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FROM THE FUNCTION-SHEAF DICTIONARY TO QUASICHARACTERS OF p-ADIC TORI

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Abstract We consider the rigid monoidal category of character sheaves on a smooth commutative group scheme G over a finite field k, and expand the scope of the function-sheaf dictionary from connected commutative algebraic groups to this setting. We find the group of isomorphism classes of character sheaves on G, and show that it is an extension of the group of characters of $G(k)$ by a cohomology group determined by the component group scheme of G. We also classify all morphisms in the category character sheaves on G. As an application, we study character sheaves on Greenberg transforms of locally finite type Néron models of algebraic tori over local fields. This provides a geometrization of quasicharacters of p-adic tori.

Keywords: character sheaves; p-adic tori; Néron models; Greenberg functor; geometrization; quasi-character sheaves

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Introduction

As Deligne explained in [14, Sommes trig.], if G is a connected commutative algebraic group over a finite field k, then the trace of Frobenius provides a bijection between the group $G(k)^*$ of ℓ-adic characters of $G(k)$ and isomorphism classes of those rank-one ℓ-adic local systems $\mathcal{E}$ on G for which

$$m^*\mathcal{E} \cong \mathcal{E} \boxtimes \mathcal{E}, \tag{1}$$

where $m : G \times G \to G$ is the multiplication map. If one wishes to make a category from this class of local systems, one is led to consider morphisms $\mathcal{E} \to \mathcal{E}'$ of sheaves which are compatible with particular choices of (1) for $\mathcal{E}$ and $\mathcal{E}'$. *A priori*, the composition $\mathcal{E} \to \mathcal{E}' \to \mathcal{E}''$ of two such morphisms need not be compatible with the choices of (1) for $\mathcal{E}$ and $\mathcal{E}''$. However, for connected G, the isomorphism (1) is unique, if it exists, and there is no impediment to making the dictionary categorical.

If G is a commutative algebraic group over k which is not connected, however, then the isomorphism (1) need not be unique. In order to track the choice of isomorphism, consider the category $\mathcal{CS}_0(G)$ of pairs $(\mathcal{E}, \mu_{\mathcal{E}})$, where $\mathcal{E}$ is a rank-one local system on G and $\mu_{\mathcal{E}} : m^*\mathcal{E} \to \mathcal{E} \boxtimes \mathcal{E}$ is a chosen isomorphism of local systems on $G \times G$. In this

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case, the trace of Frobenius provides an epimorphism from isomorphism classes of objects in $\mathcal{CS}_0(G)$ to characters of $G(k)$, but the epimorphism need not be injective; consequently, every character of $G(k)$ may be geometrized as a pair $(\mathcal{E}, \mu_{\mathcal{E}})$, but perhaps not uniquely. Indeed, it follows from a special case of the main result of this paper that the kernel of the trace of Frobenius $\mathcal{CS}_0(G) \to G(k)^*$ trivial if and only if the group scheme of connected components of G is cyclic. The defect in the function-sheaf dictionary for characters of commutative algebraic groups over finite fields may be addressed with the following observation: if $(\mathcal{E}, \mu_{\mathcal{E}})$ and $(\mathcal{E}', \mu'_{\mathcal{E}})$ determine the same character of $G(k)$ then $\mathcal{E} \cong \mathcal{E}'$ as local systems on G.

Motivated by an application to quasicharacters of algebraic tori over local fields, in this paper we extend the function-sheaf dictionary from commutative algebraic groups over finite fields to smooth commutative group schemes G over k. In order to do this, we replace the local system $\mathcal{E}$ on G with a Weil local system while retaining the extra structure $\mu_{\mathcal{E}}$. In this way we are led to the category $\mathcal{CS}(G)$ of *character sheaves* on G (§ 1.2): objects in $\mathcal{CS}(G)$ are triples $(\bar{\mathcal{L}}, \mu, \phi)$, where $\bar{\mathcal{L}}$ is a rank-one local system on $\bar{G} := G \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$, and $\phi : \mathrm{Fr}_G^* \bar{\mathcal{L}} \to \bar{\mathcal{L}}$ and $\mu : \bar{m}^* \bar{\mathcal{L}} \cong \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}$ are isomorphisms of sheaves satisfying certain compatibility conditions; morphisms in $\mathcal{CS}(G)$ are then morphisms of Weil sheaves which are compatible with the extra structure. This paper establishes the basic properties of category $\mathcal{CS}(G)$, using the group homomorphism

