition (b).*

(a) *For every point $s \in S$, the fibre X_s satisfies the condition* $(\mathrm{R}_{k-c(s)})$.

(b) *X satisfies the condition* (R_k).

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Proof. (1) Let $x \in X$ be a point and put $s = f(x) \in S$. Let $t_1, \dots, t_c \in \mathfrak{m}_s$ be a regular system of parameters where $c = c(s)$. Since $f : X \to S$ is flat, $f^*t_1, \dots, f^*t_c \in \mathfrak{m}_x$ is a regular sequence of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X_s,x} = \mathcal{O}_{X,x}/(f^*t_1, \dots, f^*t_c)$. Hence, we have equalities $\dim \mathcal{O}_{X_s,x} = \dim \mathcal{O}_{X,x} - c(s)$ [**11**, Chapter 0, Proposition (16.3.7)] and $\operatorname{prof} \mathcal{O}_{X_s,x} = \operatorname{prof} \mathcal{O}_{X,x} - c(s)$ [**11**, Proposition (16.4.6) (ii)] and the assertion follows.

(2) Further, $\mathcal{O}_{X,x}$ is regular if $\mathcal{O}_{X_s,x}$ is regular [**11**, Proposition (17.3.3) (ii)]. □

Corollary 1.5. *Let $S = \operatorname{Spec} \mathcal{O}_K$ be the spectrum of a discrete valuation ring and $f : X \to S$ be a normal scheme of finite type with smooth generic fibre.*

(1) *Assume $X \to S$ has geometrically reduced fibres. Then, for any surjection $S' \to S$ of spectra of discrete valuation rings, the base change $X \times_S S'$ is normal.*

(2) *There exists a surjection of spectra $S' = \operatorname{Spec} \mathcal{O}_{K'} \to S$ of discrete valuation rings such that K' is a finite extension of K and that the normalization X' of $X \times_S S'$ has geometrically reduced fibres over S'.*

Proof. (1) Since the closed fibre of $X_{S'}$ is reduced, it satisfies the conditions (R_0) and (S_1). Since the generic fibre is regular, $X_{S'}$ satisfies the conditions (R_1) and (S_2) by Lemma 1.4.

(2) We may assume that the residue field F is algebraically closed since there exists an inductive system $(\mathcal{O}_{K_i})_{i \in I}$ of finite extensions of discrete valuation rings of ramification index 1 such that the limit $\varinjlim_i F_i$ of the residue fields is an algebraic closure of F. We apply a variant [**19**, Appendix, Théorème 2] of Epp's theorem [**9**] corrected in [**18**] to the generic points of the irreducible components of the closed fibre of $X \to S$. Then, we find a surjection $S' = \operatorname{Spec} \mathcal{O}_{K'} \to S$ of spectra of discrete valuation rings and an open subscheme U of the normalization X' of the base change $X \times_S S'$ such that K' is a finite extension of K and that U is smooth over S' and contains the generic point of every irreducible component of the closed fibre.

We show that X' has geometrically reduced fibres. Since the generic fibre is smooth, it suffices to show that the geometric closed fibre is reduced. Since X' is normal, it satisfies the condition (S_2). By Lemma 1.4 (1), the closed fibre satisfies (S_1) and hence the geometric closed fibre also satisfies (S_1). Since the geometric closed fibre has a dense open subscheme smooth over the base field, it also satisfies the condition (R_0). Hence the geometric closed fibre is reduced. □

Let X be a normal scheme of finite type over $S = \operatorname{Spec} \mathcal{O}_K$. Assume that the generic fibre of X is smooth and that the closed geometric fibre is reduced. Then, the formal completion $\hat{X}$ along the closed fibre is the stable integral model of the affinoid variety defined by $\hat{X}$ itself. Thus, Corollary 1.5 implies the finiteness theorem of Grauert and Remmert [**6**, Theorem 1.2] for algebraizable formal schemes.

Applying Corollary 1.5 to $P_T^{(r)} \to S$, we obtain the following.

Corollary 1.6. *Let $T \to P$ be a closed immersion of a finite flat scheme to a smooth scheme over $S = \operatorname{Spec} \mathcal{O}_K$ and $r > 0$ be a rational number.*

(1) *There exists a surjection of spectra $S' = \operatorname{Spec} \mathcal{O}_{K'} \to S$ of discrete valuation rings of ramification index e such that K' is a finite extension of K and that $P'^{(er)}_{T'} \to S'$ has geometrically reduced fibres.*

(2) *Assume $P^{(r)}_T \to S$ has geometrically reduced fibres. Then, for any surjection $S' \to S$ of spectra of discrete valuation rings of ramification index e, the canonical map $P'^{(er)}_{T'} \to P^{(r)}_T \times_S S'$ is an isomorphism.*

Definition 1.7. Let T be a finite flat scheme over S and $T \to P$ be a closed immersion to a smooth scheme over S. Let $r > 0$ be a rational number and $S' \to S$ be a surjection of spectra of discrete valuation rings of ramification index e.

We say $P'^{(er)}_{T'} \to S'$ is a stable model of $P^{(r)}_T$ if its geometric fibres are reduced. If $P'^{(er)}_{T'} \to S'$ is a stable model, we call $P'^{(er)}_{T'} \times_{S'} \operatorname{Spec} \bar F$ the stable closed fibre and write it by $\bar P^{(r)}_{T,\bar F}$.

By Corollary 1.6 (1), there exists an S' such that $P'^{(er)}_{T'} \to S'$ is a stable model. By Corollary 1.6 (2), the stable closed fibre $\bar P^{(r)}_{T,\bar F}$ is independent of the choice of such S'. The finite map $P'^{(er)}_{T'} \to P^{(r)}_T \times_S S'$ induces a finite map $\bar P^{(r)}_{T,\bar F} \to P^{(r)}_{T,\bar F}$.

Similarly as the stable closed fibre $\bar P^{(r)}_{T,\bar F}$, we define $\bar T_{\bar F}$ for a finite flat scheme T such that T_K is étale over K as follows. For $S' = \operatorname{Spec} \mathcal{O}_{K'} \to S$ such that $T \times_S \operatorname{Spec} K'$ is the disjoint union of finitely many copies of $\operatorname{Spec} K'$, the geometric fibre $\overline{T \times_S S'} \times_{S'} \bar F$ of the normalization is independent of the choice of S'. We write it by $\bar T_{\bar F}$. The condition that $T \times_S \operatorname{Spec} K'$ is the disjoint union of finitely many copies of $\operatorname{Spec} K'$ implies that the normalization $\overline{T \times_S S'}$ is the disjoint union of finitely many copies of S'.

Definition 1.8. Let T be a finite flat scheme over S such that T_K is étale over K.

(1) Let $r > 0$ be a rational number. Let $T \to P$ be a closed immersion to a smooth scheme over S and $S' = \operatorname{Spec} \mathcal{O}_{K'} \to S$ be a surjection of spectra of discrete valuation rings of ramification index e satisfying the following conditions: the étale covering $T_K \to \operatorname{Spec} K$ splits over K' and hence the normalization $\bar T_{S'}$ of $T \times_S S'$ is isomorphic to the disjoint union of finitely many copies of S'; the product er is an integer and the geometric fibres of $P^{(er)}_{T,S'} \to S'$ are reduced.

We say the ramification of T over S is bounded by r if, the map $\bar T_{S'} \to P^{(er)}_{T,S'}$ induces an injection
$$\bar T_{\bar F} \to \pi_0(\bar P^{(r)}_{T,\bar F})$$
of finite sets.

(2) Let $r \geqslant 0$ be a rational number. We say the ramification of T over S is bounded by $r+$ if the ramification of T is bounded by every rational number $s > r$.

By Lemma 1.2, the map $\bar T_{\bar F} \to \pi_0(\bar P^{(r)}_{T,\bar F})$ is independent of P. Let T be a finite flat scheme over S and $S' \to S$ be a surjection of spectra of discrete valuation rings of ramification e. Then, it is clear from the definition that the ramification of T over S is bounded by r if and only if the ramification of $T \times_S S'$ over S' is bounded by er.

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We will see later that Definition 1.8 is equivalent to the definition in [**1**, Definition 6.3] for finite flat $\mathcal{O}_K$-algebra locally of complete intersection.

Lemma 1.9. *For a finite flat scheme T over S, the following conditions are equivalent.*

(1) *T is locally of complete intersection.*

(2) *There exists a Cartesian diagram*

$$\begin{array}{ccc} T & \longrightarrow & Q \\ \downarrow & & \downarrow \\ S & \longrightarrow & P \end{array} \tag{1.1}$$

of schemes over S satisfying the following condition.

(CI) *The vertical arrows are quasi-finite flat and the horizontal arrows are closed immersions; the schemes P and Q are smooth over S.*

Proof. (1) $\Rightarrow$ (2). Take a surjection $\mathcal{O}_K[X_1, \ldots, X_d] \to A$ and let I denote the kernel. The closed immersion $T \to Q = \boldsymbol{A}^d_S$ is regular of codimension d and the $\mathcal{O}_T$-module I/I^2 is free of rank d. By lifting a basis, we find elements $f_1, \ldots, f_d \in I$ such that $(f_1, \ldots, f_d) = I$ on a neighbourhood of T. We define a map $Q \to P = \boldsymbol{A}^d_S$ by $f_1, \ldots, f_d$ and consider the 0-section $S \to P$. Then, shrinking Q if necessary, the diagram (1.1) is Cartesian and the map $Q \to P$ is quasi-finite and flat by [**11**, Chapter 0, Proposition (15.1.21)].

(2) $\Rightarrow$ (1). Since the immersion $S \to P$ is regular, the immersion $T \to Q$ is also regular and T is locally of complete intersection over S. □

We compute the scheme $P^{(r)}_T$ explicitly in the case where $T = S \to P$ is a section of a smooth scheme $P \to S$ of relative dimension d. The conormal sheaf $\mathcal{N}_{S/P} = \mathcal{I}_S/\mathcal{I}_S^2$ is canonically identified with the free $\mathcal{O}_K$-module $\Omega^1_{P/S} \otimes_{\mathcal{O}_P} \mathcal{O}_S$ of rank d.

Lemma 1.10. *Let $S \to P$ be a section of a smooth scheme $P \to S$ and $r > 0$ be a rational number. Let $j : P_K = P \times_S K \to P$ be the open immersion and $\mathcal{I}_S \subset \mathcal{O}_P$ be the ideal sheaf of S regarded as a subscheme of P by the section $s : S \to P$.*

(1) *The affine P-scheme $P^{(r)}_S$ is defined by the quasi-coherent $\mathcal{O}_P$-algebra*

$$\sum_{l \geqslant 0} \mathfrak{m}_K^{-[lr]} \cdot \mathcal{I}_S^l \subset j_* \mathcal{O}_{P_K}, \tag{1.2}$$

where $[lr]$ denotes the integral part.

(2) *Assume r is an integer. Then $P^{(r)}_S = P^{(r/1)}_S$ is smooth over $\mathcal{O}_K$. Further, by $\mathcal{O}_{P^{(r)}_S} = \mathcal{O}_P[\mathfrak{m}_K^{-r} \cdot \mathcal{I}_S] \subset j_* \mathcal{O}_{P_K}$, the closed fibre $P^{(r)}_{S,F}$ is identified with the F-vector space*

$$\boldsymbol{V}(\Omega^1_{P/S} \otimes_{\mathcal{O}_P} F \otimes_F \mathfrak{m}_K^{-r}/\mathfrak{m}_K^{-r+1}) = \operatorname{Spec} S^\bullet_F(\Omega^1_{P/S} \otimes_{\mathcal{O}_P} F \otimes_F \mathfrak{m}_K^{-r}/\mathfrak{m}_K^{-r+1}).$$

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Proof. (1) Let $n \geqslant 1$ be an integer such that $m = nr$ is an integer. Then, $P_S^{(r)}$ is defined by the normalization A of the quasi-coherent $\mathcal{O}_P$-algebra $\mathcal{O}_P[\mathfrak{m}_K^{-m} \cdot \mathcal{I}_S^n] \subset j_*\mathcal{O}_{P_K}$. Since $(\mathfrak{m}_K^{-[lr]} \cdot \mathcal{I}_S^l)^n \subset \mathfrak{m}_K^{-ml} \cdot \mathcal{I}_S^{nl} \subset \mathcal{O}_P[\mathfrak{m}_K^{-m} \cdot \mathcal{I}_S^n]$ for $l \geqslant 0$, we have an inclusion

$$\sum_{l \geqslant 0} \mathfrak{m}_K^{-[lr]} \cdot \mathcal{I}_S^l \subset A.$$

We show the inclusion is an equality. It suffices to show that $\sum_{l \geqslant 0} \mathfrak{m}_K^{-[lr]} \cdot \mathcal{I}_S^l$ is normal. Since the question is étale local on P, we may assume that P is isomorphic to $\boldsymbol{A}_S^n$ and that $S \to P$ is the 0-section. Or equivalently, we may assume that $S \to P$ is the 0-section of the vector bundle $P = \boldsymbol{V}(E) = \operatorname{Spec} S^\bullet_{\mathcal{O}_K}(E)$ associated to a free $\mathcal{O}_K$-module of finite rank. By taking a basis of E, we identify the symmetric algebra $S^\bullet_{\mathcal{O}_K}(E)$ with the monoid algebra $\mathcal{O}_K[M]$ where M is a free commutative monoid with a basis $e_1, \ldots, e_q$. Let $\sigma : M \to \mathbb{N}$ denotes the map of monoids sending $e_1, \ldots, e_q$ to 1 and let $e_0 \in \mathbb{N}$ denote the basis 1. Then the saturation $\tilde{M}_r = \{(a, b) \in \mathbb{Z} \times M \mid a + r \cdot \sigma(b) \geqslant 0\}$ of the submonoid $\langle e_0, ne_1 - me_0, \ldots, ne_q - me_0 \rangle \subset M \times \mathbb{Z}$ is equal to the union $\coprod_{l \geqslant 0} \{(a, b) \in \mathbb{Z} \times M \mid a \geqslant -[lr],\ \sigma(b) = l\}$. For a prime element π of K, the ring $\mathcal{O}_K[\tilde{M}_r]/(e_0 - \pi)$ is normal and we have $\mathcal{O}_K[\tilde{M}_r]/(e_0 - \pi) = \bigoplus_{l \geqslant 0} \mathfrak{m}_K^{-[lr]} S^l(E)$. Thus the assertion follows.

(2) We show $P_S^{(r)} = P_S^{(r/1)}$ is smooth over $\mathcal{O}_K$. Since the question is étale local on P, we may assume $P = \boldsymbol{V}(E) = \operatorname{Spec} S^\bullet(E)$ as above. Then, $P_S^{(r)} = P_S^{(r/1)} = \operatorname{Spec} S^\bullet(\mathfrak{m}_K^{-r} E)$ is smooth over $\mathcal{O}_K$.

We show that the closed immersion $P_{S,F}^{(r)} = P_{S,F}^{(r/1)} \to \operatorname{Spec} S^\bullet(\mathfrak{m}_K^{-r} \otimes_{\mathcal{O}_K} \mathcal{N}_{S/P} \otimes_{\mathcal{O}_K} F)$ is an isomorphism. We conclude by reducing to the case $P = \boldsymbol{V}(E) = \operatorname{Spec} S^\bullet(E)$ as above. $\square$

Let $v : \bar{K} \to \mathbb{Q} \cup \{\infty\}$ be the extension of the normalized discrete valuation $v : K \to \mathbb{Z} \cup \{\infty\}$ to a separable closure. For a rational number r, we put $\mathfrak{m}_{\bar{K}}^r = \{a \in \bar{K} \mid v(a) \geqslant r\}$ and $\mathfrak{m}_{\bar{K}}^{r+} = \{a \in \bar{K} \mid v(a) > r\}$.

Corollary 1.11. *Let $m, n > 0$ be positive integers such that $r = m/n$ and $(m, n) = 1$. Then for the reduced closed fibre $(P_{S,F}^{(r)})_{\mathrm{red}}$, we have a commutative diagram*

$$\begin{array}{ccc} \bar{P}_{S,\bar{F}}^{(r)} & \longrightarrow & \boldsymbol{V}(\Omega^1_{P/S} \otimes_{\mathcal{O}_P} \bar{F} \otimes_{\bar{F}} \mathfrak{m}_{\bar{K}}^{(-r)}/\mathfrak{m}_{\bar{K}}^{(-r)+}) \\ & & = \operatorname{Spec} S^\bullet(\Omega^1_{P/S} \otimes_{\mathcal{O}_P} \bar{F} \otimes_{\bar{F}} \mathfrak{m}_{\bar{K}}^{(-r)}/\mathfrak{m}_{\bar{K}}^{(-r)+}) \\ \downarrow & & \downarrow \\ (P_{S,F}^{(r)})_{\mathrm{red}} & \longrightarrow & \operatorname{Spec} \bigoplus_{l \geqslant 0} S^{nl} \Omega^1_{P/S} \otimes_{\mathcal{O}_P} F \otimes_F \mathfrak{m}_K^{-ml}/\mathfrak{m}_K^{-ml+1} \end{array} \tag{1.3}$$

The horizontal arrows are isomorphisms induced by (1.2) and the right vertical arrow is induced by the natural inclusion.

Proof. We show that the lower horizontal arrow is induced by the surjection

$$\mathcal{O}_{P_S^{(r)}} = \sum_{l \geqslant 0} \mathfrak{m}_K^{-[lr]} \cdot \mathcal{I}_S^l \to \mathcal{O}_{P_{S,F}^{(r)}}.$$

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If $[lr] < lr$, the image of $\mathfrak{m}_K^{-[lr]} \cdot \mathcal{I}_S^l$ is nilpotent. Similarly, the image of $\mathfrak{m}_K^{-[lr]} \cdot \mathcal{I}_S^{l+1} \subset \mathfrak{m}_K^{-[(l+1)r]} \cdot \mathcal{I}_S^{l+1}$ is also nilpotent. Thus, it induces a surjection

$$\bigoplus_{l \geqslant 0,\, lr \in \mathbb{N}} \mathfrak{m}_K^{-lr} \cdot \mathcal{I}_S^l / \mathcal{I}_S^{l+1} \otimes_{\mathcal{O}_K} F = \bigoplus_{l \geqslant 0,\, lr \in \mathbb{N}} S^l \Omega^1_{P/S} \otimes_{\mathcal{O}_P} F \otimes_F \mathfrak{m}_K^{-lr} / \mathfrak{m}_K^{-lr+1} \to \mathcal{O}_{P^{(r)}_{F,S\mathrm{red}}}$$

and define the lower horizontal arrow as a closed immersion.

The upper horizontal arrow is defined as the lower one for the base change to a finite extension of K of ramification index e such that er is an integer. It is an isomorphism by Lemma 1.10 (2). The commutativity of the diagram is clear. Since the right vertical arrow is defined by an injection of a ring, the lower horizontal arrow is an isomorphism. □

We consider a Cartesian diagram (1.1) satisfying the condition (CI) in Lemma 1.9. For positive integers $m, n > 0$, the diagram

$$\begin{array}{ccc} Q_T^{(m/n)} & \longrightarrow & Q \\ \downarrow & & \downarrow \\ P_S^{(m/n)} & \longrightarrow & P \end{array} \tag{1.4}$$

is Cartesian. Hence the canonical map $Q_T^{(m/n)} \to P_S^{(m/n)}$ is also quasi-finite and flat and induces a finite map $Q_{T,F}^{(m/n)} \to P_{S,F}^{(m/n)}$ on the closed fibres. For $r = m/n$, we have a quasi-finite morphism $Q_T^{(r)} \to P_S^{(r)}$ of schemes and a finite morphism of

$$\widehat{Q_T^{(r)}} \to \widehat{P_S^{(r)}}$$

of affine formal schemes over $\hat{S} = \operatorname{Spf} \hat{\mathcal{O}}_K$. If $Q \to P$ is étale, the diagram (1.4) with (m/n) replaced by (r) is also Cartesian.

A diagram (1.1) satisfying the condition (CI) in Lemma 1.9 naturally arises in the following ways.

Example 1.12.

(1) Let A be a finite flat $\mathcal{O}_K$-algebra locally of complete intersection and let $\mathcal{O}_K[T_1, \dots, T_n]/(f_1, \dots, f_n) \to A$ be an isomorphism over $\mathcal{O}_K$. We define a closed immersion $T = \operatorname{Spec} A \to Q = \operatorname{Spec} \mathcal{O}_K[T_1, \dots, T_n]$ by the surjection $\mathcal{O}_K[T_1, \dots, T_n] \to A$. We also define a section $S \to P = \operatorname{Spec} \mathcal{O}_K[S_1, \dots, S_n]$ by $S_1, \dots, S_n \mapsto 0$. Then, by defining $Q \to P$ by $S_i \mapsto f_i$, we obtain a Cartesian diagram (1.1) satisfying the condition (CI) in Lemma 1.9 on a neighbourhood of T.

(2) Let X be a smooth scheme over k and D be a smooth irreducible divisor of X. We consider the local ring $\mathcal{O}_K = \mathcal{O}_{X,\xi}$ at the generic point ξ of D. Let $f : Y \to X$ be a quasi-finite flat morphism of smooth schemes over k and assume $V = Y \times_X U \to U = X \setminus D$ is étale. We assume $T = Y \times_X S$ is finite over S. We put $P = X \times_k S$ and $Q = Y \times_k S$ and let $Q \to P$ be $f \times 1_S$. We consider the immersions $S \to P$ and $T \to Q$ defined by the natural maps $S \to X$ and $T \to Y$. Then we obtain a Cartesian diagram (1.1) satisfying the condition (CI) in Lemma 1.9.

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Lemma 1.13. *Let T be a finite flat scheme over S of degree d such that T_K is étale over K. We consider a Cartesian diagram*

$$\begin{array}{ccc} T & \longrightarrow & Q \\ \downarrow & & \downarrow \\ S & \longrightarrow & P \end{array}$$

satisfying the condition (CI) *in Lemma 1.9. We consider the following conditions.*

(1) *The ramification of T is bounded by r.*

(2) *The number of connected components of the scheme $Q^{(r)}_{T,\bar F}$ is d.*

(3) *The scheme $Q^{(r)}_{T,\bar F}$ over $P^{(r)}_{S,\bar F}$ is isomorphic to the disjoint union of d copies of $P^{(r)}_{S,\bar F}$.*

(4) *The map $Q^{(r)}_{T,\bar F} \to P^{(r)}_{S,\bar F}$ is finite and étale.*

(5) *The induced map $\bar T \to Q^{(r)}_T$ is a closed immersion.*

(6) *The ramification of T is bounded by $r+$.*

Then, we have implications (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6)*. If $Q_K \to P_K$ is finite étale, we have* (4) $\Leftrightarrow$ (5)*.*

Proof. (1) $\Rightarrow$ (3). We may assume that the map $Q^{(r)}_T \to P^{(r)}_S$ is finite flat of degree d on the generic fibre. Assume the ramification of T is bounded by r. For each $t \in \bar T_{\bar F}$, let $Q^{(r),t}_{T,\bar F}$ denote the connected component containing the image of t by the map $\bar T \to Q^{(r)}_T$. Then, we have an open and closed immersion

$$\coprod_{t\in \bar T_{\bar F}} Q^{(r),t}_{T,\bar F} \to Q^{(r)}_{T,\bar F}.$$

Since the number of the points in every geometric fibre of the map $Q^{(r)}_T \to P^{(r)}_S$ is at most d, we obtain an equality

$$\coprod_{t\in \bar T_{\bar F}} Q^{(r),t}_{T,\bar F} = Q^{(r)}_{T,\bar F}$$

and the map $Q^{(r),t}_{T,\bar F} \to P^{(r)}_{S,\bar F}$ is finite flat of degree 1 for every $t \in \bar T_{\bar F}$.

(3) $\Rightarrow$ (2). It follows from the fact that $P^{(r)}_{S,\bar F}$ is connected.

(2) $\Rightarrow$ (1). By [**7**, Chapter V, § 2.4, Theorem 3], the image of every connected component of $\bar Q^{(r)}_{T,\bar F}$ is $\bar P^{(r)}_{S,\bar F}$. Hence the inclusion $\bar T_{\bar F} \to \bar Q^{(r)}_{T,\bar F}$ of the inverse image of $0 \in \bar P^{(r)}_{S,\bar F}$ defines a surjection $\bar T_{\bar F} \to \pi_0(\bar Q^{(r)}_{T,\bar F})$. Since the cardinalities are the same, it is a bijection.

(3) $\Rightarrow$ (4). Clear.

(4) ⇒ (5). We may assume that the map $Q_T^{(r)} \to P_S^{(r)}$ is finite étale. Then, the diagram

$$\begin{array}{ccc} \bar{T} & \longrightarrow & Q_T^{(r)} \\ \big\downarrow & & \big\downarrow \\ S & \longrightarrow & P_S^{(r)} \end{array} \tag{1.5}$$

is Cartesian and the upper horizontal arrow is a closed immersion.

(5) ⇒ (6). Let $s > r$ be a rational number. Then, we have a commutative diagram

$$\begin{array}{ccccccc} \bar{T}_{\bar{F}} & \longrightarrow & (Q_{T,\bar{F}}^{(s)})_{\mathrm{red}} & \longrightarrow & \bar{T}_{\bar{F}} & \longrightarrow & Q_T^{(r)} \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ \operatorname{Spec} \bar{F} & \longrightarrow & P_{S,\bar{F}}^{(s)} & \longrightarrow & \operatorname{Spec} \bar{F} & \longrightarrow & P_S^{(r)} \end{array}$$

Since the composition of the left two upper horizontal arrows is the identity, the map $\bar{T}_{\bar{F}} \to \pi_0(Q_{T,\bar{F}}^{(s)})$ is an injection.

(5) ⇒ (4). If $Q_K \to P_K$ is finite étale, the condition (5) implies that the map $Q_{T'}^{(er)} \to P_{S'}^{(er)}$ of stable models is finite étale on a neighborhood of the image of $\bar{T}$. Hence the assertion follows from the purity of Zariski–Nagata. □

The equivalence (1) ⇔ (2) means that Definition 1.8 (1) is equivalent to that in [**1**, Definition 6.3] if A is locally of complete intersection. Under the assumption that $Q_K \to P_K$ is finite étale, we have an equivalence (4) ⇔ (5) ⇔ (6) (cf. [**2**, Corollary 4.12]). The author does not know how to prove the implication (6) ⇒ (4) without using rigid geometry.

Corollary 1.14. *Let T be a finite flat scheme locally of complete intersection over S and $T \to P$ a closed immersion to a smooth scheme over S. Assume T_K is étale over K. Then, there exists a positive rational number r such that the ramification of T over S is bounded by r.*

Proof. By Lemma 1.13 (5) ⇒ (6), it is a consequence of Lemma 1.3. □

1.2. Logarithmic variant

We keep the notation in the previous subsection. We consider a logarithmic variant of the constructions in the previous section, without using log geometry. We work with Cartier divisors to replace log structures.

Let $D_S \subset S = \operatorname{Spec} \mathcal{O}_K$ be the Cartier divisor $\operatorname{Spec} F$. Let T be a flat scheme of finite type over S and D_T be a Cartier divisor of T satisfying the following condition.

(D) For each $t \in T$, there exists an integer $e_t \geqslant 1$ such that the pull-back of D_S is equal to $e_t D_T$ on a neighbourhood of t.

The condition (D) implies that the complement $T \setminus D_T$ is equal to the generic fibre T_K. If P is a regular flat scheme of finite type over S and if the reduced closed fibre $D_P = (P \times_S D_S)_{\mathrm{red}}$ is regular, then the Cartier divisor D_P satisfies the condition (D). For (T, D_T) satisfying the condition (D), let e_T denote the least common multiple $\mathrm{lcm}_{t\in T}\, e_t$. The condition $e_T = 1$ is equivalent to that D_T is the pull-back of D_S.

Let T be a flat scheme of finite type over S and D_T be a Cartier divisor of T satisfying the condition (D). For a surjection $S' = \mathrm{Spec}\, \mathcal{O}_{K'} \to S$ of the spectra of discrete valuation rings of ramification index $e' = e_{K'/K}$, we define the log base change or the log product $T' = T \times_S^{\log} S'$ as follows. First, we consider the case where we have $e_t = e$ for every $t \in T$ and there exists a generator f of the ideal of D_T. Let π' be a prime element of K'. We define $u \in \Gamma(T, \mathcal{O}_T^\times)$ and $v \in \mathcal{O}_{K'}^\times$ by $\pi = uf^e$ and $\pi = v\pi'^{e'}$ and a morphism $T \times_S S' \to \mathrm{Spec}\, \mathbb{Z}[X, Y, U^{\pm 1}, V^{\pm 1}]/(UX^e - VY^{e'})$ by $X \mapsto f$, $Y \mapsto \pi'$, $U \mapsto u$, $V \mapsto v$. Let $d = (e, e')$ be the greatest common divisor and put $e = de_1$ and $e' = de'_1$. Let a and b be integers satisfying $d = ae + be'$. We define

$$\begin{aligned} T \times_S^{\log} S' &= (T \times_S S') \times_{\mathrm{Spec}\, \mathbb{Z}[X,Y,U^{\pm 1},V^{\pm 1}]/(UX^e - VY^{e'})} \mathrm{Spec}\, \mathbb{Z}[Z, W^{\pm 1}, U^{\pm 1}] \\ &= (T \times_S S')[Z, W^{\pm 1}]/(f - Z^{e'_1}W^a, \pi' - Z^{e_1}W^{-b}, v - uW^d), \end{aligned} \tag{1.6}$$

where $\mathbb{Z}[X, Y, U^{\pm 1}, V^{\pm 1}]/(UX^e - VY^{e'}) \to \mathbb{Z}[Z, W^{\pm 1}, U^{\pm 1}]$ is defined by $X \mapsto Z^{e'_1}W^a$, $Y \mapsto Z^{e_1}W^{-b}$, $V \mapsto UW^d$. This is independent of the choices and is well defined. In the general case, we define $T \times_S^{\log} S'$ by patching.

The canonical map $T' = T \times_S^{\log} S' \to T \times_S S'$ is finite. If $e_T = 1$, the canonical map $T' = T \times_S^{\log} S' \to T \times_S S'$ is an isomorphism.

If T' is flat over S', we define a Cartier divisor $D_{T'}$ locally to be that defined by Z in (1.6). Then, the divisor $D_{T'}$ satisfies the condition (D) by putting $e_{t'} = e_t/\gcd(e_t, e_{K'/K})$ for $t' \in T'$ above $t \in T$. We have $e_{T'} = e_T/\gcd(e_T, e_{K'/K})$. In particular, if $e_{K'/K}$ is divisible by e_T, we have $e_{T'} = 1$ and the divisor $D_{T'}$ is the pull-back of $D_{S'}$.

Definition 1.15. Let K be a discrete valuation field and let D_S be the Cartier divisor $\mathrm{Spec}\, F$ of $S = \mathrm{Spec}\, \mathcal{O}_K$.

