J. Plasma Phys. (2016), *vol.* 82, 905820502
doi:10.1017/S0022377816000805

Properties and evolution of anisotropic structures in collisionless plasmas

A. R. Karimov[1,2,†], **M. Y. Yu**[3,4] **and L. Stenflo**[5]

[1]Institute for High Temperatures, Russian Academy of Sciences, Izhorskaya 13/19, Moscow, 127412, Russia

[2]Department of Electrophysical Facilities, National Research Nuclear University MEPhI, Kashirskoye shosse 31, Moscow, 115409, Russia

[3]Department of Physics, Zhejiang University, 310027 Hangzhou, China

[4]Institut für Theoretische Physik I, Ruhr-Universität Bochum, D-44780 Bochum, Germany

[5]Department of Physics, Linköping University, SE-58183 Linköping, Sweden

(Received 20 July 2016; revised 30 August 2016; accepted 30 August 2016)

A new class of exact electrostatic solutions of the Vlasov–Maxwell equations based on the Jeans's theorem is proposed for studying the evolution and properties of two-dimensional anisotropic plasmas that are far from thermodynamic equilibrium. In particular, the free expansion of a slab of electron–ion plasma into vacuum is investigated.

Key words: plasma expansion, plasma flows, plasma nonlinear phenomena

1. Introduction

There has been much interest in the properties of quasi-stationary structures containing particles whose interaction is governed by long-range, such as the gravitational or electrostatic, forces (Holloway & Dorning 1991; Buchanan & Dorning 1993; Lancellotti & Dorning 1998; Schamel 2004; Kozlov 2008; Campa *et al.* 2014; Levin *et al.* 2014). Such structures are associated with many phenomena, including solitons, shocks, vortices, nonlinear waves, etc. in nature and the laboratory, and can often exist even far from thermodynamic equilibrium because of the absence collisional or turbulent relaxation within the time scale of interest. In particular, they can appear in highly rarefied plasmas not near thermodynamic equilibrium. However, depending on its initial distribution, a collisionless plasma can often still evolve into quasi-stationary states because of the presence of the self-consistent, or averaged, electrostatic field of the individual charged particles (Holloway & Dorning 1991; Buchanan & Dorning 1993; Lancellotti & Dorning 1998; Schamel 2004; Kozlov 2008; Levin *et al.* 2014). However, unlike collisional relaxation, which tends to drive the system towards thermodynamic equilibrium, in collisionless relaxation any initial energy imbalance among the different degrees of freedom can be preserved, causing the system to evolve in a preferred direction in the physical or phase space. Moreover, in plasmas the motion of both the electrons and ions can play important

† Email address for correspondence: alexanderkarimov999@gmail.com

roles in the evolution, even though they are on very different time scales because of the much larger ion mass. This is because the initial or short-time behaviour can often determine the pathway and thus the asymptotic behaviour of the highly nonlinear evolution (see, e.g. Schamel (2004), Luque & Schamel (2005), Eliasson & Shukla (2006) and the references therein). Complex behaviour can also be expected for magnetized plasmas, which are anisotropic and a large number of different modes of collective motion can exist (Clemmow & Dougherty 1969; Lancellotti & Dorning 1998; Eliasson & Shukla 2006; Levin *et al.* 2014).

In order to investigate the evolution and properties of collisionless plasmas, we shall construct time-dependent non-Maxwellian distribution functions satisfying the two-dimensional Vlasov–Maxwell equations. In general, for initial states far from equilibrium, the convection terms in the governing equations are not small (Kuznetsov 1996; Bohr *et al.* 1998; Kiessling 2003; Schamel 2004). They in fact determine the asymptotic behaviour, which (if it exists) is usually still not near equilibrium (see, e.g. Taranov 1976; Lewis & Symon 1984; Majda, Majda & Zheng 1994; Dorozhkina & Semenov 1998; Karimov & Lewis 1999; Karimov 2001; Kovalev & Bychenkov 2003; Karimov 2013; Schamel 2015). For such problems it is necessary to use a fully nonlinear formulation. In this paper, we shall invoke the Jeans's theorem (Clemmow & Dougherty 1969; Lewis & Symon 1984; Kiessling 2003; Agren *et al.* 2005; Agren & Moiseenko 2006; Pecseli 2012) to consider the two-dimensional (2-D) evolution of a collisionless electron–ion plasma slab, in particular, its expansion into vacuum.

2. Formulation of the kinetic problem

We are interested in the non-relativistic electrostatic evolution of a finite 2-D unmagnetized electron–ion plasma slab. The distribution functions $f_s = f_s(t, \boldsymbol{r}, \boldsymbol{v})$, where $s = e, i$ indicate the electrons and ions, respectively, are governed by the corresponding Vlasov and reduced Maxwell equations

$$\partial_t f_s + \boldsymbol{v}\cdot\nabla_r f_s + \frac{q_s}{m_s}\boldsymbol{E}\cdot\nabla_v f_s = 0, \tag{2.1}$$

$$\nabla\cdot\boldsymbol{E} = 4\pi\sum_s q_s n_s, \tag{2.2}$$

$$\nabla\times\boldsymbol{E} = 0, \tag{2.3}$$

$$\partial_t\boldsymbol{E} = -4\pi\sum_s q_s\boldsymbol{j}_s, \tag{2.4}$$

where $\boldsymbol{E}$ is the electrostatic field and $n_s = \int f_s(t, \boldsymbol{r}, \boldsymbol{v})\,\mathrm{d}\boldsymbol{v}$ the number density, $\boldsymbol{j}_s = \int \boldsymbol{v} f_s(t, \boldsymbol{r}, \boldsymbol{v})\,\mathrm{d}\boldsymbol{v}$ the flux and q_s the charge, of the s particles. The reduced (no displacement current and magnetic field perturbation) Maxwell equations correspond to the Darwin approach for open-boundary electrostatic problems in plasma physics (see, e.g. Nielson & Lewis 1976; Degond & Raviart 1992; Birdsall & Langdon 2004). The approach is particularly useful for considering electrostatic phenomena in complex laboratory and space plasmas since by using the reduced current equation, one can avoid solving Poisson's equation, which requires stringent boundary conditions (Arfken & Weber 2006). In the Darwin approach, the electrostatic nature of the problem is preserved by proper formulation of the initial condition.