$$\mathrm{Tr}_G : \mathcal{CS}(G)_{/\mathrm{iso}} \to G(k)^*$$

provided by the trace of Frobenius to find the relation between character sheaves on G and characters of $G(k)$. Then we return to our motivating application, and use character sheaves to geometrize and categorify quasicharacters algebraic tori over local fields.

We begin our study of category $\mathcal{CS}(G)$ by returning to the case when G is a connected commutative algebraic group over k, revisiting Deligne's function-sheaf dictionary (§ 1). We consider character sheaves that arise via base change to $\bar{k}$ from local systems on G (§ 1.4) and those that appear in a pushforward from a constant sheaf along a discrete isogeny $H \to G$ (§ 1.5). While these constructions make sense even for non-connected G, in the connected case we show that every character sheaf can be described in both of these ways (§ 1.6). We use this fact to prove that $\mathrm{Tr}_G : \mathcal{CS}(G) \to G(k)^*$ is an isomorphism for connected commutative algebraic groups G. We also determine the automorphism groups of character sheaves on such G. These facts are well known.

Next, we consider character sheaves on étale commutative group schemes G over k (§ 2). Étale group schemes form a counterpoint to connected algebraic groups, since the component group of any smooth group scheme is an étale group scheme. Our key tools for understanding the trace of Frobenius in the étale case are a reinterpretation of $\mathcal{CS}(G)$ in terms of stalks (§ 2.1) and the Hochschild–Serre spectral sequence (§ 2.2) for $\mathcal{W} \ltimes \bar{G}$, where $\mathcal{W} \subset \mathrm{Gal}(\bar{k}/k)$ is the Weil group for k. We define (§ 2.3) an isomorphism S_G from $\mathcal{CS}(G)_{/\mathrm{iso}}$ to the second cohomology of the total space of the spectral sequence

$$E_2^{p,q} := \mathrm{H}^p(\mathcal{W}, \mathrm{H}^q(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) \Rightarrow \mathrm{H}^{p+q}(\mathcal{W} \ltimes \bar{G}, \bar{\mathbb{Q}}_\ell^\times).$$

Paired with the short exact sequence

$$0 \to \mathrm{H}^0(\mathcal{W}, \mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) \to \mathrm{H}^2(E_G^\bullet) \to \mathrm{H}^1(\mathcal{W}, \mathrm{H}^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) \to 0$$

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arising from the spectral sequence, the isomorphism S_G allows us to show (§ 2.4) that the group homomorphism $\mathrm{Tr}_G : \mathcal{CS}(G)_{/\mathrm{iso}} \to G(k)^*$ is surjective with kernel $\mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}}$. The necessity of using Weil local systems on G in the definition of $\mathcal{CS}(G)$ already appears here: if one were to use local systems on G instead, the group homomorphism Tr_G would not then be surjective (§ 2.5). Moreover, as examples show (§ 2.12), the kernel of Tr_G is non-trivial in general.

Having understood $\mathcal{CS}(G)$ in two extreme cases (for connected commutative algebraic groups and for étale commutative group schemes), we turn to the case of smooth commutative group schemes (§ 3) using the component group sequence

$$0 \to G^0 \to G \to \pi_0(G) \to 0.$$

Using pullbacks of character sheaves, we obtain the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{CS}(\pi_0(G))_{/\mathrm{iso}} & \longrightarrow & \mathcal{CS}(G)_{/\mathrm{iso}} & \longrightarrow & \mathcal{CS}(G^0)_{/\mathrm{iso}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Tr}_{\pi_0(G)}} & & \downarrow{\scriptstyle \mathrm{Tr}_G} & & \downarrow{\scriptstyle \mathrm{Tr}_{G^0}} & & \\
0 & \longrightarrow & \pi_0(G)(k)^* & \longrightarrow & G(k)^* & \longrightarrow & G^0(k)^* & \longrightarrow & 0
\end{array}$$