(1) Let T be a flat scheme of finite type over S and D_T be a Cartier divisor of T satisfying the condition (D). We say (T, D_T) is log flat over S, if, for an arbitrary surjection $S' = \mathrm{Spec}\, \mathcal{O}_{K'} \to S$ of the spectra of discrete valuation rings, the log base change $T' = T \times_S^{\log} S' \to S'$ is flat.

(2) Let P be a regular flat scheme of finite type over S such that the reduced closed fibre $D_P = (P \times_S D_S)_{\mathrm{red}}$ is regular. We say P is log smooth over S, if étale locally on P, there exists a smooth map $P \to P_e$ for some $e \geqslant 1$ where

$$P_e = \begin{cases} \mathrm{Spec}\, \mathcal{O}_K[t]/(t^e - \pi) & \text{if } e \in \mathcal{O}_K^\times, \\ \mathrm{Spec}\, \mathcal{O}_K[t, u^{\pm 1}]/(ut^e - \pi) & \text{otherwise,} \end{cases}$$

and π is a prime element of K.

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(3) Let $T \to P$ be a closed immersion of flat schemes over S and D_T and D_P be Cartier divisors satisfying the condition (D). If $D_T = D_P \times_P T$, we say $T \to P$ is an exact closed immersion.

Lemma 1.16. *Let P be a regular flat scheme of finite type over S such that $D_P = (P \times_S D_S)_{\rm red}$ is regular and that P is log smooth over S. Let $S' = \operatorname{Spec} \mathcal{O}_{K'} \to S$ be a surjection of the spectra of discrete valuation rings. We put $P' = P \times_S^{\log} S'$.*

(1) *The scheme P' is regular and flat over S', $D_{P'} = (P' \times_{S'} D_{S'})_{\rm red}$ is regular and P' is log smooth over S'. If the ramification index $e' = e_{K'/K}$ is divisible by e_P, the map $P' \to S'$ is smooth.*

(2) *Let T be a finite flat scheme over S and $T \to P$ be a regular exact closed immersion. Then, T is log flat and $T' = T \times_S^{\log} S' \to P'$ is also a regular exact closed immersion.*

Proof. (1) It suffice to prove the case where $P = P_e$ for an integer $e \geqslant 1$. If e is invertible in $\mathcal{O}_K$, in the notation of (1.6), the log product $P_e \times_S^{\log} S'$ is given by

$$\operatorname{Spec} \mathcal{O}_{K'}[t]/(t^e - \pi)[Z, W^{\pm 1}]/(t - Z^{e'_1} W^a, \pi' - Z^{e_1} W^{-b}, v - W^d) \\ = \operatorname{Spec} \mathcal{O}_{K'}[W, Z]/(W^d - v, Z^{e_1} - W^b \pi').$$

Since $W^b \pi'$ is a prime element of the unramified extension $\mathcal{O}_{K'}[W]/(W^d - v)$, the assertion follows. Assume e is not invertible in $\mathcal{O}_K$. Then, in the notation of (1.6), $P_e \times_S^{\log} S'$ is given by

$$\operatorname{Spec} \mathcal{O}_{K'}[t, u^{\pm 1}]/(ut^e - \pi)[Z, W^{\pm 1}]/(t - Z^{e'_1} W^a, \pi' - Z^{e_1} W^{-b}, v - uW^d) \\ = \operatorname{Spec} \mathcal{O}_{K'}[Z, W^{\pm 1}]/(W^{-b} Z^{e_1} - \pi').$$

First, we consider the case where e_1 is invertible in $\mathcal{O}_K$. In this case, the étale covering $P_e \times_S^{\log} S'[V]/(V^{e_1} - W) = \operatorname{Spec} \mathcal{O}_{K'}[Z, V^{\pm 1}]/((V^{-b} Z)^{e_1} - \pi')$ of $P_e \times_S^{\log} S'$ is smooth over $P'_{e_1} = \mathcal{O}_{K'}[T]/(T^{e_1} - \pi')$. Assume e_1 is not invertible in $\mathcal{O}_K$. Then, by the definition of b, we have $(b, e_1) = 1$ and b is invertible in $\mathcal{O}_K$. Hence $P_e \times_S^{\log} S' = \operatorname{Spec} \mathcal{O}_{K'}[Z, W^{\pm 1}]/(W^{-b} Z^{e_1} - \pi')$ is étale over $P'_{e_1} = \operatorname{Spec} \mathcal{O}_{K'}[Z, V^{\pm 1}]/(V Z^{e_1} - \pi')$.

If e' divides e, we have $e_1 = 1$ and $P_e \times_S^{\log} S'$ is smooth over $P'_1 = S'$.

(2) By the definition of the base change, the map $T' \to P'$ is a closed immersion and T' is finite over S'. Since the ideal $\mathcal{I}_{T'} \subset \mathcal{O}_{P'}$ is locally generated by d elements where d is the relative dimension of P' over S', the immersion $T' \to P'$ is regular and T' is flat over S'. By the definition of $D_{T'}$, the immersion $T' \to P'$ is a regular and exact closed immersion. □

Corollary 1.17. *Let P be a regular flat scheme of finite type over S such that $D_P = (P \times_S D_S)_{\rm red}$ is irreducible and regular and that P is log smooth over S. Let $S' = \operatorname{Spec} \mathcal{O}_{K'} \to P$ be the localization at the generic point ξ of D_P.*

Let L be a finite separable extension of K, $T = \operatorname{Spec} \mathcal{O}_L$ and $D_T = (T \times_S D_S)_{\rm red}$. Then, $T' = T \times_S^{\log} S'$ is equal to $\operatorname{Spec} \mathcal{O}_{L \otimes_K K'}$ and we have $D_{T'} = (T' \times_{S'} D_{S'})_{\rm red}$.

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Proof. It suffices to show that $T' = T \times_S^{\log} S'$ is regular and that $D_{T'}$ is defined by the reduced closed point at each closed point $\xi' \in T'$. Let $t \in T$ be the image of ξ' and T_t be the localization at t. Then, the localization of T' at ξ' is equal to a localization of $P \times_S^{\log} T_t$ and the assertion follows from Lemma 1.16 (1). □

For the convenience of a reader familiar with the terminologies on log geometry as in [**16**, § 4], we include a lemma, not used in the sequel, showing that the Definition 1.15 above is a special case of the standard definitions.

Lemma 1.18. *We consider $S = \operatorname{Spec} \mathcal{O}_K$ as a log scheme with the log structure defined by D_S.*

(1) *Let T be a flat scheme of finite type over S and D_T be a Cartier divisor satisfying the condition* (D)*. Then, the following conditions are equivalent.*

  (a) *The log scheme T with the log structure defined by D_T is log flat over S.*

  (b) *(T, D_T) is log flat over S in the sense of Definition 1.15 (1).*

(2) *Let P be a regular flat scheme of finite type over S such that the reduced closed fibre $D_P = (P \times_S D_S)_{\mathrm{red}}$ is regular. Then, the following conditions are equivalent.*

  (a) *The log scheme P with the log structure defined by D_P is log smooth over S.*

  (b) *(P, D_P) is log smooth over S in the sense of Definition 1.15 (2).*

Proof. (1) $(a) \Rightarrow (b)$. Let $S' = \operatorname{Spec} \mathcal{O}_{K'} \to S$ be a surjection of the spectra of discrete valuation rings and we show that the base change $T' = T \times_S^{\log} S' \to S'$ is flat at each closed point $t' \in T'$. We put $e' = e_{t'}$. Let S'_1 be the localization of $P_{e'}$ over S' and consider the Cartesian diagram

$$\begin{array}{ccc} T' & \longleftarrow & T'_1 = T' \times_{S'}^{\log} S'_1 \\ \Big\downarrow & & \Big\downarrow \\ S' & \longleftarrow & S'_1 \end{array} \tag{1.7}$$

Since $e' = e_{t'}$, the map $T'_1 \to S'_1$ is strict on a neighbourhood V'_1 of the inverse image of t'. Since $T' \to S'$ is log flat, the map $V'_1 \to S'_1$ is log flat and strict and hence is flat. Since $S'_1 \to S$ is log flat, the map $V'_1 \to T'$ is also log flat and strict and hence is flat. Hence the map $T' \to S'$ is flat.

$(b) \Rightarrow (a)$. Let $t \in T$ be a closed point and put $e = e_t$. Let S_1 be the localization of P_e and consider the Cartesian diagram (1.7) with $'$ removed everywhere. Then, as above, there exists an open neighbourhood $V_1 \subset T_1$ of the inverse image of t such that $V_1 \to T$ and $V_1 \to S_1$ are flat. Hence by [**16**, Proposition 4.3.10], the map $T \to S$ is log flat.

(2) $(b) \Rightarrow (a)$. Since P_e is log smooth over S, the assertion follows.

$(a) \Rightarrow (b)$ We consider the ring homomorphism $\mathbb{Z}[\mathbb{N}] \to \mathcal{O}_K$ sending $1 \in \mathbb{N}$ to a prime element π. The question is étale local. Hence, we may assume that $P = \operatorname{Spec} \mathcal{O}_K \otimes_{\mathbb{Z}[\mathbb{N}]} \mathbb{Z}[M]$ for a morphism $\mathbb{N} \to M$ of fs-monoids such that the map $\mathbb{Z} \to M^{\mathrm{gp}}$ is an injection and that the order of the torsion part of the cokernel is invertible in $\mathcal{O}_K$. Further $\bar{M} = M/M^\times$ is isomorphic to $\mathbb{N}$. We may assume M^{gp} is torsion free.

If the order e of the cokernel of $\mathbb{Z} \to \bar{M}^{\mathrm{gp}}$ is invertible in $\mathcal{O}_K$, we may assume $M = \mathbb{N}$. In this case, we have $P = P_e$. Assume e is not invertible. In this case, we may assume $M = \mathbb{N} \times \mathbb{Z}$ and the map $\mathbb{N} \to M$ sends 1 to $(e, 1)$. Then, we also have $P = P_e$. □

In this subsection, T denotes a finite flat scheme over S and D_T denotes a Cartier divisor of T satisfying the condition (D) such that (T, D_T) is log flat over S. Recall that e_T denotes the least common multiple of the integers $e_t \geqslant 1$ for closed points $t \in T$.

Definition 1.19. Let T be a finite flat scheme over S such that T_K is étale over K and let D_T denote a Cartier divisor of T satisfying the condition (D) such that (T, D_T) is log flat over S.

(1) For a rational number $r > 0$, we say that the log ramification of (T, D_T) over S is bounded by r if, for one (and hence for any) surjection $S' = \operatorname{Spec} \mathcal{O}_{K'} \to S$ of spectra of discrete valuation rings such that $e = e_{K'/K}$ is divisible by e_T, the ramification of the finite flat scheme $T \times_S^{\log} S'$ over S' is bounded by er.

(2) For a rational number $r \geqslant 0$, we say that the log ramification of (T, D_T) over S is bounded by $r+$ if the log ramification of (T, D_T) over S is bounded by s for every rational number $s > r$.

(3) Let L be a finite étale K-algebra, $T = \operatorname{Spec} \mathcal{O}_L$ and $D_T = \operatorname{Spec}(\mathcal{O}_L \otimes_{\mathcal{O}_K} F)_{\mathrm{red}}$. Then, we say that the log ramification of L over K is bounded by r (respectively by $r+$) if the log ramification of (T, D_T) is bounded by r (respectively by $r+$).

Let (T, D_T) be as in Definition 1.19 and $S' \to S$ be a surjection of spectra of discrete valuation rings of ramification index e. Then, it is clear from the definition that the log ramification of T over S is bounded by r if and only if the ramification of $T \times_S^{\log} S'$ over S' is bounded by er.

Lemma 1.20. *Let P be a regular flat scheme of finite type over S such that $D_P = (P \times_S D_S)_{\mathrm{red}}$ is irreducible and regular and that P is log smooth over S and let ξ be the generic point of D_P. We put $\mathcal{O}_{K'} = \mathcal{O}_{P,\xi}$ and consider the surjection $S' = \operatorname{Spec} \mathcal{O}_{K'} \to S$ of ramification index e.*

Then, for a finite separable extension L of K, the log ramification of L over K is bounded by r if and only the log ramification of $L \otimes_K K'$ over K' is bounded by er.

Proof. Clear from Corollary 1.17 and the above remark. □

Let T be a finite flat scheme over S and let D_T be a Cartier divisor satisfying the condition (D). We consider an exact closed immersion $T \to P$ to a log smooth scheme P over S. Let $S' = \operatorname{Spec} \mathcal{O}_{K'} \to S$ be a surjection of spectra of discrete valuation rings of ramification index e. Then, the base change $T' = T \times_S^{\log} S' \to P' = P \times_S^{\log} S'$ is an exact closed immersion to a log smooth scheme over S'. Assume e is divisible by the integer

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e_P. Then, the map $P' \to S'$ is smooth. Thus for positive integers $m, n > 0$ and $r = m/n$, we apply the construction in § 1.1 to define $P'^{[em/n]}_{T'}$, $P'^{(em/n)}_{T'}$, $P'^{[er]}_{T'}$, $P'^{(er)}_{T'}$, $\widehat{P'^{(er)}_{T'}}$, etc.

Example 1.21. Assume K is complete.

(1) Let L be a finite separable extension of K and $\mathcal{O}_K[X_1, \dots, X_n]/(f_1, \dots, f_n) \to \mathcal{O}_L$ be an isomorphism. Let $m \leqslant n$ be an integer such that the images $z_1, \dots, z_m$ of $X_1, \dots, X_m$ are non-zero and that z_i is a prime element of L for some $1 \leqslant i \leqslant m$. We define a map $\mathbb{N}^{m+1} \to \mathbb{N}$ by sending the standard basis of $\mathbb{N}^{m+1}$ to $e_{L/K}, v_L(z_1), \dots, v_L(z_m)$. Let M be the inverse image of $\mathbb{N}$ by the induced map $\mathbb{Z}^{m+1} \to \mathbb{Z}$. We define $\mathbb{N}^{m+1} \to \mathcal{O}_K[X_1, \dots, X_n]$ by sending the standard basis to π, $X_1, \dots, X_m$ where π is a prime element of K.

We put $P = \operatorname{Spec} \mathcal{O}_K[X_1, \dots, X_n] \otimes_{\mathbb{Z}[\mathbb{N}^{m+1}]} \mathbb{Z}[M]$. Then, P is regular, the reduced closed fibre of P is regular and P is log smooth over S. Further the isomorphism $\mathcal{O}_K[X_1, \dots, X_n]/(f_1, \dots, f_n) \to \mathcal{O}_L$ induces an exact closed immersion $T = \operatorname{Spec} \mathcal{O}_L \to P$. For a finite extension K' over K with ramification index e divisible by $e_{L/K}$, the affinoid variety over K' defined by the formal $\mathcal{O}_{K'}$-scheme $\widehat{P'^{(er)}_{T'}}$ is the affinoid variety $Y^a_{Z,I,P}$ defined in [**1**, 3.1] for $a = r$ and $I = \{1, \dots, n\} \supset P = \{1, \dots, m\}$.

(2) Assume $\operatorname{Spf} \boldsymbol{A}$ is the completion of P at $T = \operatorname{Spec} A$. For a finite extension K' over K with ramification index e divisible by $e_{L/K}$, the affinoid K'-variety defined by the formal $\mathcal{O}_{K'}$-scheme $\widehat{P'^{(er)}_{T'}}$ is the affinoid variety $X^j_{\log}(\boldsymbol{A} \to A)'_K$ defined in [**2**, § 4.2] for $j = r$.

We consider a Cartesian diagram

$$\begin{array}{ccc} T & \longrightarrow & Q \\ \downarrow & & \downarrow \\ S & \longrightarrow & P \end{array} \tag{1.8}$$

of schemes over S satisfying the following condition.

(LCI) The vertical arrows are quasi-finite and flat and the horizontal arrows are closed immersions. The scheme P is smooth over S, Q is regular flat over S, $D_Q = (Q \times_S D_S)_{\text{red}}$ is smooth over F and Q is log smooth over S.

We consider the Cartier divisor $D_T = D_Q \times_Q T$. Then, by Lemma 1.16 (2), the pair (T, D_T) is log flat over S.

Let $S' \to S$ be a surjection of the spectra of discrete valuation rings of ramification index e. We assume that e_Q divides e. Then by Lemma 1.16, the log product $Q' =$

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$Q \times_S^{\log} S'$ is smooth and the immersion $T' = T \times_S^{\log} S' \to Q'$ is a regular immersion. Hence the Cartesian diagram

$$\begin{array}{ccc} T' & \longrightarrow & Q' \\ \downarrow & & \downarrow \\ S' & \longrightarrow & P' = P \times_S S' \end{array} \tag{1.9}$$

satisfies the condition (CI) in Lemma 1.9.

1.3. Logarithmic ramification groups

In [**1**, Definitions 3.4 and 3.12], we introduced two filtrations, the non-logarithmic one and the logarithmic one, by ramification groups of the absolute Galois group. In this paper, we will only be interested in the logarithmic filtration.

Assume K is a henselian discrete valuation field. Let $\bar{K}$ be a separable closure and $G_K = \mathrm{Gal}(\bar{K}/K)$ be the absolute Galois group. In [**1**, Definition 3.12], we define a decreasing filtration by logarithmic ramification groups $G^r_{K,\log} \subset G_K$ indexed by positive rational numbers $r \geqslant 0$. By Example 1.2.7.1, for a finite étale algebra L over K, the logarithmic ramification of L is bounded by r in the sense of Definition 1.19 (3) if and only if the action of $G^r_{K,\log}$ on the finite set $\mathrm{Hom}_K(L, \bar{K})$ is trivial. We put

$$G^{r+}_{K,\log} = \overline{\bigcup_{q>r} G^q_{K,\log}} \quad \text{and} \quad \mathrm{Gr}^r_{\log} G_K = G^r_{K,\log}/G^{r+}_{K,\log}.$$

For $r = 0$, $G^{0+}_{K,\log} \subset G^0_{K,\log}$ are equal to the inertia subgroup and its pro-p Sylow subgroup $P \subset I$.

We consider the opposite category (FE/K) of finite étale K-algebras. We identify the category (FE/K) with that of finite discrete sets with continuous action of the absolute Galois group G_K by the fibre functor $X \mapsto X(\bar{K})$. For a rational number $r \geqslant 0$, the étale K-algebras L such that the log ramification is bounded by $r+$ form a Galois subcategory $(\mathrm{FE}/K)^{r+}$ of (FE/K) corresponding to a normal closed subgroup $G^{r+}_{K,\log} \subset G_K = \mathrm{Gal}(\bar{K}/K)$. For an extension of discrete valuation field K' over K of ramification index e, the natural map $G_{K_1} \to G_K$ sends $G^{er}_{K',\log}$ into $G^r_{K,\log}$.

In the rest of this section, we assume that K satisfies the following condition.

(Geom) There exist a smooth scheme X over k, an irreducible divisor D smooth over k with the generic point ξ and an isomorphism $S \to \mathrm{Spec}\, \mathcal{O}^h_{X,\xi}$ to the henselization of the local ring.

Let $\Omega^1_F(\log)$ denote the F-vector space $\Omega^1_{X/k}(\log D)_\xi \otimes_{\mathcal{O}_{X,\xi}} F$. It fits in an exact sequence $0 \to \Omega^1_{F/k} \to \Omega^1_F(\log) \xrightarrow{\mathrm{res}} F \to 0$. We extend the normalized discrete valuation $v_K : K \to \mathbb{Z} \cup \{\infty\}$ to $v_K : \bar{K} \to \mathbb{Q} \cup \{\infty\}$. Let $r > 0$ be a rational number. We put $\mathfrak{m}^r_{\bar{K}} = \{a \in \bar{K} \mid v_K(a) \geqslant r\}$ and $\mathfrak{m}^{r+}_{\bar{K}} = \{a \in \bar{K} \mid v_K(a) > r\}$. Let $\Theta^{(r)}_{\log} = \Theta^{(r)}_{F,\log}$ denote the $\bar{F}$-vector space $\boldsymbol{V}(\Omega^1_F(\log) \otimes_F \mathfrak{m}^{(-r)}_{\bar{K}}/\mathfrak{m}^{(-r)+}_{\bar{K}})$.

Let $P' = (X \times_k S)'$ be the blow-up of $X \times_k S$ at $D \times_k D_S$ and define the log product $P = (X \times_k S)^\sim \subset P'$ to be the complement of the proper transforms of $D \times_k S$ and of

$X \times_k D_S$. Then, P is smooth over S and the canonical map $S \to X$ induces a section $S \to P$. Thus, for a rational number $r > 0$, applying the construction in § 1.1, we define the schemes $P_S^{(r)}$, $\bar{P}_{S,\bar{F}}^{(r)}$, etc. Since $\mathcal{N}_{S/P} = \Omega^1_{X/k}(\log D)_\xi$, we have a canonical isomorphism

$$\bar{P}_{S,\bar{F}}^{(r)} \to \Theta_{\log}^{(r)} \tag{1.10}$$

by Lemma 1.10.

Under the condition (Geom), a canonical surjection $\pi_1^{\mathrm{ab}}(\Theta_{\log}^{(r)}) \to \mathrm{Gr}^r_{\log} G_K$ is defined in [**2**, (5.12.1)]. We recall the construction. Let L be a finite étale algebra over K. After replacing X by an étale neighbourhood of ξ if necessary, there exists a finite flat morphism $f : Y \to X$ of smooth schemes over k such that $V = Y \times_X U \to U = X \setminus D$ is étale and that $Y \times_X S = T = \operatorname{Spec} \mathcal{O}_L$. We also assume that $V \subset Y$ is the complement of a smooth divisor E.

Similarly as the construction of $P = (X \times_k S)^\sim$, let $Q' = (Y \times_k S)'$ be the blow-up of $Y \times_k S$ at $E \times_k D_S$ and $Q = (Y \times_k S)^\sim \subset Q'$ be the complement of the proper transforms of $E \times_k S$ and of $Y \times_k D_S$. We consider the immersions $S \to P$ and $T \to Q$ defined by the natural maps $S \to X$ and $T \to Y$. Then we obtain a Cartesian diagram (1.8) satisfying the condition (LCI).

Let K' be a finite extension such that the ramification index e' is divisible by $e_{L/K}$. We put $S' = \operatorname{Spec} \mathcal{O}_{K'}$ and consider the diagram

$$\begin{array}{ccc} T' = T \times_S^{\log} S' & \longrightarrow & Q' = Q \times_S^{\log} S' \\ \downarrow & & \downarrow \\ S' & \longrightarrow & P' = P \times_S S' \end{array}$$

satisfying the condition (CI). Assume that the log ramification of L over K is bounded by $r+$. Then, the conditions (4) and (6) in Lemma 1.13 are equivalent in this case and the induced map

$$\bar{Q}'^{(er)}_{T',\bar{F}} \to \bar{P}'^{(er)}_{S',\bar{F}} = \Theta^{(r)}_{F,\log} \tag{1.11}$$

is finite étale. This construction defines a functor $(\mathrm{FE}/K)^{r+} \to (\mathrm{FE}/\Theta_{\log}^{(r)})$ to the category of finite étale schemes over $\Theta_{\log}^{(r)}$ and hence a morphism $\pi_1(\Theta_{\log}^{(r)}) \to G_K/G_{K,\log}^{r+}$. In [**2**, Theorem 5.12.1], it is proved that it factors through the abelian quotient and induces a surjection

$$\pi_1^{\mathrm{ab}}(\Theta_{\log}^{(r)}) \to \mathrm{Gr}^r_{\log} G_K. \tag{1.12}$$

The surjectivity is a consequence of the fact that the surjection $\bar{T}_{\bar{F}} \to \pi_0(\bar{Q}'^{(er)}_{T',\bar{F}})$ induces a bijection $\bar{T}_{\bar{F}}/G^{r+}_{K,\log} \to \pi_0(\bar{Q}'^{(er)}_{T',\bar{F}})$, as in the proof of [**2**, Theorem 2.15]. In Theorem 1.24, we will give a refinement of the surjection (1.12).

We give a compatibility of the map (1.12) with a log smooth base change. Let $S \to X$ be as above. Let t be a uniformizer of $D \subset X$ and $e_1 \geqslant 1$ be an integer. Let X_1 be a scheme smooth over

$$Z_{e_1} = \begin{cases} X[T]/(T^{e_1} - t) & \text{if } e_1 \text{ is invertible in } k, \\ X[T, U^{\pm 1}]/(UT^{e_1} - t) & \text{if } e_1 \text{ is 0 in } k, \end{cases}$$

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and assume $D_1 = (D \times_X X_1)_{\mathrm{red}}$ is irreducible. Let $\mathcal{O}_{K_1}$ be the henselization $\mathcal{O}^h_{X_1,\xi_1}$ at the generic point ξ_1 of D_1.

Lemma 1.22. *Let $S_1 = \operatorname{Spec} \mathcal{O}_{K_1} \to S = \operatorname{Spec} \mathcal{O}_K$ be the surjection of the spectra of discrete valuation rings of ramification index e_1 above and let $r > 0$ be a rational number. Let F_1 denote the residue field of K_1 and let $\pi : \Theta^{(e_1 r)}_{F_1,\log} \to \Theta^{(r)}_{F,\log}$ be the map induced by $F_1 \otimes_F \Omega_F(\log) \to \Omega_{F_1}(\log)$.*

Then, the map $\operatorname{Gr}^{e_1 r}_{\log} G_{K_1} \to \operatorname{Gr}^r_{\log} G_K$ induced by the canonical map $G^{e_1 r}_{K_1,\log} \to G^r_{K,\log}$ is a surjection and the diagram

$$\begin{array}{ccc} \pi_1^{\mathrm{ab}}(\Theta^{(e_1 r)}_{F_1,\log}) & \xrightarrow{\pi_*} & \pi_1^{\mathrm{ab}}(\Theta^{(r)}_{F,\log}) \\ \downarrow & & \downarrow \\ \operatorname{Gr}^{e_1 r}_{\log} G_{K_1} & \longrightarrow & \operatorname{Gr}^r_{\log} G_K \end{array} \tag{1.13}$$

is commutative.

Proof. The natural map $F_1 \otimes_F \Omega_F(\log) \to \Omega_{F_1}(\log)$ is injective and hence $\pi : \Theta^{(e_1 r)}_{F_1,\log} \to \Theta^{(r)}_{F,\log} \times_{\bar F} \bar F_1$ is a surjection of vector spaces and admits a section. Thus, it induces a surjection $\pi_* : \pi_1^{\mathrm{ab}}(\Theta^{(e_1 r)}_{F_1,\log}) \to \pi_1^{\mathrm{ab}}(\Theta^{(r)}_{F,\log})$. Hence, it suffices to show the commutativity of the diagram (1.13).

Let $Y \to X$ and $Q = (Y \times S)^\sim \to P = (X \times S)^\sim$ be finite coverings and $S' \to S$ be a finite surjection that appeared in the construction of the map (1.11). The normalization Y_1 of the fibre product $Y \times_X X_1$ is smooth over k and $V_1 = V \times_U U_1$ is the complement of a smooth divisor $D_{Y_1} \subset Y_1$. By Corollary 1.17, the log product $S' \times^{\log}_S S_1$ is normal and is a finite disjoint union of spectra of discrete valuation rings. Let $S'_1 = \operatorname{Spec} \mathcal{O}_{K'_1}$ be a connected component and e' be the ramification index $e_{K'_1/K}$. Applying the construction of the map (1.11) to $Y_1 \to X_1$ and $S'_1 \to S_1$, we obtain a finite étale covering

$$\bar Q'^{(e' r)}_{T'_1, \bar F_1} \to \bar P'^{(e' r)}_{S'_1, \bar F_1} = \Theta^{(e' r)}_{F_1,\log}. \tag{1.14}$$

It suffices to show that the diagram

$$\begin{array}{ccc} \bar Q'^{(e' r)}_{T'_1, \bar F_1} & \longrightarrow & \bar P'^{(e' r)}_{S'_1, \bar F_1} \\ \downarrow & & \downarrow \\ \bar Q'^{(e r)}_{T', \bar F} & \longrightarrow & \bar P'^{(e r)}_{S', \bar F} \end{array}$$

is Cartesian.

By the construction, it suffices to show that the map $P_1 \to P \times_S S_1$ is smooth. Since $P_1 = (X_1 \times S_1)^\sim \to (Z_{e_1} \times S_1)^\sim$ is smooth, it is reduced to the case where $X_1 = Z_{e_1}$. First, we consider the case e_1 is invertible in k. Let $\pi \in \mathcal{O}_K$ be the image of t and $\pi_1 \in \mathcal{O}_{K_1}$ be the image of T. Then, $P_1 = (X[T]/(T^{e_1} - t) \times S_1)^\sim = X \times S_1[T]/(T^{e_1} - t)[V^{\pm 1}](T - V\pi_1)$ equals $X \times S_1[V^{\pm 1}]/(V^{e_1}\pi - t)$. This is étale over

$$P \times_S S_1 = (X \times S)^\sim \times_S S_1 = X \times S_1[W^{\pm 1}]/(t - W\pi).$$

We assume e_1 is not invertible in k. Let $\pi \in \mathcal{O}_K$ be the image of t and let $\pi_1, u \in \mathcal{O}_{K_1}$ be the image of T, U. Then, $P_1 = (X[T, U^{\pm 1}]/(UT^{e_1} - t) \times S_1)^{\sim}$ equals

$$X \times S_1[T, U^{\pm 1}]/(UT^{e_1} - t)[V^{\pm 1}](T - V\pi_1) = X \times S_1[U^{\pm 1}, V^{\pm 1}]/(UV^{e_1}\pi_1^{e_1} - t).$$

This is smooth over $P \times_S S_1 = X \times S_1[W^{\pm 1}]/(t - W\pi)$. □

For an $\bar{F}$-vector space V of finite dimension, we introduce a quotient $\pi_1^{\mathrm{alg}}(V)$ of $\pi_1^{\mathrm{ab}}(V)$ annihilated by p. We regard V as a smooth group scheme $\mathrm{Spec}\, S^{\bullet}V^{\vee}$ over $\bar{F}$. For a finite abelian group A, we identify the group $H^1(V, A)$ of isomorphism classes of A-torsors with $\mathrm{Hom}(\pi_1^{\mathrm{ab}}(V), A)$ and the extension group $\mathrm{Ext}(V, A)$ in the category of smooth algebraic groups over $\bar{F}$ as a subgroup of $H^1(V, A)$. Then, the quotient $\pi_1^{\mathrm{alg}}(V)$ is defined by the equality $\mathrm{Hom}(\pi_1^{\mathrm{alg}}(V), A) = \mathrm{Ext}(V, A) \subset H^1(V, A)$ for finite abelian groups A (cf. [**20**, § 6.3, Proposition 6]). The pro-finite group $\pi_1^{\mathrm{alg}}(V)$ is the Pontrjagin dual of $\mathrm{Ext}(V, \mathbb{F}_p)$.