As mentioned, we shall consider the moving-boundary problem of the expansion of a 2-D plasma slab far from thermal equilibrium, i.e. we look for solutions of (2.1)–(2.4) in the form

$$f_s = f_s(t, x, y, v_x, v_y) > 0 \tag{2.5}$$

defined in the region $\Gamma = \{(x, y), |x| \leqslant X_s(t), |y| \leqslant Y_s(t)\}$, where $X_s(t)$ and $Y_s(t)$ denote the average fronts, or boundaries (to be defined more precisely later), separating the s particles from the vacuum. The initial slab can thus be defined by $X_s(t=0) = Y(t=0) = L$, where $L > 0$ is the initial dimension of the plasma slab in the x and y directions. Accordingly, as initial condition we take

$$f_s(t=0, \boldsymbol{r}, \boldsymbol{v}) = \begin{cases} f_{0s}(v_x, v_y) > 0, & |x| \leqslant L, |y| \leqslant L \\ 0, & |x| > L, |y| > L, \end{cases} \tag{2.6}$$

where the initial distribution function $f_{0s}(v_x, v_y)$ has finite moments

$$\left| \int_{-\infty}^{+\infty} \boldsymbol{v}^k f_{0s}(\boldsymbol{r}, \boldsymbol{v})\, \mathrm{d}\boldsymbol{v} \right| < \infty, \quad k = 0, 1, 2, \ldots . \tag{2.7}$$

The plasma is assumed to be initially neutral and at rest, so that we have

$$\int_{-\infty}^{+\infty} f_{0s}(\boldsymbol{r}, \boldsymbol{v})\, \mathrm{d}\boldsymbol{v} = 1 \quad \text{and} \quad \int_{-\infty}^{+\infty} \boldsymbol{v} f_{0s}(\boldsymbol{r}, \boldsymbol{v})\, \mathrm{d}\boldsymbol{v} = 0. \tag{2.8a,b}$$

It is convenient to normalize (2.1)–(2.4) by

$$\bar{t} = \omega_{pe} t, \quad \bar{\boldsymbol{r}} = \frac{\boldsymbol{r}}{L}, \quad \bar{\boldsymbol{v}} = \frac{\boldsymbol{v}}{v_0}, \quad \bar{\boldsymbol{E}} = \frac{\boldsymbol{E}}{E_0}, \quad \bar{n}_s = \frac{n_s}{n_0}, \tag{2.9a–e}$$

where $\omega_{pe} = \sqrt{4\pi n_0 e^2/m_e}$ is the electron plasma frequency, $E_0 = 4\pi e n_0 L$, $v_0 = L\omega_{pe}$, n_0 is the initial plasma density ($n_0 = n_e = n_i$) and $-e$ and m_e are the charge and mass of the electron, respectively. For clarity, in the following we shall omit the overhead bars. The normalized equations are then

$$\partial_t f_s + \boldsymbol{v} \cdot \boldsymbol{\nabla}_r f_s + \frac{Q_s}{M_s} \boldsymbol{E} \cdot \boldsymbol{\nabla}_v f_s = 0, \tag{2.10}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \sum_s Q_s n_s \tag{2.11}$$

$$\partial_t \boldsymbol{E} = -\sum_s Q_s \boldsymbol{j}_s, \tag{2.12}$$

where $Q_e = -1$, $M_e = 1$ and $Q_i = 1$, $M_i = m_i/m_e$ and m_i is the ion mass.

The plasma is assumed to be symmetric with respect to $(x=0, y=0)$. It is therefore sufficient to consider only the upper half of Γ. Accordingly, we can write

$$f_s(t=0, x, y, v_x, v_y) = f_{0s}(a^0_{xs} v_x + b^0_{xs} v_y, a^0_{ys} v_x + b^0_{ys} v_y), \tag{2.13}$$

where a^0_{xs}, b^0_{xs}, a^0_{ys} and b^0_{ys} are constants, defined in the initial region $\Gamma(t=0) = \{(x, y), 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$. We see that the system is not in thermodynamic (Maxwellian) equilibrium and there is a preferred direction for the evolution, given by the constant coefficients a^0_{xs}, b^0_{xs}, a^0_{ys} and b^0_{ys}. The latter are determined by the self-consistent electrostatic field as well as external fields, if any.

3. The invariants

According to the Jeans's theorem, solutions of the Vlasov–Maxwell equation can be written as

$$f_s = f_s(I_{1s}, I_{2s}, \ldots, I_{Ks}), \tag{3.1}$$

where $I_{1s}, I_{2s}, \ldots, I_{Ks}$ are the invariants of motion, i.e. they remain constant along the trajectory of a particle, or along a phase-space characteristic of (2.10), even though they can be functions of time, space and velocity.

We start from a simple case, where the invariants I_{ls} are linear functions of $\boldsymbol{v}$, namely

$$I_{ls} = a_{ls}(t)v_x + b_{ls}(t)v_y + c_{ls}(t)x + d_{ls}(t)y + h_{ls}(t), \tag{3.2}$$

where the coefficients $a_{ls}(t)$, $b_{ls}(t)$, $c_{ls}(t)$, $d_{ls}(t)$ and $h_{ls}(t)$ depend only on time. The ansatz (3.2) corresponds to presetting the spatial structures of the self-consistent and the external fields (if any), and thereby also the particle densities, currents, etc. These parameters have to be obtained by trial and error, such that $d_t I_{ls} = 0$ along the particle trajectory (see, e.g. Lewis & Leach 1982; Struckmeier & Riedel 2001). That is, the time-dependent coefficients should exist and satisfy (2.10)–(2.12).

The equations for the coefficients $a_{ls}(t)$ to $h_{ls}(t)$ can be obtained by substituting (3.1) into (2.10):

$$\sum_l \left[G_s(\boldsymbol{x}, \boldsymbol{v}) + \frac{Q_s}{M_s}(a_{ls}E_x + b_{ls}E_y) \right] \partial_{I_{ls}} f_s = 0, \tag{3.3}$$

where

$$G_s(\boldsymbol{x}, \boldsymbol{v}) = \dot{a}_{ls}v_x + \dot{b}_{ls}v_y + \dot{c}_{ls}x + \dot{d}_{ls}y + c_{ls}v_x + d_{ls}v_y + \dot{h}_{ls}, \tag{3.4}$$

and the overhead dot denotes the time derivative. Since $\partial_{I_{ls}} f_s$ should be independent for $s = e, i$, equation (3.3) is satisfied if

$$G_s(\boldsymbol{x}, \boldsymbol{v}) + \frac{Q_s}{M_s}(a_{ls}E_x + b_{ls}E_y) = 0. \tag{3.5}$$