We show that the rows of this diagram are exact (§§ 3.1, 3.2), so we may apply the snake lemma to prove the main theorem of the paper,

Theorem (Theorem 3.6). *If G is a smooth commutative group scheme over k then the trace of Frobenius gives a short exact sequence*

$$0 \longrightarrow \mathrm{H}^2(\pi_0(\bar{G}), \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}} \longrightarrow \mathcal{CS}(G)_{/iso} \xrightarrow{\mathrm{Tr}_G} G(k)^* \longrightarrow 0$$

If the component group scheme $\pi_0(\bar{G})$ is cyclic, then the kernel of Tr_G will be trivial, and each character of $G(k)$ will uniquely determine a character sheaf on G, up to isomorphism. But when $\pi_0(\bar{G})$ is large (see Remark 2.9), G will admit *invisible character sheaves* with trivial trace of Frobenius.

We also illuminate the nature of the category $\mathcal{CS}(G)$ by showing that every morphism in this category is either an isomorphism or trivial, and by showing the following.

Theorem (Theorem 3.9). *If G is a smooth commutative group scheme over k then*

$$\mathrm{Aut}(\mathcal{L}) \cong \mathrm{H}^1(\pi_0(\bar{G}), \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}}$$

for all quasicharacter sheaves $\mathcal{L}$ on G.

Application to quasicharacters of p-adic tori and abelian varieties. As indicated above, our interest in the function-sheaf dictionary for smooth commutative group schemes over finite fields comes from an application to p-adic representation theory, specifically to quasicharacters (§ 4.2) of p-adic tori. However, we found that our method of passing from p-adic tori to group schemes over k applies more generally to any local field K with finite residue field k and to any commutative algebraic group over K that admits a Néron model X. This class of algebraic groups over K includes abelian varieties and unipotent K-wound groups, in addition to the algebraic tori we initially considered.

In this paper we show that if X is as above then quasicharacters of $X(K)$ are geometrized and categorified by character sheaves on the Greenberg transform $\mathrm{Gr}_R(X)$ of the Néron model X. Although not locally of finite type, $\mathrm{Gr}_R(X)$ is a commutative group scheme over k and also a projective limit of smooth commutative group schemes $\mathrm{Gr}_n^R(X)$. This structure allows us to adapt our work on character sheaves on smooth group schemes over finite fields to construct (§ 4.4) a category $\mathcal{QCS}(X)$ of *quasicharacter sheaves* for X, which are certain sheaves on $\mathrm{Gr}_R(X) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$, with extra structure. The ability to generalize the function-sheaf dictionary to non-connected group schemes plays a crucial role in this application.

Having defined quasicharacter sheaves on Néron models of commutative algebraic groups over K and character sheaves on commutative group schemes over k, we consider how these categories are related as K and k vary. We describe (§ 3.7) functors between categories of quasicharacter sheaves that model restriction and norm homomorphisms of character groups $G(k')^* \to G(k)^*$ and $G(k)^* \to G(k')^*$, and describe how quasicharacter sheaves behave under Weil restriction (§ 4.6). We also give (§ 4.7) a categorical version of a result of Chai and Yu [12], relating quasicharacter sheaves for tori over different local fields, even local fields with different characteristic.

Finally, specializing to the case that $X = T$ is the Néron model of an algebraic torus over K (§ 4.5), we give a canonical short exact sequence

$$0 \to \mathrm{H}^2(X_*(T)_{\mathcal{I}_K}, \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}} \to \mathcal{QCS}(T)_{/\mathrm{iso}} \to \mathrm{Hom}(T(K), \bar{\mathbb{Q}}_\ell^\times) \to 0,$$

where $X_*(T)_{\mathcal{I}_K}$ is the group of coinvariants of the cocharacter lattice $X_*(T)$ of the algebraic torus T_K by the action of the inertia group $\mathcal{I}_K$ of K, and where $\mathrm{Hom}(T(K), \bar{\mathbb{Q}}_\ell^\times)$ denotes the group of quasicharacters of $T(K)$. We further show that automorphism groups in $\mathcal{QCS}(T)$ are given, for every quasicharacter sheaf $\mathcal{F}$ for T, by

$$\mathrm{Aut}(\mathcal{F}) \cong (\check{T}_\ell)^{\mathcal{W}_K},$$

where $\mathcal{W}_K$ is the Weil group for K and $\check{T}_\ell$ is the ℓ-adic dual torus to T.

