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CO-H-SPACES AND ALMOST LOCALIZATION

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Abstract Apart from simply connected spaces, a non-simply connected co-H-space is a typical example of a space X with a coaction of $B\pi_1(X)$ along $r^X\colon X \to B\pi_1(X)$, the classifying map of the universal covering. If such a space X is actually a co-H-space, then the fibrewise p-localization of r^X (or the 'almost' p-localization of X) is a fibrewise co-H-space (or an 'almost' co-H-space, respectively) for every prime p. In this paper, we show that the converse statement is true, i.e. for a non-simply connected space X with a coaction of $B\pi_1(X)$ along r^X, X is a co-H-space if, for every prime p, the almost p-localization of X is an almost co-H-space.

Keywords: co-H-space; fibrewise localization; Lusternik–Schnirelmann category; LS category

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1. Fundamentals and results

We assume that spaces have the homotopy type of CW-complexes and are based, and that maps and homotopies preserve base points. A space X is a co-H-space if there exists a comultiplication, say $\mu^X\colon X \to X \vee X$, satisfying $j^X \circ \mu^X \simeq \Delta^X$, where $j^X\colon X \vee X \hookrightarrow X \times X$ is the inclusion and $\Delta^X\colon X \to X \times X$ is the diagonal. As with spaces, we say that a group G is a co-H-group if there exists a homomorphism $G \to G * G$ such that the composition with the first and second projections are the identity of G. Thus, the fundamental group of a co-H-space is a co-H-group and the classifying space of a co-H-group is a co-H-space.

When X is a simply connected co-H-space, the p-localization $X_{(p)}$ is also a co-H-space for any prime p. The immediate problem is whether the converse statement holds. In [**3**], the first author settles the answer in the positive when X is a finite simply connected complex. In this paper, we extend the above result to non-simply connected spaces in terms of fibrewise p-localization (see [**1**,**14**]) or, paraphrasing the second author, in terms of almost p-localization (see [**10**]), rather than a usual p-localization, since the only nilpotent non-simply connected co-H-space is the circle in [**8**].

From now on, we assume that a space X is the total space of a fibrewise space $r^X\colon X \to B\pi_1(X)$, where r^X is the classifying map of $c^X\colon \tilde{X} \to X$, the universal covering of X.

A coaction of $B\pi_1(X)$ along r^X is a map $\nu\colon X \to B\pi_1(X)\vee X$ such that when composed with the first projection $B\pi_1(X)\vee X \to B\pi_1(X)$ we obtain r^X, and when composed with the second projection $B\pi_1(X)\vee X \to X$ we obtain the identity 1_X. In other words, it consists of a copairing $\nu\colon X \to B\pi_1(X)\vee X$ with co-axes r^X and 1_X in the sense of Oda [**15**]. Since the fundamental group of a space X with a coaction of $B\pi_1(X)$ along r^X is clearly a co-H-group, $\pi_1(X)$ is free by [**5**] or [**11**], and hence $B\pi_1(X)$ has the homotopy type of a bunch of circles, say B. Thus, X is a fibrewise space over a bunch of circles $B = B\pi_1(X)$. Let $s^X\colon B \to X$ represent the generators of $\pi_1(X)$ associated with circles. We may assume that $r^X \circ s^X \simeq 1_B$ and so X is a fibrewise-pointed space over B.

For such a space admitting a coaction of B along $r^X\colon X \to B$, a retraction map $\rho\colon B\vee D(X) \to X$ is constructed in [**11**, Theorem 3.3], where $D(X)$ is a simply connected finite complex. If we factorize $\rho_{|D(X)}\colon D(X) \to X$ through $\tilde{X}$, we deduce that if there exists a coaction of B along r^X, then X is dominated by $B \vee \tilde{X}$. The space B is a co-H-space and, if $\tilde{X}$ is a co-H-space, X is dominated by a co-H-space and is hence a co-H-space itself. On the other hand, recall that co-H-spaces are spaces with Lusternik–Schnirelmann (LS) category less than or equal to 1. Since the LS category of $\tilde{X}$ cannot exceed the LS category of X [**6**], if X is a co-H-space, then $\tilde{X}$ is also a co-H-space. Gathering this all together and taking into consideration that the almost localization is natural, we obtain the following result.

