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Optimal premium pricing policy in a competitive insurance market environment

Athanasios A. Pantelous*
Institute for Financial and Actuarial Mathematics (IFAM), Department of Mathematical Sciences, University of Liverpool, UK

Eudokia Passalidou
Institute for Financial and Actuarial Mathematics (IFAM), Department of Mathematical Sciences, University of Liverpool, UK

Abstract
In this paper, we propose a model for the optimal premium pricing policy of an insurance company into a competitive environment using Dynamic Programming into a stochastic, discrete-time framework when the company is *expected* to drop part of the market. In our approach, the *volume* of business which is related to the past year experience, the *average* premium of the market, the company's *premium* which is a control function and a linear stochastic *disturbance,* have been considered. Consequently, maximizing the total expected linear discounted utility of the wealth over a finite time horizon, the optimal premium strategy is defined *analytically* and *endogenously.* Finally, considering two different strategies for the *average* premium of the market, the optimal premium policy for a company with an expected decreasing volume of business is derived and fully investigated. The results of this paper are further evaluated by using data from the Greek Automobile Insurance Industry.

JEL (classification): G22; C61

Subject (classification): IM20; IM30

Keywords
Optimal Premium Strategies; Competitive Markets; Volume of Business, Break-Even Premium Rate; Greek Automobile Insurance Industry.

1. Introduction

Nowadays, the number of products from different insurance companies has been significantly increased because of several micro and macro economical challenges, of the strong market competition and of the boosting securitization needs of the new era after the last (global) financial crisis. However, there is still little literature available in actuarial science on modelling how insurance premiums should be determined in *competitive* market environments, and how the competition actually affects the determination of the company's premiums; see for further discussion Daykin *et al.* (1994) and Emms *et al.* (2007).

*Correspondence to: Athanasios A. Pantelous, Institute for Financial and Actuarial Mathematics (IFAM), Department of Mathematical Sciences, University of Liverpool, L69 7ZL, Liverpool, UK. Tel. +44 151 79 45079. E-mail: A.Pantelous@liverpool.ac.uk

It is well-known in the insurance industry that the fair pricing process for non-life products is a crucial issue for every General Insurance company, especially within the unfolding of the time-bound de-tariffing road map by *Insurance Regulatory and Development Authority* (IRDA) which is once again under a great concern and publicity; see the recent article in *Insurance Chronicle*, Ramana (2006). Consequently, the failure of a uniform and global price in any Insurance Market, which can be based only on the premium rates, the policy terms and the conditions applicable to a particular portfolio of risks, force the insurance companies to provide more competitive prices. Especially, nowadays because of the global financial crisis, the premium strategy must be determined more accurately and competitively in order to ensure the viability of each company and to increase the volume of business in a long-term.

Inevitably, several questions can arise. For instance, in this part of the paper, we would like to mention just a few of them: "*What is the optimal premium strategy for an individual insurance company and for a specific portfolio of homogeneous or/and heterogeneous risks?*"; "*how is this related to the competitive market?*"; "*how does the volume of business affect the premium strategy?*" are only some of the questions that can be stated, and with non-trivial or straightforward answers.

The first attempt towards this direction was carried out by Taylor (1986, 1987) who investigated the relation between the market's behaviour and the optimal response of an individual insurer. Actually, he has assumed that this relation depends upon various factors including:

(a) the predicted time which will elapse before a return of market rates into profitability,

(b) the price elasticity of demand for the insurance product under consideration, and

(c) the rate of return required on the capital supporting the insurance operation.

Taylor (1986) investigates the appropriate response of an insurer, and he maximizes the expected present value of the wealth arising over a pre-defined finite time horizon. Additionally, he assumes that the insurance products display a positive price-elasticity of demand. Thus, if the market as a whole begins underwriting at a loss, any attempt by a particular insurer to maintain profitability will result in a reduction of his volume of business.

Consequently, following Taylor's (1986) ideas, for a given sequence of average market prices over fixed years to the planning horizon, the demand function $f_k(\cdot)$ is given by a relation of the following type $f_k : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ (where $\mathbb{R}_+ \triangleq (0, \infty]$),

$$f_k(V_{k-1}, p_k, \bar{p}_k, \theta_k), \tag{1.1}$$

where p_k denotes the premium rate charged by the insurer in year k, $\bar{p}_k$ denotes the average premium rate charged by all insurers in the market in year k, V_{k-1} denotes the company's volume of business for the previous year, i.e. $k-1$ and θ_k denotes the set of all other variables considered to be relevant to the demand function in year k. Obviously, as the demand function (1.1) is too general, Taylor (1986), and Emms *et al.* (2007) considered and investigated some special cases. Moreover, they assumed that the optimal pricing strategy prescribes a sequence of prices (premium rates) over the k years such as to maximize the expected profit of those k years discounted at rate of return per annum.

Taylor (1986) made several assumptions, however one of them can be further relaxed here in order to make the model a little more realistic. Thus, the assumption that the discarding of the unspecified set of variables amounts effectively to treating the sequence of market rates $\bar{p}_k$ as given, *exogenously* to the strategy of the insurer is under consideration here. Additionally, Taylor (1986) considered two different demand functions and he assumed a constant price for the elasticity demand. Thus, according to his results, it is found that the optimal strategies do not follow what someone might consider as obvious rules. For instance, it is not the case that profitability is best served by following the market during a period of premium rate depression. In particular, the optimal strategy may well involve underwriting for important profit margins at times when the average market premium rate is well short of breaking even.

The very interesting paper by Emms *et al*. (2007) can be considered as an extension of Taylor's (1986, 1987) ideas into a continuous-time stochastic framework, since they have used a stochastic process for modelling the market average premium, $\bar{p}$. In particular, they adapt Taylor's demand function and they model $\bar{p}$ using a geometric Brownian motion, i.e. $\frac{d\bar{p}}{\bar{p}} = \mu dt + \sigma dZ$ where Z is a Wiener process and both the drift μ and the volatility σ are assumed to be constant. Emms *et al.* (2007) handled the problem as a stochastic optimal control problem assuming that the premium policy is a control function where $q(t)$ denotes the volume of exposure at time t and $p(t)$ denotes the premium rate (per unit of exposure) charged by the insurer at time t. Therefore, the demand process is described by $\frac{dp}{q} = \log f(p, \bar{p})dt$, where $p = p(\bar{p}, \pi, t)$ is the premium at time t. Moreover, the utility function takes the linear form $U(w,t) = e^{-\beta t}w$, where β is the *inter-temporal* discount rate, and finally they defined the linear maximization problem $\max_p \mathbb{E}\int_0^T U(w(t), t)dt$ over a choice of strategies p and a finite time horizon T.