By any measure, there are more quasicharacter sheaves for T than quasicharacters of $T(K)$. In this regard, we are reminded of the work of Vogan [35], in which he finds a geometrization of complete Langlands parameters for p-adic groups and, in the process, is led to study the representations of all the pure rational forms of the p-adic group, simultaneously. A similar phenomenon appears in recent work by Joseph Bernstein in which his geometric Ansatz leads to the study of certain sheaves on the stacky classifying space of the p-adic group, resulting in a category which appears to be tied to the representations of all the pure rational forms of the p-adic group [3]. Indeed, Bernstein has suggested to us that our category of quasicharacters for T may be tied to quasicharacters of all the pure rational forms of T. It would be interesting to pursue this idea.

Relation to other work. The main use of the term character sheaf is due to Lusztig. It is applied to certain perverse sheaves on connected reductive algebraic groups over algebraically closed fields in [29, Definition 2.10], and to certain perverse sheaves on certain reductive groups over algebraically closed fields in the series of papers beginning with [30]. When commutative, it is not difficult to relate Frobenius-stable character sheaves to our character sheaves (Remark 3.11). The new features that we have found pertaining to Weil sheaves and $\mathrm{H}^2(\pi_0(\bar{G}), \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}}$ do not arise in that context because, for such groups, Weil sheaves are unnecessary (§ 3.6) and $\mathrm{H}^2(\pi_0(\bar{G}), \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}} = 0$ (Remark 2.9).

For a connected commutative algebraic group over a finite field, it is not uncommon to refer to local systems satisfying (1) as character sheaves; see, for example, [26, Introduction]. Our definition of character sheaves on smooth commutative group schemes over finite fields evolved from this notion, with an eye towards quasicharacters of p-adic groups. The process of creating a category from the group of quasicharacters of a p-adic torus informs our choice of the term quasicharacter sheaf in this paper.

We anticipate that future work on quasicharacter sheaves will make use of [32, 33], and will clarify the relation between this project and other attempts to geometrize admissible distributions on p-adic groups, such as [13] (limited to quasicharacters of $\mathbb{Z}_p^\times$) and [1] (limited to characters of depth-zero representations). We are actively pursuing the question of how to extend the notion of quasicharacter sheaves to provide a geometrization of admissible distributions on connected reductive algebraic groups over p-adic fields, not just commutative ones.

1. Definitions and recollections

1.1. Notation

Throughout this paper, G is a smooth commutative group scheme over a finite field k, and $m : G \times G \to G$ is its multiplication morphism.

We will make use of the short exact sequence of smooth group schemes defining the component group scheme for G:

$$0 \longrightarrow G^0 \xrightarrow{\iota_0} G \xrightarrow{\pi_0} \pi_0(G) \longrightarrow 0$$

Then G^0 is a connected algebraic group and $\pi_0(G)$ is an étale commutative group scheme. In contrast to the case of algebraic varieties, the component group scheme $\pi_0(G)$ for G need not be finite.

It follows from the smoothness of G that the structure morphism $G \to \mathrm{Spec}(k)$ is locally of finite type, being smooth. If the structure morphism $G \to \mathrm{Spec}(k)$ is also étale, then G is an étale group scheme; this does not imply that $\pi_0(G)$ is finite. An algebraic group over k is a smooth group scheme of finite type, in which case its component group scheme is finite.

We fix an algebraic closure $\bar{k}$ of k, and write $\bar{G}$ for the smooth commutative group scheme $G \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$ over $\bar{k}$ obtained by base change from k. The multiplication morphism for $\bar{G}$ will be denoted by $\bar{m}$.