Since the space and velocity coordinates are independent in the phase space, we obtain $c_{ls} = -\dot{a}_{ls}$, $d_{ls} = -\dot{b}_{ls}$, and

$$\ddot{a}_{ls}x + \ddot{b}_{ls}y - \dot{h}_{ls} = \frac{Q_s}{M_s}(a_{ls}E_x + b_{ls}E_y). \tag{3.6}$$

The initial conditions (2.13)

$$a_{ls}(t=0) = a^0_{ls}, \quad b_{ls}(t=0) = b^0_{ls}, \quad h_{ls}(t=0) = \dot{a}_{ls}(t=0) = \dot{b}_{ls}(t=0) = 0 \quad (3.7a–c)$$

allow us to define the number K (here not more than four) of constants of motion. In fact, the set (3.2) is a system of linear algebraic equations relating I_{ls} to v_x and v_y. Accordingly, the four invariants $I_{ls} \neq 0$ at any time uniquely determine v_x and v_y. However, from (3.7) we have $d_s(t=0) = c_s(t=0) = h_{ls}(t=0) = 0$ at $t = 0$, so that the rank of the matrix with $a_{ls}, b_{ls}, c_{ls}, d_{ls}$ and h_{ls} is not more than two. That is, there are only two independent equations. The distribution functions can then be rewritten as functions of the invariants

$$f_s(t, \boldsymbol{r}, \boldsymbol{v}) = f_s(I_{xs}, I_{ys}), \tag{3.8}$$

where $l = x, y$.

The case $K = 1$ is singular: I_{xs} and I_{ys} are not linearly independent. Nevertheless, it is still realistic and shall thus be separately considered later.

4. The case $K=2$

We first investigate the case $K=2$, and use (3.8) to determine a_{ls}, b_{ls} and $\boldsymbol{E}$. The particle densities and fluxes can be expressed as integrals in I_{xs} and I_{ys} (in place of v_x and v_y) (see appendix A)

$$n_s=\frac{1}{\Lambda_s}, \tag{4.1}$$

which verifies the ansatz that the densities n_s are functions of time only. Moreover, we have

$$j_{xs}=\frac{\dot{a}_{xs}b_{ys}-b_{xs}\dot{a}_{ys}}{\Lambda_s^2}x+\frac{\dot{b}_{xs}b_{ys}-b_{xs}\dot{b}_{ys}}{\Lambda_s^2}y+\frac{b_{xs}h_{ys}-b_{ys}h_{xs}}{\Lambda_s^2}, \tag{4.2}$$

$$j_{ys}=\frac{\dot{a}_{ys}a_{xs}-a_{ys}\dot{a}_{xs}}{\Lambda_s^2}x+\frac{a_{xs}\dot{b}_{ys}-a_{ys}\dot{b}_{xs}}{\Lambda_s^2}y+\frac{a_{ys}h_{xs}-a_{xs}h_{ys}}{\Lambda_s^2}, \tag{4.3}$$

where $\Lambda_s=a_{xs}b_{ys}-a_{ys}b_{xs}$.

From (2.11) and (4.1), we obtain

$$\partial_x E_x+\partial_y E_y=\sum_s q_s\Lambda_s^{-1}, \tag{4.4}$$

where the right-hand side is only a function of t. One can then write

$$E_x=A(t)x+B(t)y+H(t),\quad E_y=C(t)x+D(t)y+F(t), \tag{4.5a,b}$$

where the time-dependent functions A, B, C, D, H and F still have to be determined. Substituting (4.5) into (2.3) we get

$$C(t)=B(t). \tag{4.6}$$

Equation (3.6) then becomes

$$\ddot{a}_{ls}x+\ddot{b}_{ls}y-\dot{h}_{ls}=\frac{Q_s}{M_s}[(a_{ls}A+b_{ls}B)x+(a_{ls}B+b_{ls}D)y+(a_{ls}H+b_{ls}F)]. \tag{4.7}$$

The terms involving the space coordinates x and y can now be separated. Accordingly, we have

$$\ddot{a}_{ls}=\frac{Q_s}{M_s}(a_{ls}A+b_{ls}B),\quad \ddot{b}_{ls}=\frac{Q_s}{M_s}(a_{ls}B+b_{ls}D),\quad \dot{h}_{ls}=-\frac{Q_s}{M_s}(a_{ls}H+b_{ls}F). \tag{4.8a–c}$$

One can verify that Ampere's law without the displacement current is identically satisfied.

Similarly, from (4.2), (4.3) and (4.5) with (4.6) in (2.12), one obtains

$$\dot{A}x+\dot{B}y+\dot{H}=-x\sum_s Q_s\frac{\dot{a}_{xs}b_{ys}-b_{xs}\dot{a}_{ys}}{\Lambda_s^2}-y\sum_s Q_s\frac{\dot{b}_{xs}b_{ys}-b_{xs}\dot{b}_{ys}}{\Lambda_s^2}-\sum_s Q_s\frac{\dot{b}_{xs}h_{ys}-b_{ys}\dot{h}_{xs}}{\Lambda_s^2} \tag{4.9}$$

and

$$\dot{B}x+\dot{D}y+\dot{F}=-x\sum_s Q_s\frac{\dot{a}_{ys}a_{xs}-a_{ys}\dot{a}_{xs}}{\Lambda_s^2}-y\sum_s Q_s\frac{a_{xs}\dot{b}_{ys}-a_{ys}\dot{b}_{xs}}{\Lambda_s^2}-\sum_s Q_s\frac{\dot{a}_{ys}h_{xs}-a_{xs}\dot{h}_{ys}}{\Lambda_s^2}. \tag{4.10}$$

https://doi.org/10.1017/S0022377816000805 Published online by Cambridge University Press