Rather than Taylor (1986, 1987), Emms *et al.* (2007) studied fixed premium strategies and the sensitivity of the model to its parameters involved. In their approach, the important parameters which determine the optimal strategies are the ratio of initial market average premium to break-even premium, the measure of the inverse elasticity of the demand function and the non-dimensional drift of the market average premium.

In our new approach, we introduce a stochastic demand function for the volume of business of an insurance company into a discrete-time framework extending further Taylor's (1986, 1987) ideas. Additionally, using a linear discounted function for the wealth process of the company, see also Emms *et al*. (2007), we provide an analytical, *endogenous* formula for the optimal premium strategy of the insurance company when it is expected to lose part of the market. Mathematically speaking, we create a maximization problem for the wealth process of a company, which is solved using stochastic dynamic programming. Thus, the optimal controller (i.e. the premium) is defined endogenously by the market as the company struggles to increase its volume of business into a competitive environment with the same characteristics as in Emms & Haberman (2005); Emms *et al.* (2007) and Taylor (1986, 1987). Finally, we consider two different strategies for the average premium of the market, and the optimal premium policy is derived and fully investigated. The results of this paper are further evaluated by using data from the Greek Automobile Insurance Industry.

The paper is organized as follows: In section 2 a discrete-time model for the insurance market is constructed. We discuss appropriate values for the model parameters and adopt suitable parameterizations. The next section considers each strategy in turn: we find analytical forms for the optimal strategies. In Premium Strategy I, the *average* premium of the market is calculated considering all the competitors of the market, and their proportions regarding the volume of business. In Premium Strategy II, the *average* premium of market is calculated considering the top 5 competitors of the market. Finally we summarize these results and make suggestions for modelling improvements in section 4.

2. Model Formulation

2.1 Basic Notation and Assumptions

Following Taylor's (1986, 1987) and Emms *et al.* (2007) approaches, we propose a stochastic, discrete-time premium pricing model to describe a competitive insurance market. Thus, the following notation is needed:

V_k: denotes the *volume of business* (or *exposure*) underwritten by the insurer in year $[k, k+1)$. This volume may be measured in any meaningful unit, e.g. number of claims incurred, total man-hours at risk (for workers' compensation insurance). In our paper, we consider the number of claims incurred as the volume of exposure.

π_k: denotes the *break-even* premium in year $[k, k+1)$, i.e. risk premium plus expenses per unit exposure.

p_k: denotes the *premium* charged by the insurer in year $[k, k+1)$. This is our *control* parameter.

$\bar{p}_k$: denotes the "*average*" *premium* charged by the market in year $[k, k+1)$. We further assume that this process is stochastic, see also Emms *et al.* (2007). Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the probability space and $\{\bar{p}_k | k = 1, 2, \ldots\}$ be the sequence of random variables defined on this probability space.

r: denotes the *rate of return* on equity required by shareholders of the insurer whose strategy is under consideration. We further assume that this rate is deterministic.

υ: denotes the corresponding discount factor, $\upsilon = (1+r)^{-1}$.

θ_k: denotes the set of all other stochastic variables (which are assumed to be independently distributed in time and Gaussian) and it is considered to be relevant to the demand function in year $[k, k+1)$, such as *inflation*, *interest* rate, *exchange* rate, *marketing* etc. In other words, this stochastic parameter tries to number the contracts that the company loses or gains due to these disturbances, which actually affect the volume of business of the insurance company. However, for the purposes of the present version of the paper, further analysis of the micro and macro economics parameters that get involved in θ_k is omitted. We will leave it as a future direction to our research.

In this paper, as in Emms *et al.* (2007) and Taylor (1986), we make the following assumptions.

Assumption 1: There is positive price-elasticity of demand, i.e. if the market as a whole begins underwriting at a loss, any attempt by a particular insurer to maintain profitability will result in a reduction of his volume of business.

Assumption 2: There is a finite time horizon.

Assumption 3: Demand in year $k+1$ is assumed to be proportional to demand in the preceding year k.

Assumption 4: θ_k affects the volume of business in a *linear* way (i.e. additive noise).

Additionally, extending Taylor's (1986, 1987) assumptions, we assume that the demand function is stochastic (because of θ_k and $\bar{p}_k$). Here, we denote the wealth process w_k as the insurer's capital at time $[k, k+1)$, following Emms *et al.* (2007) ideas, so we obtain

$$w_{k+1} = -a_k w_k + (p_k - \pi_k) V_k, \tag{2.1}$$

where $a_k \in [0,1]$ denotes the *excess return* on capital (i.e. return on capital required by the shareholders of the insurer whose strategy is under consideration). Thus, $-a_k w_k$ is the *cost* of holding w_k in the time interval $[k,k+1)$.

Following Taylor (1986, 1987), the volume of business (or exposure) of an insurance company for a given sequence of average market prices over the k^{th} year is given by a relation of the following type

$$V_k \triangleq f_k(V_{k-1}, p_k, \bar{p}_k, \theta_k), \tag{2.2}$$

where p_k is the controller and θ_k denotes the set of all other random variables (disturbances) which are considered to be relevant to the demand function. Under this assumption V_k is a stochastic variable and depends on k.

Our aim is to determine the strategy which maximizes the expected total utility of the wealth at time k over a *finite* time horizon T. As it has been also considered by Emms *et al.* (2007), we use a linear discounted function (of wealth).

Analytically, we want to maximize

$$\max_{p_k} \mathbb{E}\left[\sum_{k=0}^{T} U(w_k, k)\right], \tag{2.3}$$

where $U(w_k, k) = v^k w_k$ is the present value of the wealth w_k.