Let Fr denote the geometric Frobenius element in $\mathrm{Gal}(\bar{k}/k)$ as well as the corresponding automorphism of $\mathrm{Spec}(\bar{k})$. The Weil group $\mathcal{W} \subset \mathrm{Gal}(\bar{k}/k)$ is the subgroup generated by Fr. Let $\mathrm{Fr}_G := \mathrm{id}_G \times \mathrm{Fr}$ be the Frobenius automorphism of $\bar{G} = G \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$.

We fix a prime ℓ, invertible in k. We will work with constructible ℓ-adic sheaves [15, § 1.1]; [25, Exposés V, VI] on schemes locally of finite type over k, employing the

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standard formalism. We also make extensive use of the external tensor product of ℓ-adic sheaves, defined as follows: if $\mathcal{F}$ and $\mathcal{G}$ are constructible ℓ-adic sheaves on schemes X and Y and $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the projections, then $\mathcal{F} \boxtimes \mathcal{G} := p_X^* \mathcal{F} \otimes p_Y^* \mathcal{G}$.

For any commutative group A, we will write A^* for the dual group $\mathrm{Hom}(A, \bar{\mathbb{Q}}_\ell^\times)$.

1.2. Character sheaves on commutative group schemes over finite fields

Definition 1.1. A *character sheaf on G* is a triple $\mathcal{L} := (\bar{\mathcal{L}}, \mu, \phi)$, where the following hold.

(CS.1) $\bar{\mathcal{L}}$ is a rank-one ℓ-adic local system on $\bar{G}$, by which we mean a constructible ℓ-adic sheaf on $\bar{G}$, *lisse* on each connected component of $\bar{G}$, whose stalks are one-dimensional $\bar{\mathbb{Q}}_\ell$-vector spaces.

(CS.2) $\mu : \bar{m}^* \bar{\mathcal{L}} \to \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}$ is an isomorphism of sheaves on $\bar{G} \times \bar{G}$ such that the following diagram commutes, where $m_3 := m \circ (m \times \mathrm{id}) = m \circ (\mathrm{id} \times m)$.

$$\begin{array}{ccc} \bar{m}_3^* \bar{\mathcal{L}} & \xrightarrow{(\bar{m} \times \mathrm{id})^* \mu} & \bar{m}^* \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}} \\ {\scriptstyle (\mathrm{id} \times \bar{m})^* \mu} \downarrow & & \downarrow {\scriptstyle \mu \boxtimes \mathrm{id}} \\ \bar{\mathcal{L}} \boxtimes \bar{m}^* \bar{\mathcal{L}} & \xrightarrow{\mathrm{id} \boxtimes \mu} & \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}} \end{array}$$

(CS.3) $\phi : \mathrm{Fr}_G^* \bar{\mathcal{L}} \to \bar{\mathcal{L}}$ is an isomorphism of constructible ℓ-adic sheaves on $\bar{G}$ compatible with μ in the sense that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Fr}_{G \times G}^* \bar{m}^* \bar{\mathcal{L}} & \xrightarrow{\mathrm{Fr}_{G \times G}^* \mu} & \mathrm{Fr}_{G \times G}^* (\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}) \\ \| & & \| \\ \bar{m}^* \mathrm{Fr}_G^* \bar{\mathcal{L}} & & \mathrm{Fr}_G^* \bar{\mathcal{L}} \boxtimes \mathrm{Fr}_G^* \bar{\mathcal{L}} \\ {\scriptstyle \bar{m}^* \phi} \downarrow & & \downarrow {\scriptstyle \phi \boxtimes \phi} \\ \bar{m}^* \bar{\mathcal{L}} & \xrightarrow{\mu} & \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}} \end{array}$$

Morphisms of character sheaves are defined in the natural way.