Equating the terms in equations (4.9) and (4.10) of similar spatial dependence, we get

$$\dot{A}=\sum_s Q_s\Lambda_s^{-2}(b_{xs}\dot{a}_{ys}-\dot{a}_{xs}b_{ys}), \tag{4.11}$$

$$\dot{B}=\sum_s Q_s\Lambda_s^{-2}(b_{xs}\dot{b}_{ys}-\dot{b}_{xs}b_{ys}), \tag{4.12}$$

$$\dot{D}=\sum_s Q_s\Lambda_s^{-2}(a_{ys}\dot{b}_{xs}-a_{xs}\dot{b}_{ys}), \tag{4.13}$$

$$\dot{H}=\sum_s Q_s\frac{b_{ys}\dot{h}_{xs}-\dot{b}_{xs}h_{ys}}{\Lambda_s^2}, \tag{4.14}$$

$$\dot{F}=\sum_s Q_s\frac{a_{xs}\dot{h}_{ys}-\dot{a}_{ys}h_{xs}}{\Lambda_s^2}, \tag{4.15}$$

and from (4.6) the condition

$$\sum_s Q_s\Lambda_s^{-2}(b_{xs}\dot{b}_{ys}-\dot{b}_{xs}b_{ys})=\sum_s Q_s\Lambda_s^{-2}(a_{ys}\dot{a}_{xs}-\dot{a}_{ys}a_{xs}). \tag{4.16}$$

From the mathematical point of view, the equations (4.8)–(4.16) form a closed set that depends on the parameters A, B and D. We now consider the physical meanings of these parameters and the relation (4.16). Accordingly, we first evaluate

$$\Omega_s=\nabla\times\boldsymbol{j}_s. \tag{4.17}$$

Inserting (4.2) and (4.3), we get

$$\Omega_s=\frac{b_{xs}\dot{b}_{ys}+\dot{a}_{ys}a_{xs}-\dot{b}_{xs}b_{ys}-a_{ys}\dot{a}_{xs}}{\Lambda_s^2}. \tag{4.18}$$

It follows that (4.16) corresponds to the condition for vortex-free motion. The functions $B(t)$ and $C(t)$ are then related to the vortex component of the electrical field in (4.5) by $C-B=\sum Q_s\Omega_s$.

On the other hand, combining (4.11) and (4.13) and integrating with respect to time, we get

$$A+D=\sum Q_s n_s, \tag{4.19}$$

which shows that $A(t)$ and $D(t)$ are related to the action of the electrostatic field.

Finally, we should define the moving boundaries $X_s(t)$ and $Y_s(t)$ for the expanding electron and ion fluids by requiring that the total number

$$N_s=\int_0^{X_s}\int_0^{Y_s} n_s\,\mathrm{d}x\,\mathrm{d}y \tag{4.20}$$

of each species of particles is constant since there is no loss or source of particles in the evolving plasma volume $\Gamma(t)$. For spatially homogeneous plasma, we have $N_s=X_sY_sn_s$.

https://doi.org/10.1017/S0022377816000805 Published online by Cambridge University Press

For convenience, we set $N_e(t=0)=N_i(t=0)=1$ in the initial volume $\Gamma(t=0)$. From the particle conservation condition $d_t N_s = 0$ we obtain

$$Y_s n_s \left[\dot{X}_s + \frac{\dot{n}_s}{2n_s} X_s\right] + X_s n_s \left[\dot{Y}_s + \frac{\dot{n}_s}{2n_s} Y_s\right] = 0. \tag{4.21}$$

Since $X_s(t)$ and $Y_s(t)$ are independent, we can set

$$\dot{X}_s + \frac{\dot{n}_s}{2n_s} X_s = 0, \quad \dot{Y}_s + \frac{\dot{n}_s}{2n_s} Y_s = 0 \tag{4.22a,b}$$

and find

$$X_s(t) = Y_s(t) = n_s^{-1/2}, \tag{4.23}$$

which in view of (4.1) becomes

$$X_s(t) = Y_s(t) = \Lambda_s^{1/2}. \tag{4.24}$$

5. Existence of solutions

We now show that there indeed exist non-trivial solutions of (4.8)–(4.16). Let us consider the invariants with the coefficients

$$b_{xs} = a_{ys}, \quad b_{ys} = a_{xs}, \quad h_{xs} = h_{ys} = 0, \tag{5.1a–c}$$

so that (4.16) is satisfied and (4.12) becomes

$$\dot{B} = \sum_s \frac{Q_s}{a_{xs}^2 - a_{ys}^2} \left[\frac{\dot{a}_{xs} + \dot{a}_{ys}}{a_{xs} + a_{ys}} - \frac{\dot{a}_{xs} - \dot{a}_{ys}}{a_{xs} - a_{ys}}\right]. \tag{5.2}$$

Equations (4.11) and (4.13) become identical:

$$\dot{A} = \dot{D} = \sum_s \frac{Q_s}{a_{xs}^2 - a_{ys}^2} \left[\frac{\dot{a}_{xs} + \dot{a}_{ys}}{a_{xs} + a_{ys}} + \frac{\dot{a}_{xs} - \dot{a}_{ys}}{a_{xs} - a_{ys}}\right], \tag{5.3}$$

so that $A(t) = D(t)$ if $A(t=0) = D(t=0)$. As a result, equations (4.8) reduce to

$$\ddot{a}_{ls} = \frac{Q_s}{M_s}(Aa_{xs} + Ba_{ys}), \quad \ddot{a}_{ys} = \frac{Q_s}{M_s}(Aa_{ys} + Ba_{xs}). \tag{5.4a,b}$$

Equation (4.13) can be integrated to

$$A = D = \frac{1}{2}\sum_s \frac{Q_s}{a_{ys}^2 - a_{xs}^2} \equiv \frac{1}{2}\sum_s Q_s n_s \neq 0. \tag{5.5}$$

As mentioned, the functions $A(t)$ and $D(t)$ are associated with the action of electrostatic field $\boldsymbol{E}$. However, for (5.1) there are no non-trivial quasi-neutral states, i.e. the plasma layer always remains charged. To verify this we start by assuming the opposite, namely $n_e = n_i$. In view of (5.1), we can then write

$$a_{ye}^2 - a_{xe}^2 \equiv a_{yi}^2 - a_{xi}^2, \tag{5.6}$$

https://doi.org/10.1017/S0022377816000805 Published online by Cambridge University Press

and set $A\equiv 0$ in (5.4). From the reduced equations (5.4) one gets

$$\frac{M_s}{Q_s}(a_{ys}\ddot{a}_{xs}-a_{xs}\ddot{a}_{ys})=(a_{ys}^2-a_{xs}^2). \tag{5.7}$$

The condition (5.6) requires that the right-hand side of (5.7) does not depend on s, so that we can set

$$a_{ls}=\sqrt{\frac{Q_s}{M_s}}\alpha_l(t), \tag{5.8}$$

where $\alpha_l(t)$ is a function of t. However, this form of a_{ls} cannot satisfy the quasi-neutrality condition (5.6). Accordingly, the choice (5.1) cannot describe quasi-neutral expansion of the plasma slab.