The definition of the quotient $\pi_1^{\mathrm{alg}}(V)$ can be rephrased as follows. Let $(\mathrm{FE}/V)^{\mathrm{alg}}$ be the full subcategory of (FE/V) whose objects are finite étale morphisms $f: X \to V$ such that there exists a structure of commutative algebraic group scheme on X and that f is a morphism of algebraic groups. Then $\pi_1^{\mathrm{alg}}(V)$ is the quotient of $\pi_1^{\mathrm{ab}}(V)$ corresponding to the subcategory $(\mathrm{FE}/V)^{\mathrm{alg}}$.

The map $V^{\vee} = \mathrm{Hom}_{\bar{F}}(V, \bar{F}) \to \mathrm{Ext}(V, \mathbb{F}_p)$ sending a linear form $f: V \to \mathbb{A}^1_{\bar{F}}$ to the pull-back by f of the Artin–Schreier sequence $0 \to \mathbb{F}_p \to \mathbb{A}^1_{\bar{F}} \xrightarrow{t \mapsto t^p - t} \mathbb{A}^1_{\bar{F}} \to 0$ is an isomorphism by [**20**, § 8.3, Proposition 3]. Thus we have defined a canonical isomorphism

$$V^{\vee} \to \mathrm{Hom}(\pi_1^{\mathrm{alg}}(V), \mathbb{Q}/\mathbb{Z}) = \mathrm{Ext}(V, \mathbb{F}_p). \tag{1.15}$$

Lemma 1.23. *Let V be an $\bar{F}$-vector space of finite dimension. For a continuous character $\chi : \pi_1^{\mathrm{ab}}(V) \to \mathbb{Q}/\mathbb{Z}$ of finite order, the following conditions are equivalent.*

(1) *χ factors through the quotient $\pi_1^{\mathrm{alg}}(V)$.*

(2) *$-^*\chi = \mathrm{pr}_1^*\,\chi - \mathrm{pr}_2^*\,\chi$ in $\mathrm{Hom}(\pi_1^{\mathrm{ab}}(V \times V), \mathbb{Q}/\mathbb{Z})$.*

Proof. (1) $\Rightarrow$ (2). If $[\chi] \in \mathrm{Ext}(V, \mathbb{Q}/\mathbb{Z})$ denotes the corresponding extension class, we have $-^*[\chi] = \mathrm{pr}_1^*[\chi] - \mathrm{pr}_2^*[\chi]$.

(2) $\Rightarrow$ (1). Let $\chi : \pi_1^{\mathrm{ab}}(V) \to \mathbb{Q}/\mathbb{Z}$ be a character satisfying $-^*\chi = \mathrm{pr}_1^*\,\chi - \mathrm{pr}_2^*\,\chi$. Taking the pull-back by the injection into the second component $V \to V \times V$, we obtain $(-1)^*\chi = -\chi$. Hence we have $+^*\chi = \mathrm{pr}_1^*\,\chi + \mathrm{pr}_2^*\,\chi$. By induction on n, we have $n \cdot \chi = [n]^*\chi$. Hence, we have $p \cdot \chi = 0$.

Let $f : X \to V$ be the $\mathbb{F}_p$-torsor corresponding to χ. By $+^*\chi = \mathrm{pr}_1^*\,\chi + \mathrm{pr}_2^*\,\chi$, we have an isomorphism $(X \times X)/\mathbb{F}_p \to X \times_V (V \times V)$ of $\mathbb{F}_p$-torsors on $V \times V$. We consider the composition $\tilde{+} : X \times X \to (X \times X)/\mathbb{F}_p \to X \times_V (V \times V) \to X$. Take a point $\tilde{0} \in f^{-1}(0)$. By shifting by the $\mathbb{F}_p$-action, we may assume $\tilde{0} \,\tilde{+}\, \tilde{0} = \tilde{0}$. Then we can easily verify that $\tilde{+}$ defines a group structure on X and the map $f : X \to V$ is compatible with the group structure. □

We will prove the following theorem in the next subsection.

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Theorem 1.24. *Let K be a henselian discrete valuation field satisfying the condition* (Geom). *The graded quotient* $\mathrm{Gr}^r_{\log} G_K$ *is annihilated by* p *and the surjection (1.12) induces a surjection*

$$\pi_1^{\mathrm{alg}}(\Theta^{(r)}_{\log}) \to \mathrm{Gr}^r_{\log} G_K. \tag{1.16}$$

By the isomorphism (1.15), Theorem 1.24 has the following corollary.

Corollary 1.25. *The dual of the surjection* $\pi_1^{\mathrm{ab}}(\Theta^{(r)}_{\log}) \to \mathrm{Gr}^r_{\log} G_K$ *defines an injection*

$$\mathrm{rsw} : \mathrm{Hom}(\mathrm{Gr}^r_{\log} G_K, \mathbb{F}_p) \to \mathrm{Hom}(\pi_1^{\mathrm{ab}}(\Theta^{(r)}_{\log}), \mathbb{F}_p) = \Omega^1_F(\log) \otimes_F \mathfrak{m}^{(-r)}_{\bar K}/\mathfrak{m}^{(-r)+}_{\bar K}.$$

For a character $\chi : \mathrm{Gr}^r_{\log} G_K \to \mathbb{F}_p$, we call the image $\mathrm{rsw}\,\chi \in \Omega^1_F(\log) \otimes_F \mathfrak{m}^{(-r)}_{\bar K}/\mathfrak{m}^{(-r)+}_{\bar K}$ the refined Swan character of χ. This definition generalizes that of Kato in the abelian case in [**14**, Definition (5.3)] and [**15**, (3.4.2)].

Theorem 1.24 implies the prime-to-p part of the Hasse–Arf theorem. Let V be an ℓ-adic representation V of G_K. Since $P = G^{0+}_{K,\log}$ is a pro-p group, there exists a unique direct sum decomposition $V = \bigoplus_{q \geqslant 0, q \in \mathbb{Q}} V^{(q)}$ by G_K-submodules such that the $G^{r+}_{K,\log}$-fixed part is given by $V^{G^{r+}_{K,\log}} = \bigoplus_{q \geqslant r} V^{(q)}$. We put $\mathrm{Sw}_K V = \sum_r r \cdot \mathrm{rank}\, V^{(r)} \in \mathbb{Q}$.

Corollary 1.26.

$$\mathrm{Sw}_K V \in \mathbb{Z}\left[\frac{1}{p}\right].$$

Proof. It suffices to show that $\dim V \cdot r \in \mathbb{Z}[1/p]$ assuming $V = V^{(r)}$. This is equivalent to that $\dim V$ is divisible by the prime-to-p part m of the denominator of r. Let $\chi : \mathrm{Gr}^r_{\log} G_K \to \mu_p \subset \bar{\mathbb{Q}}_\ell^\times$ be a character appearing in the restriction of V to $G^r_{K,\log}$. The injection $\mathrm{Hom}(\mathrm{Gr}^r_{\log} G_K, \mathbb{F}_p) \to \mathrm{Hom}_{\bar F}(\mathfrak{m}^r_{\bar K}/\mathfrak{m}^{r+}_{\bar K}, \Omega^1_F(\log) \otimes \bar F)$ is compatible with the action of $I \subset G_K$ on $\mathrm{Gr}^r_{\log} G_K$ by the conjugacy. Since the action of I on $\mathfrak{m}^r_{\bar K}/\mathfrak{m}^{r+}_{\bar K}$ is by the multiplication through the quotient $I \to \mu_m$, there are m conjugates of χ appearing with the same multiplicities in V. Thus the assertion follows. □

By the same limit argument as in the proof of [**2**, Theorem 2.15], Theorem 1.24 and Corollary 1.26 imply the following.

Corollary 1.27. *Let K be an arbitrary henselian discrete valuation field K of characteristic $p > 0$.*

(1) *The pro-finite abelian group* $\mathrm{Gr}^r_{\log} G_K$ *is annihilated by* p.

(2) *For an ℓ-adic representation V of G_K, we have* $\mathrm{Sw}_K V \in \mathbb{Z}[1/p]$.

In the mixed characteristic case, one can prove results analogous to Theorem 1.24 and Corollaries 1.25 and 1.26. The author plans to discuss them in a paper in preparation.

1.4. Nearby cycles

Let X be a smooth scheme over k, D be a smooth irreducible divisor of X and $U = X \setminus D$ be the complement. Let ξ be the generic point of D and $\mathcal{O}_K = \mathcal{O}^h_{X,\xi}$ be the henselization of the local ring at ξ. We put $S = \operatorname{Spec} \mathcal{O}_K$ and let $\eta = \operatorname{Spec} K$ be the generic point. We consider the log product $P = (X \times_k S)^\sim$ as in the last subsection and the section $S \to P$ induced by the canonical map $S \to X$.

For a rational number $r \geqslant 0$, we consider the Cartesian diagram

$$\begin{array}{ccccccc}
P^{(r)}_{S,F} & \xrightarrow{i^{(r)}} & P^{(r)}_S & \xleftarrow{j^{(r)}} & P^{(r)}_{S,\eta} = U \times \eta & \xrightarrow{\mathrm{pr}_1} & U \\
\downarrow & & \downarrow{\scriptstyle p^{(r)}} & & \downarrow{\scriptstyle \mathrm{pr}_2} & & \\
\operatorname{Spec} F & \xrightarrow{i} & S & \xleftarrow{j} & \eta = \operatorname{Spec} K & &
\end{array}$$

Let $s^{(r)} : S \to P^{(r)}_S$ be the section induced by $S \to P$. By abuse of notation, we will also write $p^{(r)} : P^{(r)}_{S,F} \to \operatorname{Spec} F$ and $s^{(r)} : \operatorname{Spec} F \to P^{(r)}_{S,F}$ for the maps induced on the closed fibres. For $r = 0$, we have $P^{(0)}_S = P = (X \times_k S)^\sim$. Let $\psi^{(r)}$ be the nearby cycle functor $R\psi$ for $p^{(r)} : P^{(r)}_S \to S$ and ψ be the nearby cycle functor for the identity $S \to S$. For a sheaf $\mathcal{F}_\eta$ on η, we identify $\psi(\mathcal{F}_\eta)$ with the G_K-module $\mathcal{F}_{\bar\eta}$.

Definition 1.28. Let $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on $U = X \setminus D$. The stalk $\mathcal{F}_{\bar\eta}$ defines a representation of the absolute Galois group G_K.

For a rational number $r > 0$, we say that the log ramification of $\mathcal{F}$ at ξ along D is bounded by r if $G^r_{K,\log}$ acts trivially on $\mathcal{F}_{\bar\eta}$. Similarly, for a rational number $r \geqslant 0$, we say that the log ramification of $\mathcal{F}$ along D is bounded by $r+$ if $G^{r+}_{K,\log}$ acts trivially on $\mathcal{F}_{\bar\eta}$.

Since $P = G^{0+}_{K,\log}$ is a pro-p group, there exists a unique direct sum decomposition

$$\mathcal{F}_{\bar\eta} = \bigoplus_{q \geqslant 0,\, q \in \mathbb{Q}} \mathcal{F}^{(q)}_{\bar\eta} \tag{1.17}$$

by G_K-submodules such that the $G^{r+}_{K,\log}$-fixed part is given by

$$\mathcal{F}_{\bar\eta}^{G^{r+}_{K,\log}} = \bigoplus_{q \leqslant r} \mathcal{F}^{(q)}_{\bar\eta}.$$

Replacing X by an étale neighbourhood of ξ if necessary, we may assume that there exists a direct sum decomposition $\mathcal{F} = \bigoplus_{q \geqslant 0} \mathcal{F}^{(q)}$ inducing (1.17).

We identify the stable closed fibre $\bar{P}^{(r)}_{S,\bar F}$ with the $\bar F$-vector space $\Theta^{(r)}_{\log}$ by the isomorphism in Corollary 1.11.

Proposition 1.29. *Let $r > 0$ be a rational number and let $\pi^{(r)} : \bar{P}^{(r)}_{S,\bar F} \to P^{(r)}_{S,\bar F}$ be the canonical map. Let $\mathcal{F}$ be a smooth sheaf on U. We assume that $\mathcal{F}_{\bar\eta} = \mathcal{F}^{(q)}_{\bar\eta}$ for a rational number $q \geqslant 0$.*

(1) *Assume $q = r$. Let $\mathcal{F}_{\bar\eta} = \bigoplus_\chi \mathcal{F}^{(\chi)}_{\bar\eta}$ be the decomposition by characters $\chi : \mathrm{Gr}^r_{\log} G_K \to \Lambda^\times$. Let $\mathcal{L}_\chi$ be the smooth sheaf of rank 1 on $\bar{P}^{(r)}_{S,\bar F} = \Theta^{(r)}_{\log}$ defined by the composition $\pi_1(\Theta^{(r)}_{\log})^{\mathrm{ab}} \to \mathrm{Gr}^r_{\log} G_K \to \Lambda^\times$.*

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Then, there exists a canonical isomorphism

$$\psi^{(r)}(\mathrm{pr}_1^*\,\mathcal{F}) \to \bigoplus_\chi \pi_*^{(r)}\mathcal{L}_\chi \otimes p^{(r)*}\mathcal{F}_{\bar\eta}^{(\chi)} \tag{1.18}$$

on $P^{(r)}_{S,\bar F}$.

(2) *If $q < r$, then there exists a canonical isomorphism*

$$\psi^{(r)}(\mathrm{pr}_1^*\,\mathcal{F}) \to \pi_*^{(r)}\Lambda \otimes p^{(r)*}\psi(\mathcal{F}_\eta) \tag{1.19}$$

on $P^{(r)}_{S,\bar F}$.

Proof. Let $V \to U$ be the finite étale covering trivializing $\mathcal{F}$. Replacing X by an étale neighbourhood of ξ, we may assume that V is the complement of a smooth irreducible divisor of the normalization Y of X in V as in the previous subsection. We put $T = Y \times_X \operatorname{Spec}\mathcal{O}_K = \operatorname{Spec}\mathcal{O}_L$. By the assumption, the log ramification of L is bounded by $q+$.

We consider the diagram (1.8). Since $q \leqslant r$, there exists a finite extension K' of K of ramification index e such that er is an integer and that, for the base change by $S' = \operatorname{Spec}\mathcal{O}_{K'} \to S$, the map $Q^{(er)}_{T'} \to P^{(er)}_{S'}$ is finite étale by Lemma 1.13. Further, if $q < r$, the finite étale covering $Q^{(er)}_{T',\bar F} \to P^{(er)}_{S',\bar F}$ is trivial.

The pull-back of $\mathrm{pr}_1^*\,\mathcal{F}$ to $U \times \operatorname{Spec} K' \subset P^{(er)}_{S'}$ is trivialized by the restriction of the finite étale covering $Q^{(er)}_{T'} \to P^{(er)}_{S'}$. Hence, it is extended to a smooth sheaf $\mathcal{G}$ on $P^{(er)}_{S'}$. By the definition of the surjection $\pi_1(\Theta^{(r)}_{\log})^{\mathrm{ab}} \to \mathrm{Gr}^r_{\log} G_K$, the pull-back of $\mathcal{G}$ to $\Theta^{(r)}_{\log} = \bar P^{(r)}_{S,\bar F} = P^{(er)}_{S'} \times_{S'} \bar F$ is defined by the induced action of $\pi_1(\Theta^{(r)}_{\log})^{\mathrm{ab}}$ on $\mathcal{F}_{\bar\eta} = \bigoplus_\chi \mathcal{F}^{(\chi)}_{\bar\eta}$. Hence, it is isomorphic to $\bigoplus_\chi \mathcal{L}_\chi^{\operatorname{rank}\mathcal{F}^{(\chi)}}$ if $q = r$. If $q < r$, the pull-back of $\mathcal{G}$ to $\bar P^{(r)}_{S,\bar F}$ is constant.

We consider the nearby cycle functor ψ' for the smooth map $p'^{(er)} : P^{(er)}_{S'} \to S'$. Let $s' : S' \to P^{(er)}_{S'}$ be the section induced by $S \to P^{(r)}_S$. Then $\psi'(\mathrm{pr}_1^*\,\mathcal{F})$ is the restriction of $\mathcal{G}$ on $P^{(er)}_{S',\bar F}$ and the base change map

$$s'^*\psi'(\mathrm{pr}_1^*\,\mathcal{F}) = \mathcal{G}_0 \to \psi(\mathcal{F}) \tag{1.20}$$

is an isomorphism, where $0 \in \Theta^{(r)}_{\log}$ denotes the origin. Thus we obtain a canonical isomorphism

$$\psi'(\mathrm{pr}_1^*\,\mathcal{F}) \to \begin{cases} \bigoplus_\chi \mathcal{L}_\chi \otimes p'^{(er)*}\mathcal{F}^{(\chi)}_{\bar\eta} & \text{if } q = r, \\ p'^{(er)*}\psi(\mathcal{F}_\eta) & \text{if } q < r. \end{cases} \tag{1.21}$$

Since $\psi^{(r)}(\mathrm{pr}_1^*\,\mathcal{F}) = \pi_*^{(r)}\psi'(\mathrm{pr}_1^*\,\mathcal{F})$, the isomorphism (1.21) induces isomorphisms (1.18) and (1.19). □

Corollary 1.30. *Let $r \geqslant 0$ be a rational number.*

(1) *If the log ramification of $\mathcal{F}$ is bounded by $r+$, then the base change map*

$$s^{(r)*}\psi^{(r)}(\mathrm{pr}_1^*\,\mathcal{F}) \to \psi(\mathcal{F}_\eta) \tag{1.22}$$

is an isomorphism.

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(2) *The base change map* (1.22) *induces an isomorphism*

$$s^{(r)*}R^0\psi^{(r)}(\mathrm{pr}_1^*\mathcal{F}) \to \mathcal{F}_{\bar\eta}^{G^{r+}_{K,\log}} \subset \mathcal{F}_{\bar\eta} = \psi(\mathcal{F}_\eta) \tag{1.23}$$

from the degree 0*-part to the* $G^{r+}_{K,\log}$*-fixed part.*

Proof. (1) We may assume $\mathcal{F} = \mathcal{F}^{(q)}$ for some rational number $0 \leqslant q \leqslant r$. First, we consider the case $r > 0$. We use the notation of the proof of Proposition 1.29. Since the inverse image $\pi^{(r)-1}(0)$ of $0 \in \Theta^{(r)}_{\log} = \bar P^{(r)}_{S,\bar F}$ consists of the image of the geometric closed point by the section $S \to P^{(r)}_S$, the isomorphism (1.20) shows that the base change map (1.22) is an isomorphism.

Assume $r = q = 0$. Then, the smooth sheaf $\mathrm{pr}_1^*\mathcal{F}$ on $U \times \eta \subset P = P^{(0)}_S$ is tamely ramified along $P \times_S \operatorname{Spec} F$. By Abhyankar's lemma, the projections $U \times \eta \to U$ and $U \times \eta \to \eta$ induce isomorphisms on the tame inertia. Hence, étale locally on P, it is isomorphic to the pull-back of a sheaf on η. Since P is smooth over S, the assertion follows.

(2) We may assume $\mathcal{F} = \mathcal{F}^{(q)}$ for some rational number $q \geqslant 0$. By 1, it suffices to consider the case $q > r$. Since the base change map $R^0\psi^{(r)}(\mathrm{pr}_1^*\mathcal{F})_{\bar s} \to R^0\psi(\mathcal{F}_\eta)$ is injective, it suffices to show that the base change map is the 0-map.

Let $f_{rq} : P^{(q)}_S \to P^{(r)}_S$ be the canonical map. The sheaf $\psi^{(q)}(\mathrm{pr}_1^*\mathcal{F})$ has no non-trivial geometrically constant subsheaf, by Proposition 1.29. Since the image $f_{rq}(P^{(q)}_{S,\bar F})$ is a point, the base change map $f_{rq}^*\psi^{(r)}(\mathrm{pr}_1^*\mathcal{F}) \to \psi^{(q)}(\mathrm{pr}_1^*\mathcal{F})$ is the 0-map. Thus the composition $\psi^{(r)}(\mathrm{pr}_1^*\mathcal{F})_{\bar s} \to \psi^{(q)}(\mathrm{pr}_1^*\mathcal{F})_{\bar s} \to \psi(\mathcal{F}_\eta)$ is also the 0-map as required. □

We consider $\mathcal{H} = \mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}_\eta, \mathrm{pr}_1^*\mathcal{F})$ on $P^{(r)}_{S,\eta} = U \times \eta$ and the base change map with respect to the diagram

$$\begin{array}{ccc} U \times \eta & \xrightarrow{j^{(r)}} & P^{(r)}_S \\ \uparrow & & \uparrow{\scriptstyle s^{(r)}} \\ \eta & \xrightarrow{j} & S \end{array}$$

Corollary 1.31. *Let $r \geqslant 0$ be a rational number.*

(1) *The following conditions are equivalent.*

(a) *The log ramification of $\mathcal{F}$ is bounded by $r+$.*

(b) *The base change map*

$$s^{(r)*}j^{(r)}_*\mathcal{H} \to j_*\mathcal{E}nd(\mathcal{F}_\eta)$$

is an isomorphism.

(c) *The identity $1 \in \mathrm{End}_{G_K}(\mathcal{F}_{\bar\eta}) = \Gamma(S, j_*\mathcal{E}nd(\mathcal{F}_\eta))$ is in the image of the base change map*

$$\Gamma(S, s^{(r)*}j^{(r)}_*\mathcal{H}) \to \Gamma(S, j_*\mathcal{E}nd(\mathcal{F}_\eta)).$$

(2) *Assume that the $G^{r+}_{K,\log}$-fixed part $\mathcal{F}_{\bar\eta}^{G^{r+}_{K,\log}}$ is 0. Then we have $i^*s^{(r)*}j^{(r)}_*\mathcal{H} = 0$.*

Proof. (1) $(a) \Rightarrow (b)$. It suffices to show the isomorphism for the geometric closed fibre at $\bar{s} = \operatorname{Spec} \bar{F} \to S$. By the assumption (a), we have

$$\mathcal{F}_{\bar{\eta}} = \mathcal{F}_{\bar{\eta}}^{G^{r+}_{K,\log}}$$

and the base change map (1.23) induces an isomorphism

$$\begin{aligned} s^{(r)*} R^0\psi^{(r)}(\mathcal{H})_{\bar{s}} &= \operatorname{Hom}(\psi(\mathcal{F}_\eta), s^{(r)*}R^0\psi^{(r)}(\operatorname{pr}_1^*\mathcal{F})) \\ &\to \operatorname{Hom}(\psi(\mathcal{F}_\eta), \psi(\mathcal{F}_\eta)) = \operatorname{End}(\psi(\mathcal{F}_\eta)). \end{aligned}$$

Taking the fixed parts by the inertia subgroup $I \subset G_K$, we obtain an isomorphism

$$(s^{(r)*} j_*^{(r)} \mathcal{H})_{\bar{s}} = s^{(r)*} R^0\psi^{(r)}(\mathcal{H})^I \to \operatorname{End}_I(\psi(\mathcal{F}_\eta)) = (j_* \mathcal{E}nd(\mathcal{F}_\eta))_{\bar{s}}$$

as required.

$(b) \Rightarrow (c)$. Clear.

$(c) \Rightarrow (a)$. We consider the direct sum decomposition $\mathcal{F}_{\bar{\eta}} = \bigoplus_q \mathcal{F}_{\bar{\eta}}^{(q)}$. It suffices to show that the identity is not in the image assuming $\mathcal{F}_{\bar{\eta}}^{(q)} \neq 0$ for some $q > r$. Thus it is reduced to the assertion (2).

(2) Assume $\mathcal{F}_{\bar{\eta}}^{G^{r+}_{K,\log}} = 0$. Then, similarly as in the proof of (1) $(a) \Rightarrow (b)$ above, we have $(s^{(r)*} j_*^{(r)} \mathcal{H})_{\bar{s}} = s^{(r)*} R^0\psi^{(r)}(\mathcal{H})^I = 0$. □

Corollary 1.32. *Assume that $r > 0$ is an integer and that the restriction to $G^r_{K,\log}$ of the action on $\mathcal{F}_{\bar{\eta}}$ is by the multiplication by a character $\chi : \operatorname{Gr}^r_{\log} G_K \to \Lambda^\times$.*

(1) *There exists a canonical isomorphism*

$$\psi^{(r)}(\operatorname{pr}_1^* \mathcal{F}) \to \mathcal{L}_\chi \otimes p^{(r)*}\psi(\mathcal{F}_\eta) \tag{1.24}$$

on $P^{(r)}_{S,\bar{F}}$.

(2) *There exists a canonical isomorphism*

$$i^{(r)*} j_*^{(r)} \mathcal{H} \to \mathcal{L}_\chi \otimes p^{(r)*} i^* j_* \mathcal{E}nd(\mathcal{F}_\eta) \tag{1.25}$$

on $P^{(r)}_{S,F}$.

Proof. (1) Clear from Proposition 1.29 (1).

(2) By (1), we have an isomorphism

$$\begin{array}{c} \operatorname{Hom}(p^{(r)*}\psi(\mathcal{F}_\eta), \psi^{(r)}(\operatorname{pr}_1^*\mathcal{F})) = \psi^{(r)}(\mathcal{H}) \\ \downarrow \\ \operatorname{Hom}(p^{(r)*}\psi(\mathcal{F}_\eta), \mathcal{L}_\chi \otimes p^{(r)*}\psi(\mathcal{F}_\eta)) = \mathcal{L}_\chi \otimes p^{(r)*}\psi(\mathcal{E}nd(\mathcal{F}_\eta)) \end{array} \tag{1.26}$$

We have canonical isomorphisms $R\Gamma(I, \psi^{(r)}) \to i^{(r)*} Rj_*^{(r)}$ and $R\Gamma(I, \psi) \to i^* Rj_*$ of functors. Thus, we obtain the isomorphism (1.25) by taking the inertia fixed parts in (1.26). □

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The following geometric construction is crucial in the proof of Theorem 1.24. Let $(X \times X)' \to X \times X$ be the blow-up at $D \times D$ and let $(X \times X)^{\sim} \subset (X \times X)'$ be the complement of the proper transforms of $D \times X$ and of $X \times D$. We call the immersion $\tilde{\delta} : X \to (X \times X)^{\sim}$ induced by the diagonal $\delta : X \to X \times X$ the log diagonal. Let $\mathcal{J}_X \subset (X \times X)^{\sim}$ be the ideal defining the log diagonal and let $\tilde{j} : U \times U \to (X \times X)^{\sim}$ be the open immersion. For an integer $r \geqslant 0$, we define a scheme $(X \times X)^{(r)}$ affine over $(X \times X)^{\sim}$ by the quasi-coherent $\mathcal{O}_{(X \times X)^{\sim}}$-algebra $\sum_{l \geqslant 0} \mathcal{I}_D^{-lr} \cdot \mathcal{J}_X^l \subset \tilde{j}_* \mathcal{O}_{U \times U}$.

The fibre product $(X \times X)^{(r)} \times_X D$ with respect to the second projection is canonically identified with the vector bundle $\boldsymbol{V}(\Omega_X^1(\log D)(rD)) \times_X D$, similarly as in Corollary 1.11. Hence the map $(X \times S)^{(r)} \to (X \times X)^{(r)}$ defined by the canonical map $S \to X$ induces an isomorphism

$$\begin{array}{c} (X \times S)^{(r)} \times_X \operatorname{Spec} \bar{F} = \Theta_{\log}^{(r)} \\ \downarrow \\ (X \times X)^{(r)} \times_X \operatorname{Spec} \bar{F} = \boldsymbol{V}(\Omega_X^1(\log D)(rD)) \times_X \operatorname{Spec} \bar{F} \end{array} \tag{1.27}$$

Lemma 1.33. *Let $r > 0$ be an integer.*

(1) *There exists a unique map $\mu : (X \times S)^{(r)} \times_S (X \times S)^{(r)} \to (X \times X)^{(r)}$ that makes the diagram*

$$\begin{array}{ccc} (X \times S)^{(r)} \times_S (X \times S)^{(r)} & \xrightarrow{\mu} & (X \times X)^{(r)} \\ \downarrow & & \downarrow \\ (X \times S) \times_S (X \times S) = X \times X \times S & \xrightarrow{\mathrm{pr}_{12}} & X \times X \end{array} \tag{1.28}$$

commutative.

(2) *Under the identification (1.27) $(X \times X)^{(r)} \times_X \operatorname{Spec} \bar{F} = \Theta_{\log}^{(r)}$, the map*

$$\mu : (X \times S)^{(r)} \times_S (X \times S)^{(r)} \to (X \times X)^{(r)}$$

induces the difference $- : \Theta_{\log}^{(r)} \times_{\bar{F}} \Theta_{\log}^{(r)} \to \Theta_{\log}^{(r)}$ on the fibre over $\operatorname{Spec} \bar{F}$.

Proof. (1) We put $P = (X \times S)^{\sim} \times_S (X \times S)^{\sim}$. Applying the basic construction to the smooth scheme P and the diagonal section $S \to P$, we define $q : P_S^{(r)} \to P$ and a section $s^{(r)} : S \to P_S^{(r)}$. The projections $P \to (X \times S)^{\sim}$ induce $P_S^{(r)} \to (X \times S)^{(r)}$. We show that the product

$$P_S^{(r)} \to (X \times S)^{(r)} \times_S (X \times S)^{(r)} \tag{1.29}$$

is an isomorphism. The ideal defining the closed subscheme $S \subset P$ is generated by the two pull-backs of the ideal defining the closed subscheme $S \subset (X \times S)^{\sim}$. Hence, the map (1.29) is a closed immersion. Since both $P_S^{(r)}$ and $(X \times S)^{(r)} \times_S (X \times S)^{(r)}$ are smooth over S of the same dimension, the closed immersion (1.29) is an open immersion. Since the map (1.29) is an isomorphism on each fibre, it is an isomorphism.

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Let $D_{(X\times S)^\sim} \subset (X\times S)^\sim$ be the pull-back $\mathrm{pr}_1^* D = \mathrm{pr}_2^* D_S$. Since $\mathrm{pr}_1^* D_{(X\times S)^\sim} = \mathrm{pr}_2^* D_{(X\times S)^\sim}$ on P, there exists a unique map $\lambda : P \to (X\times X)^\sim$ that makes the diagram (1.28) with $\mu : P^{(r)} = (X\times S)^{(r)} \times_S (X\times S)^{(r)} \to (X\times X)^{(r)}$ replaced by $\lambda : P \to (X\times X)^\sim$ commutative. By the commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ P & \xrightarrow{\ \lambda\ } & (X\times X)^\sim \end{array}$$

the pull-back $\lambda^*(\mathcal{I}_D^{-lr}\cdot \mathcal{J}_X^l)$ is contained in $\mathfrak{m}_K^{-lr}\cdot \mathcal{I}_S^l$. Hence the assertion follows.