Proposition 1.1. *Let $r^X\colon X \to B\pi_1(X)$ be a fibrewise space over $B = B\pi_1(X)$ with a coaction of B along r^X. The following two statements then hold.*

(1) *The space X is a co-H-space if and only if $\tilde{X}$ is a co-H-space.*

(2) *The almost p-localization of X for a prime p, $X^B_{(p)}$, is also a fibrewise space over B with a coaction of B along the classifying map of the universal covering.*

Following earlier authors (see, for example, [**7**, **9**–**11**]), we paraphrase the fibrewise property (see [**12**, **13**]) for a fibrewise based space $r^X\colon X \to B\pi_1(X)$ as the 'almost' property for a space X. Recall that X is an 'almost' co-H-space if there exists a map $\mu\colon X \to X \vee_B X$ such that, when composed with each projection, the identity of X is obtained. Here $X \vee_B X$ is the pushout of the folding map $\nabla_B\colon B \vee B \to B$ and $s^X \vee s^X\colon B \vee B \to X \vee X$, where s^X is a section of the classifying map.

Our main result is the following theorem. The rest of this paper will be devoted to proving it.

Theorem 1.2. *Let $r^X\colon X \to B\pi_1(X)$ be a fibrewise space over $B = B\pi_1(X)$ with a coaction of B along r^X. If X is a connected finite complex whose almost p-localization is an almost co-H-space for every prime p, then X is a co-H-space.*

Some notation is required. For a set of primes P, the almost P-localization of $r^X\colon X \to B$ (in other words, a fibrewise P-localization of X) is a map $l^B_{(P)}\colon X \to X^B_{(P)}$ that commutes with projections to B. The map $l^B_{(P)}$ induces an isomorphism of fundamental groups and acts as a standard P-localization on the fibre $\tilde{X}$, so $\tilde{X}^B_{(P)} = \tilde{X}_{(P)}$.

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Most of the proofs in this paper follow by induction on the dimension of the space. Henceforth, it will be useful to work with the almost localization introduced by the second author in [**10**].

2. The 'almost' version of Zabrodsky mixing

To show the main result, we need a fibrewise version of a result of Zabrodsky [**16**, Proposition 4.3.1]. Let Π denote the set of all primes, and let P and Q denote the disjoint sets with $P \sqcup Q = \Pi$. We first give a lemma.

Lemma 2.1. *Let $M = B \vee M_0$ be the wedge sum of B, a bouquet of one-dimensional spheres, and let M_0 be a simply connected finite CW-complex. Let X be a fibrewise pointed space over B. Then,*

$$[M, X]_B = \text{pullback}\{[M, X^B_{(P)}]_B \to [M, X^B_{(0)}]_B \leftarrow [M, X^B_{(Q)}]_B\}.$$

Proof. Since we have the following equivalence between homotopy sets,

$$[M, X^B_{(P)}]_B = [B \vee M_0, X^B_{(P)}]_B = [M_0, X^B_{(P)}] = [M_0, \tilde{X}^B_{(P)}] = [M_0, \tilde{X}_{(P)}],$$

it is equivalent to prove that

$$[M_0, \tilde{X}] = \text{pullback}\{[M_0, \tilde{X}_{(P)}] \to [M_0, \tilde{X}_{(0)}] \leftarrow [M_0, \tilde{X}_{(Q)}]\}.$$

This is clear by the fracture square lemma of Bousfield–Kan [**2**, 6.3.(ii)], since M_0 is finite and $\tilde{X}$ is simply connected. □

Proposition 2.2. *Let M satisfy the conditions of the previous lemma and let $f\colon M \to X$ be a fibrewise pointed rational equivalence of fibrewise pointed spaces over $B = B\pi_1(M) = B\pi_1(X)$ commuting with coactions of B along the classifying maps of the universal coverings r^M and r^X, respectively. We then have the following.*

(1) *There exists a unique fibrewise pointed space $M(P, f)$ over B and fibrewise maps $\psi(P, f)\colon M \to M(P, f)$ and $\phi(Q, f)\colon M(P, f) \to X$ such that $\psi(P, f)$ is a fibrewise P-equivalence, $\phi(Q, f)$ is a fibrewise Q-equivalence and the following diagram is homotopy commutative:*

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & X \\ & {\scriptstyle \psi(P,f)} \searrow \quad \nearrow {\scriptstyle \phi(Q,f)} & \\ & M(P, f) & \end{array}$$

(2) *The fibrewise pointed space $M(P, f)$ over B and the fibrewise maps $\psi(P, f)$ and $\phi(Q, f)$ are natural with respect to f, i.e. for a homotopy commutative square*

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & X_1 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ M_2 & \xrightarrow[f_2]{} & X_2 \end{array}$$