Thus, the ordinary differential equations (ODEs) (5.2)–(5.4), together with the initial conditions on the distribution functions, fully determine the evolution of the plasma, which remains non-neutral for all t. Given the initial values of a_{ls} and b_{ls}, one can numerically integrate these ODEs. The evolution of the distribution functions is then determined when the explicit forms of the initial distribution functions $f_s(I_{xs}(t=0), I_{ys}(t=0))$ are specified.

6. The reduced case $K=1$

We now consider the degenerate case, where the rank of the matrix of the algebraic equations (3.2) is unity, or when the equations are linearly dependent. For simplicity, we shall concentrate on the case where the distribution function depends only on one invariant, say I_s, or

$$f_s=f_s(I_s). \tag{6.1}$$

A simple but physically relevant exact solution can be obtained if we also set $b_s=\lambda a_s$, where λ is an arbitrary constant. Then I_s becomes

$$I_s=a_s v_x+\lambda a_s v_y-\dot{a}_s x-\lambda\dot{a}_s y+h_s, \tag{6.2}$$

where we have omitted the subscript l (i.e. $a_s=a_{ls}$, $b_s=b_{ls}$ and $h_s=h_{ls}$). We note that the problem remains exact and two-dimensional, even though we have used only one invariant and a specific choice of parameters.

From (B 3) and (B 8*a*,*b*) one can get the particle densities and fluxes (see appendix B)

$$n_s=\frac{1}{\lambda a_s^2},\quad j_{xs}=\frac{\dot{a}_s x-h_s}{\lambda a_s^3},\quad j_{ys}=\frac{\dot{a}_s}{\lambda a_s^3}y. \tag{6.3a–c}$$

The relations (4.9) and (4.10) then become

$$\dot{A}x+\dot{B}y+\dot{H}=-\sum_s Q_s\frac{\dot{a}_s x-h_s}{\lambda a_s^3}, \tag{6.4}$$

$$\dot{B}x+\dot{D}y+\dot{F}=-\sum_s Q_s\frac{\dot{a}_s y}{\lambda a_s^3}. \tag{6.5}$$

It follows that

$$\dot{B}=\dot{F}=0,\quad \dot{A}=\dot{D}=-\sum_s Q_s\frac{\dot{a}_s}{\lambda a_s^3},\quad \dot{H}=\sum_s Q_s\frac{h_s}{\lambda a_s^3}. \tag{6.6a–c}$$

Integrating the first two relations in (6.6) with respect to t, we find

$$B = F = 0, \quad A = D = \frac{1}{2\lambda}\sum_s \frac{Q_s}{a_s^2}, \tag{6.7a,b}$$

so that equations (4.8) become

$$\ddot{a}_e = \left(\frac{1}{a_e^2} - \frac{1}{a_i^2}\right)\frac{a_e}{\lambda}, \quad \ddot{a}_i = -\delta\left(\frac{1}{a_e^2} - \frac{1}{a_i^2}\right)\frac{a_i}{\lambda}, \tag{6.8a,b}$$

and

$$\dot{h}_s = -\frac{Q_s a_s}{M_s} H, \tag{6.9}$$

where $\delta = m_e/m_i$. Substituting (6.9) into the third equation of (6.6) we obtain

$$\dot{H} = -\sum_s \frac{1}{\lambda M_s a_s^2} H, \tag{6.10}$$

which can be integrated to

$$H = H_0 \exp\left[-\frac{1}{\lambda}\int_0^t \left(\frac{1}{a_e^2} + \frac{\delta}{a_i^2}\right) \mathrm{d}t'\right], \tag{6.11}$$

where H_0 is an arbitrary constant to be determined by the initial conditions. Finally, combining (6.3) and (4.23), we get

$$X_s(t) = Y_s(t) = a_s \lambda^{-1/2}, \tag{6.12}$$

where λ is determined by the initial value $N_s(t=0)$. The evolution of the distribution functions are thereby fully determined by their initial values $f_s(I_s(t=0))$, where the invariants I_s are given by the solutions of (4.8). We note that the coefficients a_e and a_i are the functions describing the moving boundaries of the electron and ion fluids. That is, equations (6.8) are the equations of motion for the corresponding fronts.

7. The behaviour at short and long times

The ODEs (6.8) can be solved numerically when δ and λ as well as a_e and a_i and their time derivatives at $t = 0$ are given, so that the solutions depend only on these initial conditions. Typical solutions are shown in figure 1: (*a*) free expansion of the plasma slab, (*b*) expansion with large-amplitude oscillations and (*c*) contracting plasma slab with oscillating electron front. Numerical investigation also allows us to obtain an empirical relation

$$D_- a_i < a_e < D_+ a_i, \tag{7.1}$$

where $D_- > 0$ and $D_+ > 0$ are constants. This relation reflects the electrostatic interaction between the ion and electron fluids.

One can give a qualitative analysis of the expansion dynamics at large times. Combining the equations (6.8) we get

$$\frac{\ddot{a}_e}{a_e} + \frac{1}{\delta}\frac{\ddot{a}_i}{a_i} = 0, \tag{7.2}$$