(2) Let $\mathcal{J}_X \subset \mathcal{O}_{(X\times X)^\sim}$ and $\mathcal{J}_S \subset \mathcal{O}_{(X\times S)^\sim}$ be the ideals defining the closed subschemes $X \subset (X\times X)^\sim$ and $S \subset (X\times S)^\sim$ respectively. By the identification in Corollary 1.11, the map $\Theta^{(r)}_{\log} \times_F \Theta^{(r)}_{\log} \to \Theta^{(r)}_{\log} \subset (X\times X)^{(r)}$ is defined by $\mathcal{I}_D^{-r}\mathcal{J}_X \to \mathfrak{m}_K^{-r}\cdot \mathcal{J}_S \oplus \mathfrak{m}_K^{-r}\cdot \mathcal{J}_S$. Hence, it is a linear map of vector bundles. Thus it suffices to show that the composition with the injections $i_1, i_2 : \Theta^{(r)}_{\log} \to \Theta^{(r)}_{\log} \times_F \Theta^{(r)}_{\log}$ of the two factors are the identity of $\Theta^{(r)}_{\log}$ and the multiplication by -1 respectively.

Let $s : S \to (X\times S)^{(r)}$ be the map induced by the canonical map $S \to X$. We consider the map $\iota_1 = (\mathrm{id}_{(X\times S)^{(r)}}, s\circ \mathrm{pr}_2) : (X\times S)^{(r)} \to (X\times S)^{(r)} \times_S (X\times S)^{(r)}$. Then, its restriction $\Theta^{(r)}_{\log} \to \Theta^{(r)}_{\log} \times_{\bar F} \Theta^{(r)}_{\log}$ to the closed fibre is the injection into the first component. The composition $\mu\circ \iota_1$ is the map $(X\times S)^{(r)} \to (X\times X)^{(r)}$ induced by the canonical map $S \to X$. Hence the composition $\mu \circ i_1$ is the identity of $\Theta^{(r)}_{\log}$. Similarly, we consider the map $\iota_2 = (s\circ \mathrm{pr}_2, \mathrm{id}_{(X\times S)^{(r)}}) : (X\times S)^{(r)} \to (X\times S)^{(r)} \times_S (X\times S)^{(r)}$. Then the composition $\mu\circ\iota_2 : (X\times S)^{(r)} \to (X\times X)^{(r)}$ is the composition of the canonical map $(X\times S)^{(r)} \to (X\times X)^{(r)}$ and the map $(X\times X)^{(r)} \to (X\times X)^{(r)}$ switching the two factors. Hence the composition $\mu\circ i_2$ is the multiplication by -1 of $\Theta^{(r)}_{\log}$. Hence the assertion is proved. □

Proof of Theorem 1.24. We start with some reduction steps. For each non-trivial character $\chi : \mathrm{Gr}^r_{\log} G_K \to \Lambda^\times$, the surjection (1.12) defines a locally constant sheaf $\mathcal{L}_\chi$ of Λ-modules of rank 1 on $\Theta^{(r)}_{\log}$. By Lemma 1.23, in order to prove Theorem 1.24, it suffices to show that, for every character $\chi : \mathrm{Gr}^r_{\log} G_K \to \Lambda^\times$, there exists an isomorphism $-^*\mathcal{L}_\chi \to \mathcal{H}om(p_2^*\mathcal{L}_\chi, p_1^*\mathcal{L}_\chi)$ assuming Λ is a finite field.

We reduce it to the case where r is an integer. Let $e > 0$ be an integer such that er is an integer and let K_1 be an extension of K of ramification index e as in Lemma 1.22. Then the construction of $\mathcal{L}_\chi$ commutes with the base change $K \to K_1$. Hence, it is reduced to the case where r is an integer.

We further reduce it to the case where the restriction to $G^r_{K,\log}$ of the action on $\mathcal{F}_{\bar\eta}$ is by the multiplication by a character $\chi : \mathrm{Gr}^r_{\log} G_K \to \Lambda^\times$. By the same argument as in the last paragraph, we may replace K by a tamely ramified extension. Hence we may assume the restriction to $G^{0+}_{K,\log}$ is irreducible. By [**2**, Theorem 5.12.1], $\mathrm{Gr}^r_{\log} G_K$ is in the centre of $G^{0+}_{K,\log}/G^{r+}_{K,\log}$. Hence, the action on $\mathcal{F}_{\bar\eta}$ to $G^r_{K,\log}$ is by the multiplication by a character $\chi : \mathrm{Gr}^r_{\log} G_K \to \Lambda^\times$.

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We assume that $r > 0$ is an integer and the restriction of $\mathcal{F}_{\bar\eta}$ to $G^r_{K,\log}$ is the multiplication by a non-trivial character $\chi : \mathrm{Gr}^r_{\log} G_K \to \Lambda^\times$. We consider the commutative diagram

$$\begin{array}{ccccc}
\Theta^{(r)}_{\log} \times_F \Theta^{(r)}_{\log} & \xrightarrow{\ i\ } & (X \times S)^{(r)} \times_S (X \times S)^{(r)} & \xleftarrow{\ j\ } & (U \times \eta) \times_\eta (U \times \eta) \\
{\scriptstyle -}\downarrow & & & & = U \times U \times \eta \\
\Theta^{(r)}_{\log} & & {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{12}} \\
\downarrow & & & & \\
(X \times X)^{(r)} \times_X D & \xrightarrow{\ i'\ } & (X \times X)^{(r)} & \xleftarrow{\ j'\ } & U \times U
\end{array}$$

The left square is commutative by Lemma 1.33 (2). We consider the base change map

$$-^*((i'^* j'_* \mathcal{H})|_{\Theta^{(r)}_{\log}}) \to i^* j_* \,\mathrm{pr}^*_{12}\, \mathcal{H} \tag{1.30}$$

for $\mathcal{H} = \mathcal{H}om(\mathrm{pr}^*_2 \mathcal{F}, \mathrm{pr}^*_1 \mathcal{F})$ on $U \times U$.

First, we compute $i^* j_* \,\mathrm{pr}^*_{12}\, \mathcal{H}$. We have $\psi(\mathrm{pr}^*_{12}\, \mathcal{H}) = \psi(\mathcal{H}om(\mathrm{pr}^*_2 \mathcal{F}, \mathrm{pr}^*_1 \mathcal{F}))$ where $\mathrm{pr}_i : U \times U \times \eta \to U$ denote the projections. Further, we have

$$\psi(\mathcal{H}om(\mathrm{pr}^*_2 \mathcal{F}, \mathrm{pr}^*_1 \mathcal{F})) = \mathcal{H}om(\psi(\mathrm{pr}^*_2 \mathcal{F}), \psi(\mathrm{pr}^*_1 \mathcal{F})) = \mathcal{H}om(\mathrm{pr}^*_2\, \psi^{(r)} \mathcal{F}, \mathrm{pr}^*_1\, \psi^{(r)} \mathcal{F}),$$

where $\mathrm{pr}_i : \Theta^{(r)}_{\log} \times \Theta^{(r)}_{\log} \to \Theta^{(r)}_{\log}$ denote the projections in the right-hand side. By Proposition 1.29, it is further identified with

$$\mathcal{H}om(\mathrm{pr}^*_2\, \mathcal{L}_\chi \otimes \psi(\mathcal{F}_\eta), \mathrm{pr}^*_1\, \mathcal{L}_\chi \otimes \psi(\mathcal{F}_\eta)) \to \mathcal{H}om(\mathrm{pr}^*_2\, \mathcal{L}_\chi, \mathrm{pr}^*_1\, \mathcal{L}_\chi) \otimes \psi(\mathcal{E}nd(\mathcal{F}_\eta)).$$

Here and in the following, $\psi(\mathcal{F}_\eta)$, etc., on the base also denote their pull-backs by abuse of notation. Thus, similarly as Corollary 1.32, we obtain an isomorphism $i^* j_* \,\mathrm{pr}^*_{12}\, \mathcal{H} \to \mathcal{H}om(p^*_2 \mathcal{L}_\chi, p^*_1 \mathcal{L}_\chi) \otimes \mathrm{End}_I(\mathcal{F}_\eta)$ by taking the inertia fixed parts.

Next, we compute the restriction $(i'^* j'_* \mathcal{H})|_{\Theta^{(r)}_{\log}}$. This is the same as $i^* j^{(r)}_* \mathcal{H}$ computed in Corollary 1.32. Hence it is canonically isomorphic to $\mathcal{L}_\chi \otimes \mathrm{End}_I(\mathcal{F}_\eta)$. Hence, the map (1.30) induces a map

$$-^* \mathcal{L}_\chi \otimes \mathrm{End}_I(\mathcal{F}_\eta) \to \mathcal{H}om(p^*_2 \mathcal{L}_\chi, p^*_1 \mathcal{L}_\chi) \otimes \mathrm{End}_I(\mathcal{F}_\eta)$$

of smooth sheaves. Since, this is an isomorphism at the origin, it is an isomorphism on $\Theta^{(r)}_{\log} \times_F \Theta^{(r)}_{\log}$. By evaluating at the identity of $\mathcal{F}_\eta$, we obtain an isomorphism $-^* \mathcal{L}_\chi \to \mathcal{H}om(p^*_2 \mathcal{L}_\chi, p^*_1 \mathcal{L}_\chi)$ as required. □

2. Ramification along a divisor

We introduce the notion of additive sheaves on vector bundles and its generalization in § 2.1. In § 2.2, we study a global variant of the basic construction in § 1.1. After these preliminaries, we study the ramification of smooth sheaves on the complement of a divisor with normal crossings along the divisor in § 2.3.

2.1. Additive sheaves on vector bundles and generalizations

We recall the definition of the Fourier–Deligne transform [**17**]. Let X be a scheme over $\mathbb{F}_p$. Let $E = \boldsymbol{V}(\mathcal{E}) \to X$ be a vector bundle of rank d and let $E^\vee = \boldsymbol{V}(\mathcal{E}^\vee) \to X$ be the dual. The canonical pairing defines a map $\langle\cdot\,,\cdot\rangle : E \times_X E^\vee \to \boldsymbol{A}^1$. We consider the diagram

$$\begin{array}{ccccc} E & \xleftarrow{\mathrm{pr}_1} & E \times_X E^\vee & \xrightarrow{\langle\cdot,\cdot\rangle} & \boldsymbol{A}^1 \\ & & \Big\downarrow{\scriptstyle \mathrm{pr}_2} & & \\ & & E^\vee & & \end{array}$$

where pr_i denote the projections.

We fix a non-trivial character $\psi : \mathbb{F}_p \to \Lambda^\times$ and let $\mathcal{L}_\psi$ be the smooth rank 1 Artin–Schreier sheaf on $\boldsymbol{A}^1 = \operatorname{Spec} k[t]$ defined by the $\mathbb{F}_p$-torsor $\boldsymbol{A}^1 \to \boldsymbol{A}^1 : t \mapsto t^p - t$ and by ψ. For a sheaf $\mathcal{G}$ on the dual $E^\vee$ of a vector bundle E, we define the naive Fourier transform $F_\psi(\mathcal{G})$ on E by

$$F_\psi(\mathcal{G}) = R\,\mathrm{pr}_{1!}(\mathrm{pr}_2^*\,\mathcal{G} \otimes \langle\cdot\,,\cdot\rangle^*\mathcal{L}_\psi).$$

For a sheaf $\mathcal{H}$ on E, we define the inverse Fourier transform $F'_{\psi'}(\mathcal{H})$ by

$$F'_{\psi'}(\mathcal{H}) = R\,\mathrm{pr}_{2!}(\mathrm{pr}_1^*\,\mathcal{H} \otimes \langle\cdot\,,\cdot\rangle^*\mathcal{L}_{\psi'})(d)[2d],$$

where $\psi' : \mathbb{F}_p \to \Lambda^\times$ denotes the inverse of ψ.

We have canonical isomorphisms

$$\mathcal{H} \to F_\psi F'_{\psi'}\mathcal{H}, \qquad \mathcal{G} \to F'_{\psi'} F_\psi \mathcal{G}. \tag{2.1}$$

Let $f : E \to F$ be a linear morphism of vector bundles over X and $f^\vee : F^\vee \to E^\vee$ be the dual. Then, we have a canonical isomorphism

$$f^* F_\psi \mathcal{G} \to F_\psi R f^\vee_! \mathcal{G} \tag{2.2}$$

for a sheaf $\mathcal{G}$ on $F^\vee$. Similarly, we have a canonical isomorphism

$$F_\psi f^{\vee *}\mathcal{G} \to R f_! F_\psi \mathcal{G} \tag{2.3}$$

for a sheaf $\mathcal{G}$ on $E^\vee$. Dually, we have a canonical isomorphism

$$R f_* F_\psi \mathcal{G} \to F_\psi R f^{\vee !}\mathcal{G} \tag{2.4}$$

for a sheaf $\mathcal{G}$ on $E^\vee$.

We introduce the notion of additive sheaves on vector bundles.

Definition 2.1. Let $E = \boldsymbol{V}(\mathcal{E})$ be a vector bundle over a scheme X over k and let $\mathcal{H}$ be a constructible sheaf on E. Let $\mathcal{G} = F'_{\psi'}\mathcal{H}$ be the inverse Fourier transform and define a constructible subset $S \subset E^\vee$ to be the support of $\mathcal{G}$.

We say $\mathcal{H}$ on E is additive if, for every point x of X, the fibre $S \times_X x$ is finite. For an additive constructible sheaf $\mathcal{H}$ on E, we call the support $S = S_\mathcal{H} \subset E^\vee$ of the inverse Fourier transform $\mathcal{G} = F'_{\psi'}\mathcal{H}$ the dual support of $\mathcal{H}$. We say an additive constructible sheaf is non-degenerate if the intersection of the *closure* of the dual support $S_\mathcal{H}$ with the 0-section is empty.

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Example 2.2. Let f be a linear form on a vector bundle $E \to X$ and $\mathcal{H}$ be the Artin–Schreier sheaf on E defined by the equation $T^p - T = f$ and by ψ. Then, $\mathcal{H}$ is the naive Fourier transform $F_\psi(\Lambda_S)$ of the constant sheaf on the image S of the corresponding section $X \to E^\vee$. Hence $\mathcal{H}$ is additive and its dual support is S. It is non-degenerate if and only if the intersection of S with the 0-section is empty.

A constructible sheaf $\mathcal{H}$ on a vector bundle E is additive if and only if, for every geometric point $\bar{x} \to X$, the pull-back $\mathcal{H}|_{E_{\bar{x}}}$ is additive, by the proper base change theorem. If $X = \operatorname{Spec} \bar{F}$ is the spectrum of an algebraically closed field, a constructible sheaf $\mathcal{H}$ on a vector space E is additive if and only if $\mathcal{H}$ is a direct sum of rank 1 Artin–Schreier sheaves defined by linear forms by the isomorphism (2.1). A constructible subsheaf $\mathcal{H}'$ of an additive constructible sheaf $\mathcal{H}$ is additive if and only if it is smooth on each fibre.

We have the following elementary properties on additive sheaves.

Lemma 2.3.

(1) *Let $f : E' \to E$ be a linear map of vector bundles over X and $f^\vee : E^\vee \to E'^\vee$ be the dual. If $\mathcal{H}$ is additive, then $f^*\mathcal{H}$ is additive and we have $S_{f^*\mathcal{H}} = f^\vee(S_\mathcal{H})$.*

Assume $f : E' \to E$ is surjective and identify $E^\vee$ with the image $f^\vee(E^\vee)$ by the closed immersion $f^\vee : E^\vee \to E'^\vee$. Then, conversely, $\mathcal{H}$ is additive if $f^*\mathcal{H}$ is additive.

(2) *Let $f : E \to X$ be a vector bundle and let $\mathcal{H}$ be an additive constructible sheaf. If $\mathcal{H}$ is non-degenerate, we have $Rf_*\mathcal{H} = Rf_!\mathcal{H} = 0$.*

Proof. (1) Clear from (2.2).

(2) Clear from (2.2) and (2.4). □

An additive sheaf is uniquely determined by the restriction to the complement of the 0-section.

Proposition 2.4. *Let E be a vector bundle over X and $\mathcal{H}$ be an additive sheaf on E. Let $E^0 = E \setminus 0(X)$ be the complement of the 0-section and $g : E^0 \to E$ be the open immersion. Then, the canonical map $\mathcal{H} \to g_*g^*\mathcal{H}$ is an isomorphism.*

Proof. By devissage, we may assume that the dual support $S = S_\mathcal{H} \subset E^\vee$ is locally closed and normal and that the inverse Fourier transform $\mathcal{G} = F'_{\psi'}(\mathcal{H})$ is locally constant on S. We have an isomorphism $\mathcal{H} \to \mathrm{pr}_{1!}(\mathrm{pr}_2^*\mathcal{G} \otimes \mu^*\mathcal{L}_\psi)$ where μ denote the composition of the inclusion $E \times_X S \to E \times_X E^\vee$ with $\langle \cdot , \cdot \rangle : E \times_X E^\vee \to \boldsymbol{A}^1$. Since the canonical map $\mathrm{pr}_2^*\mathcal{G} \to (g \times 1)_*(g \times 1)^*\mathrm{pr}_2^*\mathcal{G}$ is an isomorphism, the map $\mathrm{pr}_2^*\mathcal{G} \otimes \mu^*\mathcal{L}_\psi \to (g \times 1)_*(g \times 1)^*(\mathrm{pr}_2^*\mathcal{G} \otimes \mu^*\mathcal{L}_\psi)$ is also an isomorphism.

Since $S \to X$ is quasi-finite, there exists a normal scheme $\bar{S}$ finite over X and containing S as a dense open subscheme, by Zariski's main theorem. Let $j : S \to \bar{S}$ denote the open immersion. Then, the isomorphism $\mathrm{pr}_2^*\mathcal{G} \otimes \mu^*\mathcal{L}_\psi \to (g \times 1)_*(g \times 1)^*(\mathrm{pr}_2^*\mathcal{G} \otimes \mu^*\mathcal{L}_\psi)$ is extended to an isomorphism $(1 \times j)_!(\mathrm{pr}_2^*\mathcal{G} \otimes \mu^*\mathcal{L}_\psi) \to (g \times 1)_*(g \times 1)^*(1 \times j)_!(\mathrm{pr}_2^*\mathcal{G} \otimes \mu^*\mathcal{L}_\psi)$. Hence the assertion follows by the proper base change theorem for the finite map $\mathrm{pr}_1 : E \times_X \bar{S} \to E$. □

Proposition 2.5. *Let X be a scheme over k and $E \to X$ be a vector bundle. For a constructible sheaf $\mathcal{H}$ on E, the following conditions are equivalent.*

(1) *$\mathcal{H}$ is additive.*

(2) *For every geometric point $\bar{x} \in X$ and for every closed point $a \in E_{\bar{x}}$, there exists an isomorphism $(+a)^*(\mathcal{H}|_{E_{\bar{x}}}) \to \mathcal{H}|_{E_{\bar{x}}}$.*

Proof. We may assume k is algebraically closed and $X = \operatorname{Spec} k$. Let $\mathcal{G} = F'_{\psi'}\mathcal{H}$ be the inverse Fourier transform and $S \subset E^\vee$ be the support of $\mathcal{G}$. A closed point $a \in E$ defines a linear form $\langle a, \cdot\rangle : E^\vee \to \boldsymbol{A}^1$. The conditions (1) and (2) are equivalent to the following conditions respectively.

(1′) For every closed point $a \in E$, the image of S by the map $\langle a, \cdot\rangle : E^\vee \to \boldsymbol{A}^1$ is finite.

(2′) For every closed point $a \in E$, there exists an isomorphism $\mathcal{G} \otimes \langle a, \cdot\rangle^*\mathcal{L}_\psi \to \mathcal{G}$.

The condition (1′) implies (2′) since the restriction of $\langle a, \cdot\rangle^*\mathcal{L}_\psi$ on S is constant. We show that (2′) implies (1′). Let $U \subset E^\vee$ be a normal integral locally closed subscheme supported in S such that the restriction $\mathcal{G}|_U$ is locally constant. Let $\pi : V \to U$ be a connected finite étale covering such that $\pi^*\mathcal{G}|_U$ is constant. Then, by the condition (2′), $\pi^*\langle ca, \cdot\rangle^*\mathcal{L}_\psi$ is constant on V for every $a \in E$ and $c \in k$. Namely, the Artin–Schreier coverings $T^p - T = ct$ of $\boldsymbol{A}^1 = \operatorname{Spec} k[t]$ for all $c \in k$ are trivialized by the pull-back by the map $\langle a, \cdot\rangle \circ \pi : V \to \boldsymbol{A}^1$. If this map was dominant, the function field $k(V)$ would contain infinitely many linearly disjoint extensions of $k(t)$. Therefore, the image of $\langle a, \cdot\rangle \circ \pi : V \to \boldsymbol{A}^1$ collapses to a point. Hence the condition (2′) implies (1′). □

For a vector bundle E over X let $+ : E \times_X E \to E$ denote the sum. Its dual is the diagonal map $\delta : E^\vee \to E^\vee \times_X E^\vee$.

Proposition 2.6. *Let X be a scheme over k and $E \to X$ be a vector bundle. Let $\mathcal{H}$ be an additive constructible sheaf on E and $\mathcal{K}$ be a constructible sheaf on E. Let $\mathcal{H}|_0$ denote the restriction on the 0-section $X \subset E$ and let $e \in \Gamma(X, \mathcal{H}|_0)$ be a section. Let $u : \mathcal{H} \boxtimes \mathcal{K} \to +^*\mathcal{K}$ be a map such that the composition*

$$u|_{0\times E} \circ (e \otimes 1_{\mathcal{K}}) : \mathcal{K} \to \mathcal{H}|_0 \otimes \mathcal{K} \to \mathcal{K} \tag{2.5}$$

is the identity of $\mathcal{K}$. Then $\mathcal{K}$ is additive and the support $S_{\mathcal{M}} \subset E^\vee$ of $\mathcal{M} = F_\psi\mathcal{K}$ is a subset of the support $S_{\mathcal{G}} \subset E^\vee$ of $\mathcal{G} = F_\psi\mathcal{H}$.

Proof. We regard e as a global section $e \in \Gamma(E^\vee, \mathcal{G}) = \Gamma(X, \mathcal{H}|_0)$. By (2.3), the map u induces $\mathcal{G} \boxtimes \mathcal{M} \to \delta_*\mathcal{M}$ on the Fourier transform and hence a bilinear map $v : \mathcal{G} \otimes \mathcal{M} \to \mathcal{M}$ by adjunction. We show that the composition

$$v \circ (e \otimes 1_{\mathcal{M}}) : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M} \to \mathcal{M} \tag{2.6}$$

is the identity of $\mathcal{M}$. Let $\tilde{e} : \Lambda_X(-d)[-2d] \to \mathcal{H}$ be the cup product of $e : \Lambda_X \to \mathcal{H}|_0$ with the map $\Lambda_X(-d)[-2d] \to \Lambda_E$ defined by the cycle class of the 0-section $X \subset E$. We consider the map

$$u \circ (\tilde{e} \boxtimes 1_{\mathcal{K}}) : \Lambda_X(-d)[-2d] \boxtimes \mathcal{K} \to \mathcal{H} \boxtimes \mathcal{K} \to +^*\mathcal{K}. \tag{2.7}$$

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By the assumption that the composition of (2.5) is the identity, the induced map $+_*(\Lambda_X(-d)[-2d] \boxtimes \mathcal{K}) = \mathcal{K}(-d)[-2d] \to +_* +^* \mathcal{K} = \mathcal{K}(-d)[-2d]$ is the identity map. Therefore, the Fourier transform

$$F_\psi u \circ (e \boxtimes 1_{\mathcal{M}}) : \Lambda_E \boxtimes \mathcal{M} \to \mathcal{G} \boxtimes \mathcal{M} \to \delta_* \mathcal{M}$$

of (2.7) induces the identity in (2.6).

Since the composition in (2.6) is the identity of $\mathcal{M}$, the support $S_{\mathcal{M}}$ is a subset of the support of $e \in \Gamma(E^\vee, \mathcal{G})$. Hence we have $S_{\mathcal{M}} \subset S_{\mathcal{G}}$ and $\mathcal{K}$ is additive. □

Lemma 2.7. *Let X be a normal scheme over k and $E \to X$ be a vector bundle. Let $\mathcal{H}$ be a constructible sheaf on E satisfying the following condition: for every point $x \in X$, the restriction $\mathcal{H}|_{E_x}$ is locally constant and there exists a dense open subscheme $U \subset X$ such that, if $j : E_U = E \times_X U \to E$ denotes the open immersion, the pull-back $\mathcal{H}_U = j^*\mathcal{H}$ is an additive locally constant sheaf and that the canonical map $\mathcal{H} \to j_*j^*\mathcal{H}$ is injective.*

Then, the sheaf $\mathcal{H}$ is additive and we have $S_{\mathcal{H}} \subset \overline{S_{\mathcal{H}_U}} \subset E^\vee$.

Proof. Let $S \subset E^\vee$ be the support of $F'_{\psi'}\mathcal{H}$. It suffices to show that, for each $x \in X \setminus U$, the fibre S_x is a finite set and that we have $S_x \subset \overline{S_{\mathcal{H}_U}}$. Let $f : X' \to X$ be the normalization of the blowing-up of X at the closure $\overline{\{x\}}$ and $j' : U' = f^{-1}(U) \to X'$ be the open immersion. Let $\mathcal{H}'$ be the pull-back of $\mathcal{H}$ on $E' = E \times_X X'$ and $S' \subset E'^\vee$ be the support of $F'_{\psi'}\mathcal{H}'$. Then we have $S' = f^{-1}(S)$ and $S = f(S')$ where $f : E'^\vee \to E^\vee$ also denotes the induced map by abuse of notation. Since the base change map $f^*j_*j^*\mathcal{H} \to j'_*j'^*\mathcal{H}'$ is injective by Lemma 2.9 below, the canonical map $\mathcal{H}' \to j'_*j'^*\mathcal{H}'$ is also injective. Hence, it suffices to show the assertion for the generic point of the exceptional divisor and we may assume $\mathcal{O}_{X,x}$ is a discrete valuation ring.

We may assume X is integral. Let η be the generic point of X and $K = \kappa(\eta)$ be the fraction field of X. By replacing X by the normalization in a finite extension of K, we may assume that the fibre $S_\eta \subset E^\vee_\eta$ consists of finitely many K-rational points. Then, we may assume $\mathcal{H}_U = \bigoplus_{f \in S_\eta} \mathcal{L}_f \otimes \mathcal{F}_f$ is the direct sum of the tensor product of the rank one sheaves $\mathcal{L}_f$ defined by the Artin–Schreier equations $T^p - T = f$ for linear forms $f \in S_\eta$ on E_U with a constant sheaf $\mathcal{F}_f$. Let $S_{\eta,x} \subset S_\eta$ denote the subset consisting of the linear forms regular at x and, for $f \in S_{\eta,x}$, let $\bar{f}$ denotes the reduction at x. Then, the following lemma and the purity imply that the restriction $j_*j^*\mathcal{H}|_{E_x}$ on the fibre is $\bigoplus_{f \in S_{\eta,x}} \mathcal{L}_{\bar{f}} \otimes \mathcal{F}_f$. Since $\mathcal{H}|_{E_x}$ is a smooth subsheaf of $j_*j^*\mathcal{H}|_{E_x} = \bigoplus_{f \in S_{\eta,x}} \mathcal{L}_{\bar{f}} \otimes \mathcal{F}_f$, it is reduced to the following lemma.

Lemma 2.8. *Let K be a discrete valuation field of characteristic $p > 0$ and we consider the valuation v_L of $L = K(t_1, \dots, t_n)$ defined by the prime ideal $\mathfrak{m}_K \cdot \mathcal{O}_K[t_1, \dots, t_n]$ of the polynomial ring. Then, for a linear form $f \in Kt_1 + \dots + Kt_n \subset L$, the Artin–Schreier extension of L defined by $T^p - T = f$ is unramified with respect to v_L if and only if $f \in \mathcal{O}_K t_1 + \dots + \mathcal{O}_K t_n$.*

Proof. It suffices to show that the Artin–Schreier extension is ramified assuming $v_L(f) = -m < 0$. If $p \nmid m$, it is a totally ramified extension. If $p|m$, the residue field extension is the purely inseparable extension generated by the pth root of the non-zero linear form $\overline{\pi^{m/p} f}$. □

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Lemma 2.9. *Let $f : X \to Y$ be a morphism of normal schemes and $V \subset Y$ be a dense open subscheme such that $U = X \times_Y V \subset X$ is a dense open subscheme. Let $j : V \to Y$ and $j' : U \to X$ be open immersions and $f' : U \to V$ be the restriction of f. Then, for a locally constant sheaf on V, the base change map $f^* j_* \mathcal{F} \to j'_* f'^* \mathcal{F}$ is an injection.*

Proof. Exercise. □

We introduce a generalization of vector bundles.

Definition 2.10. Let X be a scheme and let $\mathcal{L}$ and $\mathcal{E}$ be an invertible $\mathcal{O}_X$-module and a locally free $\mathcal{O}_X$-module of finite rank, respectively, and let $n \geqslant 1$ be an integer. We call the vector bundloid of degree n associated to $(\mathcal{E}, \mathcal{L})$ the affine X-scheme

$$E = \boldsymbol{V}_n(\mathcal{E}, \mathcal{L})$$

defined by the quasi-coherent $\mathcal{O}_X$-algebra $\bigoplus_{l \geqslant 0} S^{nl}\mathcal{E} \otimes \mathcal{L}^{\otimes l}$. We call $E^\vee = \boldsymbol{V}_n(\mathcal{E}^\vee, \mathcal{L}^\vee)$ the dual of E.

The grading defines a natural action of the multiplicative group $\boldsymbol{G}_m$ on $\boldsymbol{V}_n(\mathcal{E}, \mathcal{L})$. For $n = 1$, we have $\boldsymbol{V}_1(\mathcal{E}, \mathcal{L}) = \boldsymbol{V}(\mathcal{E} \otimes \mathcal{L})$. For $m = nr$, the inclusion $\bigoplus_{l \geqslant 0} S^{nrl}\mathcal{E} \otimes \mathcal{L}^{\otimes rl} \subset \bigoplus_{l \geqslant 0} S^{nl}\mathcal{E} \otimes \mathcal{L}^{\otimes l}$ defines a finite surjection

$$\pi_{mn} : \boldsymbol{V}_n(\mathcal{E}, \mathcal{L}) \to \boldsymbol{V}_m(\mathcal{E}, \mathcal{L}^{\otimes r}).$$

It induces an isomorphism $\boldsymbol{V}_n(\mathcal{E}, \mathcal{L})/\mu_r \to \boldsymbol{V}_m(\mathcal{E}, \mathcal{L}^{\otimes r})$ with respect to the action restricted to the group $\mu_r \subset \boldsymbol{G}_m$ of rth roots of unity. If X is a scheme over $\mathbb{F}_p$ and if r is a power of p, the map $\pi_{mn} : \boldsymbol{V}_n(\mathcal{E}, \mathcal{L}) \to \boldsymbol{V}_m(\mathcal{E}, \mathcal{L}^{\otimes r})$ induces an isomorphism on the étale site. If $\mathcal{L} \to \mathcal{O}_X$ is an isomorphism, the map π_{n1} defines a finite surjection $\boldsymbol{V}(\mathcal{E}) = \boldsymbol{V}_1(\mathcal{E}, \mathcal{O}_X) \to \boldsymbol{V}_n(\mathcal{E}, \mathcal{O}_X) \to \boldsymbol{V}_n(\mathcal{E}, \mathcal{L})$. If $\mathcal{E} = \mathcal{O}_X$, we have $\boldsymbol{V}_n(\mathcal{O}_X, \mathcal{L}) = \boldsymbol{V}(\mathcal{L})$.