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with f_1 and f_2 fibrewise rational equivalences, there exists $k(P; f_1, f_2)\colon M(P, f_1) \to M(P, f_2)$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccccc} M_1 & \xrightarrow{\psi(P,f_1)} & M(P,f_1) & \xrightarrow{\phi(Q,f_1)} & X_1 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle k(P;f_1,f_2)} & & \downarrow{\scriptstyle t} \\ M_2 & \xrightarrow[\psi(P,f_2)]{} & M(P,f_2) & \xrightarrow[\phi(Q,f_2)]{} & X_2 \end{array}$$

(3) *If, furthermore, M and X have the integral homology of a finite space, then $M(P, f)$ has as well.*

Proof. We define

$$\alpha(P,f) = \ell^B_P \circ f^B_{(P)} = f^B_{(0)} \circ \ell^B_P\colon M^B_{(P)} \to X^B_{(0)} \quad \text{and} \quad \beta(Q,f) = \ell^B_Q\colon X^B_{(Q)} \to X^B_{(0)},$$

where by $\ell^B_P\colon Y \to Y^B_{(0)}$ we denote the almost rationalization of a fibrewise based P-local space $r^Y : Y \to B$. With the above maps, we obtain the following commutative diagram:

$$\begin{array}{ccccc} M^B_{(Q)} & \xrightarrow{f^B_{(Q)}} & X^B_{(Q)} & = & X^B_{(Q)} \\ {\scriptstyle \ell^B_Q}\downarrow & & {\scriptstyle \beta(Q,f)}\downarrow & & {\scriptstyle \ell^B_Q}\downarrow \\ M^B_{(0)} & \xrightarrow{f^B_{(0)}} & X^B_{(0)} & = & X^B_{(0)} \\ {\scriptstyle \ell^B_P}\uparrow & & {\scriptstyle \alpha(P,f)}\uparrow & & {\scriptstyle \ell^B_P}\uparrow \\ M^B_{(P)} & = & M^B_{(P)} & \xrightarrow[f^B_{(P)}]{} & X^B_{(P)} \end{array} \tag{2.1}$$

Now, taking the universal covering spaces, we get pullback diagrams by using the Bousfield–Kan fibre square of localizations [**2**, p. 127]:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{\ell}_{(Q)}} & \tilde{M}_{(Q)} \\ {\scriptstyle \tilde{\ell}_{(P)}}\downarrow & & \downarrow{\scriptstyle \tilde{\ell}_Q} \\ \tilde{M}_{(P)} & \xrightarrow[\tilde{\ell}_P]{} & \tilde{M}_{(0)} \end{array} \qquad \begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\ell}_{(Q)}} & \tilde{X}_{(Q)} \\ {\scriptstyle \tilde{\ell}_{(P)}}\downarrow & & \downarrow{\scriptstyle \tilde{\ell}_Q} \\ \tilde{X}_{(P)} & \xrightarrow[\tilde{\ell}_P]{} & \tilde{X}_{(0)} \end{array}$$

Then, by Theorem 6.3 in [**4**], we obtain that the following square diagrams are fibrewise homotopy equivalent to fibrewise pullback diagrams:

$$\begin{array}{ccc} M & \xrightarrow{\ell^B_{(Q)}} & M^B_{(Q)} \\ {\scriptstyle \ell^B_{(P)}}\downarrow & & \downarrow{\scriptstyle \ell^B_Q} \\ M^B_{(P)} & \xrightarrow[\ell^B_P]{} & M^B_{(0)} \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\ell^B_{(Q)}} & X^B_{(Q)} \\ {\scriptstyle \ell^B_{(P)}}\downarrow & & \downarrow{\scriptstyle \ell^B_Q} \\ X^B_{(P)} & \xrightarrow[\ell^B_P]{} & X^B_{(0)} \end{array}$$

Let the pair of maps $\hat{\alpha}(P,f)\colon M(P,f) \to X^B_{(Q)}$ and $\hat{\beta}(Q,f)\colon M(P,f) \to M^B_{(P)}$ be the fibrewise pullback of $\alpha(P,f)$ and $\beta(Q,f)$ so that $M(P,f)^B_{(P)} = M^B_{(P)}$ and $M(P,f)^B_{(Q)} = X^B_{(Q)}$:

$$\begin{array}{ccc} M(P,f) & \xrightarrow{\hat{\alpha}(P,f)} & X^B_{(Q)} \\ {\scriptstyle \hat{\beta}(Q,f)}\big\downarrow & & \big\downarrow{\scriptstyle \beta(Q,f)} \\ M^B_{(P)} & \xrightarrow[\alpha(P,f)]{} & X^B_{(0)} \end{array}$$