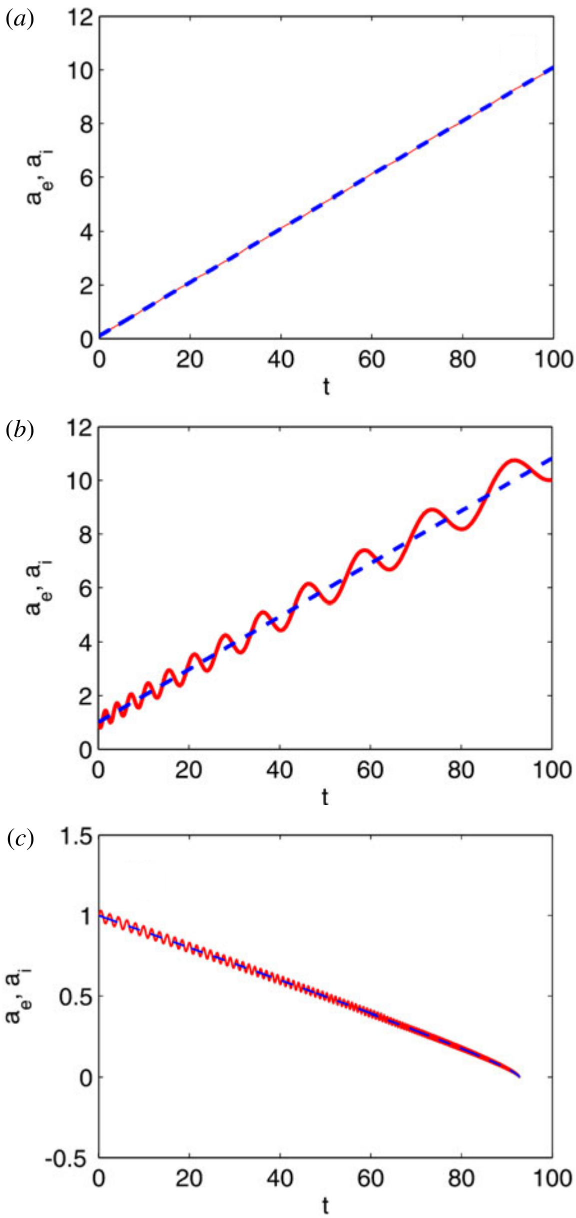

FIGURE 1. Evolution of the ion and electron fronts $a_i(t)$ (blue dotted curve) and $a_e(t)$ (red solid curve) respectively for the different initial data: (*a*) $a_i(0)=a_e(0)=0.1$, $\dot{a}_e(0)=10^{-6}$, $\dot{a}_i(0)=0.1$; (*b*) $a_i(0)=a_e(0)=1$, $\dot{a}_e(0)=-0.7$, $\dot{a}_i(0)=0.1$; (*c*) $a_i(0)=a_e(0)=1$, $\dot{a}_e(0)=0.1$, $\dot{a}_i(0)=-0.01$.

which after integration with the initial conditions (3.7) yields

$$\frac{\dot{a}_e}{a_e}+\frac{1}{\delta}\frac{\dot{a}_i}{a_i}+\int_0^t\left[\left(\frac{\dot{a}_e}{a_e}\right)^2+\frac{1}{\delta}\left(\frac{\dot{a}_i}{a_i}\right)^2\right]\mathrm{d}t'=0. \tag{7.3}$$

https://doi.org/10.1017/S0022377816000805 Published online by Cambridge University Press

Further integration gives

$$a_e a_i^{1/\delta} = \exp\left[-\int_0^t \theta(a_e, a_i)\,\mathrm{d}t'\right], \tag{7.4}$$

where

$$\theta(a_e, a_i) = \int_0^t \left[\left(\frac{\dot{a}_e}{a_e}\right)^2 + \frac{1}{\delta}\left(\frac{\dot{a}_i}{a_i}\right)^2\right]\mathrm{d}t' > 0. \tag{7.5}$$

Taking into account (7.1), we can rewrite (7.4) as

$$a_i < D_-^{-\delta/1+\delta} \exp\left[-\frac{\delta}{1+\delta}\int_0^t \theta(a_e, a_i)\mathrm{d}t'\right]. \tag{7.6}$$

This relation implies that a_e and a_i, and thus X_s and Y_s, are always bounded. In view of the Chaplygin comparison theorems (see, e.g. Yang, Shi & Li 2011), similar behaviour for the more general case $K = 2$ can be expected.

8. Discussion and conclusion

In contrast to the asymptotic stationary solution, namely the Maxwell distribution, of the Boltzmann and other equations including collision or velocity–space diffusion effects, the Vlasov equation can have an infinite number of asymptotic states (Clemmow & Dougherty 1969), depending on the initial distribution. For example, the works of Demeio & Zweifel (1990), Demeio & Holloway (1991), Manfredi (1997) showed that even though small-amplitude electric fields can be damped and eventually vanish, sufficiently large-amplitude perturbations can evolve into wave-like or other states. The results here belong to the latter class. The Vlasov system possesses such a property because it precludes direct particle–particle collisions that tend to randomize the particle velocities, and the interaction via the self-consistent electrostatic field cannot change the system entropy. However, one can still compare the macroscopic quantities (velocity–space moments of the distribution function) and the electrostatic field with that obtained from the corresponding fluid models. In fact, some of our results on the evolution of initially confined plasmas are similar to phenomena predicted by the latter (Karimov & Godin 2009; Karimov, Stenflo & Yu 2009*a*,*b*; Karimov, Yu & Stenflo 2011, 2012; Wang *et al.* 2016).

In this paper we have considered the properties and expansion of a collisionless plasma slab with anisotropic non-equilibrium particle distributions. We obtained fully nonlinear time-dependent, 2-D solutions of the Vlasov–Maxwell equations by invoking the Jeans' theorem. In contrast to most existing works invoking the latter, here the invariants of motion used to construct the distribution function are linear combinations of the phase-space variables, but the coefficients are time dependent and governed by ODEs. That is, they are not related to the traditional conservation laws such as that for energy and momentum. The plasma density, flux, as well as the electrostatic field then depend on the form of the invariants as well as how they appear in the distribution function, as can be seen from the relations (4.5), (4.1), (4.2) and (4.3). The solutions then describe non-equilibrium plasma flows, where imbalance among the different degrees of freedom leads to a preferred direction of evolution in the phase space.

We emphasize that the solutions, including the highly simplified but physically non-trivial case $K = 1$, are mathematically exact and are also valid for open systems, including those with non-conservative space- and time-dependent external forces. One can expect that similar results can also be found for higher-dimensional systems. Finally, we note that by using polynomial (instead of linear) forms of the invariants, the dynamics of other systems of physical interest (Lewis & Leach 1982; Struckmeier & Riedel 2001; Agren *et al.* 2005) can also be considered, such as that of a vortex system (Eyink & Sreenivasan 2006; Saffman 2006; Chavanis 2012).

Acknowledgements

We would like to dedicate our work to the centenary birthday of Professor A. E. Sheindlin (member of Russian Academy of Sciences). The present paper stems from the questions put by a Journal of Plasma Physics referee on Karimov, Yu & Stenflo (2016). The authors would like to express their profound gratitude to the referee for the valuable remarks and suggestions on the lines of further researches. M.Y.Y. was supported by the National Natural Science Foundation of China (11374262, 11475147) and the State Key Laboratory of High Field Laser Physics at SIOM.