We call the section $X \to E = \boldsymbol{V}_n(\mathcal{E}, \mathcal{L})$ defined by the augmentation $\bigoplus_{l \geqslant 0} S^{nl}\mathcal{E} \otimes \mathcal{L}^{\otimes l} \to \mathcal{O}_X$ the 0-section of E. We identify X with a closed subscheme of E by the 0-section. On the complement $E^0 = E \setminus X$ of the 0-section, we have a natural map

$$\varphi : E^0 \to \boldsymbol{P}(\mathcal{E}) = \mathcal{P}roj(S^\bullet \mathcal{E})$$

since $\boldsymbol{P}(\mathcal{E})$ is canonically identified with $\mathcal{P}roj(\bigoplus_{l \geqslant 0} S^{nl}\mathcal{E} \otimes \mathcal{L}^{\otimes l})$. It induces an isomorphism $E^0/\boldsymbol{G}_m \to \boldsymbol{P}(\mathcal{E})$. The finite map $\pi_{mn} : \boldsymbol{V}_n(\mathcal{E}, \mathcal{L}) \to \boldsymbol{V}_m(\mathcal{E}, \mathcal{L}^{\otimes r})$ is compatible with the map $\varphi : E^0 \to \boldsymbol{P}(\mathcal{E})$.

Lemma 2.11. *Let $\mathcal{O}(n)$ be the tautological sheaf on $\boldsymbol{P}(\mathcal{E})$. Then, there exists a canonical isomorphism $\varphi^* \mathcal{O}(n) \to \mathcal{L}^\vee$ on E^0.*

Proof. The invertible sheaf $\mathcal{O}(n)$ on $\boldsymbol{P}(\mathcal{E})$ is the pull-back of $\mathcal{O}(1)$ on $\boldsymbol{P}(S^n\mathcal{E})$ by the Veronese embedding $\boldsymbol{P}(\mathcal{E}) \to \boldsymbol{P}(S^n\mathcal{E})$. We put $E = \boldsymbol{V}_n(\mathcal{E}, \mathcal{L})$ and let $p : E \to X$ denote the projection. Then, we have a tautological map $p^*(S^n\mathcal{E} \otimes \mathcal{L}) \to \mathcal{O}_E$. On E^0, this is a surjection and defines a surjection $p^* S^n\mathcal{E} \to p^*\mathcal{L}^\vee$. Since the composition $E^0 \to \boldsymbol{P}(\mathcal{E}) \to \boldsymbol{P}(S^n\mathcal{E})$ is defined by the surjection $p^* S^n\mathcal{E} \to p^*\mathcal{L}^\vee$, the assertion follows. □

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Lemma 2.12. *Let $E = \boldsymbol{V}_n(\mathcal{E},\mathcal{L}) \to X$ be a vector bundloid on a scheme X over k. Let $\mathcal{M}$ be an invertible $\mathcal{O}_X$-module and $\mathcal{L} \to \mathcal{M}^{\otimes n}$ be an isomorphism and let $\pi : \tilde{E} = \boldsymbol{V}(\mathcal{E} \otimes \mathcal{M}) = \boldsymbol{V}_1(\mathcal{E},\mathcal{M}) \to E = \boldsymbol{V}_n(\mathcal{E},\mathcal{L})$ be the induced map. Let $g : E^0 = E \setminus X \to E$ and $\tilde{g} : \tilde{E}^0 = \tilde{E} \setminus X \to \tilde{E}$ be the open immersions of the complements of the 0-section and $\pi^0 : \tilde{E}^0 \to E^0$ be the restriction of $\pi : \tilde{E} \to E$. We consider the following condition on a constructible sheaf $\mathcal{H}$ on E.*

(P) *The canonical map $\mathcal{H} \to g_* g^* \mathcal{H}$ is an isomorphism and the sheaf $\tilde{g}_* \pi^{0*} g^* \mathcal{H}$ on $\tilde{E}$ is additive.*

(1) *Let $\mathcal{M}'$ be another invertible $\mathcal{O}_X$-module and $\mathcal{L} \to \mathcal{M}'^{\otimes n}$ be an isomorphism. We define $\pi' : \tilde{E}' \to E$ etc. as above. Then the condition (P) for $\mathcal{H}$ with respect to $\pi : \tilde{E} \to E$ is equivalent to that for $\pi' : \tilde{E}' \to E$.*

Assume $\mathcal{H}$ satisfies the equivalent conditions and put $\tilde{\mathcal{H}} = \tilde{g}_* \pi^{0*} g^* \mathcal{H}$ on $\tilde{E}$ and $\tilde{\mathcal{H}}' = \tilde{g}'_* \pi'^{0*} g'^* \mathcal{H}$ on $\tilde{E}'$. Then $S_{\tilde{\mathcal{H}}} \subset \tilde{E}^\vee$ and $S_{\tilde{\mathcal{H}}'} \subset \tilde{E}'^\vee$ have the same images in $E^\vee$.

(2) *Let n' be the prime-to-p part of n and assume that k contains a primitive n'th root of 1. We consider the natural action of $G = \mu_{n'}$ on $\tilde{E}$ over E. Then, the condition (P) for $\mathcal{H}$ is equivalent to the following condition.*

(P′) *There exist an additive constructible sheaf $\tilde{\mathcal{H}}$ on $\tilde{E}$ with an action of G and an isomorphism $\mathcal{H} \to (\pi_* \tilde{\mathcal{H}})^G$.*

Proof. (1) The assertion is étale local on X. Let $n = n'n''$ be the decomposition into the prime-to-p part and the p-primary part. Replacing X by the covering defined by the equation $T^{n''} - u$ for a unit u does not change the étale topology. Hence, we may assume there exists an isomorphism $\mathcal{M} \to \mathcal{M}'$ compatible with $\mathcal{L} \to \mathcal{M}^{\otimes n}$ and $\mathcal{L} \to \mathcal{M}'^{\otimes n}$. Then the assertion is clear.

(2) On the restriction on E^0, the canonical map $g^* \mathcal{H} \to (\pi^0_* \pi^{0*} g^* \mathcal{H})^G$ to the G-fixed part is an isomorphism. Hence, it induces an isomorphism

$$g_* g^* \mathcal{H} \to (g_* \pi^0_* \pi^{0*} g^* \mathcal{H})^G \to (\pi_* \tilde{g}_* \pi^{0*} g^* \mathcal{H})^G. \tag{2.8}$$

(P) $\Rightarrow$ (P′). We put $\tilde{\mathcal{H}} = \tilde{g}_* \pi^{0*} g^* \mathcal{H}$. Then, if the canonical map $\mathcal{H} \to g_* g^* \mathcal{H}$ is an isomorphism, we obtain an isomorphism $\mathcal{H} \to (\pi_* \tilde{\mathcal{H}})^G$ by the isomorphism (2.8).

(P′) $\Rightarrow$ (P). Let $\mathcal{H} \to (\pi_* \tilde{\mathcal{H}})^G$ be an isomorphism. Then, it induces an isomorphism $\pi^{0*} g^* \mathcal{H} \to \tilde{g}^* \tilde{\mathcal{H}}$ compatible with the G-action. By Proposition 2.4, it induces an isomorphism $\tilde{g}_* \pi^{0*} g^* \mathcal{H} \to \tilde{\mathcal{H}}$ and $\tilde{g}_* \pi^{0*} g^* \mathcal{H}$ is additive. By the isomorphism (2.8), the isomorphism $\mathcal{H} \to (\pi_* \tilde{\mathcal{H}})^G$ implies that the canonical map $\mathcal{H} \to g_* g^* \mathcal{H}$ is an isomorphism. □

We generalize the notion of additive sheaves on vector bundloids.

Definition 2.13. Let $E = \boldsymbol{V}_n(\mathcal{E}, \mathcal{L}) \to X$ be a vector bundloid of degree n over a scheme X over k. We say a constructible sheaf $\mathcal{H}$ on E is potentially additive if it satisfies the condition (P) in Lemma 2.12 Zariski locally on X.

Let $\mathcal{H}$ be a potentially additive constructible sheaf on E. Then, we define a constructible subset $S_{\mathcal{H}}$ of the dual $E^\vee$ as the image of the dual support $S_{\tilde{\mathcal{H}}}$ of the additive sheaf $\tilde{\mathcal{H}} = \tilde{g}_* \pi^{0*} g^* \mathcal{H}$ in the notation of Lemma 2.12 Zariski locally on X and call $S_{\mathcal{H}}$ the dual support of $\mathcal{H}$. We say a potentially additive constructible sheaf $\mathcal{H}$ on E is non-degenerate if the intersection of the closure of $S_{\mathcal{H}} \subset E^\vee$ with the 0-section is empty.

Lemma 2.14. *Let $p : E = \boldsymbol{V}_n(\mathcal{E}, \mathcal{L}) \to X$ be a vector bundloid of degree n over a scheme X over k. Let $\mathcal{H}$ be a potentially additive constructible $\mathbb{Q}_\ell$-sheaf on E. If it is non-degenerate, then we have $Rp_! \mathcal{H} = Rp_* \mathcal{H} = 0$.*

Proof. Since the assertion is Zariski local on X, we may use the notation in Lemma 2.12. Let $\tilde{p} : \tilde{E} \to X$ denote the structural map. Since $\mathcal{H}$ is assumed non-degenerate, we have $R\tilde{p}_! \tilde{\mathcal{H}} = R\tilde{p}_* \tilde{\mathcal{H}} = 0$ by Lemma 2.3 (2). Therefore, $Rp_! \mathcal{H} = (R\tilde{p}_! \tilde{\mathcal{H}})^G$ and $Rp_* \mathcal{H} = (R\tilde{p}_* \tilde{\mathcal{H}})^G$ are 0. □

2.2. Global basic construction

We study the basic construction in § 1.1 in a global setting. Let X be a smooth scheme over k, D be a divisor with simple normal crossings and $j : U = X \setminus D \to X$ be the open immersion of the complement. Let $p : P \to X$ be a smooth morphism of relative dimension d and $s : X \to P$ be a section. By the section s, we regard X as a closed subscheme of P.

Let $D_1, \ldots, D_m$ be the irreducible components of D. We consider an effective divisor $R = r_1 D_1 + \cdots + r_m D_m$ with rational coefficients $r_1, \ldots, r_m \geqslant 0$. For an integer $l \geqslant 0$, let $[lR]$ denote the integral part of lR and $\mathcal{I}_{[lR]} \subset \mathcal{O}_X$ be the ideal sheaf of the effective divisor $[lR]$. Let $\mathcal{I}_X \subset \mathcal{O}_P$ be the ideal sheaf of $X \subset P$ and $j_P : P_U = P \times_X U \to P$ be the open immersion. We define an affine P-scheme $q : P^{(R)} \to P$ by the quasi-coherent $\mathcal{O}_P$-algebra

$$\sum_{l \geqslant 0} p^* \mathcal{I}_{[lR]}^{-1} \cdot \mathcal{I}_X^l \subset j_{P*} \mathcal{O}_{P_U}. \tag{2.9}$$

Let $p^{(R)} : P^{(R)} \to X$ be the canonical map and $s^{(R)} : X \to P^{(R)}$ be the section induced by $s : X \to P$. We also regard $D \subset X$ as closed subschemes of $P^{(R)}$ by the section $s^{(R)}$.

Here is an alternative construction of $q : P^{(R)} \to P$. Let $n > 0$ be an integer such that $M = nR$ has integral coefficients. Let $\bar{q} : P^{[M/n]} \to P$ be the blow-up by the ideal $p^* \mathcal{I}_M + \mathcal{I}_X^n \subset \mathcal{O}_P$ and $P^{(M/n)} \subset P^{[M/n]}$ be the complement of the support of $\bar{q}^* (p^* \mathcal{I}_M + \mathcal{I}_X^n) / \bar{q}^* p^* \mathcal{I}_M$. The morphism $P^{(M/n)} \to P$ is affine and $P^{(M/n)}$ is defined by the quasi-coherent $\mathcal{O}_P$-subalgebra $\mathcal{O}_P[p^* \mathcal{I}_M^{-1} \cdot \mathcal{I}_X^n] \subset j_{P*} \mathcal{O}_{P_U}$. Then, similarly as Lemma 1.10 (1), $P^{(R)}$ is identified with the normalization of $P^{(M/n)}$.

We put $I^+ = \{i \mid 1 \leqslant i \leqslant m,\ r_i > 0\}$ and $D^+ = \sum_{i \in I^+} D_i$. We describe the structure of the inverse image $E^+ = P^{(R)} \times_X D^+$ in terms of vector bundloids introduced in the previous subsection.

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Lemma 2.15. *Let $D_1, \dots, D_m$ be the irreducible components of D and put $I^+ = \{i \mid 1 \leqslant i \leqslant m,\ r_i > 0\}$.*

(1) *Let $I \subset I^+$ be a non-empty subset and $n_I \geqslant 1$ be the minimum integer n such that the coefficients in nR of D_i are integers for all $i \in I$. Let D_I be the intersection $\bigcap_{i\in I} D_i$ and put $D_I^\circ = D_I \setminus \bigcup_{i\in I^+\setminus I}(D_i \cap D_I)$.*
Then, there exists a canonical isomorphism

$$E_I^\circ = (P^{(R)} \times_X D_I^\circ)_{\mathrm{red}} \to \boldsymbol{V}_{n_I}(\mathcal{N}_{X/P}, \mathcal{O}(n_I R)) \times_X D_I^\circ$$

over D_I°. The restriction $D_I^\circ \to E_I^\circ$ of the section $s^{(R)} : X \to P^{(R)}$ corresponds to the 0-section of the right-hand side.

(2) *Let $R^* = X \times_P (P^{(R)} \setminus X)$ be the inverse image of $X = s(X) \subset P$ by the restriction of the canonical map $q : P^{(R)} \to P$ on the complement $P^{(R)} \setminus X$ of the section $s^{(R)}$. Then R^* is a divisor of $P^{(R)} \setminus X$ and satisfies $R^* = p^{(R)*}R$.*

(3) *Assume the coefficients of R are integers. Then, the map $p^{(R)} : P^{(R)} \to X$ is smooth. The inverse image $E^+ = P^{(R)} \times_X D^+$ of $D^+ = \sum_{i\in I^+} D_i$ is canonically isomorphic to the vector bundle $\boldsymbol{V}(\mathcal{N}_{X/P} \otimes \mathcal{O}(R)) \times_X D^+$.*

Proof. (1) We may assume $I = I^+$. Similarly as the definition of the lower horizontal arrow of (1.3), a surjection

$$\bigoplus_{l\geqslant 0,\, n_I \mid l} (\mathcal{O}(lR) \otimes S^l \mathcal{N}_{X/P}) \otimes_{\mathcal{O}_X} \mathcal{O}_{D_I^\circ} \to \mathcal{O}_{E_I^\circ}$$

is defined by using the definition (2.9) of $P^{(R)}$. In other words, we have a closed immersion $E_I^\circ \to \boldsymbol{V}_{n_I}(\mathcal{N}_{X/P}, \mathcal{O}(n_I R)) \times_X D_I^\circ$. We show this is an isomorphism. Since the question is étale local on P, we may assume $P = \boldsymbol{V}(\mathcal{E})$ is a vector bundle defined by a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank d and $s : X \to P$ is the 0-section. Then $P^{(R)}$ is the affine scheme over X defined by the $\mathcal{O}_X$-algebra $\bigoplus_{l\geqslant 0} S^l\mathcal{E} \otimes \mathcal{O}([lR])$. Since the image of the l-component in $\mathcal{O}_{E_I^\circ}$ is 0 unless $[lR] = lR$, the assertion follows.

(2) Let $n > 0$ be an integer such that $M = nR$ has integral coefficients. Since the question is local on P, we may assume the ideal $\mathcal{I}_X \subset \mathcal{O}_P$ is generated by d sections $e_1, \dots, e_d$ and $\mathcal{I}_{nR}$ has a basis l. Then, on the open subscheme of $P^{(R)}$ where $f_i = l^{-1}e_i^n$ is invertible, the pull-back of the ideal $\mathcal{I}_X = (e_1, \dots, e_d)$ is generated by e_i since $e_j = e_i \cdot l^{-1} e_j e_i^{n-1}/f_i$. Since the support of the closed subscheme of $P^{(R)}$ defined by the ideal $(f_1, \dots, f_d)$ is $s^{(R)}(X)$, the assertion follows.

(3) We show that the scheme $P^{(R)}$ is smooth over X. Since the question is étale local on P, we may assume $P = \boldsymbol{V}(\mathcal{E})$ is a vector bundle defined by a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank d and $s : X \to P$ is the 0-section as in the proof of (1). Then $P^{(R)}$ is the vector bundle $\boldsymbol{V}(\mathcal{E} \otimes \mathcal{O}(R))$ and the assertion follows.

Similarly as in the proof of (1), we obtain a closed immersion $E^+ = E \times_X D^+ \to \boldsymbol{V}(\mathcal{N}_{X/P} \otimes \mathcal{O}(R)) \times_X D^+$ and we see that this is an isomorphism. □

We have the following functoriality of the construction of $P^{(R)}$.

Lemma 2.16. *We consider a commutative diagram*

$$\begin{array}{ccccc} Y & \xrightarrow{t} & Q & \xrightarrow{q} & Y \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{s} & P & \xrightarrow{p} & X \end{array}$$

of smooth schemes over k. We assume that $s : X \to P$ and $t : Y \to Q$ are sections of smooth maps $p : P \to X$ and $q : Q \to Y$ respectively. Let D be a divisor of X with simple normal crossings. Assume that the divisor $D_Y = (D \times_X Y)_{\rm red}$ has simple normal crossings. Let $R = \sum_i r_i D_i \geqslant 0$ be an effective divisor with rational coefficients $r_i \geqslant 0$ and let $R_Y = f^*R$ be the pull-back.

(1) *There exists a unique map $g^{(R)} : Q^{(R_Y)} \to P^{(R)}$ lifting $g : Q \to P$.*

(2) *Suppose that the coefficients of R are integral. Let D^+ and D_Y^+ be the supports of R and of R_Y respectively. We identify $E^+ = P^{(R)} \times_X D^+$ with $\boldsymbol{V}(\mathcal{N}_{X/P} \otimes \mathcal{O}(R)) \times_X D^+$ and $E_Y^+ = Q^{(R_Y)} \times_Y D_Y^+$ with $\boldsymbol{V}(\mathcal{N}_{Y/Q} \otimes \mathcal{O}(R_Y)) \times_Y D_Y^+$ as in Lemma 2.15 (1). Then the restriction*

$$E_Y^+ = \boldsymbol{V}(\mathcal{N}_{Y/Q} \otimes \mathcal{O}(R_Y)) \times_Y D_Y^+ \to E^+ = \boldsymbol{V}(\mathcal{N}_{X/P} \otimes \mathcal{O}(R)) \times_X D^+$$

of $g^{(R)} : Q^{(R_Y)} \to P^{(R)}$ is the linear map of vector bundles induced by the canonical map $f^*\mathcal{N}_{X/P} \to \mathcal{N}_{Y/Q}$.

(3) *Suppose further that $f : Y \to X$ is the identity of X and $g : Q \to P$ is smooth. Then the induced map $g^{(R)} : Q^{(R_Y)} \to P^{(R)}$ is smooth.*

Proof. (1) We have $g^*\mathcal{I}_X \subset \mathcal{I}_Y$ since the left square is commutative. By the inequalities $g^*[lR] \leqslant [lR_Y] \leqslant lg^*R$, we have $g^*\mathcal{I}_{[lR]}^{-1} \subset \mathcal{I}_{[lR_Y]}^{-1}$. Hence we have $g^*(\mathcal{I}_{[lR]}^{-1} \cdot \mathcal{I}_X^l) \subset \mathcal{I}_{[lR_Y]}^{-1} \cdot \mathcal{I}_Y^l$ and the assertion follows from the definition of $Q^{(R_Y)}$.

(2) The restriction $E_Y^+ \to E^+$ is induced by the linear map $g^* : \mathcal{I}_R^{-1} \cdot \mathcal{I}_X \to \mathcal{I}_{R_Y}^{-1} \cdot \mathcal{I}_Y$ and the assertion follows.

(3) On the complements of the inverse images of D^+, the maps $P^{(R)} \to P$ and $Q^{(R)} \to Q$ are isomorphisms. Hence, the assumption that $f : Q \to P$ is smooth implies that the restriction on the complements of the inverse images of D^+ is smooth. Since the coefficients of R are assumed integral, the maps $P^{(R)} \to X$ and $Q^{(R)} \to X$ are smooth by Lemma 2.15 (1). Hence, it suffices to show that the induced map $E^{+\prime} = Q^{(R)} \times_X D^+ \to E^+ = P^{(R)} \times_X D^+$ is smooth by [**11**, Proposition (17.8.1)].

By (2), it is identified with the map $\boldsymbol{V}(\mathcal{N}_{X/Q} \otimes \mathcal{O}(R)) \times_X D^+ \to \boldsymbol{V}(\mathcal{N}_{X/P} \otimes \mathcal{O}(R)) \times_X D^+$ of vector bundles induced by the canonical map $f^*\mathcal{N}_{X/P} \to \mathcal{N}_{X/Q}$. Since $Q \to P$ is assumed smooth, the map $f^*\mathcal{N}_{X/P} \to \mathcal{N}_{X/Q}$ is a locally splitting injection and the map $\boldsymbol{V}(\mathcal{N}_{X/Q} \otimes \mathcal{O}(R)) \times_X D^+ \to \boldsymbol{V}(\mathcal{N}_{X/P} \otimes \mathcal{O}(R)) \times_X D^+$ is smooth. □

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Corollary 2.17. *Let P and Q be smooth schemes over X and $s : X \to P$ and $t : X \to Q$ be sections. Similarly as $P^{(R)}$ and $Q^{(R)}$, we define $(P \times_X Q)^{(R)}$ by the section $(s, t) : X \to P \times_X Q$.*

Assume the coefficients of R are integers. Then the maps $(P \times_X Q)^{(R)} \to P^{(R)}$ and $(P \times_X Q)^{(R)} \to Q^{(R)}$ induces an isomorphism

$$(P \times_X Q)^{(R)} \to P^{(R)} \times_X Q^{(R)}. \tag{2.10}$$

Proof. The ideal defining the closed subscheme $X \subset P \times_X Q$ is generated by the pullbacks of those defining $X \subset P$ and $X \subset Q$. Hence the map (2.10) is a closed immersion. Since the both schemes $(P \times_X Q)^{(R)}$ and $P^{(R)} \times_X Q^{(R)}$ are smooth of the same dimension over X, the closed immersion (2.10) is an open immersion. By Lemma 2.15 (3), it induces an isomorphism on the fibres over D^+. Hence the assertion follows. □

We establish some cohomological properties of $P^{(R)}$.

Proposition 2.18.

(1) *The cycle class defines an isomorphism*

$$\mathbb{Q}_\ell(d)[2d] \to Rp^{(R)!}\mathbb{Q}_\ell. \tag{2.11}$$

(2) *Define the cycle class $[X] \in H^{2d}_X(P^{(R)}, \mathbb{Q}_\ell(d))$ to be the inverse image of $1 \in H^0(X, \mathbb{Q}_\ell) = H^0_X(P^{(R)}, Rp^{(R)!}\mathbb{Q}_\ell)$ by the isomorphism (2.11). Then, for the pullback $s^{(R)*}[X] = (X, X)_{P^{(R)}} \in H^{2d}(X, \mathbb{Q}_\ell(d))$, we have*

$$\begin{aligned}(X, X)_{P^{(R)}} &= (X, X)_P - (c(\mathcal{N}_{X/P})^* \cap (1 + R)^{-1} \cap [R])_{\deg d} \\ &= (-1)^d (c_d(\mathcal{N}_{X/P}) + (c(\mathcal{N}_{X/P}) \cap (1 - R)^{-1} \cap [R])_{\deg d}). \end{aligned} \tag{2.12}$$

Proof. (1) Since the question is étale local, we may assume there exists a smooth map $X \to \boldsymbol{A}^m_k$ such that D is the inverse image of the union of the coordinate hyperplanes. Let $n_1, \ldots, n_m > 0$ be integers such that $n_1 r_1, \ldots, n_m r_m$ are integers and let $\pi : \tilde{X} = X \times_{\boldsymbol{A}^m_k} \boldsymbol{A}^m_k \to X$ be the base change by the map $\boldsymbol{A}^m_k \to \boldsymbol{A}^m_k$ defined by $t_i \mapsto t_i^{n_i}$. We put $\tilde{P} = P \times_X \tilde{X}$ and $\tilde{R} = \pi^* R$. We consider the commutative diagram

$$\begin{array}{ccc} P^{(R)} & \xleftarrow{\ \pi\ } & \tilde{P}^{(\tilde{R})} \\ {\scriptstyle p^{(R)}}\downarrow & & \downarrow{\scriptstyle \tilde{p}^{(\tilde{R})}} \\ X & \xleftarrow{\ \pi\ } & \tilde{X} \end{array} \tag{2.13}$$

Here and in the followings, let π also denote the base changes of π by abuse of notation. The map $\tilde{p}^{(\tilde{R})} : \tilde{P}^{(\tilde{R})} \to \tilde{X}$ is smooth by Lemma 2.15. Let $n'_1, \ldots, n'_m$ be the prime-to-p parts of $n_1, \ldots, n_m$ and we consider the action of $G = \mu_{n'_1} \times \cdots \times \mu_{n'_m}$ on $\boldsymbol{A}^m$ by the multiplication on the coordinates. We also consider the induced actions of G on $\tilde{X}$, $\tilde{P}^{(\tilde{R})}$, etc. The induced map $\tilde{P}^{(\tilde{R})}/G \to P^{(R)}$ defines an isomorphism on the étale sites.

We put $\mathcal{K}_{P^{(R)}} = Rp^{(R)!}\mathbb{Q}_\ell$ and $\mathcal{K}_{\tilde{P}^{(\tilde{R})}} = R(p^{(R)} \circ \pi)^! \mathbb{Q}_\ell$. The trace map $\pi_! \pi^* \mathcal{K}_{P^{(R)}} = \pi_* \pi^* \mathcal{K}_{P^{(R)}} \to \mathcal{K}_{P^{(R)}}$ [**8**, Théorème 6.2.3] defines its adjoints $\pi^* \mathcal{K}_{P^{(R)}} \to \pi^! \mathcal{K}_{P^{(R)}} = \mathcal{K}_{\tilde{P}^{(\tilde{R})}}$

and $\mathcal{K}_{P^{(R)}} \to \pi_*\mathcal{K}_{\tilde{P}(\tilde{R})}$. We also have the adjunction map $\pi_*\mathcal{K}_{\tilde{P}(\tilde{R})} = \pi_!\pi^!\mathcal{K}_{P^{(R)}} \to \mathcal{K}_{P^{(R)}}$. The composition $\mathcal{K}_{P^{(R)}} \to \pi_*\mathcal{K}_{\tilde{P}(\tilde{R})} \to \mathcal{K}_{P^{(R)}}$ is the multiplication by the degree $[\tilde{P}^{(\tilde{R})} : P^{(R)}]$ by [**8**, Théorème 6.2.3 (Var 4)]. Hence $\mathcal{K}_{P^{(R)}}$ is a direct summand of the G-fixed part $(\pi_*\mathcal{K}_{\tilde{P}(\tilde{R})})^G$.

We consider the commutative diagram

$$\begin{array}{ccc} \mathbb{Q}_\ell(d)[2d] & \longrightarrow & \mathcal{K}_{P^{(R)}} \\ \downarrow & & \downarrow \\ (\pi_*\mathbb{Q}_\ell(d)[2d])^G & \longrightarrow & (\pi_*\mathcal{K}_{\tilde{P}(\tilde{R})})^G \end{array}$$

where the horizontal arrows are defined by the cycle classes. Since $\tilde{P}^{(\tilde{R})}$ is smooth over k, the lower horizontal arrow is an isomorphism. Since the left vertical arrow is an isomorphism, $(\pi_*\mathcal{K}_{\tilde{P}(\tilde{R})})^G$ is a direct summand of $\mathcal{K}_{P^{(R)}}$. Thus the assertion is proved.

(2) First, we reduce it to the case where P is a vector bundle over X and $s : X \to P$ is the 0-section, by the deformation to the normal bundle. We put $\tilde{X} = X \times \boldsymbol{A}^1$ and $\tilde{D} = D \times \boldsymbol{A}^1$. Let $\bar{P}$ be the blow-up of $P \times \boldsymbol{A}^1$ at $X \times \{0\}$ and $\tilde{P} \subset \bar{P}$ be the complement of the proper transform of $P \times \{0\}$. Then, the map $\tilde{p} : \tilde{P} \to \tilde{X}$ is smooth. We consider the Cartesian diagram

$$\begin{array}{ccccc} V & \longrightarrow & \tilde{P} & \longleftarrow & P \times \mathbb{G}_m \\ \downarrow & & \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p\times\mathrm{id}} \\ X & \longrightarrow & \tilde{X} & \longleftarrow & X \times \mathbb{G}_m \\ \downarrow & & \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \boldsymbol{A}^1 & \longleftarrow & \mathbb{G}_m \end{array}$$

where $V = \boldsymbol{V}(\mathcal{N}_{X/P})$ denotes the normal bundle.