(CS.4) If $\mathcal{L} = (\bar{\mathcal{L}}, \mu, \phi)$ and $\mathcal{L}' = (\bar{\mathcal{L}}', \mu', \phi')$ are character sheaves on G, then a morphism $\rho : \mathcal{L} \to \mathcal{L}'$ is a map $\bar{\rho} : \bar{\mathcal{L}} \to \bar{\mathcal{L}}'$ of constructible ℓ-adic sheaves on $\bar{G}$ such that the following diagrams both commute.

$$\begin{array}{ccc} \mathrm{Fr}_G^* \bar{\mathcal{L}} & \xrightarrow{\mathrm{Fr}_G^* \bar{\rho}} & \mathrm{Fr}_G^* \bar{\mathcal{L}}' \\ {\scriptstyle \phi} \downarrow & & \downarrow {\scriptstyle \phi'} \\ \bar{\mathcal{L}} & \xrightarrow{\bar{\rho}} & \bar{\mathcal{L}}' \end{array} \qquad \begin{array}{ccc} \bar{m}^* \bar{\mathcal{L}} & \xrightarrow{\bar{m}^* \bar{\rho}} & \bar{m}^* \bar{\mathcal{L}}' \\ {\scriptstyle \mu} \downarrow & & \downarrow {\scriptstyle \mu'} \\ \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}} & \xrightarrow{\rho \boxtimes \rho} & \bar{\mathcal{L}}' \boxtimes \bar{\mathcal{L}}' \end{array}$$

The category of character sheaves on G will be denoted by $\mathcal{CS}(G)$.

Category $\mathcal{CS}(G)$ is a rigid monoidal category [19, § 1.10] under the tensor product $\mathcal{L} \otimes \mathcal{L}'$ defined by $(\bar{\mathcal{L}} \otimes \bar{\mathcal{L}}', \mu \otimes \mu', \phi \otimes \phi')$, with duals given by applying the sheaf hom

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functor $\mathscr{H}om(-, \bar{\mathbb{Q}}_\ell)$. This rigid monoidal category structure for $\mathcal{CS}(G)$ gives the set $\mathcal{CS}(G)_{/\mathrm{iso}}$ of isomorphism classes in $\mathcal{CS}(G)$ the structure of a group.

Remark 1.2. The category of character sheaves on G is not abelian since it is not closed under direct sums; thus $\mathcal{CS}(G)$ is not a tensor category in the sense of [16, 0.1]. We suspect that requiring that μ be injective rather than an isomorphism and dropping the condition that the stalks be one dimensional would yield an abelian category.

We will describe the group $\mathcal{CS}(G)_{/\mathrm{iso}}$ in Theorem 3.6 and the sets $\mathrm{Hom}(\mathcal{L}, \mathcal{L}')$ in Theorem 3.9; in this way we provide a complete description of the category $\mathcal{CS}(G)$. In the meantime, we make an elementary observation about $\mathrm{Hom}(\mathcal{L}, \mathcal{L}')$.

Lemma 1.3. *Let G be a smooth commutative group scheme over k. If $\mathcal{L}$ and $\mathcal{L}'$ are character sheaves on G, then every $\rho \in \mathrm{Hom}(\mathcal{L}, \mathcal{L}')$ is either trivial (zero on every stalk) or an isomorphism.*

1.3. Trace of Frobenius

In this section we introduce two tools which will help us understand isomorphism classes of objects in $\mathcal{CS}(G)$: the map $\mathcal{CS}(G)_{/\mathrm{iso}} \to G(k)^*$ given by trace of Frobenius, and the pullback functor $\mathcal{CS}(G) \to \mathcal{CS}(H)$ associated to a morphism $H \to G$ of smooth group schemes over k.