Taking the fibrewise pullback of the vertical arrows in diagram (2.1), we immediately obtain the existence of maps $\psi(P,f)\colon M \to M(P,f)$ and $\phi(Q,f)\colon M(P,f) \to X$, which fit within the following commutative diagram:

$$\begin{array}{ccccc} M^B_{(Q)} & \xrightarrow{f^B_{(Q)}} & X^B_{(Q)} & = & X^B_{(Q)} \\ {\scriptstyle \ell^B_{(Q)}}\big\uparrow & & {\scriptstyle \hat{\alpha}(P,f)}\big\uparrow & & {\scriptstyle \ell^B_{(Q)}}\big\uparrow \\ M & \xrightarrow{\psi(P,f)} & M(P,f) & \xrightarrow{\phi(Q,f)} & X \\ {\scriptstyle \ell^B_{(P)}}\big\downarrow & & {\scriptstyle \hat{\beta}(Q,f)}\big\downarrow & & {\scriptstyle \ell^B_{(P)}}\big\downarrow \\ M^B_{(P)} & = & M^B_{(P)} & \xrightarrow[f^B_{(P)}]{} & X^B_{(P)} \end{array}$$

We then have that f and $\phi(Q,f) \circ \psi(P,f)$ have the same almost P-localization and Q-localization, hence they are equal by Lemma 2.1. Since f is a rational equivalence, $f^B_{(P)}$ is a Q-equivalence and, hence, so is $\phi(Q,f)$. Similarly, we obtain that $\psi(P,f)$ is a P-equivalence. Thus, Proposition 2.2 (1) is proved. The verification of (2) and (3) is straightforward and we leave it to the reader. □

Proposition 2.3. *For any prime p, we have $M(\bar{p},f) = M(\bar{p},\ell^B_{(p)} \circ f)$.*

Proof. By the unique existence, Proposition 2.2 (1), the proposition follows from

$$M \xrightarrow{\psi(\bar{p},f)} M(\bar{p},f) \xrightarrow{\ell^B_{(p)} \circ \phi(p,f)} X^B_{(p)},$$

since $\ell^B_{(p)}$ is a p-equivalence. □

We now give a result that will be used in the proof of the main theorem.

Proposition 2.4. *Let $f\colon M \to X$ be a fibrewise pointed rational equivalence satisfying the conditions of the previous proposition. If f is a fibrewise co-H-map, then $\psi(P,f)\colon M \to M(P,f)$ is a fibrewise co-H-map.*

Proof. We consider the following diagram, where ρ^Y_k is the projection onto the kth factor. Then the compositions of the vertical arrows on the left and of the vertical arrows

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on the right are the identity of M and X, respectively. The existence of the dotted arrow is given by Proposition 2.2 (1):

$$\begin{array}{ccccc}
M & \xrightarrow{\psi(P,f)} & M(P,f) & \xrightarrow{\phi(Q,f)} & X \\
\downarrow{\scriptstyle \nu_M} & & \vdots{\scriptstyle \nu_{M(P,f)}} & & \downarrow{\scriptstyle \nu_X} \\
M \vee_B M & \xrightarrow[\psi(P,f)\vee\psi(P,f)]{} & M(P,f) \vee_B M(P,f) & \xrightarrow[\phi(Q,f)\vee\phi(Q,f)]{} & X \vee_B X \\
\downarrow{\scriptstyle \rho_k^M} & & \downarrow{\scriptstyle \rho_k^{M(P,f)}} & & \downarrow{\scriptstyle \rho_k^X} \\
M & \xrightarrow{\psi(P,f)} & M(P,f) & \xrightarrow{\phi(Q,f)} & X
\end{array}$$