The section $s : X \to P$ induces a section $\tilde{s} : \tilde{X} \to \tilde{P}$. By applying the basic construction, we define $\tilde{p}^{(R)} : \tilde{P}^{(R)} \to \tilde{X}$ and $p_0^{(R)} : V^{(R)} \to X$ and their sections $\tilde{s}^{(R)} : \tilde{X} \to \tilde{P}^{(R)}$ and $s_0^{(R)} : X \to V^{(R)}$. Similarly as $s^{(R)*}[X] \in H^{2d}(X, \mathbb{Q}_\ell(d))$, the classes $\tilde{s}^{(R)*}[\tilde{X}] \in H^{2d}(\tilde{X}, \mathbb{Q}_\ell(d))$ and $s_0^{(R)*}[X] \in H^{2d}(X, \mathbb{Q}_\ell(d))$ are defined. The pull-backs $\sigma_0^*, \sigma_1^* : H^{2d}(\tilde{X}, \mathbb{Q}_\ell(d)) \to H^{2d}(X, \mathbb{Q}_\ell(d))$ by the 0- and 1-sections $\sigma_0, \sigma_1 : X \to \tilde{X}$ are isomorphisms and we have $\sigma_0^*(\tilde{s}^{(R)*}[\tilde{X}]) = s_0^{(R)*}[X]$ and $\sigma_1^*(\tilde{s}^{(R)*}[\tilde{X}]) = s^{(R)*}[X]$ respectively. Hence the assertion for (X, P) is reduced to that for (X, V). Thus we may assume P is a vector bundle over X and $s : X \to P$ is the 0-section.

Let $q : P^{(R)} \to P$ denote the canonical map. It suffices to show the equality

$$[s^{(R)}(X)] = q^*[s(X)] - p^{(R)*}((c(\mathcal{N}_{X/P})^* \cap (1+R)^{-1} \cap [R])_{\deg d})$$

in $H^{2d}_{q^{-1}(s(X))}(P^{(R)}, \mathbb{Q}_\ell(d))$. For a closed subscheme F of $P^{(R)}$, let F° denote the complement $F \setminus (F \cap D^+)$. Let R^* be the divisor of $P^{(R)} \setminus X$ defined in Lemma 2.15 (2). Then, we have $q^{-1}(s(X)) = s^{(R)}(X) \cup \bar{R}^*$ and $s^{(R)}(X) \cap \bar{R}^* = D^+$. Hence, by (1), the space $H^{2d}_{s^{(R)}(X)\cap\bar{R}^*}(P^{(R)}, \mathbb{Q}_\ell(d)) = H^{2d}_{D^+}(X, s^{(R)!}\mathbb{Q}_\ell(d))$ is isomorphic to $H^0_{D^+}(X, \mathbb{Q}_\ell) = 0$. By the exact sequence

$$0 = H^{2d}_{s^{(R)}(X)\cap \bar R^*}(P^{(R)}, \mathbb{Q}_\ell(d)) \to H^{2d}_{q^{-1}(s(X))}(P^{(R)}, \mathbb{Q}_\ell(d)) \to H^{2d}_{X^\circ}(P^{(R)\circ}, \mathbb{Q}_\ell(d)) \oplus H^{2d}_{R^*}(P^{(R)\circ}, \mathbb{Q}_\ell(d)),$$

the restriction map

$$H^{2d}_{q^{-1}(s(X))}(P^{(R)}, \mathbb{Q}_\ell(d)) \to H^{2d}_{X^\circ}(P^{(R)\circ}, \mathbb{Q}_\ell(d)) \oplus H^{2d}_{R^*}(P^{(R)\circ}, \mathbb{Q}_\ell(d))$$

is an injection. Therefore, it suffices to show that the components of the restriction of $q^*[s(X)]$ are $[s^{(R)}(X)^\circ]$ and $p^{(R)*}(c(\mathcal{N}_{X/P})^* \cap (1+R)^{-1} \cap [R])_{\deg d}$ respectively.

This is clear for the first component $[s^{(R)}(X)^\circ]$. By the excess intersection formula [**10**, Theorem 6.3], the second component is $(c(p^{(R)*}\mathcal{N}_{X/P})^* \cap c(\mathcal{N}_{R^*/P^{(R)\circ}})^{*-1} \cap [R^*])_{\deg d}$. Hence, the assertion follows by Lemma 2.15 (2). □

2.3. Ramification along a divisor

We globalize the constructions in §§ 1.1 and 1.2 and the computations in § 1.4. They generalize those in [**3**, § 4] and allows denominators and higher rank. The construction of $(X \times X)^{(r)}$ and $\mathcal{H}$ in § 1.4 is the special case of that in this subsection.

Let X be a smooth scheme of dimension d over k and D be a divisor with simple normal crossings. Let $D_1, \ldots, D_m$ be the irreducible components of D. We put $U = X \setminus D$ and let $j : U \to X$ denote the open immersion.

We define the log blow up $(X \times X)' \to X \times X$ to be the blow-up at $D_1 \times D_1, D_2 \times D_2, \ldots, D_m \times D_m$. Namely the blow-up by the product $\mathcal{I}_{D_1\times D_1} \cdot \mathcal{I}_{D_2\times D_2} \cdots \mathcal{I}_{D_m\times D_m} \subset \mathcal{O}_{X\times X}$ of ideal sheaves. We define the log product $(X \times X)^\sim \subset (X \times X)'$ to be the complement of the proper transforms of $D\times X$ and $X\times D$. The diagonal map $X \to X\times X$ induces a closed immersion $\tilde\delta : X \to (X \times X)^\sim \subset (X \times X)'$ called the log diagonal map. The scheme $(X \times X)^\sim$ is affine over $X \times X$ and is defined by the quasi-coherent $\mathcal{O}_{X\times X}$-algebra

$$\mathcal{O}_{X\times X}[\mathrm{pr}_1^* \mathcal{I}_{D_i}^{-1} \cdot \mathrm{pr}_2^* \mathcal{I}_{D_i}, \mathrm{pr}_1^* \mathcal{I}_{D_i} \cdot \mathrm{pr}_2^* \mathcal{I}_{D_i}^{-1};\ i = 1, \ldots, m] \subset j^\times_* \mathcal{O}_{U\times U},$$

where $j^\times : U\times U \to X\times X$ is the open immersion. The projections $p_1, p_2 : (X\times X)^\sim \to X$ are smooth. The conormal sheaf $\mathcal{N}_{X/(X\times X)^\sim}$ is canonically identified with the locally free $\mathcal{O}_X$-module $\Omega^1_X(\log D)$ of rank d.

Let $R = r_1D_1 + \cdots + r_mD_m$ be an effective divisor with rational coefficients $r_1, \ldots, r_m \geqslant 0$. We apply the construction of § 2.1 to the smooth map $p_2 : P = (X \times X)^\sim \to X$ and its section $\tilde\delta : X \to (X \times X)^\sim$. Then, we obtain $P^{(R)} = (X \times X)^{(R)} \to X$ and its section $\delta^{(R)} : X \to (X \times X)^{(R)}$. Thus, we have constructed a diagram

$$\begin{array}{ccccc} X \times X & \longleftarrow & (X \times X)' & \longleftarrow & (X \times X)^{[R]} \\ & & \uparrow & & \uparrow \\ & & (X \times X)^\sim & \longleftarrow & (X \times X)^{(R)} \end{array}$$

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where the vertical arrows are open immersions. For $R = 0$, we have $(X \times X)^{(R)} = (X \times X)^{\sim}$.

We consider the Cartesian diagram

$$\begin{array}{ccc} U \times U & \xrightarrow{j^{(R)}} & (X \times X)^{(R)} \\ {\scriptstyle \delta_U}\uparrow & & \uparrow{\scriptstyle \delta^{(R)}} \\ U & \xrightarrow{j} & X \end{array}$$

where the horizontal arrows are open immersions and the vertical arrows are the diagonal immersions.

Definition 2.19. Let $\mathcal{F}$ be a smooth sheaf on $U = X \setminus D$. We define a smooth sheaf $\mathcal{H}$ on $U \times U$ by $\mathcal{H} = \mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, \mathrm{pr}_1^* \mathcal{F})$. Let $R = \sum_i r_i D_i \geqslant 0$ be an effective divisor with rational coefficients and we consider the open immersion $j^{(R)} : U \times U \to (X \times X)^{(R)}$. We identify $\delta_U^* \mathcal{H} = \mathcal{E}nd(\mathcal{F})$ and regard the identity $\mathrm{id}_{\mathcal{F}} \in \mathrm{End}_U(\mathcal{F})$ as a section of $\Gamma(U, \mathcal{E}nd(\mathcal{F})) = \Gamma(X, j_* \mathcal{E}nd(\mathcal{F})) = \Gamma(X, j_* \delta_U^* \mathcal{H})$.

We say that the log ramification of $\mathcal{F}$ along D is bounded by $R+$ if the identity $\mathrm{id}_{\mathcal{F}} \in \mathrm{End}_U(\mathcal{F}) = \Gamma(X, j_* \delta_U^* \mathcal{H})$ is in the image of the base change map

$$\Gamma(X, \delta^{(R)*} j_*^{(R)} \mathcal{H}) \to \Gamma(X, j_* \delta_U^* \mathcal{H}) = \mathrm{End}_U(\mathcal{F}). \tag{2.14}$$

We compare Definition 2.19 with Definition 1.28.

Lemma 2.20. *Let $\mathcal{F}$ be a smooth sheaf on $U = X \setminus D$ and let $R = \sum_i r_i D_i \geqslant 0$ be an effective divisor with rational coefficients. We consider the smooth sheaf $\mathcal{H} = \mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, \mathrm{pr}_1^* \mathcal{F})$ on $U \times U \subset (X \times X)^{(R)}$.*

(1) *We consider the following conditions.*

(a) *The log ramification of $\mathcal{F}$ along D is bounded by $R+$.*

(b) *For every irreducible component D_i of D, the log ramification of $\mathcal{F}$ along D is bounded by r_i+ at the generic point ξ_i of D_i.*

(c) *There exists an open subscheme $X' \subset X$ such that $X' \supset U$, that $D' = X' \cap D$ is dense in D and that the base change map*

$$\delta^{(R)*} j_*^{(R)} \mathcal{H} \to j_* \delta_U^* \mathcal{H}$$

is an isomorphism on D'.

(d) *The base change map*

$$\delta^{(R)*} j_*^{(R)} \mathcal{H} \to j_* \delta_U^* \mathcal{H}$$

is an isomorphism.

Then, we have implications (d) $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

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(2) *Let D_i be a component of D satisfying $r_i > 0$ and let $E_i = (X \times X)^{(R)} \times_X D_i$ be the inverse image. Then, the vanishing*

$$\mathcal{F}_{\bar{\eta}_i}^{G_{K,\log}^{r_i+}} = 0$$

implies $j_*\mathcal{H}|_{E_i} = 0$.

Proof. (1) The implication $(a) \Rightarrow (b)$ follows from Corollary 1.31 $(c) \Rightarrow (a)$. The implication $(b) \Rightarrow (c)$ follows from Corollary 1.31 $(a) \Rightarrow (b)$. The implication $(d) \Rightarrow (a)$ is obvious.

(2) Let ξ_i be the generic point of D_i. It suffices to show $j_*\mathcal{H}|_{E_{i,\xi_i}} = 0$. Hence, it follows from Corollary 1.31. □

The author does not know a counterexample for the implication $(a) \Rightarrow (d)$. The conditions (a)–(d) are equivalent, if the rank of $\mathcal{F}$ is 1.

In the tamely ramified case, we have the following equivalence for $R = 0$.

Corollary 2.21. *The following conditions are equivalent.*

(1) *The log ramification of $\mathcal{F}$ along D is bounded by $0+$.*

(2) *$\mathcal{F}$ is tamely ramified along D.*

(3) *The base change map*

$$\tilde{\delta}^*\tilde{j}_*\mathcal{H} \to j_*\delta_U^*\mathcal{H}$$

is an isomorphism on D.

Proof. By Lemma 2.20 (1) $(a) \Rightarrow (b)$, the condition (1) implies (2).

Assume $\mathcal{F}$ is tamely ramified along D. Then $\mathcal{H}$ on $U \times U$ is tamely ramified along $(X \times X)^\sim \setminus (U \times U)$. Hence, by Abhyankar's lemma, étale locally on $(X \times X)^\sim$, it is isomorphic to the pull-back of a sheaf on U with respect to the second projection $(X \times X)^\sim \to X$. Since the projection is smooth, the condition (3) is satisfied.

It is clear that (3) implies (1). □

We have the following stability under the pull-back.

Lemma 2.22. *Let Y be a smooth scheme over k and $f : Y \to X$ be a morphism over k. Assume that the reduced inverse image $D_Y = (D \times_X Y)_{\mathrm{red}}$ is a divisor with simple normal crossings and let R_Y be the pull-back f^*R.*

Let $\mathcal{F}$ be a smooth sheaf on $U = X \setminus D$ and $\mathcal{F}_V$ be the pull-back to $V = U \times_X Y = Y \setminus D_Y$. If the log ramification of $\mathcal{F}$ is bounded by $R+$, then the log ramification of $\mathcal{F}_V$ is bounded by R_Y+.

Proof. We show that the map $f \times f : Y \times Y \to X \times X$ is lifted to $(f \times f)^{(R)} : (Y \times Y)^{(R_Y)} \to (X \times X)^{(R)}$. For each irreducible component D_i of D, the pull-backs of $\mathrm{pr}_1 D_i$ and $\mathrm{pr}_2 D_i$ are equal on the log product $(Y \times Y)^\sim$. Hence, the map $f \times f$:

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$Y \times Y \to X \times X$ is uniquely lifted to $(f \times f)^{\sim} : (Y \times Y)^{\sim} \to (X \times X)^{\sim}$. This is uniquely lifted to $(Y \times Y)^{(R_Y)} \to (X \times X)^{(R)}$ by Lemma 2.16 (1).

Let $j^{(R)} : U \times U \to (X \times X)^{(R)}$ and $j^{(R_Y)} : V \times V \to (Y \times Y)^{(R_Y)}$ be the open immersions and $f_U : V \to U$ be the restriction of $f : Y \to X$. We put $\mathcal{F}_V = f_U^* \mathcal{F}$, $\mathcal{H} = \mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, \mathrm{pr}_1^* \mathcal{F})$ and $\mathcal{H}' = \mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}_V, \mathrm{pr}_1^* \mathcal{F}_V)$. Then, the base change map

$$(f \times f)^{(R)^*} j_*^{(R)} \to j_*^{(R_Y)} (f_U \times f_U)^*$$

defines a commutative diagram

$$\begin{array}{ccc} \Gamma(X, \delta^{(R)*} j_*^{(R)} \mathcal{H}) & \longrightarrow & \mathrm{End}_U(\mathcal{F}) \\ \downarrow & & \downarrow \\ \Gamma(Y, \delta^{(R_Y)*} j_*^{(R_Y)} \mathcal{H}') & \longrightarrow & \mathrm{End}_V(\mathcal{F}_V) \end{array}$$

By the assumption that the log ramification of $\mathcal{F}$ is bounded by $R+$, the identity of $\mathcal{F}$ is in the image of the upper horizontal arrow. Hence, the identity of $\mathcal{F}_V$ is in the image of the lower horizontal arrow and the log ramification of $\mathcal{F}_V$ is bounded by R_Y+. □

We consider the restrictions of $\mathcal{F}$ on smooth curves in X and compare them.

Proposition 2.23. *Let $\mathcal{F}$ be a smooth sheaf on $U = X \setminus D$ such that the log ramification of $\mathcal{F}$ is bounded by $R+$. Let C and C' be smooth curves in X and x be a closed point in $C \cap C' \cap D$. We assume that $C \cap U$ and $C' \cap U$ are not empty and let $\mathcal{F}_C$ and $\mathcal{F}_{C'}$ denote the restrictions of $\mathcal{F}$ on $C \cap U$ and $C' \cap U$ respectively. Assume that the following conditions are satisfied:*

(1) *For every irreducible component D_i of D, we have $(C, D_i)_x = (C', D_i)_x$.*

(2) $\mathrm{length}_x\, \mathcal{O}_{C \cap C', x} \geqslant (C, R + D)_x$.

Then, étale locally at x, there exist an isomorphism $f : C \to C'$ and an isomorphism $f^* \mathcal{F}|_{C'} \to \mathcal{F}|_C$.

The author thanks the referee for pointing out a similarity with [**5**, Théorème 4.3.1].

Proof. It suffices to consider the case $C \neq C'$. Since the assertion is étale local, we may assume $C \cap D = C' \cap D = C \cap C' = \{x\}$ set theoretically and the residue field of x is k. We put $n = \mathrm{length}_x\, \mathcal{O}_{C \cap C', x}$. Take an isomorphism $k[t]/(t^n) \to \mathcal{O}_{C \cap C', x}$ and lift it to étale morphisms $C \to \boldsymbol{A}_k^1$ and $C' \to \boldsymbol{A}_k^1$. Since the assertion is étale local, we may assume there exists an isomorphism $f : C \to C'$ inducing the identity on $C \cap C'$.

We consider the graph of f

$$g = (1, f) : C \to C \times C' \subset X \times X.$$

The intersection with the diagonal defines an isomorphism $C \times_{X \times X} X \to (C \times C') \times_{X \times X} X = C \cap C'$ since $f : C \to C'$ induces the identity on $C \cap C'$. By the assumption (1) and by the universal property of the log blow-up, the immersion $g : C \to X \times X$

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is uniquely lifted to an immersion $\tilde{g} : C \to (X \times X)^{\sim}$ to the log product. We put $C \cap^{\log} C' = C \times_{(X\times X)^{\sim}} X \subset C \cap C'$. We show

$$\operatorname{length}_x \mathcal{O}_{C\cap^{\log} C',x} = \operatorname{length}_x \mathcal{O}_{C\cap C',x} - (C, D)_x. \tag{2.15}$$

Let $\mathcal{I}_X \subset \mathcal{O}_{X\times X}$ and $\mathcal{J}_X \subset \mathcal{O}_{(X\times X)^{\sim}}$ be the ideal sheaves of $X \subset X \times X$ and of $X \subset (X \times X)'$ respectively and let $\mathcal{I}_E \subset \mathcal{O}_{(X\times X)^{\sim}}$ be the ideal sheaves of $E = p^*D$. Then, we have $\mathcal{I}_X \mathcal{O}_{(X\times X)^{\sim}} = \mathcal{J}_X \cdot \mathcal{I}_E$. By pulling it back by $\tilde{g}$, we obtain the equality (2.15).

By the assumption (2) and by (2.15), we have $\operatorname{length}_x \mathcal{O}_{C\times_{(X\times X)^{\sim}} X,x} \geqslant (C, R)_x$. In other words, we have inclusions $\mathcal{J}_X^l \mathcal{O}_C \subset \mathcal{I}_{[lR]} \mathcal{O}_C$ for every integer $l \geqslant 0$. By the definition of $(X \times X)^{(R)}$ (2.9), the immersion $\tilde{g} : C \to (X \times X)^{\sim}$ is further lifted to $h : C \to (X \times X)^{(R)}$. We consider the Cartesian diagram

$$\begin{array}{ccc} C \cap U & \xrightarrow{h_U} & U \times U \\ {\scriptstyle j_C}\downarrow & & \downarrow{\scriptstyle j^{(R)}} \\ C & \xrightarrow{h} & (X \times X)^{(R)} \end{array}$$

where the vertical arrows are open immersions. We also consider the base change maps

$$\begin{aligned} h^* j_*^{(R)} \mathcal{H} &\to j_{C*} h_U^* \mathcal{H} = j_{C*}\, \mathcal{H}om(f^*\mathcal{F}_{C'}, \mathcal{F}_C), \\ \delta^{(R)*} j_*^{(R)} \mathcal{H} &\to j_* \delta_U^* \mathcal{H} = j_*\, \mathcal{E}nd(\mathcal{F}). \end{aligned} \tag{2.16}$$

Let K denote the fraction field of the henselization $\mathcal{O}^h_{C,x}$ and let $\bar{\eta}$ denote the geometric point of C defined by an algebraic closure $\bar{K}$ of K. Let G_K be the absolute Galois group $\operatorname{Gal}(\bar{K}/K)$. By the assumption that the log ramification is bounded by $R+$, we have a unique element e in $\Gamma(x, (\delta^{(R)*} j_*^{(R)} \mathcal{H})|_x) = \Gamma(x, (h^* j_*^{(R)} \mathcal{H})|_x)$ whose image in $\Gamma(x, (j_*\, \mathcal{E}nd(\mathcal{F}))|_x)$ is the identity of $\mathcal{F}_{\bar{\eta}}$. The image of e in $\Gamma(x, (j_{C*}\, \mathcal{H}om(f^*\mathcal{F}_{C'}, \mathcal{F}_C))|_x)$ defines a G_K-homomorphism $\varphi : \mathcal{F}_{f(\bar{\eta})} \to \mathcal{F}_{\bar{\eta}}$. Switching the two factors, we obtain a G_K-homomorphism $\psi : \mathcal{F}_{\bar{\eta}} \to \mathcal{F}_{f(\bar{\eta})}$. Since the construction is compatible with the composition, the maps φ and ψ are the inverse of each other. □

We study the higher direct image $R^q j_*^{(R)} \mathcal{H}$. We put $I^+ = \{i \mid 1 \leqslant i \leqslant m,\ r_i > 0\}$ and $D^+ = \bigcup_{i\in I^+} D_i$. First, we consider the case where the coefficients of $R = \sum_i r_i D_i$ are integers. If the coefficients of R are integers, the inverse image $E^+ = (X \times X)^{(R)} \times_X D^+$ is identified with the vector bundle $\boldsymbol{V}(\Omega^1_X(\log D)(R)) \times_X D^+$ over D^+ by Lemma 2.15 (3). We prepare a global analogue of Lemma 1.33. In the following lemma and proposition, we consider the fibre product

$$(X \times X) \underset{p_2 \searrow X \swarrow p_1}{\times} (X \times X)$$

with respect to the second and the first projections $X \times X \to X$ and identify it naturally with the triple product $X \times X \times X$. To ease the notation, we drop $p_2 \searrow\!\swarrow p_1$. Similarly, we will also consider the product $(X \times X)^{(R)} \times_X (X \times X)^{(R)}$, etc., with respect to the second and the first projections.

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Lemma 2.24. *Assume the coefficients of R are integers.*

(1) *There exists a smooth map $\mu : (X\times X)^{(R)} \times_X (X\times X)^{(R)} \to (X\times X)^{(R)}$ that makes the diagram*

$$\begin{array}{ccc} (X\times X)^{(R)} \times_X (X\times X)^{(R)} & \xrightarrow{\mu} & (X\times X)^{(R)} \\ \downarrow & & \downarrow \\ (X\times X)\times_X (X\times X) = X\times X\times X & \xrightarrow{\mathrm{pr}_{13}} & X\times X \end{array}$$

commutative.

(2) *Let D^+ be the support of R and we identify $E^+ = (X\times X)^{(R)} \times_X D^+$ with the vector bundle $\boldsymbol{V}(\Omega^1_X(\log D)\otimes \mathcal{O}(R))\times_X D^+$ as above. The restriction of μ defines the addition $E^+ \times_{D^+} E^+ \to E^+$ of the vector bundle $E^+ = \boldsymbol{V}(\Omega^1_X(\log D)(R))\times_X D^+$.*

Proof. (1) Let $P = (X\times X)^\sim \times_X (X\times X)^\sim$ be the fibre product with respect to the second and the first projections $(X\times X)^\sim \to X$. We define $P^{(R)} \to P$ by applying the construction in § 2.2 to the smooth map $P = (X\times X)^\sim \times_X (X\times X)^\sim \to X$ and the diagonal section $X \to P$. The projections $P \to (X\times X)^\sim$ induce an isomorphism $P^{(R)} \to (X\times X)^{(R)} \times_X (X\times X)^{(R)}$ by Corollary 2.17.

On $P = (X\times X)^\sim \times_X (X\times X)^\sim$, the pull-backs of $\mathrm{pr}_1^* D_i$ and $\mathrm{pr}_3^* D_i$ are equal for each component D_i of D. Hence the map $\mathrm{pr}_{13} : (X\times X)\times_X (X\times X) \to X\times X$ is lifted to $P = (X\times X)^\sim \times_X (X\times X)^\sim \to (X\times X)^\sim$. This is uniquely lifted to a smooth map $P^{(R)} \to (X\times X)^{(R)}$ by Lemma 2.16.

(2) The restriction $E^+ \times_{D^+} E^+ \to E^+$ is a linear map of vector bundles by Lemma 2.16 (2). Hence, it suffices to show that the compositions with the injections $i_1, i_2 : E^+ \to E^+ \times_{D^+} E^+$ of the two factors are the identity of E^+. We consider the map $\iota_1 : (X\times X)^{(R)} \to (X\times X)^{(R)} \times_X (X\times X)^{(R)}$ defined by the identity of $(X\times X)^{(R)}$ and $\delta^{(R)}\circ \mathrm{pr}_2$. Then, its restriction $E^+ \to E^+ \times_{D^+} E^+$ is the injection of the first factor. Since the composition $\mu\circ\iota_1$ is the identity, the composition $\mu\circ i_1 : E^+ \to E^+ \times_{D^+} E^+ \to E^+$ is the identity. Similarly, by considering the map $\iota_2 : (X\times X)^{(R)} \to (X\times X)^{(R)} \times_X (X\times X)^{(R)}$ defined by $\delta^{(R)}\circ\mathrm{pr}_1$ and the identity of $(X\times X)^{(R)}$, we see that $\mu\circ i_2$ is the identity. Hence the assertion follows. □

Proposition 2.25. *Let X be a smooth scheme over k and $\mathcal{F}$ be a smooth sheaf on the complement $U = X\setminus D$ of a divisor with simple normal crossings. Let $R = \sum_i r_i D_i \geqslant 0$ be an effective divisor with integral coefficients $r_i \geqslant 0$. Assume that the log ramification of $\mathcal{F}$ is bounded by $R+$. We put $D^+ = \bigcup_{i:r_i>0} D_i$ and $E^+ = \boldsymbol{V}(\Omega^1_X(\log D)\otimes\mathcal{O}(R))\times_X D^+$. Let $j^{(R)} : U\times U \to (X\times X)^{(R)}$ be the open immersion.*

(1) *For every integer $q \geqslant 0$, the restriction of $R^q j_*^{(R)}\mathcal{H}$ on E^+ is additive.*

(2) *Let $S^q \subset E^{+\vee} = \boldsymbol{V}(\Omega^1_X(\log D)^\vee \otimes \mathcal{O}(-R))\times_X D^+$ be the dual support of $R^q j_*^{(R)}\mathcal{H}|_{E^+}$. Then, we have $S^q \subset S^0$.*

(3) *Let D_i be an irreducible component of D^+ and ξ_i be the generic point. Then, the intersection $S^0 \cap E_i^\vee$ with $E_i^\vee = E^{+\vee} \times_{D^+} D_i$ is a subset of the closure $\overline{S^0_{\xi_i}}$ of the generic fibre.*

Proof. Since $\mu : (X \times X)^{(R)} \times_X (X \times X)^{(R)} \to (X \times X)^{(R)}$ is smooth, the base change map $\mu^* Rj_*^{(R)}\mathcal{H} \to Rj_{3*}\,\mathrm{pr}_{13}^*\,\mathcal{H}$ is an isomorphism, where $j_3 : U \times U \times U \to (X \times X)^{(R)} \times_X (X \times X)^{(R)}$ denotes the open immersion. Hence, the composition

$$\mathcal{H} \boxtimes \mathcal{H} = \mathcal{H}om(\mathrm{pr}_2^*\,\mathcal{F}, \mathrm{pr}_1^*\,\mathcal{F}) \otimes \mathcal{H}om(\mathrm{pr}_3^*\,\mathcal{F}, \mathrm{pr}_2^*\,\mathcal{F}) \to \mathcal{H}om(\mathrm{pr}_3^*\,\mathcal{F}, \mathrm{pr}_1^*\,\mathcal{F}) = \mathrm{pr}_{13}^*\,\mathcal{H}$$

induces

$$Rj_*^{(R)}\mathcal{H} \boxtimes Rj_*^{(R)}\mathcal{H} \to Rj_{3*}\,\mathrm{pr}_{13}^*\,\mathcal{H} = \mu^* Rj_*^{(R)}\mathcal{H}. \tag{2.17}$$

Let $\bar{x}$ be an arbitrary geometric point of D^+. We show that the restriction of $R^q j_*^{(R)}\mathcal{H}$ on the fibre $E_{\bar{x}}^+$ satisfies the condition (2) in Proposition 2.5. By the assumption that the log ramification of $\mathcal{F}$ is bounded by $R+$, we have a unique section $e \in \Gamma(X, \delta^{(R)*} j_*^{(R)}\mathcal{H})$ lifting the identity $\mathrm{id}_{\mathcal{F}} \in \Gamma(X, j_*\delta_U^*\mathcal{H})$. Take an étale neighbourhood $V \to (X \times X)^{(R)}$ of $\bar{x}$ and a section $\tilde{e} \in \Gamma(V, j_*^{(R)}\mathcal{H})$ whose stalk in $(j_*^{(R)}\mathcal{H})_{\bar{x}} = (\delta^{(R)*} j_*^{(R)}\mathcal{H})_{\bar{x}}$ is the stalk of e above.

Since e is a lifting of the identity, the pairing (2.17) with the restriction of $\tilde{e}$ is an isomorphism $\mathrm{pr}_1^*\,\mathcal{H} \to \mathrm{pr}_{13}^*\,\mathcal{H}$ on $(U \times U) \times_X ((U \times U) \times_{(X \times X)^{(R)}} V)$. It is uniquely extend to an isomorphism

$$\mathrm{pr}_1^*\, Rj_*^{(R)}\mathcal{H} = Rj_*\,\mathrm{pr}_1^*\,\mathcal{H} \to \mu^* Rj_*^{(R)}\mathcal{H} = Rj_*\,\mathrm{pr}_{13}^*\,\mathcal{H} \tag{2.18}$$

on $(X \times X)^{(R)} \times_X V$. It is equal to the map defined as the pairing (2.17) with $\tilde{e}$.

For a closed point $a \in E_{\bar{x}}$ in the image of $V \times_X \bar{x} \to (X \times X)^{(R)} \times_X \bar{x} = E_{\bar{x}}$, the isomorphism (2.18) defines an isomorphism $Rj_*^{(R)}\mathcal{H}|_{E_{\bar{x}}} \to (+a)^*(Rj_*^{(R)}\mathcal{H}|_{E_{\bar{x}}})$ on the restriction to $E_{\bar{x}}$. Since $E_{\bar{x}}$ is generated by the image of $V \times_X \bar{x}$, there is an isomorphism $Rj_*^{(R)}\mathcal{H}|_{E_{\bar{x}}} \to (+a)^*(Rj_*^{(R)}\mathcal{H}|_{E_{\bar{x}}})$ for every closed point $a \in E_{\bar{x}}$. Thus the sheaf $R^q j_*^{(R)}\mathcal{H}|_{E_{\bar{x}}}$ satisfies the condition (2) in Proposition 2.5 and hence is additive for every $q \geqslant 0$.

(2) It suffices to apply Proposition 2.6 to $j_*^{(R)}\mathcal{H} \boxtimes R^q j_*^{(R)}\mathcal{H} \to \mu^* R^q j_*^{(R)}\mathcal{H}$.