Let $(\bar{\mathcal{L}}, \phi)$ be a Weil sheaf on G. Every $g \in G(k)$ determines a geometric point $\bar{g}$ fixed by Fr_G. Together with the canonical isomorphism $(\mathrm{Fr}_G^* \bar{\mathcal{L}})_{\bar{g}} \cong \bar{\mathcal{L}}_{\mathrm{Fr}_G(\bar{g})}$, the automorphism ϕ determines an automorphism $\phi_{\bar{g}}$ of the $\bar{\mathbb{Q}}_\ell$-vector space $\bar{\mathcal{L}}_{\bar{g}}$. Let $\mathrm{Tr}(\phi_{\bar{g}}; \bar{\mathcal{L}}_{\bar{g}})$ be the trace of $\phi_{\bar{g}} \in \mathrm{Aut}_{\bar{\mathbb{Q}}_\ell}(\bar{\mathcal{L}}_{\bar{g}})$, and let $t_{(\bar{\mathcal{L}},\phi)} : G(k) \to \bar{\mathbb{Q}}_\ell$ be the function defined by

$$t_{(\bar{\mathcal{L}},\phi)}(g) := \mathrm{Tr}(\phi_{\bar{g}}; \bar{\mathcal{L}}_{\bar{g}}), \tag{2}$$

commonly called the *trace of Frobenius of* $(\bar{\mathcal{L}}, \phi)$. Note that, if $(\bar{\mathcal{L}}, \phi) \cong (\bar{\mathcal{L}}', \phi')$ as Weil sheaves, then $t_{(\bar{\mathcal{L}},\phi)} = t_{(\bar{\mathcal{L}}',\phi')}$ as functions on $G(k)$.

Now suppose that $\mathcal{L} = (\bar{\mathcal{L}}, \mu, \phi)$ is a character sheaf on G. Then the isomorphism $\bar{m}^* \bar{\mathcal{L}} \cong \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}$ and the diagram of (CS.3) guarantee that the function $t_{(\bar{\mathcal{L}},\phi)} : G(k) \to \bar{\mathbb{Q}}_\ell^\times$ is a group homomorphism, which we will also denote by $t_{\mathcal{L}}$. Moreover, this homomorphism depends only on the isomorphism class of $\mathcal{L}$, so we obtain a map

$$\begin{aligned} \mathrm{Tr}_G : \mathcal{CS}(G)_{/\mathrm{iso}} &\to G(k)^*, \\ \mathcal{L} &\mapsto t_{\mathcal{L}}. \end{aligned}$$

Since tensor products on the stalks of $\mathcal{L}$ induce pointwise multiplication on the trace of Frobenius, Tr_G is a group homomorphism.

The next two results follow easily from the definitions.

Lemma 1.4. *If $f : H \to G$ is a morphism of smooth commutative group schemes over k, then*

$$\begin{aligned} f^* : \mathcal{CS}(G) &\to \mathcal{CS}(H) \\ (\bar{\mathcal{L}}, \mu, \phi) &\mapsto (\bar{f}^* \bar{\mathcal{L}}, (\bar{f} \times \bar{f})^* \mu, \bar{f}^* F) \end{aligned}$$

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defines a monoidal functor dual to $f : H(k) \to G(k)$ in the sense that

$$\begin{array}{ccc} \mathcal{CS}(G)_{/iso} & \xrightarrow{f^*} & \mathcal{CS}(H)_{/iso} \\ \downarrow{\scriptstyle \mathrm{Tr}_G} & & \downarrow{\scriptstyle \mathrm{Tr}_H} \\ G(k)^* & \longrightarrow & H(k)^* \end{array}$$

is a commutative diagram of groups. Moreover, $(f \circ g)^* = g^* \circ f^*$.

If G_1 and G_2 are smooth commutative group schemes over k, then characters of $(G_1 \times G_2)(k)$ all take the form $\chi_1 \otimes \chi_2$ for characters χ_1 of $G_1(k)$ and χ_2 of $G_2(k)$. The next lemma shows that character sheaves on G enjoy an analogous property.