Consequently, substituting (2.2) into (2.1), the wealth process w_k is given by (2.4)

$$w_{k+1} = -a_k w_k + (p_k - \pi_k) f_k(V_{k-1}, p_k, \bar{p}_k, \theta_k), \tag{2.4}$$

and w_0, V_0, V_{-1} (the volume of business now and for the previous year) a_0 and π_0 are the initial conditions.

Extending Taylor's (1986, 1987) ideas, who assumed that the volume of business in year $k+1$ is proportional to the demand of the preceding year, in this paper we propose that the volume of business is proportional to the *average* premium charged by the market (see Assumption 3), but reverse proportional to the premium rate charged by the insurer in year k. Empirically speaking, this new approach might be considered as a little more realistic, since it is true that whenever the average premium stays unchanged and the premium charged by the insurer increases, unavoidably the company's volume of business might decrease. On the other hand, whenever the premium calculated by the insurer stays unchanged and the average premium decreases, the volume of business might decrease as well. These thoughts lead to the assumption that the volume of business should be proportional to the rate $\frac{\bar{p}_k}{p_k}$.

Additionally, it is realistic to assume that there might be an unexpected set of parameters, which can modify (i.e. decrease or increase) the volume of business. Consequently, we can assume that this set of parameters can be modelled using the stochastic variable θ_k, which can take either *positive* or *negative* values. In this paper, since we are more interested in investigating the premium strategy of an insurance company when it is expected to lose part of the market, we assume that the expected values of θ_k is *positive* (i.e. $\mathbb{E}(\theta_k) > \mu$, where $\mu > 0$ is a deductible parameter which can be pre-defined by the managerial team), and then the volume of business is *strictly* decreasing, i.e. losing part of the competitive market. Obviously, within the next lines, the case $\mathbb{E}(\theta_k) < \mu$ is also discussed, however this case is not very interested since it implies that the insurance company is increasing gradually its volume, and any change in its premium policy might affect it negatively.

Consequently, we can assume that the volume of business is given by

$$V_k = V_{k-1}\frac{\bar{p}_k}{p_k} - \theta_k, \tag{2.5}$$

where θ_k is being involved as an additive white noise.

2.2 Calculation of the Optimal Premium

After the basic notations, and the mathematical formulation of the problem, we need to calculate the optimal premium, which maximize the expected total utility of the wealth (2.3).

Following the general ideas about stochastic dynamic programming and control theory in a discrete-time framework, see for instance the classical books by Bertzekas (2000) and Kushner (1970), we determine the strategy which maximises the expected total utility of wealth (2.3) over a finite time horizon T, and over a choice of strategies p. This is similar to the objective function used by Taylor (1986, 1987), and Emms *et al.* (2007).

The next Theorem provides us with the optimal premium strategy for the finite time horizon maximization problem (2.3)–(2.5), see also Jacobson (1974) and Kushner (1970).

Theorem 1 *For the wealth process* $\{w_k\}_{k\in 0,1,\ldots,T-1}$ *given by*

$$w_{k+1} = -a_k w_k + (p_k - \pi_k)\left(V_{k-1}\frac{\bar{p}_k}{p_k} - \theta_k\right), \tag{2.6}$$

where $\mathbb{E}(\theta_k) > \mu$, $\mu > 0$, *and for the maximization problem defined by*

$$\max_{p_k} \mathbb{E}\left[\sum_{i=k}^{T-1} \upsilon^i w_i\right], \tag{2.7}$$

with initial conditions w_0, V_0, V_{-1}, a_0, *and* π_0, *the optimal strategy process* p_k^* *is given by*

$$p_k^* = \left(\frac{1}{\mathbb{E}(\theta_k)}\pi_k V_{k-1}\mathbb{E}(\bar{p}_k)\right)^{1/2} \text{ for } k = 0, 1, \ldots, T-1, \tag{2.8}$$

where $\bar{p}_k$, π_k *is the "average" and the break-even premium respectively, in year* k; V_{k-1} *is the volume of exposure underwritten by the insurer in year* $k-1$, *and* $\mathbb{E}(\theta_k)$ *is the expectation of the (stochastic) disturbance* θ_k *in year* k, *and the maximum value of (2.7) is given by*

$$w_0 d_0 + e_0. \tag{2.9}$$

Moreover, we define

$$d_k = \upsilon^k - a_k d_{k+1} > 0\,, \text{ and } d_T = 0, \tag{2.10}$$

$$\begin{aligned} e_k = {} & -d_{k+1}\left(\left(\frac{1}{\mathbb{E}(\theta_k)}\pi_k V_{k-1}\mathbb{E}(\bar{p}_k)\right)^{1/2}\mathbb{E}(\theta_k) - V_{k-1}\mathbb{E}(\bar{p}_k)\right) \\ & + d_{k+1}\pi_k\left(\mathbb{E}(\theta_k) - V_{k-1}\mathbb{E}(\bar{p}_k)\left(\frac{1}{\mathbb{E}(\theta_k)}\pi_k V_{k-1}\mathbb{E}(\bar{p}_k)\right)^{-1/2}\right) + e_{k+1}\,, \text{ and } e_T = 0. \end{aligned} \tag{2.11}$$

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Proof Define

$$J_k(w_k) \triangleq \max_{p_k, p_{k+1}, \ldots, p_{T-1}} \mathbb{E}_{|w_k}\left[\sum_{i=k}^{T-1} \upsilon^i w_i\right]. \tag{2.12}$$

Then, as it is known [6], the optimal performance criterion satisfied the Bellman equation

$$\begin{aligned} J_k(w_k) &= \max_{p_k} \mathbb{E}_{|w_k}\left\{\upsilon^k w_k + J_{k+1}(w_{k+1})\right\} \\ &= \max_{p_k}\left\{\upsilon^k w_k + \mathbb{E}_{|w_k} J_{k+1}(w_{k+1})\right\} \end{aligned} \tag{2.13}$$

where $\mathbb{E}_{|w_k}(\bar{p}_k) = \mathbb{E}(\bar{p}_k)$ and $\mathbb{E}_{|w_k}(\theta_k) = \mathbb{E}(\theta_k) > \mu > 0$, and $J_T(w_T) = w_T d_T + e_T = 0$; see (2.10), and (2.11).