In order to see that $M(P,f)$ is a fibrewise co-H-space, it is sufficient to show that $\rho_k^{M(P,f)} \circ \nu_{M(P,f)}$ is a fibrewise homotopy equivalence for $k = 1, 2$. By the commutativity of the diagram, we can easily see that $\rho^{M(P,f)}\nu_{M(P,f)}$ induces both a homology $\mathbb{Z}_{(P)}$-equivalence and a homology $\mathbb{Z}_{(Q)}$-equivalence on the fibre. Since the fibre of $M(P,f) \to B$ is simply connected, $\rho_k^{M(P,f)}\nu_{M(P,f)}$ induces a homotopy equivalence on the fibre. Then, by a theorem of Dold [**4**], we can conclude that $\rho_k^{M(P,f)}\nu_{M(P,f)}$ is a fibrewise homotopy equivalence. Thus, $M(P,f)$ is an almost co-H-space and the difference between $\rho_k^{M(P,f)}\nu_{M(P,f)}$ and the identity of $M(P,f)$ is defined as a fibrewise map $d\colon M(P,f) \to M(P,f)$. Again using P and Q localization, we see that d is trivial and, hence, $\rho_k^{M(P,f)} \circ \nu_{M(P,f)}$ is homotopic to the identity. Thus, $\psi(P,f)\colon M \to M(P,f)$ is a fibrewise co-H-map. □

From now on, we fix a space X that is a finite complex with a coaction of $B = B\pi_1(X)$ along $r^X\colon X \to B$, the classifying map of the universal covering of X. Hence, X is a fibrewise pointed space as we mentioned in § 1. From [**10**], a co-H-space X has a homology decomposition $\{X_i;\, i \geqslant 1\}$ such that

$$B = X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n = X,$$

together with a cofibration sequence $S_i \xrightarrow{h_i} X_i \hookrightarrow X_{i+1}$, $i \geqslant 1$, where S_i stands for a Moore space of type $(H_{i+1}(X), i)$.

3. Fibrewise co-*H*-structures on almost localizations

By the assumption on a finite complex X, $X^B_{(p)}$ is an almost p-local co-H-space, which also implies that $X^B_{(0)}$ is an almost rational co-H-space. An almost rational co-H-space $X^B_{(0)}$ is a wedge sum of B and finitely many rational spheres of dimension not less than 2 [**7**]. Hence, the k'-invariants are all of finite order and we can prove the following lemma.

Lemma 3.1. *There exist a fibrewise pointed space $M = M(X)$ that is a wedge sum of a finite number of spheres and an almost rational equivalence $f = f(X)\colon M \to X$ that satisfies, for any prime p, that there are fibrewise co-H-structures on M such that $\ell^B_{(p)} \circ f\colon M \to X^B_{(p)}$ is a fibrewise co-H-map.*

Proof. We construct M and f by induction on the homological dimension of X. When $X = X_1$ we have nothing to do. We may therefore assume that $X = X_{i+1}$ and that we have constructed $M_i = M(X_i)$ and a fibrewise rational equivalence $f_i = f(X_i)\colon M_i \to X_i$ that satisfies the lemma. Let d_i be the order of the k'-invariant $h_i\colon S_i \to X_i$ and $M_{i+1} = M_i \vee \Sigma S_i^0$, where $S_i^0 \subseteq S_i$ is the Moore space of type $(H_{i+1}(X;\mathbb{Z})/\text{torsion}, i)$. We denote the multiplication by d_i by $d_i\colon S_i \to S_i$. Let $\nu_p\colon X_{(p)}^B \to X_{(p)}^B \vee_B X_{(p)}^B$ be the given co-H-structure on $X_{(p)}^B$ and let $\nu_i^M\colon M_i \to M_i \vee_B M_i$ be the co-H-structure such that $\ell_{(p)}^B \circ f_i$ is a fibrewise co-H-map. Since $(X_i)_{(p)}^B$ gives the homology decomposition of $X_{(p)}^B$, ν_p induces a fibrewise co-H-structure ν_p^i on $(X_i)_{(p)}^B$ by the arguments of [**10**]. Recall that the k'-invariants are all of finite order, hence $h_i \circ d_i|_{S_i^0} = *$, and thus we obtain the following commutative diagrams:

$$\begin{array}{ccc} S_i^0 & \xrightarrow{\;*\;} & M_i \\ {\scriptstyle d_i|_{S_i^0}}\downarrow & & \downarrow{\scriptstyle f_i} \\ S_i & \xrightarrow[h_i]{} & X_i \end{array} \qquad\qquad \begin{array}{ccc} M_i & \xrightarrow{\;\nu_i^M\;} & M_i \vee_B M_i \\ {\scriptstyle \ell_{(p)}^B\circ f_i}\downarrow & & \downarrow{\scriptstyle \ell_{(p)}^B\circ f_i \vee \ell_{(p)}^B\circ f_i} \\ (X_i)_{(p)}^B & \xrightarrow[\nu_p^i]{} & (X_i)_{(p)}^B \vee_B (X_i)_{(p)}^B \end{array}$$

By taking cofibres of the horizontal arrows of the left-hand square, we get the following commutative diagram:

$$\begin{array}{ccccc} M_i & \hookrightarrow & M_{i+1} & \longrightarrow & \Sigma S_i^0 \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f'_{i+1}} & & \downarrow{\scriptstyle \Sigma d_i^0|_{\Sigma S_i^0}} \\ X_i & \hookrightarrow & X_{i+1} & \longrightarrow & \Sigma S_i \end{array}$$

where $M_{i+1} = M_i \vee \Sigma S_i^0$ and f'_{i+1} is a rational equivalence, since f_i and $\Sigma d_i^0|_{\Sigma S_i^0}$ are rational equivalences.