(3) It follows immediately from Lemma 2.7. □

We consider the general case. Namely, we drop the assumption that the coefficients of R are integers. For a non-empty subset $I \subset I^+$, we put $D_I = \bigcap_{i \in I} D_i$ and $D_I^\circ = D_I \backslash \bigcup_{i \in I^+ \backslash I}(D_i \cap D_I)$. Recall that n_i denotes the denominator of $r_i = m_i/n_i$ and n_I is the least common multiple of n_i for $i \in I$. The inverse image $E_I^\circ = ((X \times X)^{(R)} \times_X D_I^\circ)_{\mathrm{red}}$ is identified with $\boldsymbol{V}_{n_I}(\Omega_X^1(\log D), \mathcal{O}(n_I R)) \times_X D_I^\circ$ by Lemma 2.15 (1).

Proposition 2.26. *Let the notation be as in Proposition 2.25 except that we do not assume the coefficients of R are integers. Assume that the log ramification of $\mathcal{F}$ is bounded by $R+$. Let $I \subset I^+ = \{i \mid 1 \leqslant i \leqslant m,\ r_i > 0\}$ be a non-empty subset and $E_I^\circ = \boldsymbol{V}_{n_I}(\Omega_X^1(\log D), \mathcal{O}(n_I R)) \times_X D_I^\circ$ be the reduced inverse image.*

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(1) *For every integer $q \geqslant 0$, the restriction of $R^q j_*^{(R)}\mathcal{H}$ on E_I° is potentially additive.*

(2) *Let $S_I^q \subset E_I^{\circ\vee} = \boldsymbol{V}_{n_I}(\Omega^1_X(\log D)^\vee, \mathcal{O}(-n_I R)) \times_X D_I^\circ$ be the dual support of $(R^q j_*^{(R)}\mathcal{H})|_{E_I^\circ}$. Then, we have $S_I^q \subset S_I^0$.*

(3) *For $i \in I$, let ξ_i be the generic point of the irreducible component D_i and $F_i = \kappa(\xi_i)$ be the function field of D_i. We consider the canonical map*

$$E_i^\vee \times_{D_i} D_I^\circ = \boldsymbol{V}_{n_i}(\Omega^1_X(\log D)^\vee, \mathcal{O}(-n_i R)) \times_X D_I^\circ \\ \to E_I^{\circ\vee} = \boldsymbol{V}_{n_I}(\Omega^1_X(\log D)^\vee, \mathcal{O}(-n_I R)) \times_X D_I^\circ.$$

Then, S_I^0 is a subset of the image of the intersection $\overline{S^0_{i,\xi_i}} \times_{D_i} D_I^\circ \subset E_i^\vee \times_{D_i} D_I^\circ$ of the closure of the generic fibre.

Proof. (1) By replacing X by $X \setminus \bigcup_{i \in I^+ \setminus I} D_i$, we may assume $I^+ = I$, $E_I = E_I^\circ$ and $n = n_I$. Since the assertion is Zariski local, we may take a smooth map $X \to A = \boldsymbol{A}^m = \operatorname{Spec} k[T_1, \dots, T_m]$ such that D is the inverse image of the union of coordinate hyperplanes. We put $A' = \boldsymbol{A}^m \times \boldsymbol{G}_m^m = \operatorname{Spec} k[T_1, \dots, T_m, U_1^{\pm 1}, \dots, U_m^{\pm 1}]$ and $\tilde{A} = \operatorname{Spec} k[S_1, \dots, S_m, U_1^{\pm 1}, \dots, U_m^{\pm 1}]$ and define a map $\tilde{A} \to A'$ by $T_i \mapsto U_i S_i^{n_i}$. We consider the base change $X \leftarrow X' \leftarrow \tilde{X}$ of $A \leftarrow A' \leftarrow \tilde{A}$.

We define schemes $(X \times X')^\sim$ and $(X \times \tilde{X})^\sim$ by the Cartesian diagram

$$\begin{array}{ccccc} (X \times X)^\sim & \longleftarrow & (X \times X')^\sim & \longleftarrow & (X \times \tilde{X})^\sim \\ {\scriptstyle p_2}\downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & X' & \longleftarrow & \tilde{X} \end{array}$$

and consider the sections $X' \to (X \times X')^\sim$ and $\tilde{X} \to (X \times \tilde{X})^\sim$ induced by the log diagonal $X \to (X \times X)^\sim$. The map $\tilde{X} \to X$ induces $(\tilde{X} \times \tilde{X})^\sim \to (X \times \tilde{X})^\sim$. By applying the construction in § 2.2 to the pull-backs R' and $\tilde{R}$ of R to X' and to $\tilde{X}$, we obtain the commutative diagram

$$\begin{array}{ccccccc} (X \times X)^{(R)} & \xleftarrow{f} & (X \times X')^{(R')} & \xleftarrow{g} & (X \times \tilde{X})^{(\tilde{R})} & \xleftarrow{h} & (\tilde{X} \times \tilde{X})^{(\tilde{R})} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \tilde{p}} & & \\ X & \longleftarrow & X' & \longleftarrow & \tilde{X} & & \end{array} \tag{2.19}$$

Since $X' \to X$ is smooth, the left square of (2.19) is Cartesian and the horizontal arrow $f : (X \times X')^{(R')} \to (X \times X)^{(R)}$ is smooth. Since $\tilde{R}$ has integral coefficients, the right vertical arrow $\tilde{p} : (X \times \tilde{X})^{(\tilde{R})} \to \tilde{X}$ is smooth. We show that the map $h : (\tilde{X} \times \tilde{X})^{(\tilde{R})} \to (X \times \tilde{X})^{(\tilde{R})}$ is smooth. By Lemma 2.16 (3), it suffices to show that the map $(\tilde{X} \times \tilde{X})^\sim \to (X \times \tilde{X})^\sim$ is smooth. Thus, it is reduced to showing that the map

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$(\tilde{A} \times \tilde{A})^{\sim} \to (A \times \tilde{A})^{\sim}$ is smooth. Since the map

$$\begin{aligned}
(\tilde{A} \times \tilde{A})^{\sim} &= \operatorname{Spec} k[S_i, U_i^{\pm 1}, S_i', U_i'^{\pm 1}, V_i^{\pm 1}(i = 1, \dots, m)]/(S_i' - V_i S_i(i = 1, \dots, m)) \\
&= \operatorname{Spec} k[S_i, U_i^{\pm 1}, U_i'^{\pm 1}, V_i^{\pm 1}(i = 1, \dots, m)] \\
&\downarrow \\
(A \times \tilde{A})^{\sim} &= \operatorname{Spec} k[T_i, S_i', U_i'^{\pm 1}, W_i^{\pm 1}(i = 1, \dots, m)]/(U_i' S_i'^{n_i} - W_i T_i(i = 1, \dots, m)) \\
&= \operatorname{Spec} k[S_i', U_i'^{\pm 1}, W_i^{\pm 1}(i = 1, \dots, m)]
\end{aligned}$$

is defined by $W_i \mapsto V_i^{n_i} U_i'/U_i$, it is smooth.

We put $U' = U \times_X X'$ and $\tilde{U} = U \times_X \tilde{X}$ and consider the diagram

$$\begin{array}{ccccccc}
(X \times X)^{(R)} & \xleftarrow{f} & (X \times X')^{(R')} & \xleftarrow{g} & (X \times \tilde{X})^{(\tilde{R})} & \xleftarrow{h} & (\tilde{X} \times \tilde{X})^{(\tilde{R})} \\
{\scriptstyle j^{(R)}}\uparrow & & {\scriptstyle j^{(R)\prime}}\uparrow & & {\scriptstyle j^{(R)\sim}}\uparrow & & {\scriptstyle \tilde{j}^{(R)}}\uparrow \\
U \times U & \longleftarrow & U \times U' & \longleftarrow & U \times \tilde{U} & \longleftarrow & \tilde{U} \times \tilde{U}
\end{array}$$

where the vertical arrows are open immersions. We consider the pull-backs $\mathcal{H}'$, $\mathcal{H}^{\sim}$ and $\tilde{\mathcal{H}}$ of $\mathcal{H}$ respectively on $U \times U'$, $U \times \tilde{U}$ and on $\tilde{U} \times \tilde{U}$.

Since $\tilde{R}$ is integral, the restriction of $R^q \tilde{j}_*^{(R)} \tilde{\mathcal{H}}$ on $\tilde{E}^+$ is additive by Proposition 2.25 for every $q \geqslant 0$. Since h is smooth, the base change map $h^* R^q j_*^{(R)\sim} \mathcal{H}^{\sim} \to R^q \tilde{j}_*^{(R)} \tilde{\mathcal{H}}$ is an isomorphism. The conormal sheaves $\mathcal{N}_{\tilde{X}/(X \times \tilde{X})^{\sim}}$ and $\mathcal{N}_{\tilde{X}/(\tilde{X} \times \tilde{X})^{\sim}}$ are canonically identified with $\Omega^1_X(\log D) \otimes \mathcal{O}_{\tilde{X}}$ and with $\Omega^1_{\tilde{X}}(\log \tilde{D})$ respectively. Since the map $\Omega^1_X(\log D) \otimes \mathcal{O}_{\tilde{X}} \to \Omega^1_{\tilde{X}}(\log \tilde{D})$ is a locally splitting injection, the restriction of $R^q j_*^{(R)\sim} \mathcal{H}^{\sim}$ is additive by Lemma 2.3.

To study the restriction of $R^q j_*^{(R)\prime} \mathcal{H}'$, we introduce some notations. For $i \in I$, let $D_i' \subset X'$ be the inverse image of D_i and $\tilde{D}_i$ be the divisor defined by S_i. We put $D_I' = \bigcap_{i \in I} D_i'$ and $\tilde{D}_I = \bigcap_{i \in I} \tilde{D}_i$. The natural map $\tilde{D}_I \to D_I'$ is an isomorphism. Let $E_I' \subset (X \times X')^{(R')}$ and $\tilde{E}_I \subset (X \times \tilde{X})^{(\tilde{R})}$ be the reduced inverse images of D_I' and of $\tilde{D}_I$. Recall $n = n_I$ is the least common multiple of the denominators n_i for $i \in I$. By Lemma 2.15 (1), we have a canonical isomorphism $E_I' \to \boldsymbol{V}_n(\Omega^1_X(\log D), \mathcal{O}(-nR'))_{D_I'}$ and $\tilde{E}_I \to \boldsymbol{V}_1(\Omega^1_X(\log D), \mathcal{O}(-\tilde{R}))_{\tilde{D}_I}$. The natural map $\pi_I : \tilde{E}_I \to E_I'$ is then identified with the canonical map $\pi_n : \boldsymbol{V}_1(\Omega^1_X(\log D), \mathcal{O}(-\tilde{R}))_{D_I'} \to \boldsymbol{V}_n(\Omega^1_X(\log D), \mathcal{O}(-nR'))_{D_I'}$.

Let n_i' be the prime-to-p part of n_i and n' be the prime-to-p part of $n = n_I$. We consider the natural action of $G = \prod_{i \in I} \mu_{n_i'}$ on $\tilde{X}$ over X'. Since the map $\tilde{D}_I \to D_I'$ is an isomorphism, the action of G on $\tilde{D}_I$ is trivial. The action of G on $\tilde{E}_I$ factors through the product map $G \to \mu_{n'}$ and the action of $\mu_{n'}$ on $\tilde{E}_I$ is by the multiplication.

We show that the restriction of $R^q j_*^{(R)\prime} \mathcal{H}'$ on E_I' is potentially additive. The canonical map $R^q j_*^{(R)\prime} \mathcal{H}' \to g_*(R^q j_*^{(R)\sim} \mathcal{H}^{\sim})^G$ is an isomorphism. Let G_1 be the kernel of $G \to \mu_{m'}$. Then since the restriction of $R^q j_*^{(R)\sim} \mathcal{H}^{\sim}$ on $\tilde{E}_I$ is additive, its G_1-fixed part $(R^q j_*^{(R)\sim} \mathcal{H}^{\sim})|_{\tilde{E}_I}^{G_1}$ is also additive. Hence by Lemma 2.12, the $\mu_{n'}$-fixed part

$$\pi_{n*}((R^q j_*^{(R)^{\sim}} \mathcal{H}^{\sim})|_{\tilde{E}_I}^{G_1})^{\mu_{n'}} = g_*(R^q j_*^{(R)\sim} \mathcal{H}^{\sim})^G|_{E_I'}$$

is potentially additive. Thus the restriction $(R^q j_*^{(R)\prime} \mathcal{H}')|_{E_I'}$ is potentially additive.

Since the map $X' \to X$ is smooth, the base change map $f^* R^q j_*^{(R)} \mathcal{H} \to R^q j_*^{(R)\prime} \mathcal{H}'$ is an isomorphism. Since the map $X' \to X$ admits a section, the restriction of $R^q j_*^{(R)} \mathcal{H}$ on E_I is also potentially additive.

(2) Similarly as in the proof of (1), we may assume $I = I^+$ and $D_I = D_I^\circ$. We show the inclusion $S_I^q \subset S_I^0$. Let $S_I^{q\sim} \subset E_I^{\sim\vee}$ be the dual support of the additive sheaf $(R^q j_*^{(R)^\sim} \mathcal{H}^\sim)^{G_1}$. We apply Proposition 2.6 to the map

$$(j_*^{(R)^\sim} \mathcal{H}^\sim)^{G_1} \boxtimes (R^q j_*^{(R)^\sim} \mathcal{H}^\sim)^{G_1} \to \mu^* (R^q j_*^{(R)^\sim} \mathcal{H}^\sim)^{G_1}$$

and the pull-back to $\Gamma(\tilde{X}, \tilde{\delta}^{(R)*} j_*^{(R)^\sim} \mathcal{H}^\sim)$ of the section $e \in \Gamma(X, \delta^{(R)*} j_*^{(R)} \mathcal{H})$ lifting the identity of $\mathcal{F}$. Then, we obtain the inclusion $S_I^{q\sim} \subset S_I^{0\sim}$.

Since the dual support $S_I^{q\prime} \subset E_I^{\prime\vee}$ of the potentially additive sheaf $R^q j_*^{(R)\prime} \mathcal{H}'$ is the image of $S_I^{q\sim}$ by the canonical map

$$\begin{aligned} E_I^{\sim *} &= \boldsymbol{V}_1(\Omega_X^1(\log D)^\vee, \mathcal{O}_{\tilde{X}}(-\tilde{R})) \times_{\tilde{X}} \tilde{D}_I \\ \to E_I^{\prime\vee} &= \boldsymbol{V}_{n_I}(\Omega_X^1(\log D)^\vee \otimes \mathcal{O}_{X'}, \mathcal{O}(-n_I R')) \times_{X'} \tilde{D}_I, \end{aligned}$$

we obtain the inclusion $S_I^{q\prime} \subset S_I^{0\prime}$. Thus, we deduce $S_I^q \subset S_I^0$ by pull-back.

(3) We have the inclusion $S_I^{0\sim} \subset \overline{S_{i,\xi_i}^{0\sim}} \times_{\tilde{D}_i} \tilde{D}_I^\circ$ by Lemma 2.7. Hence the assertion follows as in the proof of (2). □

For an integer $n > 0$ such that nR has integral coefficients, we define the dual support

$$S_{\mathcal{F}}^{(n\cdot R)} \subset E_n^\vee = \boldsymbol{V}_n(\Omega_X^1(\log D)^\vee, \mathcal{O}(-nR))_{D^+}$$

as a constructible subset as follows. Let I be a non-empty subset of $I^+ = \{i \mid 1 \leqslant i \leqslant m,\ r_i > 0\}$. Then, the restriction $\mathcal{H}_I^\circ = j_*^{(R)} \mathcal{H}|_{E_I^\circ}$ on $E_I^\circ = \boldsymbol{V}_{n_I}(\Omega_X^1(\log D), \mathcal{O}(n_I R)) \times_X D_I^\circ$ is potentially additive by Proposition 2.25. Hence the dual support $S_{\mathcal{H}_I^\circ}$ is defined as a constructible subset of the dual $\boldsymbol{V}_{n_I}(\Omega_X^1(\log D)^\vee, \mathcal{O}(-n_I R)) \times_X D_I^\circ$. Since n is divisible by n_I, the canonical map

$$\begin{aligned} \pi_{nn_I} : \boldsymbol{V}_{n_I}(\Omega_X^1(\log D)^\vee, \mathcal{O}(-n_I R)) \times_X D_I^\circ \\ \to \boldsymbol{V}_n(\Omega_X^1(\log D)^\vee, \mathcal{O}(-nR)) \times_X D_I^\circ = E_n^\vee \times_X D_I^\circ \end{aligned}$$

is defined.

Definition 2.27. Let the notation be as above. We define the dual support $S_{\mathcal{F}}^{(n\cdot R)} \subset E_n^\vee$ with respect to R as the union

$$S_{\mathcal{F}}^{(n\cdot R)} = \bigcup_{I \subset I^+, I \neq \emptyset} \pi_{nn_I}(S_{\mathcal{H}_I^\circ}).$$

We say that the log ramification of $\mathcal{F}$ is non-degenerate with respect to R if the intersection of the closure of the dual support $S_{\mathcal{F}}^{(n\cdot R)}$ with the 0-section of $E_n^\vee$ is empty, for one and hence any n.

For $n|m$, we have $S_{\mathcal{F}}^{(m\cdot R)} = \pi_{mn}(S_{\mathcal{F}}^{(n\cdot R)})$.

Corollary 2.28. *Assume that the log ramification of $\mathcal{F}$ is bounded by $R+$.*

(1) *For an irreducible component D_i of D, let ξ_i be the generic point of D_i. Then, we have*

$$S_{\mathcal{F}}^{(n\cdot R)} \subset \bigcup_{i\in I^+} \overline{S_{\mathcal{F},\xi_i}^{(n\cdot R)}}.$$

(2) *Assume that the log ramification of $\mathcal{F}$ is non-degenerate with respect to R and that Λ is a finite extension of $\mathbb{Q}_\ell$. Then $Rp_*(R^q j_*\mathcal{H}|_{D^+})$ and $Rp_!(R^q j_*\mathcal{H}|_{D^+})$ are 0 for every $q \geqslant 0$.*

Proof. (1) Clear from Proposition 2.26 (3).

(2) Clear from Lemma 2.14. □

We make explicit the relation between the dual support and the refined Swan character defined in Corollary 1.25. Let D_i be an irreducible component of D, ξ_i be the generic point of D_i and K_i be the fraction field of the henselization $\mathcal{O}_{X,\xi_i}^h$ of the local ring. The residue field F_i of K_i is the function field of D_i. Let χ be a character of $\mathrm{Gr}_{\log}^{r_i} G_{K_i}$. Recall that $\mathfrak{m}_{\bar K}^{r} \supset \mathfrak{m}_{\bar K}^{r+}$ denote $\{a \in \bar K \mid v(a) \geqslant r\} \supset \{a \in \bar K \mid v(a) > r\}$. Then, by Corollary 1.25, the refined Swan character of χ defines an $\overline{F_i}$-valued point

$$\operatorname{rsw}\chi \in \Omega^1_{F_i}(\log) \otimes \mathfrak{m}_{\bar K_i}^{(-r_i)}/\mathfrak{m}_{\bar K_i}^{(-r_i)+} = \boldsymbol{V}_1(\Omega_X(\log D)^\vee \otimes \overline{F_i}, \mathfrak{m}_{\bar K_i}^{r_i}/\mathfrak{m}_{\bar K_i}^{r_i+})(\overline{F_i}).$$

Lemma 2.29. *Assume the log ramification of $\mathcal{F}$ is bounded by $R+$. Let D_i be an irreducible component of D such that $r_i > 0$. We consider the stalk $\mathcal{F}_{\bar\eta_i}$ as a representation of G_{K_i} and the direct sum decomposition $\mathcal{F}_{\bar\eta_i} = \bigoplus_\chi \chi^{\oplus n_\chi}$ of the restriction to $G_{K_i,\log}^{r_i}$ by characters of $\mathrm{Gr}_{\log}^{r_i} G_{K_i}$. Let*

$$\pi_n : \boldsymbol{V}_1(\Omega_X(\log D)^\vee \otimes \overline{F_i}, \mathfrak{m}_{\bar K_i}^{r_i}/\mathfrak{m}_{\bar K_i}^{r_i+}) \to \boldsymbol{V}_n(\Omega_X(\log D)^\vee, \mathcal{O}(-nR))_{\xi_i} = E_{n,\xi_i}^\vee$$

be the canonical map.

Then, the generic fibre $S_{\mathcal{F},\xi_i}^{(n\cdot R)} \subset E_{n,\xi_i}^\vee = \boldsymbol{V}_n(\Omega_X(\log D)^\vee, \mathcal{O}(-nR))_{\xi_i}$ of the dual support consists of the images $\pi_n(\operatorname{rsw}\chi)$ of the refined Swan characters of χ appearing in the direct sum decomposition $\mathcal{F}_{\bar\eta_i} = \bigoplus_\chi \chi^{\oplus n_\chi}$.

Proof. It is reduced to the case where r_i is an integer, by Lemma 1.22 and by the proof of Proposition 2.26 (1). In the case where r_i is an integer, it follows from Corollary 1.32. □

Corollary 2.30. *Assume the log ramification of $\mathcal{F}$ is bounded by $R+$. The equality $\mathcal{F}_{\bar\eta_i} = \mathcal{F}_{\bar\eta_i}^{(r_i)}$ holds if and only if the generic fibre $S_{\mathcal{F},\xi_i}^{(n\cdot R)}$ does not contain 0. Also the vanishing $\mathcal{F}_{\bar\eta_i}^{(r_i)} = 0$ is equivalent to the inclusion $S_{\mathcal{F},\xi_i}^{(n\cdot R)} \subset \{0\}$.*

Proof. By Corollary 1.25, the map

$$\operatorname{rsw} : \operatorname{Hom}(\mathrm{Gr}_{\log}^r G_K, \mathbb{F}_p) \to \Omega^1_F(\log) \otimes_F \mathfrak{m}_{\bar K}^{(-r)}/\mathfrak{m}_{\bar K}^{(-r)+}$$

is injective. Hence it follows from Lemma 2.29. □

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We study the functoriality of the dual support $S_{\mathcal{F}}^{(n\cdot R)}$. Let Y be a smooth scheme over k and $f : Y \to X$ be a morphism over k. Assume that the reduced inverse image $D_Y = (D \times_X Y)_{\mathrm{red}}$ is a divisor with simple normal crossings and let R_Y be the pull-back f^*R. Let $n > 0$ be an integer such that nR is integral and we put $E_n^\vee = \boldsymbol{V}_n(\Omega^1_X(\log D)^\vee, \mathcal{O}(-nR))$ and $E_n'^\vee = \boldsymbol{V}_n(\Omega^1_Y(\log D_Y)^\vee, \mathcal{O}(-nR_Y))$. Then, the canonical map $f^*\Omega^1_X(\log D) \to \Omega^1_Y(\log D_Y)$ induces a map $\varphi : E_n^\vee \times_X Y \to E_n'^\vee$.

Lemma 2.31. *Let Y be a smooth scheme over k and $f : Y \to X$ be a morphism over k. Assume that the reduced inverse image $D_Y = (D \times_X Y)_{\mathrm{red}}$ is a divisor with simple normal crossings and let R_Y be the pull-back f^*R. Let $n > 0$ be an integer such that nR is integral and let $\varphi : E_n^\vee \times_X Y \to E_n'^\vee$ be the map defined above.*

Let $\mathcal{F}$ be a smooth sheaf on $U = X \setminus D$ and $\mathcal{F}_V$ be the pull-back to $V = U \times_X Y = Y \setminus D_Y$. Assume the log ramification of $\mathcal{F}$ is bounded by $R+$ and let $f^*S_{\mathcal{F}}^{(n\cdot R)} \subset E_n^\vee \times_X Y$ denote the inverse image of $S_{\mathcal{F}}^{(n\cdot R)} \subset E_n^\vee$. Then, we have

$$\varphi(f^*S_{\mathcal{F}}^{(n\cdot R)}) \subset S_{\mathcal{F}_V}^{(n\cdot R_Y)}.$$

Proof. Let $D_1, \dots, D_m$ and $D'_1, \dots, D'_{m'}$ be the components of D and of D_Y respectively. We put $R = \sum_{i=1}^m r_i D_i$ and $R_Y = \sum_{j=1}^{m'} r'_j D'_j$. Let J be a non-empty subset of $J^+ = \{j \mid r'_j > 0,\ j = 1, \dots, m'\}$ and put $I = \{i \mid f^{-1}(D_i) \supset D'_J,\ r_i > 0,\ i = 1, \dots, m\}$. The map $(f \times f)^{(R)} : (Y \times Y)^{(R_Y)} \to (X \times X)^{(R)}$ defined in the proof of Lemma 2.22 induces $E'^\circ_J = \boldsymbol{V}_{n_J}(\Omega^1_Y(\log D_Y), \mathcal{O}(n_J R_Y)) \to E^\circ_I = \boldsymbol{V}_{n_I}(\Omega^1_X(\log D), \mathcal{O}(n_I R))$. Since the base change map $(f \times f)^{(R)*} j_*\mathcal{H} \to j'_*(f_U \times f_U)^*\mathcal{H}$ is injective by Lemma 2.9, the assertion follows. □

For an irreducible component D_i of D, the residue map $\Omega^1_X(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_{D_i} \to \mathcal{O}_{D_i}$ defines a map

$$\begin{aligned}\mathrm{res}_i : E_n^\vee \times_X D_i = \boldsymbol{V}_n(\Omega^1_X(\log D)^\vee, \mathcal{O}(-nR)) \times_X D_i \\ \to \boldsymbol{V}_n(\mathcal{O}_{D_i}, \mathcal{O}_{D_i}(-nR)) = \boldsymbol{V}(\mathcal{O}_{D_i}(-nR)).\end{aligned}$$

Corollary 2.32. *Let the notation be as in Lemma 2.31. Let D_i be an irreducible component of D and D'_j be an irreducible component of D_Y such that the multiplicity e of D'_j in the pull-back $f^{-1}(D_i)$ is non-zero.*

(1) *Let $f^*\,\mathrm{res}_i(S_{\mathcal{F}}^{(n\cdot R)}) \subset \boldsymbol{V}(\mathcal{O}_{D'_j}(-nR))$ denote the inverse image of $\mathrm{res}_i(S_{\mathcal{F}}^{(n\cdot R)}) \subset \boldsymbol{V}(\mathcal{O}_{D_i}(-nR))$ by the map*

$$\boldsymbol{V}(\mathcal{O}_{D'_j}(-nR)) = \boldsymbol{V}(\mathcal{O}_{D_i}(-nR)) \times_{D_i} D'_j \to \boldsymbol{V}(\mathcal{O}_{D_i}(-nR))$$

induced by f. Then, we have $e \cdot f^*\,\mathrm{res}_i(S_{\mathcal{F}}^{(n\cdot R)}) \subset \mathrm{res}_j\,(S_{\mathcal{F}_V}^{(n\cdot R_Y)})$.

(2) *Assume $\mathrm{res}_i(S_{\mathcal{F}}^{(n\cdot R)}) \setminus D_i \to D_i$ is surjective and e is prime to p. Then, the log ramification of $\mathcal{F}_V$ along D'_j is not bounded by r_j.*

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Proof. (1) By the commutative diagram

$$\begin{array}{ccc} E_n^\vee \times_X D'_j & \xrightarrow{\mathrm{res}_i} & \boldsymbol{V}(\mathcal{O}_{D'_j}(-nR)) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle e} \\ E_n'^\vee \times_Y D'_j & \xrightarrow{\mathrm{res}_j} & \boldsymbol{V}(\mathcal{O}_{D'_j}(-nR_Y)) \end{array}$$

it follows from Lemma 2.31.

(2) By (1), $\mathrm{res}_j(S^{(n\cdot R_Y)}_{\mathcal{F}_V})$ is not contained in the 0-section. □

3. Characteristic cycle

We recall in § 3.1 the definition of the characteristic class and compute it under a certain assumption. We propose a definition of the characteristic cycle in some case and prove that it computes the characteristic class in § 3.2.

3.1. Characteristic class

We recall the definition of the characteristic class. For more detail on the construction, we refer to [**3**, §§ 1, 2] and [**12**, §§ 1–3]. Let X be a scheme over k and $\mathcal{F}$ be a constructible sheaf of flat Λ-modules. We put $\mathcal{K}_X = Ra^!\Lambda$ for the structural map $a : X \to \operatorname{Spec} k$ and $D_X\mathcal{F} = R\mathcal{H}om(\mathcal{F}, \mathcal{K}_X)$. We consider $\mathcal{H} = R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, R\,\mathrm{pr}_1^!\mathcal{F})$ on $X \times X$.

The canonical pairing

$$\mathcal{F} \otimes^L R\delta^!\mathcal{H} = \delta^*\mathrm{pr}_2^*\mathcal{F} \otimes^L R\delta^! R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, R\,\mathrm{pr}_1^!\mathcal{F}) \to R\delta^! R\,\mathrm{pr}_1^!\mathcal{F} = \mathcal{F}$$

induces an isomorphism

$$H^0_X(X \times X, \mathcal{H}) \to \mathrm{End}_X(\mathcal{F}). \tag{3.1}$$

Alternatively, one can apply the canonical isomorphism [**12**, (3.2.1)]. The inverse of the canonical isomorphism $\mathcal{F} \boxtimes D_X\mathcal{F} \to R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, R\,\mathrm{pr}_1^!\mathcal{F}) = \mathcal{H}$ and the canonical map $R\delta^! \to \delta^*$ induce a map

$$H^0_X(X \times X, \mathcal{H}) \to H^0(X, \mathcal{F} \otimes^L D_X\mathcal{F}). \tag{3.2}$$

The evaluation map $\mathcal{F} \otimes^L D_X\mathcal{F} \to \mathcal{K}_X$ induces a map

$$H^0(X, \mathcal{F} \otimes^L D_X\mathcal{F}) \to H^0(X, \mathcal{K}_X). \tag{3.3}$$

We define the characteristic class $C(\mathcal{F}) \in H^0(X, \mathcal{K}_X)$ to be the image of $1 \in \mathrm{End}_X(\mathcal{F})$ by the composition

$$\mathrm{End}_X(\mathcal{F}) \xrightarrow{(3.1)^{-1}} H^0_X(X \times X, \mathcal{H}) \xrightarrow{(3.2)} H^0(X, \mathcal{F} \otimes^L D_X\mathcal{F}) \xrightarrow{(3.3)} H^0(X, \mathcal{K}_X).$$

If X is proper, we have an index formula [**12**, Corollaire 4.8]

$$\chi(X_{\bar{k}}, \mathcal{F}) = \mathrm{Tr}\, C(\mathcal{F}) \tag{3.4}$$

for the Euler number $\chi(X_{\bar{k}}, \mathcal{F}) = \sum_{q=0}^{2\dim X} (-1)^q \dim H^q(X_{\bar{k}}, \mathcal{F})$.