We now show by induction that

$$J_k(w_k) = w_k d_k + e_k \tag{2.14}$$

solves (2.13) by noting that (2.14) is true for $k = T$ by assuming that (2.14) is true for $k+1$ and by proving is true for k. Substituting the assumed expression for $J_{k+1}(w_{k+1})$ into the right hand side (2.13) we obtain

$$J_k(w_k) = \max_{p_k}\left\{\upsilon^k w_k + \mathbb{E}_{|w_k} J_{k+1}(w_{k+1})\right\} = \max_{p_k}\left\{\upsilon^k w_k + \mathbb{E}_{|w_k}(w_{k+1}) d_{k+1} + e_{k+1}\right\},$$

and from (2.6) we have

$$\begin{aligned} &\max_{p_k}\left\{\upsilon^k w_k + \mathbb{E}_{|w_k}\left[-a_k w_k + (p_k - \pi_k) V_k\right] d_{k+1} + e_{k+1}\right\} \\ &= \max_{p_k}\left\{\upsilon^k w_k - a_k w_k d_{k+1} + d_{k+1}(p_k - \pi_k)\left(V_{k-1}\frac{\mathbb{E}(\bar{p}_k)}{p_k} - \mathbb{E}(\theta_k)\right)\right] + e_{k+1}\right\} \\ &= \max_{p_k}\left\{\upsilon^k w_k - a_k w_k d_{k+1} - d_{k+1}\left(p_k \mathbb{E}(\theta_k) - V_{k-1}\mathbb{E}(\bar{p}_k)\right) + d_{k+1}\pi_k\left(\mathbb{E}(\theta_k) - V_{k-1}\frac{\mathbb{E}(\bar{p}_k)}{p_k}\right) + e_{k+1}\right\} \\ &= \max_{p_k}\left\{-w_k(a_k d_{k+1} - \upsilon^k) - d_{k+1}\left(p_k \mathbb{E}(\theta_k) - V_{k-1}\mathbb{E}(\bar{p}_k)\right) + d_{k+1}\pi_k\left(\mathbb{E}(\theta_k) - V_{k-1}\frac{\mathbb{E}(\bar{p}_k)}{p_k}\right) + e_{k+1}\right\}. \end{aligned} \tag{2.15}$$

The controller that maximizes the above expression, (2.15), is given by (2.8), since

$$A = w_k\left(\upsilon^k - a_k d_{k+1}\right) + \left(V_{k-1}\mathbb{E}(\bar{p}_k) - p_k \mathbb{E}(\theta_k)\right) d_{k+1} - d_{k+1}\pi_k\left(V_{k-1}\frac{\mathbb{E}(\bar{p}_k)}{p_k} - \mathbb{E}(\theta_k)\right) + e_{k+1}.$$

The first derivative of A with respect to p_k is given

$$\frac{\partial A}{\partial p_k} = d_{k+1}\pi_k V_{k-1}\frac{\mathbb{E}(\bar{p}_k)}{p_k^2} - d_{k+1}\mathbb{E}(\theta_k) = d_{k+1}\left(\pi_k V_{k-1}\frac{\mathbb{E}(\bar{p}_k)}{p_k^2} - \mathbb{E}(\theta_k)\right).$$

If we equalize the first derivative with zero, i.e. $\frac{\partial A}{\partial p_k} = 0$, we obtain

$$d_{k+1}\left(\pi_k V_{k-1}\frac{\mathbb{E}(\bar{p}_k)}{p_k^2} - \mathbb{E}(\theta_k)\right) = 0 \overset{d_{k+1}\neq 0,\, \mathbb{E}(\theta_k) > \mu > 0}{\Leftrightarrow} \pi_k V_{k-1}\frac{\mathbb{E}(\bar{p}_k)}{p_k^2} - \mathbb{E}(\theta_k) = 0.$$

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The above expression gives the optimal strategy (2.8) as

$$\frac{\partial A}{\partial^2 p_k} = -2\pi_k V_{k-1}\mathbb{E}(\bar{p}_k)d_{k+1}\frac{1}{p_k^3} < 0,$$

where $\pi_k, V_{k-1}, \mathbb{E}(\overline{p_k}), \frac{1}{p_k^3}$ and $d_{k+1} > 0$.

Now, let's substitute the above into (2.15), we obtain

$$-w_k(a_k d_{k+1} - v^k) - d_{k+1}\left(\left(\frac{1}{\mathbb{E}(\theta_k)}\pi_k V_{k-1}\mathbb{E}(\bar{p}_k)\right)^{1/2}\mathbb{E}(\theta_k) - V_{k-1}\mathbb{E}(\bar{p}_k)\right)$$
$$+ d_{k+1}\pi_k\left(\mathbb{E}(\theta_k) - V_{k-1}\left(\frac{1}{\mathbb{E}(\theta_k)}\pi_k V_{k-1}\mathbb{E}(\bar{p}_k)\right)^{-1/2}\right) + e_{k+1}.$$

Substituting (2.10) and (2.11) in the above expression yields the fact that (2.14) is true. Thus, the proof of the Theorem 1 by induction is complete. □

Remark 1 As it is quite likely in practice, the optimal premium strategy given by (2.6) expression depends *endogenously* on the volume of business of the previous year, the break-even premium rate, the expected value of the *average* premium rate of the market and the (stochastic) variable θ_k.

Remark 2 In order to calculate the optimal premium strategy, initially we have to calculate the expectation of θ_k which models the set of all other parameters considered to be relevant to the demand function of each company, and the insurance market (i.e. financial environment, managerial policy etc); see also Assumption 4. In particular, as it has been clearly stated in the introduction; see also Remark 3, and Proposition 1, we are interested to modify the premium strategy when our volume of business is *strictly* decreasing because of the positive $\mathbb{E}(\theta_k) > \mu$. Note that as it came clear from the relation (2.5) θ_k is equal to $V_{k-1}\frac{\bar{p}_k}{p_k} - V_k$ for each previous year.