Appendix A. Particle densities and fluxes for $K=2$

Here we show how the time-dependent coefficients of the invariants I_{xs} and I_{ys} in (3.2) are related to the particle densities and fluxes of the initially bounded plasma. From (3.2), we have

$$\Lambda_s v_x(I_{xs}, I_{xs}) = b_{ys}I_{xs} - b_{xs}I_{ys} + (b_{ys}\dot{a}_{xs} - b_{xs}\dot{a}_{ys})x + (b_{ys}\dot{b}_{xs} - b_{xs}\dot{b}_{ys})y + b_{xs}h_{ys} - b_{ys}h_{xs}, \tag{A 1}$$

$$\Lambda_s v_y(I_{xs}, I_{xs}) = a_{xs}I_{ys} - a_{ys}I_{xs} + (a_{xs}\dot{a}_{ys} - a_{ys}\dot{a}_{xs})x + (a_{xs}\dot{b}_{ys} - a_{ys}\dot{b}_{xs})y + a_{ys}h_{xs} - a_{xs}h_{ys}, \tag{A 2}$$

where $\Lambda_s = a_{xs}b_{ys} - a_{ys}b_{xs}$.

In terms of the invariants, we can express the density as

$$\begin{aligned} n_s &= \int f_s(I_{xs}, I_{ys})\, \mathrm{d}v_x\, \mathrm{d}v_y \\ &= \Lambda_s^{-1} \int f_s(I_{xs}, I_{ys})\, \mathrm{d}I_{xs}\, \mathrm{d}I_{ys}, \end{aligned} \tag{A 3}$$

where we have used the transformation Jacobian $\mathcal{J} = D(v_x, v_y)/D(I_{xs}, I_{ys}) = 1/\Lambda_s$. In view of the initial or normalization condition, one can see that the plasma density is $n_s = \Lambda_s^{-1}$. That is, the plasma indeed remains homogeneous during its evolution.

Similarly, for the macroscopic flux we have

$$\begin{aligned} j_{xs} &= \int v_x(I_{xs}, I_{xs}) f_s(I_{xs}, I_{ys}) \Lambda_s^{-1}\, \mathrm{d}I_{xs}\, \mathrm{d}I_{ys} \\ &= \Lambda_s^{-2}[(\dot{a}_{xs}b_{ys} - b_{xs}\dot{a}_{ys})x + (\dot{b}_{xs}b_{ys} - b_{xs}\dot{b}_{ys})y + b_{xs}h_{ys} - b_{ys}h_{xs}], \end{aligned} \tag{A 4}$$

and

$$\begin{aligned} j_{ys} &= \int v_y(I_{xs}, I_{xs}) f_s(I_{xs}, I_{ys}) \Lambda_s^{-1}\, \mathrm{d}I_{xs}\, \mathrm{d}I_{ys} \\ &= \Lambda_s^{-2}[(\dot{a}_{ys}a_{xs} - a_{ys}\dot{a}_{xs})x + (a_{xs}\dot{b}_{ys} - a_{ys}\dot{b}_{xs})y + a_{ys}h_{xs} - a_{xs}h_{ys}], \end{aligned} \tag{A 5}$$

where we have again used the initial condition. Thus, the macroscopic flow parameters are rather complicated functions of the structure coefficients appearing in (3.2).

Appendix B. Particle densities and fluxes for $K=1$

Here we obtain the macroscopic densities and fluxes for $K=1$ by using the solution (6.1) with (6.2). Accordingly, we have

$$n_s = \int f_s(a_s v_x + \lambda a_s v_y - \dot{a}_s x - \lambda \dot{a}_s y + h_s)\,\mathrm{d}v_x\,\mathrm{d}v_y = \frac{1}{\lambda a_s^2}\int f_s(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{B 1}$$

where we have used

$$\xi = a_s v_x - \dot{a}_s x + h_s, \quad \eta = \lambda(a_s v_y - \dot{a}_s y). \tag{B 2a,b}$$

In view of the initial condition (3.7) we obtain

$$n_s = \frac{1}{\lambda a_s^2}. \tag{B 3}$$

The corresponding macroscopic fluxes are

$$j_{xs} = \int v_x f_s(a_s v_x + \lambda a_s v_y - \dot{a}_s x - \lambda \dot{a}_s y + h_s)\,\mathrm{d}v_x\,\mathrm{d}v_y \tag{B 4}$$

and

$$j_{ys} = \int v_y f_s(a_s v_x + \lambda a_s v_y - \dot{a}_s x - \lambda \dot{a}_s y + h_s)\,\mathrm{d}v_x\,\mathrm{d}v_y. \tag{B 5}$$

These can be rewritten as

$$j_{xs} = \frac{1}{\lambda a_s^3}\left(\int \xi f(\xi+\eta)\,\mathrm{d}\xi\mathrm{d}\eta + (\dot{a}_s x - h_s)\int f(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right), \tag{B 6}$$

$$j_{ys} = \frac{1}{\lambda^2 a_s^3}\left(\int \eta f(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \lambda \dot{a}_s y\int f(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right). \tag{B 7}$$

Applying the initial conditions (3.7), we find

$$j_{xs} = \frac{\dot{a}_s - h_s}{\lambda a_s^3}x, \quad j_{ys} = \frac{\dot{a}_s}{\lambda a_s^3}y. \tag{B 8a,b}$$

We note that for the distribution (6.1) with (6.2), the relation (B 3) is not unique. It is the simplest non-trivial choice. One can obtain other results for j_{xs} and j_{ys} if different ξ and η are used.

REFERENCES

AGREN, O., MOISEENKO, V., JOHANSSON, C. & SAVENKO, N. 2005 Gyro center invariant and associated diamagnetic current. *Phys. Plasmas* **12**, 122503.

AGREN, O. & MOISEENKO, V. 2006 Four motional invariants in axisymmetric tori equilibria. *Phys. Plasmas* **13**, 052501.

ARFKEN, G. B. & WEBER, H. J. 2006 *Mathematical Methods for Physicists*, 6th edn. Elsevier.

BIRDSALL, C. K. & LANGDON, A. B. 2004 *Plasma Physics via Computer Simulation*. Taylor & Francis.

BOHR, T., JENSEN, M. H., PALADIN, G. & VULPIANI, A. 1998 *Dynamics Systems Approach to Turbulence*. Cambridge University Press.

BUCHANAN, M. & DORNING, J. J. 1993 Superposition of nonlinear plasma waves. *Phys. Rev. Lett.* **70**, 3732–3735.

CAMPA, A., DAUXOIS, T., FANELLI, D. & RUFFO, S. 2014 *Physics of Long-Range Interacting Systems*. Oxford University Press.

CHAVANIS, P. H. 2012 Kinetic theory of Onsagers vortices in two-dimensional hydrodynamics. *Physica* A **391**, 3657–3679.