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Assume that X is smooth of dimension d and that $\mathcal{F}$ is a smooth sheaf of free Λ-modules of finite rank. Then, the isomorphism $\mathrm{End}_X(\mathcal{F}) \to H^0_X(X \times X, \mathcal{H})$ (3.1) is described as follows. We put $\mathcal{H}_0 = \mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, \mathrm{pr}_1^* \mathcal{F})$. By the assumptions on X and on $\mathcal{F}$, we have a canonical isomorphism $\mathcal{H}_0(d)[2d] \to \mathcal{H} = R\,\mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, R\,\mathrm{pr}_1^! \mathcal{F})$. We identify $\delta^*\mathcal{H}_0 = \mathcal{E}nd(\mathcal{F})$ and $H^0(X, \delta^*\mathcal{H}_0) = \mathrm{End}_X(\mathcal{F})$. Then, the isomorphism (3.1) is the inverse of the cup product

$$\mathrm{End}_X(\mathcal{F}) = H^0(X, \delta^*\mathcal{H}_0) \xrightarrow{\cup [X]} H^0_X(X \times X, \mathcal{H}) \tag{3.5}$$

with the cycle class $[X] \in H^0_X(X \times X, \Lambda(d)[2d])$. Further, in this case, the evaluation map $\delta^*\mathcal{H} \to \mathcal{K}_X$ is the tensor product of the trace map $\mathrm{Tr} : \delta^*\mathcal{H}_0 = \mathcal{E}nd(\mathcal{F}) \to \Lambda$ with the isomorphism $\Lambda(d)[2d] \to \mathcal{K}_X$ defined by the cycle class. Thus, in this case, we have

$$C(\mathcal{F}) = \mathrm{rank}\, \mathcal{F} \cdot (X, X)_{X \times X}$$

in $H^{2d}(X, \Lambda(d))$ where $(X, X)_{X \times X} = (-1)^d c_d(\Omega^1_X)$ is the self-intersection class.

We will compute the characteristic class in some cases. First we consider the tamely ramified case. Let X be a smooth scheme of dimension d over k and $U = X \setminus D$ be the complement of a divisor D with simple normal crossings. We consider the diagram

$$\begin{array}{ccccc} X \times X & \xleftarrow{f} & (X \times X)^\sim & \xleftarrow{\tilde{j}} & U \times U \\ & \nwarrow^{\delta} & \uparrow{\scriptstyle \tilde{\delta}} & & \uparrow{\scriptstyle \delta_U} \\ & & X & \xleftarrow[j]{} & U \end{array} \tag{3.6}$$

where $(X \times X)^\sim$ is the log product and $f : (X \times X)^\sim \to X \times X$ is the canonical map. The diagonal maps for X and U are denoted by δ and δ_U respectively and $\tilde{\delta}$ is the log diagonal map. The map $\tilde{j} : U \times U \to (X \times X)^\sim$ is the open immersion.

Proposition 3.1. *Let the notation be as in the diagram (3.6) above and let $\mathcal{F}$ be a smooth sheaf of free Λ-modules of finite rank on $U = X \setminus D$.*

We put $\mathcal{H}_0 = \mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, \mathrm{pr}_1^* \mathcal{F})$ on $U \times U$ and $\bar{\mathcal{H}} = R\,\mathcal{H}om(\mathrm{pr}_2^* j_!\mathcal{F}, R\,\mathrm{pr}_1^! j_!\mathcal{F})$ on $X \times X$. We also put $\tilde{\mathcal{H}}_0 = \tilde{j}_* \mathcal{H}_0$ and $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0(d)[2d]$ on $(X \times X)^\sim$. Let $e \in \Gamma(X, \tilde{\delta}^* \tilde{\mathcal{H}}_0)$ be the unique element that maps to the identity $\mathrm{id}_{\mathcal{F}} \in \mathrm{End}_U(\mathcal{F}) = H^0(U, \delta_U^* \mathcal{H}_0)$ and let $e \cup [X] \in H^0_X((X \times X)^\sim, \tilde{\mathcal{H}})$ be the cup product with the cycle class $[X] \in H^0_X((X \times X)^\sim, \Lambda(d)[2d])$.

(1) *There exists a unique map $f^*\bar{\mathcal{H}} \to \tilde{\mathcal{H}}$ inducing the canonical isomorphism $\mathcal{H} = R\,\mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, R\,\mathrm{pr}_1^! \mathcal{F}) \to \mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, \mathrm{pr}_1^* \mathcal{F})(d)[2d] = \mathcal{H}_0(d)[2d]$ on $U \times U$.*

(2) *Assume further that $\mathcal{F}$ is tamely ramified along D. We consider the pull-back $f^*(\mathrm{id}_{j_!\mathcal{F}}) \in H^0_{f^{-1}(X)}((X \times X)^\sim, \tilde{\mathcal{H}})$ of the identity $\mathrm{id}_{j_!\mathcal{F}} \in \mathrm{End}_X(j_!\mathcal{F}) = H^0_X(X \times X, \bar{\mathcal{H}})$. Then, we have*

$$f^*(\mathrm{id}_{j_!\mathcal{F}}) = e \cup [X] \tag{3.7}$$

in $H^0_{f^{-1}(X)}((X \times X)^\sim, \tilde{\mathcal{H}})$.

Proof. (1) Let $\bar{f} : (X \times X)' \to X \times X$ be the log blow-up. Let $(U \times X)' \subset (X \times X)'$ be the complement of the proper transform of $D \times X$ and we consider the open immersions

$$(X \times X)^\sim \xrightarrow{j_2} (U \times X)' \xrightarrow{j_1} (X \times X)'.$$

We put $\mathcal{H}' = j_{1!}Rj_{2*}\tilde{\mathcal{H}}$ on $(X \times X)'$. The log blow-up $\bar{f} : (X \times X)' \to X \times X$ is an isomorphism on the complement $U \times X$ of $D \times X$. The restriction of $\bar{\mathcal{H}}$ on $D \times X$ is 0 and the restriction $\bar{\mathcal{H}}|_{U\times X} = R\mathcal{H}om(\mathrm{pr}_2^* j_!\mathcal{F}, R\,\mathrm{pr}_1^! \mathcal{F})$ is canonically identified with the restriction $\mathcal{H}'|_{U\times X} = R(1 \times j)_*\mathcal{H}_0(d)[2d]$. Hence, there exists a unique map $\bar{f}^*\bar{\mathcal{H}} \to \mathcal{H}'$ inducing the canonical isomorphism $\mathcal{H} \to \mathcal{H}_0(d)[2d]$ on $U \times U$. The restriction of $\bar{f}^*\bar{\mathcal{H}} \to \mathcal{H}'$ on $(X \times X)^\sim$ gives the desired map $f^*\bar{\mathcal{H}} \to \tilde{\mathcal{H}}$.

(2) It suffices to show the equality $\bar{f}^*(\mathrm{id}_{j_!\mathcal{F}}) = e \cup [X]$ in $H^0_{\bar{f}^{-1}(X)}((X \times X)', \mathcal{H}')$. Since $\mathcal{F}$ is assumed tamely ramified, the adjunction $\bar{\mathcal{H}} \to R\bar{f}_*\mathcal{H}'$ of the canonical map $\bar{f}^*\bar{\mathcal{H}} \to \mathcal{H}'$ is an isomorphism, by [**3**, Lemma 2.2.4]. Hence, the pull-back $\bar{f}^* : H^0_X(X \times X, \bar{\mathcal{H}}) \to H^0_{\bar{f}^{-1}(X)}((X \times X)', \mathcal{H}')$ is an isomorphism. Since the restriction map $H^0_X(X \times X, \bar{\mathcal{H}}) = \mathrm{End}_X(j_!\mathcal{F}) \to H^0_U(U \times U, \mathcal{H}) = \mathrm{End}_U(\mathcal{F})$ is an isomorphism, the arrows in the commutative diagram

$$\begin{array}{ccc}
H^0_X(X \times X, \bar{\mathcal{H}}) & \longrightarrow & H^0_U(U \times U, \mathcal{H}) \\
\downarrow & \nearrow & \\
H^0_{\bar{f}^{-1}(X)}((X \times X)', \mathcal{H}') & &
\end{array}$$

are isomorphisms. Thus, it suffices to show the equality in $H^0_U(U \times U, \mathcal{H})$. Hence the assertion follows from the description (3.5) of the isomorphism (3.1) in the smooth case. □

Corollary 3.2. *Let the notation be as in Proposition 3.1 (2). Then, we have*

$$C(j_!\mathcal{F}) = \mathrm{rank}\,\mathcal{F} \cdot (X, X)_{(X\times X)^\sim}$$

in $H^0(X, \mathcal{K}_X)$ where $(X, X)_{(X\times X)^\sim} = (-1)^d c_d(\Omega^1_X(\log D))$ is the self-intersection class.

Proof. We consider the pull-back to $H^0(X, \tilde{\delta}^*\tilde{\mathcal{H}})$ of the equality $f^*(\mathrm{id}_{j_!\mathcal{F}}) = e \cup [X]$ by the log diagonal map $\tilde{\delta} : X \to (X \times X)^\sim$. Then, since $\delta = f \circ \tilde{\delta}$, we obtain $\delta^*(\mathrm{id}_{j_!\mathcal{F}}) = e \cup (X, X)_{(X\times X)^\sim}$ in $H^0(X, \tilde{\delta}^*\tilde{\mathcal{H}}) = H^0(X, j_*\,\mathcal{E}nd_U\,\mathcal{F}(d)[2d])$. Since the evaluation map $\tilde{\delta}^*\tilde{\mathcal{H}} \to \mathcal{K}_X$ is induced by the trace map $j_*\,\mathcal{E}nd_U\,\mathcal{F} \to \Lambda$, we obtain $C(j_!\mathcal{F}) = \mathrm{rank}\,\mathcal{F} \cdot (X, X)_{(X\times X)^\sim}$. □

Corollary 3.3. *Let X be a smooth scheme of dimension d over k and $U = X \setminus D$ be the complement of a divisor D with simple normal crossings. We keep the notation in the diagram (3.6).*

Let $D^+ \subset D$ be the union of some irreducible components and put $U^+ = X \setminus D^+ \supset U$. Let $g : \mathbb{X} \to (X \times X)^\sim$ be a morphism of schemes over k that is an isomorphism on $(U^+ \times U^+)^\sim = (U^+ \times U^+) \times_{X\times X} (X \times X)^\sim \subset (X \times X)^\sim$ and let $j^\natural : U \times U \to \mathbb{X}$ be the *open immersion. Let $\delta^\natural : X \to \mathbb{X}$ be a closed immersion satisfying $\tilde{\delta} = g \circ \delta^\natural$. We assume that the cycle map*

$$\Lambda(d)[2d] \to \mathcal{K}_{\mathbb{X}} = R(p_2 \circ f \circ g)^! \Lambda \tag{3.8}$$

is an isomorphism. Define the cycle class $[X] \in H^0_X(\mathbb{X}, \Lambda(d)[2d])$ to be the inverse image of 1 by the isomorphism $H^0_X(\mathbb{X}, \Lambda(d)[2d]) \to H^0_X(\mathbb{X}, \mathcal{K}_{\mathbb{X}}) = H^0(X, \Lambda)$ induced by the isomorphism (3.8).

Let $\mathcal{F}$ be a smooth sheaf on U of free Λ-modules of finite rank. We assume that $\mathcal{F}$ is tamely ramified along $D \cap U^+ = U^+ \setminus U$. We put $\mathcal{H}_0 = \mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, \mathrm{pr}_1^* \mathcal{F})$ on $U \times U$ and $\bar{\mathcal{H}} = R\mathcal{H}om(\mathrm{pr}_2^* j_! \mathcal{F}, \mathrm{pr}_1^! j_! \mathcal{F})$ on $X \times X$. We also put $\mathcal{H}_0^\natural = j_*^\natural \mathcal{H}_0$ on $\mathbb{X}$. Let $e \in \Gamma(X, \delta^{\natural *} \mathcal{H}_0^\natural)$ be a section such that the restriction $e|_U \in \Gamma(U, \delta_U^* \mathcal{H}_0) = \mathrm{End}_U(\mathcal{F})$ is the identity of $\mathcal{F}$. We put $E = \mathbb{X} \setminus (U^+ \times U^+)^\sim$ and assume

$$H^{2d}_E(\mathbb{X}, \mathcal{H}_0^\natural(d)) = 0. \tag{3.9}$$

(1) *There exists a unique map $(f \circ g)^* \bar{\mathcal{H}} \to \mathcal{H}^\natural = \mathcal{H}_0^\natural(d)[2d]$ inducing the canonical isomorphism $\mathcal{H} = R\mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, R\,\mathrm{pr}_1^! \mathcal{F}) \to \mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, \mathrm{pr}_1^* \mathcal{F})(d)[2d] = \mathcal{H}_0(d)[2d]$ on $U \times U$.*

(2) *We consider the pull-back $(f \circ g)^*(\mathrm{id}_{j_! \mathcal{F}}) \in H^0_{(f \circ g)^{-1}(X)}(\mathbb{X}, \mathcal{H}^\natural)$ of the identity $\mathrm{id}_{j_! \mathcal{F}} \in \mathrm{End}_X(j_! \mathcal{F}) = H^0_X(X \times X, \bar{\mathcal{H}})$. Then, we have*

$$(f \circ g)^*(\mathrm{id}_{j_! \mathcal{F}}) = e \cup [X] \tag{3.10}$$

in $H^0_{(f \circ g)^{-1}(X) \cup E}(\mathbb{X}, \mathcal{H}^\natural)$. Consequently, we have

$$C(j_! \mathcal{F}) = \mathrm{rank}\, \mathcal{F} \cdot (X, X)_{\mathbb{X}} \tag{3.11}$$

in $H^0(X, \mathcal{K}_X)$ where $(X, X)_{\mathbb{X}}$ denotes the pull-back of the cycle class $[X] \in H^{2d}_X(\mathbb{X}, \Lambda(d))$.

Proof. (1) Since the image of $(X \times X)^\sim \setminus (U \times U)$ in $X \times X$ is a subset of $D \times X$, we have $(f \circ g)^* \bar{\mathcal{H}} = j_!^\natural \mathcal{H}$ as in the proof of Proposition 3.1. Hence the assertion follows.

(2) Since $g^{-1}(U^+) = (f \circ g)^{-1}(X) \cap (U^+ \times U^+)^\sim$ is the complement of $E \cap (f \circ g)^{-1}(X)$ in $(f \circ g)^{-1}(X)$, we have an exact sequence

$$H^0_E(\mathbb{X}, \mathcal{H}^\natural) \to H^0_{(f \circ g)^{-1}(X) \cup E}(\mathbb{X}, \mathcal{H}^\natural) \to H^0_{g^{-1}(U^+)}((U^+ \times U^+)^\sim, \mathcal{H}^\natural).$$

By the assumption (3.9), we have $H^0_E(\mathbb{X}, \mathcal{H}^\natural) = 0$. Hence the restriction map

$$H^0_{(f \circ g)^{-1}(X) \cup E}(\mathbb{X}, \mathcal{H}^\natural) \to H^0_{g^{-1}(U^+)}((U^+ \times U^+)^\sim, \mathcal{H}^\natural)$$

is an injection. By Proposition 3.1, the equality $(f \circ g)^*(\mathrm{id}_{j_! \mathcal{F}}) = e \cup [X]$ holds in $H^0_{g^{-1}(U^+)}((U^+ \times U^+)^\sim, \mathcal{H}^\natural)$. Thus we obtain $(f \circ g)^*(\mathrm{id}_{j_! \mathcal{F}}) = e \cup [X]$ (3.10) in $H^0_{(f \circ g)^{-1}(X) \cup E}(\mathbb{X}, \mathcal{H}^\natural)$.

The equality (3.11) is deduced from (3.10) as in the proof of Proposition 3.1. □

Theorem 3.4. *Let X be a smooth scheme of dimension d over k and $U = X \setminus D$ be the complement of a divisor D with simple normal crossings. Let $R = \sum_i r_i D_i \geqslant 0$ be an effective divisor with rational coefficients. Let $g : (X \times X)^{(R)} \to (X \times X)^{\sim}$ and $\delta^{(R)} : X \to (X \times X)^{(R)}$ be as in § 2.3 and let $j^{(R)} : U \times U \to (X \times X)^{(R)}$ be the open immersion.*

Let $\mathcal{F}$ be a smooth sheaf on U of free Λ-modules of finite rank. We put $\mathcal{H}_0 = \mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, \mathrm{pr}_1^* \mathcal{F})$ on $U \times U$. We assume that the log ramification of $\mathcal{F}$ is bounded by $R+$ and let $e \in \Gamma(X, \delta^{(R)*} j_*^{(R)} \mathcal{H}_0)$ be the unique section whose image by the base change map in $\Gamma(X, j_* \delta_U^* \mathcal{H}_0) = \mathrm{End}_U(\mathcal{F})$ is the identity of $\mathcal{F}$. We further assume that the log ramification of $\mathcal{F}$ along D is non-degenerate with respect to R (cf. Definition 2.27).

Then, we have

$$C(j_! \mathcal{F}) = \mathrm{rank}\, \mathcal{F} \cdot (X, X)_{(X \times X)^{(R)}} \tag{3.12}$$

$$= (-1)^d \cdot \mathrm{rank}\, \mathcal{F} (c_d(\Omega_X^1(\log D)) + (c(\Omega_X^1(\log D)) \cap (1-R)^{-1} \cap [R])_{\dim 0}) \tag{3.13}$$

in $H^0(X, \mathcal{K}_X)$.

Proof. We put $D^+ = \sum_{i : r_i > 0} D_i$. We verify that $g : \mathbb{X} = (X \times X)^{(R)} \to (X \times X)^{\sim}$ satisfies the assumptions in Corollary 3.3. By the construction, the map $g : \mathbb{X} = (X \times X)^{(R)} \to (X \times X)^{\sim}$ is an isomorphism on the complement of $D^+ \subset X \subset (X \times X)^{\sim}$. The log diagonal map $\tilde{\delta} : X \to (X \times X)^{\sim}$ is lifted to a closed immersion $\delta^{(R)} : X \to (X \times X)^{(R)}$. The cycle map $\Lambda(d)[2d] \to \mathcal{K}_{(X \times X)^{(R)}}$ is an isomorphism by Proposition 2.18 (1).

By the definition of D^+ and by the assumption that the log ramification of $\mathcal{F}$ is bounded by $R+$, it follows that $\mathcal{F}$ is tamely ramified along $D \setminus D^+ = U^+ \setminus U$ by Corollary 2.21. The complement $(X \times X)^{(R)} \setminus (U^+ \times U^+)^{\sim}$ equals the inverse image E^+ of D^+. We show that $\mathcal{H}_0^{(R)} = j_*^{(R)} \mathcal{H}_0$ satisfies the assumption

$$H^{2d}_{E^+}((X \times X)^{(R)}, \mathcal{H}_0^{(R)}(d)) = 0$$

(3.9) for $\mathbb{X} = (X \times X)^{(R)}$. Let $i : E^+ \to (X \times X)^{(R)}$ be the closed immersion and $p : E^+ \to D^+$ be the projection. Since $H^{2d}_{E^+}((X \times X)^{(R)}, \mathcal{H}_0^{(R)}(d)) = H^{2d}(D^+, Rp_* Ri^! j_*^{(R)} \mathcal{H}_0(d))$, it suffices to show $Rp_* R^q i^! j_*^{(R)} \mathcal{H}_0 = 0$ for $q \geqslant 0$. Since $R^q i^! j_*^{(R)} \mathcal{H}_0$ is 0 for $q = 0, 1$ and is isomorphic to $R^{q-1} j_*^{(R)} \mathcal{H}_0$ for $q > 1$, it follows from $Rp_* R^q j_*^{(R)} \mathcal{H}_0 = 0$ proved in Lemma 2.29 (2). Thus, the assumptions in Corollary 3.3 are satisfied and we obtain the equality (3.12).

The equality (3.13) follows from the equalities (3.12) and (2.12) and the isomorphism $\mathcal{N}_{X/(X \times X)^{\sim}} \to \Omega_X^1(\log D)$. □

3.2. Characteristic cycle

Let X be a smooth scheme of dimension d over k and let D be a divisor with simple normal crossings. Let

$$T^* X(\log D) = \boldsymbol{V}(\Omega_X^1(\log D)^{\vee})$$

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be the logarithmic cotangent bundle. We regard X as a closed subscheme of $T^*X(\log D)$ by the 0-section. Let D_i be an irreducible component of D, ξ_i be the generic point of D_i and K_i be the fraction field of the henselization $\mathcal{O}^h_{X,\xi_i}$ of the local ring. The residue field F_i of K_i is the function field of D_i.

Let $r > 0$ be a rational number and $\chi : \mathrm{Gr}^r_{\log} G_{K_i} \to \mathbb{F}_p$ be a non-trivial character. The refined Swan character $\operatorname{rsw}\chi \in \Omega^1_{\bar{F}_i}(\log) \otimes \mathfrak{m}^{(-r)}_{\bar{K}_i}/\mathfrak{m}^{(-r)+}_{\bar{K}_i}$ regarded as an injection $\mathfrak{m}^r_{\bar{K}_i}/\mathfrak{m}^{r+}_{\bar{K}_i} \to \Omega^1_X(\log D)_{\xi_i} \otimes \bar{F}_i$ defines a line in the $\bar{F}_i$-vector space $\Omega^1_X(\log D)_{\xi_i} \otimes \bar{F}_i$ and hence an $\bar{F}_i$-valued point $[\operatorname{rsw}\chi] : \operatorname{Spec} \bar{F}_i \to \boldsymbol{P}(\Omega^1_X(\log D)^\vee)$. We define a reduced closed subscheme $T_\chi \subset \boldsymbol{P}(\Omega^1_X(\log D)^\vee)$ to be the Zariski closure $\overline{\{[\operatorname{rsw}\chi](\operatorname{Spec} \bar{F}_i)\}}$ of the image and let $L_\chi = \boldsymbol{V}(\mathcal{O}_{T_\chi}(1))$ be the pull-back to T_χ of the tautological sub line bundle $L \subset T^*X(\log D) \times_X \boldsymbol{P}(\Omega^1_X(\log D)^\vee)$. We have a commutative diagram

$$\begin{array}{ccccc} L_\chi & \longrightarrow & T^*X(\log D) \times_X D_i & \longrightarrow & T^*X(\log D) = \boldsymbol{V}(\Omega^1_X(\log D)^\vee) \\ \downarrow & & \downarrow & & \downarrow \\ T_\chi & \xrightarrow{\pi_\chi} & D_i & \longrightarrow & X \end{array} \tag{3.14}$$

The natural map $\pi_\chi : T_\chi \to D_i$ is generically finite.

Let $\mathcal{F}$ be a smooth ℓ-adic sheaf on $U = X \setminus D$ and $R = \sum_i r_i D_i$ be an effective divisor with rational coefficients $r_i \geqslant 0$. In the rest of the paper, we assume that $\mathcal{F}$ satisfies the following conditions.

(R) The log ramification of $\mathcal{F}$ along D is bounded by $R+$.

(C) For each irreducible component D_i of D, the closure $\overline{S^{(n\cdot R)}_{\mathcal{F},\xi_i}}$ of the generic fibre of the dual support is finite over D_i and its intersection

$$\overline{S^{(n\cdot R)}_{\mathcal{F},\xi_i}} \cap D_i$$

with the 0-section of $\boldsymbol{V}_n(\Omega^1_X(\log D)^\vee, \mathcal{O}(-nR))_{D_i}$ is empty.

By Lemma 2.29, the conditions (R) and (C) imply the following condition.

(R$'$) $\mathcal{F}_{\bar{\eta}_i} = \mathcal{F}^{(r_i)}_{\bar{\eta}_i}$ for every irreducible component D_i of D.

They also imply that the log ramification of $\mathcal{F}$ is non-degenerate with respect to R by Proposition 2.26 (3). Conversely, the condition (R$'$) implies that both (R) and (C) are satisfied on a dense open subscheme of X containing the generic points of the irreducible components of D. Under the condition (R$'$), one may expect that the closure $\overline{S^{(n\cdot R)}_{\mathcal{F},\xi_i}}$ is always finite over D_i. This is in fact proved in the case where $\operatorname{rank}\mathcal{F} = 1$ in [**14**, Theorem (7.1)]. If $\mathcal{F}$ is the rank 1 sheaf defined by a continuous character χ of $\pi_1(U)^{\mathrm{ab}}$, the condition (C) is equivalent to that χ is clean along D in the sense of [**15**, (3.4.3)]. Under the assumption $\dim X = 2$, it is proved in [**15**, Theorem 4.1] that the condition (C) is satisfied after blowing-up finitely many closed points on the boundary successively.

Lemma 3.5. *Assume that the log ramification of $\mathcal{F}$ is bounded by $R+$ and that $\mathcal{F}$ satisfies the conditions* (R) *and* (C). *Let D_i be an irreducible component of D and χ be a character of $\mathrm{Gr}^{r_i}_{\log} G_{K_i}$ appearing in the direct sum decomposition $\mathcal{F}_{\bar{\eta}_i} = \sum_\chi n_\chi \chi$.*

(1) *The scheme T_χ is finite over D_i.*

(2) *We put*

$$SS_\chi = \frac{1}{[T_\chi : D_i]} \pi_{\chi *}[L_\chi] \tag{3.15}$$

in $Z_d(T^*X(\log D) \times_X D_i)_{\mathbb{Q}}$ in the notation in (3.14). Then, we have

$$SS_\chi = (c(\Omega^1_X(\log D)) \cap (1-R)^{-1} \cap [T^*X(\log D) \times_X D_i])_{\dim d}.$$

Proof. (1) By Lemma 2.29, the generic fibre $S^{(n\cdot R)}_{\mathcal{F},\xi_i}$ of the dual support consists of the refined Swan characters rsw χ of the characters χ appearing in the direct sum decomposition $\mathcal{F}_{\bar\eta_i} = \sum_\chi n_\chi \chi$. Hence, by the condition (C), the closure $\overline{S^{(n\cdot R)}_{\mathcal{F},\xi_i}}$ is a closed subscheme of

$$\boldsymbol{V}_n(\Omega^1_X(\log D)^\vee, \mathcal{O}(-nR))_{D_i} \setminus D_i$$

finite over D_i. Hence, the union $\bigcup_\chi T_\chi$ is the image of $\overline{S^{(n\cdot R)}_{\mathcal{F},\xi_i}}$ by the canonical map

$$\varphi : \boldsymbol{V}_n(\Omega^1_X(\log D)^\vee, \mathcal{O}(-nR))_{D_i} \setminus D_i \to \boldsymbol{P}(\Omega^1_X(\log D)^\vee).$$

(2) Since the conormal sheaf of $L_\chi \subset T^*X(\log D) \times_X T_\chi$ is the pull-back of the locally free sheaf $\mathrm{Ker}(\Omega^1_X(\log D)^\vee \to \mathcal{O}(1))$ of rank $d-1$, we have

$$[L_\chi] = (-1)^{d-1} c_{d-1}(\mathrm{Ker}(\Omega^1_X(\log D)^\vee \to \mathcal{O}(1))) \cap [T^*X(\log D) \times_X T_\chi].$$

Hence we have

$$SS_\chi = (c(\Omega^1_X(\log D)) \cap c(\mathcal{O}(-1))^{-1} \cap [T^*X(\log D) \times_X D_i])_{\dim d}.$$

By Lemma 2.11, the pull-back of $\mathcal{O}(n)$ on

$$\boldsymbol{V}_n(\Omega^1_X(\log D)^\vee, \mathcal{O}(-nR))_{D_i} \setminus D_i \supset \overline{S^{(n\cdot R)}_{\mathcal{F},\xi_i}}$$

is canonically isomorphic to $\mathcal{O}(nR)$. Since the union $\bigcup_\chi T_\chi$ is the image of $\overline{S^{(n\cdot R)}_{\mathcal{F},\xi_i}}$, the assertion follows. □

Assuming the conditions (R) and (C), we define the characteristic cycle $CC(\mathcal{F})$ as a rational d-cycle on $T^*X(\log D)$.

Definition 3.6. Let $\mathcal{F}$ be a smooth Λ-sheaf on $U = X \setminus D$ satisfying the conditions (R) and (C). For an irreducible component D_i of D with $r_i > 0$, let $\mathcal{F}_{\bar\eta_i} = \sum_\chi n_\chi \chi$ be the direct sum decomposition of the representation induced on $\mathrm{Gr}^{r_i}_{\log} G_{K_i}$. We define the characteristic cycle by

$$CC(\mathcal{F}) = (-1)^d \Bigl(\mathrm{rank}\, \mathcal{F} \cdot [X] + \sum_{i, r_i > 0} r_i \cdot \sum_\chi n_\chi \cdot [SS_\chi] \Bigr) \tag{3.16}$$

in $Z^d(T^*X(\log D))_{\mathbb{Q}}$.

If $\dim X = 1$, we put $\mathrm{Sw}\,\mathcal{F} = \sum_{x\in D} \mathrm{Sw}_x\,\mathcal{F}\cdot[x] \in Z_0(X)$ and let $p : T^*X(\log D) \to X$ be the projection. Then, we have

$$CC(\mathcal{F}) = -(\operatorname{rank}\mathcal{F}\cdot[X] + p^*[\mathrm{Sw}\,\mathcal{F}]).$$

If $\mathcal{F}$ is a rank 1 sheaf defined by a continuous character χ of $\pi_1(U)^{\mathrm{ab}}$ and if $\mathcal{F}$ is clean along D, the characteristic cycle $CC(\mathcal{F})$ defined above is nothing but $\mathrm{Char}(X, U, \chi)$ defined in [**15**, (3.4.4)].

Theorem 3.7. *Let X be a smooth scheme of dimension d over k and D be a divisor with simple normal crossings. Let $\mathcal{F}$ be a smooth ℓ-adic sheaf on $U = X \setminus D$ satisfying the conditions* (R) *and* (C). *Then we have*

$$C(j_!\mathcal{F}) = [CC(\mathcal{F})]$$

in $H^{2d}(X, \Lambda(d)) = H^{2d}(T^*X(\log D), \Lambda(d))$. In other words, we have

$$C(j_!\mathcal{F}) = (CC(\mathcal{F}), X)_{T^*X(\log D)}.$$

Proof. By the assumption (C) and by Lemma 2.29, the assumption in Theorem 3.4 is satisfied. Hence the left-hand side is equal to

$$\operatorname{rank}\mathcal{F}\cdot(-1)^d\cdot(c_d(\Omega^1_X(\log D)) + (c(\Omega^1_X(\log D)) \cap (1-R)^{-1} \cap [R])_{\dim 0}).$$

By Lemma 3.5, the right-hand side is also equal to this. □

By the index formula (3.4), Theorem 3.7 implies the following.

Corollary 3.8. *Further if X is proper, we have*

$$\chi_c(U_{\bar{k}}, \mathcal{F}) = \deg(CC(\mathcal{F}), X)_{T^*X(\log D)}.$$

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