Remark 3 In a competitive market environment, we have considered that the volume of business in each company is *strictly* decreasing when the expectation of the stochastic variable (disturbance) θ_k in year $k = 0,1,\ldots,T-1$ takes positive values. Thus, the company should change the premium policy in order to enlarge its volume. On contrary, for negative or below the deductible point $\mu > 0$ values for the expectation of θ_k, i.e. $\mathbb{E}(\theta_k) < \mu$, the previous premium strategy might stay unchanged (see next corollary), since the company does not lose (significant) part of the market (i.e. by decreasing its volume).

The following proposition considers the case where the volume of business changes either *above* or *below* $\mu > 0$ (i.e., for decreasing or increasing the volume of business above or below the required level, respectively).

Remark 4 Moreover, we can show that the optimal expected wealth of the company at the year $k + 1$ is given by (2.16)

$$\mathbb{E}(w_{k+1}^*) = V_{k-1}\mathbb{E}(\bar{p}_k) + \pi_k\mathbb{E}(\theta_k) - \left\{a_k w_k + 2\left(\mathbb{E}(\theta_k)\pi_k V_{k-1}\mathbb{E}(\bar{p}_k)\right)^{1/2}\right\} \textit{ for } \mathbb{E}(\theta_k) > \mu. \quad (2.16)$$

As Taylor (1986, 1987), and Emms *et al.* (2007) propose, and in order to take benefit of the analytical formula derived by Theorem 1 for the determination of the premium strategies into a competitive environment, in the next section we use data from the Greek automobile insurance industry, see also the tables of the Hellenic Association of Insurance Companies (2010). Moreover, we assume that the premium strategies concern the price of a contract which refers to a six-month insurance for a car that is 1400cc, 10 years old and its value estimated at 5.000€.

3. Premium Strategies

3.1 Premium strategy I: Considering the Entire Market

In the first premium strategy, the expected *average* premium is calculated considering all the competitors of the market, and their proportions regarding the volume of business. In mathematical terms the expected *average* premium of the market can be estimated by

$$\mathbb{E}(\bar{p}) = \frac{1}{m}\sum_{i=1}^{K} b_{i,n}p_{i,n}, \tag{3.1}$$

where $b_{i,n} = V_{i,n}\left(\sum_{i=1}^{K} V_{i,n}\right)^{-1}$ and $\sum_{i=1}^{K} b_{i,n} = 1$ for every year n, $p_{i,n}$ is the premium of the company i^{th} for the year n; K is the number of the competitors (including also our company's premium) in the insurance market and m is the number of years for the available data (i.e. we assume that we have the uniform distribution for the weight of every year). Moreover, for the calculation of the expected values of the premium of each company and the average market premium respectively, we use the available Greek data, see next paragraphs.

Proposition 1 *Considering* (2.6) *and* (3.1), *the optimal controller (i.e. premium) for the premium strategy I is equal to*

$$p_k^* = \sqrt{\frac{1}{m\mathbb{E}(\theta_k)}\pi_k V_{k-1}\sum_{i=1}^{K} b_{i,n}p_{i,n}}, \text{ for } \mathbb{E}(\theta_k) > \mu > 0,\ k = 0, 1, \ldots, T-1. \tag{3.2}$$

Proof The proof derives straightforwardly, and it is omitted. □

This premium strategy considers the premium and the volume of business of the *entire* market. The expected *average* premium of the market is estimated using the (3.1) expression, i.e. as an expected weighted average of each competitor that gets involved in the market. Moreover, it is clear that the premium of the company with the largest volume of business affects most of the market (see also Premium Strategy II). In Table 1, the premium prices and the number of contracts for the 12 major non-life Greek insurance companies for a standard six-month cover of a 10-year old, 1400cc car (with 5.000 Euros covered amount) are presented for the years 2006, 2007, 2008 and 2009.

As we can observe in Table 1, and according to the *oligopoly* theory which began in 1838 with Cournot's oligopoly model, see for more details Friedman (1983) and the references therein, the Greek non-life insurance industry has an oligopoly market characteristic, since there are only a few main competitors, the insurance products are almost identical (with non-significant differences) and the ownership of the key inputs and barriers imposed by the government. Thus, in the case of oligopolistic market, the revenues of the firms depend on the actions of other competitors as we have considered in our premium strategies; see also Emms *et al.* (2007) and Taylor (1986, 1987).

According to the premium strategy I, the average premium of the market is equal to the weighted average of the premiums of all the companies involved in the market for every year. Moreover, the volume of business of each company for the years 2006–2009 is presented in Table 2. Finally, Table 3 summarizes the results of the (3.1) expression.

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Table 1. Premium prices in *Euros* and number of contracts for the 12 major non-life Greek insurance companies, see Hellenic Association of Insurance Companies (2010).

Insurance Companies	2006		2007		2008		2009	
	premium	# contracts	premium	# contracts	premium	# contracts	premium	# contracts
A	269.09	*298,269*	280.30	*280,991*	301.00	*261,196*	307.35	*240,698*
B	282.07	*303,673*	293.82	*308,766*	306.06	*278,362*	315.53	*250,614*
C	377.06	*282,224*	392.77	*252,630*	413.44	*258,683*	430.67	*266,414*
D	371.52	*304,609*	404.96	*255,250*	437.35	*263,510*	451.35	*278,321*
E	281.56	*295,769*	292.96	*258,181*	304.71	*274,382*	323.68	*243,294*
F	**377.83**	***796,139***	**397.71**	***687,485***	**432.30**	***726,317***	**469.89**	***779,376***
G	257.88	*298,304*	268.62	*325,836*	291.98	*273,470*	307.35	*271,487*
H	366.99	*200,135*	386.30	*182,989*	402.40	*258,534*	423.58	*267,341*
I	347.58	*211,314*	373.74	*278,174*	397.59	*283,295*	418.52	*284,889*
J	351.18	*299,690*	377.02	*318,876*	392.73	*316,556*	426.88	*338,434*
K	364.11	*299,995*	378.67	*340,898*	401.39	*344,771*	429.09	*396,112*
L	291.22	*319,453*	302.87	*287,524*	314.98	*246,976*	331.77	*241,609*

Table 2. (The volume of business, b, in % for the 12 major non-life Greek insurance companies).