Let ν'^M_{i+1} be a fibrewise co-H-structure defined by $\nu'^M_{i+1}|_{M_i} = \nu_i^M$ and

$$\nu'^M_{i+1}|_{\Sigma S_i^0}\colon \Sigma S_i^0 \xrightarrow{\nu_i^S} \Sigma S_i^0 \vee \Sigma S_i^0 \subseteq M_{i+1} \vee_B M_{i+1},$$

where ν_i^S denotes the standard co-H-structure of ΣS_i^0. Consider the following (not necessarily commutative) diagram:

$$\begin{array}{ccc} M_{i+1} & \xrightarrow{\;\nu'^M_{i+1}\;} & M_{i+1} \vee_B M_{i+1} \\ {\scriptstyle \ell_{(p)}^B\circ f'_{i+1}}\downarrow & & \downarrow{\scriptstyle \ell_{(p)}^B\circ f'_{i+1} \vee_B \ell_{(p)}^B\circ f'_{i+1}} \\ {X_{i+1}}_{(p)}^B & \xrightarrow[\nu_p^{i+1}]{} & {X_{i+1}}_{(p)}^B \vee_B {X_{i+1}}_{(p)}^B \end{array}$$

The difference between $(\ell_{(p)}^B \circ f'_{i+1} \vee_B \ell_{(p)}^B \circ f'_{i+1}) \circ \nu'^M_{i+1}$ and $\nu_p^{i+1} \circ \ell_{(p)}^B \circ f'_{i+1}$ is given by a map

$$D_{i+1}(p)\colon \Sigma S_i^0 \to \Omega_B({X_i}_{(p)}^B) *_B \Omega_B({X_i}_{(p)}^B) \subseteq \Omega_B({X_{i+1}}_{(p)}^B) *_B \Omega_B({X_{i+1}}_{(p)}^B).$$

Since f'_{i+1} is a rational equivalence, a multiple $a_{i+1}(p)D_{i+1}(p)$ can be pulled back to $\Omega_B(M_i)*_B\Omega_B(M_i)$. Now, the k'-invariants are of finite order, so X_{i+1} and M_{i+1} have the same almost p-localization except for primes in P, a finite set of primes. Let a_{i+1} be the multiple of the $a_{i+1}(p)$ for all $p \in P$. Let $f_{i+1}\colon M_{i+1} \to X_{i+1}$ be a map defined by $f_{i+1}|_{M_i} = f_i$ and by $f_{i+1}|_{\Sigma S_i^0} = a_{i+1}f'_{i+1}|_{\Sigma S_i^0}\colon \Sigma S_i^0 \to X_{i+1}$. We then choose a map

$$D_{i+1}^M(p)\colon \Sigma S_i^0 \to \Omega_B(M_i)*_B\Omega_B(M_i)$$

as a pullback of $a_{i+1}D_{i+1}(p)$ onto $\Omega_B(M_i)*_B\Omega_B(M_i)$. The co-$H$-structure defined by $\nu_{i+1}^M|_{M_i} = \nu_i^M$ and by

$$\nu_{i+1}^M|_{\Sigma S_i^0} = \nu'^M_{i+1}|_{\Sigma S_i^0} + D_{i+1}^M\colon \Sigma S_i^0 \to M_{i+1} \vee_B M_{i+1}$$

along with ν_{i+1}^p verify that $\ell_{(p)}^B \circ f_{i+1}$ is a fibrewise co-H-map at any prime $p \in P$. For a prime $q \notin P$, we may define comultiplication of $(X_{i+1})_{(q)}^B$ by that of $(M_{i+1})_{(q)}^B$, since $(X_{i+1})_{(q)}^B = (M_{i+1})_{(q)}^B$. Hence, the induction holds and we obtain the lemma. □

4. Proof of Theorem 1.2

Let P_1 and P_2 be disjoint sets of primes. We then have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\psi(\bar{P}_1,f)} & M(\bar{P}_1,f) \\ {\scriptstyle \psi(\bar{P}_2,f)}\downarrow & & \downarrow{\scriptstyle \psi(\bar{P}_1\cap\bar{P}_2,\phi(P_1,f))} \\ M(\bar{P}_2,f) & \xrightarrow[\psi(\bar{P}_1\cap\bar{P}_2,\phi(P_2,f))]{} & M(\bar{P}_1\cap\bar{P}_2,f) \end{array} \tag{4.1}$$

where the vertical arrows are P_1-equivalences and the horizontal arrows are P_2-equivalences.