CLEMMOW, P. C. & DOUGHERTY, J. P. 1969 *Electrodynamics of Particles and Plasmas*. Edison-Wesley.

DEGOND, P. & RAVIART, P. A. 1992 An analysis of the Darwin model of approximation to Maxwells equations. *Forum Math.* **4**, 13–27.

DEMEIO, L. & ZWEIFEL, P. F. 1990 Numerical simulations of perturbed Vlasov equilibria. *Phys. Fluids* B **2**, 1252–1255.

DEMEIO, L. & HOLLOWAY, J. P. 1991 Numerical simulations of BGK modes. *J. Plasma Phys.* **46**, 63–84.

DOROZHKINA, D. S. & SEMENOV, V. E. 1998 Exact solution of Vlasov equations for quasineutral expansion of plasma bunch into vacuum. *Phys. Rev. Lett.* **81**, 2691–2694.

EYINK, G. L. & SREENIVASAN, K. R. 2006 Onsager and the theory of hydrodynamic turbulence. *Rev. Mod. Phys.* **78**, 87–135.

ELIASSON, B. & SHUKLA, P. K. 2006 Formation and dynamics of coherent structures involving phase-space vortices in plasmas. *Phys. Rep.* **422**, 225–290.

HOLLOWAY, J. P. & DORNING, J. J. 1991 Undamped plasma waves. *Phys. Rev.* A **44**, 3856–3868.

KARIMOV, A. R. & LEWIS, H. R. 1999 Nonlinear solutions of the Vlasov–Poisson equations. *Phys. Plasmas* **6**, 759–761.

KARIMOV, A. R. 2001 Nonlinear solutions of a Maxwellian type for the Vlasov–Poisson equations. *Phys. Plasmas* **8**, 1533–1537.

KARIMOV, A. R., STENFLO, L. & YU, M. Y. 2009*a* Coupled azimuthal and radial flows and oscillations in a rotating plasma. *Phys. Plasmas* **16**, 062313.

KARIMOV, A. R., STENFLO, L. & YU, M. Y. 2009*b* Coupled flows and oscillations in asymmetric rotating plasmas. *Phys. Plasmas* **16**, 102303.

KARIMOV, A. R. & GODIN, S. M. 2009 Coupled radial and azimuthal oscillations in twirling cylindrical plasmas. *Phys. Scr.* **80**, 035503.

KARIMOV, A. R., YU, M. Y. & STENFLO, L. 2011 Flow oscillations in radial expansion of an inhomogeneous plasma layer. *Phys. Lett.* A **375**, 2629–2636.

KARIMOV, A. R., YU, M. Y. & STENFLO, L. 2012 Large quasineutral electron velocity oscillations in radial expansion of an ionizing plasma. *Phys. Plasmas* **19**, 092118.

KARIMOV, A. R. 2013 Coupled electron and ion nonlinear oscillations in a collisionless plasma. *Phys. Plasmas* **20**, 052305.

KARIMOV, A. R., YU, M. Y. & STENFLO, L. 2016 A new class of exact solutions for Vlasov–Poisson Plasmas. *Phys. Scr.*; (submitted).

KIESSLING, M. K. H. 2003 The 'Jeans swindle': a true story mathematically speaking. *Adv. Appl. Maths* **31**, 132–149.

KOVALEV, V. F. & BYCHENKOV, V. Y. 2003 Analytic solutions to the Vlasov equations for expanding plasmas. *Phys. Rev. Lett.* **90**, 185004.

KOZLOV, V. V. 2008 The generalized Vlasov kinetic equation. *Russian Math. Surveys* **63**, 93–130.

KUZNETSOV, E. A. 1996 Wave collapse in plasmas and fluids. *Chaos* **6**, 381–390.

LANCELLOTTI, C. & DORNING, J. J. 1998 Critical initial states in collisionless plasmas. *Phys. Rev. Lett.* **81**, 5137–5140.

LEWIS, H. R. & LEACH, P. G. L. 1982 A direct approach to finding exact invariants for one-dimensional time-dependent classical Hamiltonians. *J. Math. Phys.* **23**, 2371–2374.

LEWIS, H. R. & SYMON, K. R. 1984 Exact time-dependent solutions of the Vlasov–Poisson equations. *Phys. Fluids* **27**, 192–196.

LEVIN, Y., PAKTER, R., RIZZATO, F. B., TELES, T. N. & BENETTI, F. P. C. 2014 Nonequilibrium statistical mechanics of systems with long-range interactions. *Phys. Rep.* **535**, 1–60.

LUQUE, A. & SCHAMEL, H. 2005 Electrostatic trapping as a key to the dynamics of plasmas, fluids and other collective systems. *Phys. Rep.* **415**, 261–359.

MAJDA, A. J., MAJDA, G. & ZHENG, Y. 1994 Concentrations in the one-dimensional Vlasov–Poisson equations I: temporal development and non-unique weak solutions in the single component case. *Physica* D **74**, 268–300.

MANFREDI, G. 1997 Long-time behavior of nonlinear Landau damping. *Phys. Rev. Lett.* **79**, 2815–2828.

NIELSON, C. & LEWIS, H. R. 1976 Particle code models in the nonradiative limit. *Meth. Comput. Phys.* **16**, 367–388.

PECSELI, H. L. 2012 *Waves and Oscillations in Plasmas*. Taylor & Francis.

SAFFMAN, P. G. 2006 *Vortex Dynamics*. Cambridge University Press.

SCHAMEL, H. 2004 Lagrangian fluid description with simple applications in compressible plasma and gas dynamics. *Phys. Rep.* **392**, 279–319.

SCHAMEL, H. 2015 Particle trapping: a key requisite of structure formation and stability of Vlasov–Poisson plasmas. *Phys. Plasmas* **22**, 042301.

STRUCKMEIER, J. & RIEDEL, C. 2001 Invariants for time-dependent Hamiltonian systems. *Phys. Rev.* E **64**, 026503.

TARANOV, V. B. 1976 On the symmetry of one-dimensional high frequency motions of a collisionless plasma. *Sov. Phys. Tech. Phys.* **21**, 720–724.

WANG, Y.-M., YU, M. Y., STENFLO, L. & KARIMOV, A. R. 2016 Evolution of a cold non-neutral electron–positron plasma slab. *Chin. Phys. Lett.* **33**, 085205.

YANG, S., SHI, B. & LI, M. 2011 Mean square stability of impulsive stochastic differential systems. *Intl J. Diff. Equ.* **2011**, 613695.