Volume of Business b (%)	2006	2007	2008	2009
A	7.63%	7.44%	6.90%	6.24%
B	7.77%	8.17%	7.35%	6.49%
C	7.22%	6.69%	6.83%	6.90%
D	7.79%	6.76%	6.96%	7.21%
E	7.57%	6.83%	7.25%	6.31%
F	**20.36%**	**18.20%**	**19.18%**	**20.20%**
G	7.63%	8.63%	7.22%	7.04%
H	5.12%	4.84%	6.83%	6.93%
I	5.41%	7.36%	7.48%	7.38%
J	7.67%	8.44%	8.36%	8.77%
K	7.67%	9.02%	9.11%	10.27%
L	8.17%	7.61%	6.52%	6.26%

As it has already been mentioned above in order to calculate the optimal premium for each company first, we have to estimate the expectation of θ_k which models the set of all other parameters considered to be relevant to the demand function of each company, and the insurance market (i.e. financial environment, managerial policy etc). As it is clear from the relation (2.5) $\hat{\theta}_k$ (estimation of θ_k) can be calculated by $\hat{\theta}_k = V_{k-1}\frac{\bar{p}_k}{p_k} - V_k$.

Thus, considering the above expression, and for the available Greek data we are able to calculate $\hat{\theta}_k$ for the years 2007, 2008 and 2009 as it is shown at Table 4. So, the expected value of θ_k for the year 2010 can be given by

$$\mathbb{E}(\theta_k) = \frac{1}{m}\sum_{i=1}^{m}\hat{\theta}_i \tag{3.3}$$

Then, in Table 4, we present the expected values of θ_k (using the estimations of θ_k). As has been already mentioned before, θ_k denotes the number of contracts that the company loses or gains

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Table 3. (The expected *average* premium in *Euros* of the market for the year 2010 is given by $\mathbb{E}(\bar{p}) = \frac{1}{m}\sum_{i=1}^{K} b_{i,n}p_{i,n}.$).

Average Premium (P.S.I)	Amount in Euros
$\mathbb{E}(\overline{p})$	**364.68**

Table 4. (The values of $\hat{\theta}_k$, the change in percentage for the volume of business for the years 2007-2009, and the expected values of θ_k for the year 2010).

Companies	2006–2007		2007–2008		2008–2009		$\mathbb{E}(\theta_k)$
A	90,752	*−0.19%*	89,087	*−0.54%*	100,436	*−0.66%*	**93,255**
B	52,299	*0.41%*	100,177	*−0.82%*	103,515	*−0.86%*	**85,149**
C	−1,604	*−0.53%*	−29,404	*0.14%*	−25,304	*0.07%*	−2,503
D	7,533	*−1.03%*	−44,518	*0.20%*	−43,965	*0.25%*	−27,093
E	94,521	*−0.73%*	43,547	*0.41%*	96,982	*−0.94%*	**78,191**
F	11,841	*−2.16%*	−129,592	*0.99%*	−158,903	*1.01%*	−92,514
G	62,114	*1.00%*	145,263	*−1.40%*	85,677	*−0.19%*	**97,490**
H	−1,998	*−0.28%*	−87,902	*1.98%*	−22,335	*0.10%*	−37,508
I	−80,648	*1.96%*	−20,770	*0.12%*	−13,172	*−0.10%*	−34,994
J	−41,180	*0.78%*	−11,890	*−0.08%*	−40,763	*0.41%*	−31,426
K	−64,133	*1.35%*	−26,096	*0.08%*	−73,578	*1.16%*	−54,760
L	80,957	*−0.56%*	95,540	*−1.09%*	57,215	*−0.26%*	**77,744**

because of the parameters that affect the volume of business and they have not been included in the model. In our application, the large fluctuations in the expected values of θ_k occur due to a) the limited number of the available data, and b) the impact on each company's volume of business into the market. (Note that since we have available data for only 4 years, it is difficult to provide a good estimation for the expected values of θ_k. However, for the purpose of our application, this drawback is not crucial.)

The values of the stochastic variable θ_k can be either *above* or *below* $\mu > 0$. As we have extensively discussed in section 2, we will determine the optimal premium strategy for the year 2010 only for those companies which have positive $\mathbb{E}(\theta_k) > 0$.

These companies are A, B, E, G and L, see Figure 1.

In Table 5, we present the premium for each company for the different values of the break-even premium rate.

As it is expected, for greater values of the π_k, greater the optimal premium values become. Consequently, since the optimal premium depends on the break-even premium rate, the company should choose its competitive strategy considering the market's construction and its marginal costs; see also Emms *et al.* (2007). Thus, each company should pre-determine its break-even premium rate, in order to calculate the optimal premium strategy which will enlarge its volume of business. The results of Table 5 are shown also at Figure 1.

The results of Table 5 (see also Figure 2) are seemingly interesting. For the five insurance companies (A, B, E, G, and L) which expected to experience losses on their volume of business, for a break-even

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Table 5. (The optimal premium strategy in *Euros* for the 5 Greek insurance companies that have positive $\mathbb{E}(\theta_k)$ for the different values of the break-even premium rate).

Companies	π_k $\mathbb{E}(\theta_k)$	20%	30%	40%	50%	60%
A	*93,255*	240.52	294.58	340.15	380.30	416.60
B	*85,149*	260.24	318.73	368.03	411.47	450.75
E	*78,191*	271.01	331.92	383.27	428.50	469.40
G	*97,490*	249.83	305.98	353.32	395.02	432.73
L	*77,744*	274.21	335.84	387.79	433.56	474.94

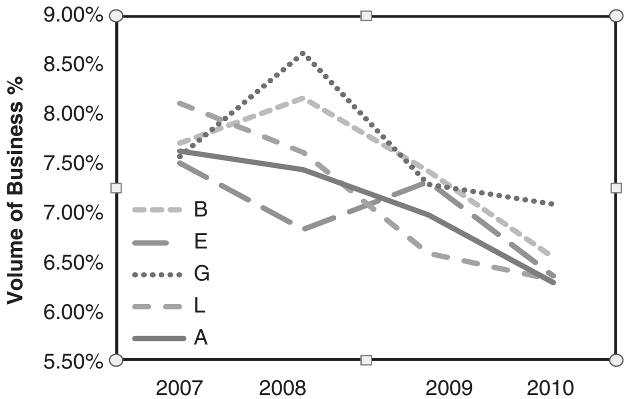

Figure 1. (The real and the expected volume of business for the 5 Greek insurance companies that have positive $\mathbb{E}(\theta_k) > \mu$).