Proposition 4.1. *Assume that M and the spaces $M(\bar{P}_i,f)$, for $i = 1,2$, admit fibrewise co-H-structures ν, ν^{P_i} for $i = 1,2$, respectively, such that $\psi(\bar{P}_i,f)$ are fibrewise co-H-maps for $i = 1,2$. Then, $M(\bar{P}_1 \cap \bar{P}_2,f)$ is a fibrewise co-H-space $(P_1 \cup P_2)$-equivalent to X.*

Proof. The commutativity of the diagram (4.1) gives a map from the pushout W of $M(\bar{P}_1,f)$ and $M(\bar{P}_2,f)$ to $M(\bar{P}_1 \cap \bar{P}_2,f)$ that induces an isomorphism of homology groups. It therefore follows that W and $M(\bar{P}_1 \cap \bar{P}_2,f)$ have the same fibre homotopy type. Since a fibrewise pushout of fibrewise co-H-structures is also a fibrewise co-H-space, we obtain that $M(\bar{P}_1 \cap \bar{P}_2,f)$ is also a fibrewise co-H-space such that $\psi(\bar{P}_1 \cap \bar{P}_2,f)\colon M \to M(\bar{P}_1 \cap \bar{P}_2,f)$ is a fibrewise co-H-map.

Now, it suffices to consider $\phi(P_1,f)\colon M(\bar{P}_1,f) \to X$ and $\phi(P_2,f)\colon M(\bar{P}_2,f) \to X$, making the diagram commutative, to conclude that the fibrewise pushout is $(P_1 \cup P_2)$-equivalent to X.

Now, given that $\phi(P_1,f)\colon M(\bar{P}_1,f) \to X$ and $\phi(P_2,f)\colon M(\bar{P}_2,f) \to X$ ensure the commutativity of the diagram, we conclude that the fibrewise pushout is $(P_1 \cup P_2)$-equivalent to X. □

Lemma 4.2. *Let $f\colon M \to X$ be the map constructed in Lemma 3.1, where M is a wedge sum of a finite number of spheres. Suppose that $X^B_{(P_1)}$ and $X^B_{(P_2)}$ are fibrewise co-H-spaces with fibrewise co-H-structures ν_1 and ν_2, respectively, such that $\ell^B_{(P_1)} \circ f$ and $\ell^B_{(P_2)} \circ f$ are fibrewise co-H-maps with respect to co-H-structures on M, ν' and ν'', respectively. There then exist fibrewise co-H-structures ν^0 on M, ν^{P_1} on $X^B_{(P_1)}$ and ν^{P_2} on $X^B_{(P_2)}$ such that $\ell^B_{(P_1)} \circ f$ and $\ell^B_{(P_2)} \circ f$ are fibrewise co-H-maps with respect to ν^0, ν^{P_1} and ν^{P_2}, respectively.*

Proof. Firstly, remark that, assuming that there exist fibrewise co-H-structures on M such that $\ell^B_{(P_1)}\circ f\colon M \to X^B_{(P_1)}$ and $\ell^B_{(P_2)}\circ f\colon M \to X^B_{(P_2)}$ are fibrewise co-H-maps is equivalent, by Proposition 2.4, to assuming that there exist fibrewise co-H-structures on M such that $\psi(\bar{P}_1, f)\colon M \to M(\bar{P}_1, f)$ and $\psi(\bar{P}_2, f)\colon M \to M(\bar{P}_2, f)$ are fibrewise co-H-maps. To simplify notation, we refer to them as ν^{P_1} on $M(\bar{P}_1, f)$ and ν^{P_2} on $M(\bar{P}_2, f)$.

Now, since M has the homotopy type of a wedge of spheres, there is a standard fibrewise co-H-structure ν on M induced from the unique co-H-structure map on each sphere. We denote by $d(\nu, \nu'')\colon M \to \Omega_B(M) *_B \Omega_B(M)$ the fibrewise difference of the fibrewise co-H-structure maps ν, ν' and ν''.