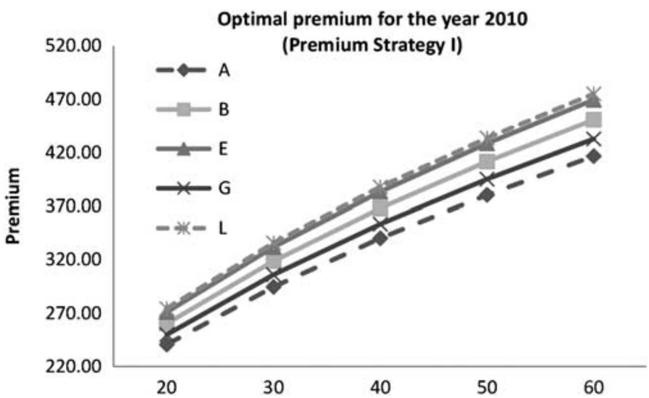

Figure 2. (The optimal premium strategy in *Euros* for the 5 Greek insurance companies that have positive $\mathbb{E}(\theta_k)$ for the different values of the break-even premium rate)

premium rate of 20–30% calculated by the formula 3.2, premiums are below the market average premium of 364.68€. Additionally, it is true that the insurance companies E and L which face similar losses (see Tables 1, 2, and 4) should provide similar premiums, which appear to be the most expensive premiums compared with the premiums of the other 3 companies.

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At this point, it should be mentioned that in this paper, we are not able to analyze further the results of Table 1, and consequently of Table 5 (and Figure 2), since the analysis of the Greek insurance market, and the micro/macro conditions that get involved for the determination of the premium strategy is far beyond the scopes of the present version of the paper. Additionally, the macro-micro economic analysis of the parameters that affect θ_k is a purpose of future research.

3.2 Premium Strategy II: Following the Leaders of the Market

In this premium strategy, the *average* premium is calculated considering the premiums of the top K^{top} competitors of the market (including the leading company of the market). In mathematical terms the expected *average* premium of the market is estimated by

$$\mathbb{E}(\bar{p}) = \frac{1}{m}\sum_{i=1}^{K^{top}} b_{i,n}^{top} p_{i,n}^{top}, \tag{3.4}$$

where $b_{i,n}^{top} = V_{i,n}\left(\sum_{i=1}^{K^{top}} V_{i,n}\right)^{-1}$ and $\sum_{i=1}^{K^{top}} b_{i,n}^{top} = 1$ for every year n, $p_{i,n}^{top}$ is the premium of the i^{th} top company for the year n; K^{top} is the number of the top competitors (including also our company's premium) in the insurance market and m is the number of years for the available data (i.e. we assume that we have the uniform distribution for the weight of every year). Next, similar to the Proposition 2, we obtain the following Proposition.

Proposition 2 *Considering* (2.6) *and* (3.4), *the optimal controller (i.e. premium) for the premium strategy II is equal to*

$$p_k^* = \sqrt{\frac{1}{m\mathbb{E}(\theta_k)}\pi_k V_{k-1}\sum_{i=1}^{K^{top}} b_{i,n}^{top} p_{i,n}^{top}}, \text{ for } \mathbb{E}(\theta_k) > \mu > 0, \text{ for } k = 0, 1, \ldots, T-1. \tag{3.5}$$

Proof. The proof derives straightforwardly, and it is omitted. □

For the purpose of this application, we consider the premium and the volume of business of the top 5 Greek insurance companies. Consequently, the expected *average* premium of the market is calculated using the (3.4) expression.

In Table 6, the premiums and the number of contracts for the 5 leading non-life Greek insurance companies are presenting for the years 2006, 2007, 2008 and 2009.

Thus, for the years 2006, we calculate the average premium considering the premium and the volume of business for the companies B, C, D, F, and L; for the year 2007 and 2008: B, F, G, J and K, and for the year 2009: D, F, I, J and K. In Table 7, the volume of business is presented.

According to the premium strategy II, the average premium of the market is equal to the weighted average of the premiums of the top 5 companies in the market for every year. Finally, Table 8 summarizes the results of the (3.4) expression.

Now, we calculate again the estimation of θ_k, since the *average* premium of the market for the years 2006, 2007, 2008 and 2009 has changed. Additionally, the average premium in the Premium Strategy I is higher than in the Premium Strategy II. So, in Table 9, we present the expected values of θ_k.

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Table 6. (Premiums prices in *Euros* and number of contracts for the top 5 non-life Greek insurance companies; see Friedman, 1983).

2006			2007			2008			2009		
Ins. Com.	premium	# contracts	Ins. Com.	premium	# contracts	Ins. Com.	premium	# contracts	Ins. Com.	premium	# contracts
B	282.07	303,673	**B**	293.82	368,766	**B**	306.06	278,362	**D**	451.35	278,321
C	377.06	312,224	**F**	397.71	687,485	**F**	432.30	726,317	**F**	468.89	779,376
D	371.52	304,609	**G**	268.62	325,836	**G**	291.98	273,470	**I**	418.52	274,889
F	377.83	796,139	**J**	377.02	318,876	**J**	392.73	316,556	**J**	426.88	338,434
L	291.22	319,453	**K**	378.67	340,898	**K**	401.39	344,771	**K**	429.09	396,112

Table 7. (The weights in % for the calculation of the average premium for the top 5 non-life Greek insurance companies).

2006	2007	2008	2009
14.91% (B)	18.06% (B)	14.35% (B)	13.46% (D)
15.33% (C)	33.67% (F)	37.45% (F)	37.70% (F)
14.96% (D)	15.96% (G)	14.10% (G)	13.30% (I)
39.10% (F)	15.62% (J)	16.32% (J)	16.37% (J)
15.69% (L)	16.70% (K)	17.78% (K)	19.16% (K)

Table 8. (The expected *average* premium in *Euros* of the market for the year 2010 is given by $\mathbb{E}(\bar{p}) = \frac{1}{m}\sum_{i=1}^{K^+} b^+_{i,n}p^+_{i,n}$.).