The proof will follow by induction on the homological decomposition of M. Since the fibrewise differences $d(\nu, \nu')$, $d(\nu, \nu'')$ and $d(\nu', \nu'')$ are trivial on $M_1 = M(X_1) = B$, we set $\nu_1 = \nu$, $\nu'_1 = \nu'$, $\nu''_1 = \nu''$, $\nu^{P_1}_1 = \nu^{P_1}$ and $\nu^{P_2}_1 = \nu^{P_2}$.

Secondly, we assume that there are fibrewise co-H-structures ν'_i and ν''_i on M, $\nu^{P_1}_i$ on $M(\bar{P}_1, f)$ and $\nu^{P_2}_i$ on $M(\bar{P}_2, f)$ such that $\psi(\bar{P}_1, f)$ and $\psi(\bar{P}_2, f)$ are fibrewise co-H-maps. We also assume that $d(\nu, \nu'_i)$ and $d(\nu, \nu''_i)$ coincide on $M_i = M(X_i)$ as induction hypotheses. Since $\psi(\bar{P}_1, f)$ is a $\bar{P}_1$-equivalence, for some integer s, there are extensions $d^{P_1}(\nu, \nu'_i)$ and $d^{P_1}(\nu, \nu''_i)$ of $sd(\nu, \nu'_i)$ and $sd(\nu, \nu''_i)$ on $M(\bar{P}_1, f)$ such that $(s, \bar{P}_1) = 1$. Similarly, for some integer t, we have extensions $d^{P_2}(\nu, \nu'_i)$ and $d^{P_2}(\nu, \nu''_i)$ of $td(\nu, \nu'_i)$ and $td(\nu, \nu''_i)$ on $M(\bar{P}_2, f)$ such that $(t, \bar{P}_2) = 1$.

Since $P_1 \cap P_2 = \emptyset$, we can choose integers n and m such that $ns + mt = 1$. Let $\nu^{P_1}_{i+1} = \nu^{P_1}_i - nd^{P_1}(\nu, \nu'_i) + nd^{P_1}(\nu, \nu''_i)$ and $\nu^{P_2}_{i+1} = \nu^{P_2}_i + md^{P_2}(\nu, \nu'_i) - md^{P_2}(\nu, \nu''_i)$, where the sum is taken by $\nu^{P_1}_i$ and $\nu^{P_2}_i$ on $M(\bar{P}_1, f)$ and $M(\bar{P}_2, f)$, respectively. Then, $\nu^{P_1}_{i+1}$ and $\nu^{P_2}_{i+1}$ give fibrewise co-H-structures on $M(\bar{P}_1, f)$ and $M(\bar{P}_2, f)$.

Similarly, let $\nu'_{i+1} = \nu'_i - nsd(\nu, \nu'_i) + nsd(\nu, \nu''_i)$ and $\nu''_{i+1} = \nu''_i + mtd(\nu, \nu'_i) - mtd(\nu, \nu''_i)$, where the sum is taken over ν'_i and ν''_i on M, respectively. By construction, $\psi(\bar{P}_1, f)$ and $\psi(\bar{P}_2, f)$ are fibrewise co-H-maps with respect to the fibrewise co-H-structures ν'_{i+1} and ν''_{i+1} on M, and $\nu^{P_1}_{i+1}$ and $\nu^{P_2}_{i+1}$, respectively. Now, using the fact that $ns + mt = 1$, and also by classical arguments of connectivity, one can prove that $d(\nu, \nu'_{i+1})$ coincides with $d(\nu, \nu''_{i+1})$ over M_{i+1}. Thus, by induction on i, we obtain the lemma. □

By Lemma 4.2, we inductively obtain that X is a fibrewise co-H-space, and therefore $\tilde{X}$ is a co-H-space. Hence, by Proposition 1.1, we obtain that X is a co-H-space. This completes the proof of Theorem 1.2.

Remark 4.3. Note that Theorem 1.2 cannot be directly obtained from [**3**, Theorem 1.1]. Indeed, following along the lines of the previous paragraph, as $X^B_{(p)}$ is a fibrewise co-H-space, $\tilde{X}_{(p)}$ is a co-H-space for every prime p. Since we are now in the simply connected case, one could be tempted to use [**3**] to conclude that $\tilde{X}$ is a co-H-space (and therefore X is as well, by Proposition 1.1). Unfortunately, results in [**3**] are obtained for finite-type spaces, while $\tilde{X}$ is not.

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