Average Premium (P.S.II)	Amount in Euros
$\mathbb{E}(\overline{p})$	**382.50**

Table 9. (The values of $\hat{\theta}_k$, the change in percentage for the volume of business for the years 2007–2009, and the expected values of θ_k for the year 2010).

Companies	2006–2007		2007–2008		2008–2009		$\mathbb{E}(\theta_k)$
A	93,507	*−0.19%*	95,824	*−0.54%*	138,074	*−0.66%*	**109,136**
B	54,974	*0.41%*	107,458	*−0.82%*	142,587	*−0.86%*	**101,674**
C	27,137	*−0.53%*	11,005	*0.14%*	−15,958	*0.07%*	**7,395**
D	9,480	*−1.03%*	−40,306	*0.20%*	−18,108	*0.25%*	−16,311
E	97,135	*−0.73%*	49,662	*0.41%*	134,526	*−0.94%*	**93,774**
F	17,023	*−2.16%*	−118,115	*0.99%*	−90,445	*1.01%*	−63,846
G	64,989	*1.00%*	153,316	*−1.40%*	125,084	*−0.19%*	**114,464**
H	−657	*−0.28%*	−84,620	*1.98%*	4,696	*0.10%*	−26,860
I	−79,184	*1.96%*	−15,720	*0.12%*	26,806	*−0.10%*	−22,700
J	−39,122	*0.78%*	−6,031	*−0.08%*	−7,920	*0.41%*	−17,691
K	−62,082	*1.35%*	−19,967	*0.08%*	−37,993	*1.16%*	−40,014
L	83,688	*−0.56%*	102,127	*−1.09%*	90,185	*−0.26%*	**92,000**

Next, we will determine the optimal premium strategy for the year 2010 only for those companies which have positive $\mathbb{E}(\theta_k)$ with $\mu > 10{,}000$ (the managerial team is not interested in modifying the premium when it expects to lose only a few thousand contracts), i.e. A, B, E, G and L.

The results of Table 10 (see also Figure 3) are also similar with those of the Premium Strategy I, since the five insurance companies have premiums significantly below the market average premium of 382.50€ for a break-even premium rate of 20–40%.

Now, if we would like to compare the findings of the two Premium Strategies, we can easily see that the Premium Strategy II is cheaper (i.e. it provides lower premiums) than the Premium Strategy I for all the A, B, E, G and L insurance companies. This result was expected, as in the Greek insurance market, the leader (dominator) companies have expensive premiums, above the average premium of the market.

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Table 10. (The optimal premium strategy in Euro for the *5* Greek insurance companies that have $\mathbb{E}(\theta_k) > 10,000$ for the different values of the break-even premium rate).

Companies	π_k / $\mathbb{E}(\theta_k)$	20	30	40	50	60
A	*109,136*	227.64	278.81	321.94	359.94	394.29
B	*101,674*	243.84	298.64	344.84	385.55	422.34
E	*93,774*	253.38	310.32	358.33	400.62	438.86
G	*114,464*	236.07	289.13	333.86	373.26	408.89
L	*92,000*	258.09	316.09	364.99	408.07	447.02

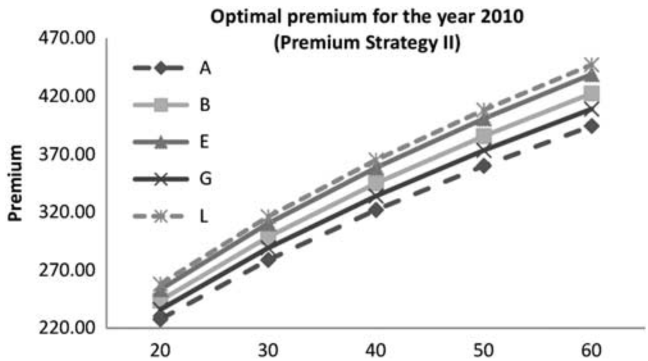

Figure 3. Optimal premium for the year 2010 (Premium Strategy II).

4. Conclusions – Further Research

In this paper, extending further the ideas proposed by Taylor (1986, 1987), Emms & Haberman (2005) and Emms *et al.* (2007), we develop a model for the optimal premium pricing policy of a non-life insurance company into a competitive market environment using elements of dynamic programming into a stochastic, discrete-time framework when the insurance company is expected to lose part of the market competition. For that reason, a stochastic demand function for the volume of business $f_k(V_{k-1}, p_k, \bar{p}_k, \theta_k) = V_{k-1}\frac{\bar{p}_k}{p_k} - \theta_k$ of an insurance company into a discrete-time has been applied. Additionally, in our approach, the *volume* of business, V_k which is related to the past year experience, the *average* premium of the market, $\bar{p}_k$, the company's *premium*, p_k, which is a control function, and a stochastic *disturbance*, θ_k, have been also considered. Thus, by maximizing the total expected linear discounted utility of the wealth $U(w_k, k) = \upsilon^k w_k$ over a finite time horizon, the optimal premium strategy is defined analytically and endogenously for $\mathbb{E}(\theta_k) > \mu > 0$.

Finally, we consider two different strategies for the *average* premium of the market. In the Premium Strategy I, the *average* premium is calculated considering all the competitors of the market, and their proportions regarding the volume of business. However, in the Premium Strategy II, the *average* premium is calculated considering the premiums of the top K^{top} competitors of the market (including the leading company of the market). The results of this paper are further evaluated by using data from the Greek Automobile Insurance Industry, which is an oligopoly market.

As a further extension of the proposed model, we would like to consider the following:

- To use a more general stochastic, non-linear demand-function for the volume of business.
- To analyze further the stochastic parameter θ_k, since several macro-micro economic parameters get involved.
- Additionally, to include into our model the inflation rate, the taxation and different other quantitative parameters, such as legislation constraints etc, regarding the insurance market environment.
- Finally, to consider a different non-linear maximization criterion.

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