Lemma 1.5. *If G_1 and G_2 are smooth commutative group schemes over k, then the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{CS}(G_1)_{/iso} \times \mathcal{CS}(G_2)_{/iso} & \xrightarrow{(\mathcal{L}_1,\mathcal{L}_2)\mapsto \mathcal{L}_1 \boxtimes \mathcal{L}_2} & \mathcal{CS}(G_1 \times G_2)_{/iso} \\ \downarrow{\scriptstyle \mathrm{Tr}_{G_1} \times \mathrm{Tr}_{G_2}} & & \downarrow{\scriptstyle \mathrm{Tr}_{G_1 \times G_2}} \\ (G_1)(k)^* \times (G_2)(k)^* & \xrightarrow{(\chi_1,\chi_2)\mapsto \chi_1 \otimes \chi_2} & (G_1 \times G_2)(k)^* \end{array}$$

Moreover, every character sheaf on $G_1 \times G_2$ is isomorphic to $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ for some character sheaves $\mathcal{L}_1$ on G_1 and $\mathcal{L}_2$ on G_2.

Using these results on pullbacks and products, we may prove a naturality property of Tr_G.

Proposition 1.6. *The homomorphism $\mathrm{Tr}_G : \mathcal{CS}(G)_{/iso} \to G(k)^*$ defines a natural transformation between the two contravariant additive functors*

$$F_1 : G \mapsto \mathcal{CS}(G)_{/iso}$$
$$F_2 : G \mapsto G(k)^*$$

from the category of smooth commutative group schemes over k to the category of commutative groups.

Proof. The first part of Lemma 1.4 shows that F_1 is a functor, while the second part shows that the trace of Frobenius is a natural transformation $T : F_1 \to F_2$. When further combined with Lemma 1.5, we see that F_1 is an additive functor and $T : F_1 \to F_2$ is a natural transformation between additive functors, concluding the proof of Proposition 1.6. □

1.4. Descent

In this section we consider a category of sheaves on G obtained by replacing the Weil sheaf $(\bar{\mathcal{L}}, \phi)$ on $\bar{G}$ in the definition of a character sheaf with an ℓ-adic local system on G itself; these will play a role in §§ 1.6 and 3.6.

Definition 1.7. Let $\mathcal{CS}_0(G)$ be the category of pairs $(\mathcal{E}, \mu_{\mathcal{E}})$, where $\mathcal{E}$ is an ℓ-adic local system on G of rank one, equipped with an isomorphism $\mu_{\mathcal{E}} : m^*\mathcal{E} \to \mathcal{E} \boxtimes \mathcal{E}$ satisfying the analogue of (CS.2) on G; morphisms in $\mathcal{CS}_0(G)$ are defined as in the second part of (CS.4).

We put a rigid monoidal structure on $\mathcal{CS}_0(G)$ in the same way as for $\mathcal{CS}(G)$.

Proposition 1.8. *Extension of scalars defines a full and faithful functor*

$$B_G : \mathcal{CS}_0(G) \to \mathcal{CS}(G).$$

Proof. Suppose that $(\mathcal{E}, \mu_{\mathcal{E}})$ is an object of $\mathcal{CS}_0(G)$. Let $b_G : \bar{G} \to G$ be the pullback of $\mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(k)$ along $G \to \mathrm{Spec}(k)$. Set $\bar{\mathcal{L}} = b_G^*\mathcal{E}$. The functor b_G^* takes local systems on G to local systems on $\bar{G}$. The local system $\bar{\mathcal{L}}$ comes equipped with an isomorphism $\phi : \mathrm{Fr}_G^* \bar{\mathcal{L}} \to \bar{\mathcal{L}}$. The resulting functor from local systems on G to Weil local systems on $\bar{G}$, given on objects by $\mathcal{E} \mapsto (\bar{\mathcal{L}}, \phi)$, is full and faithful; see [17, Exposé XIII] and [2, Proposition 5.1.2]. The isomorphism $\mu := b_{G\times G}^*\mu_{\mathcal{E}}$ satisfies (CS.2) for $\bar{\mathcal{L}}$, and ϕ is compatible with μ in the sense of (CS.3). This construction defines the functor $B_G : \mathcal{CS}_0(G) \to \mathcal{CS}(G)$ given on objects by $(\mathcal{E}, \mu_{\mathcal{E}}) \mapsto (\bar{\mathcal{L}}, \mu, \phi)$, as defined here. Because morphisms in $\mathcal{CS}_0(G)$ and $\mathcal{CS}(G)$ are morphisms of local systems on G and $\bar{G}$, respectively, satisfying condition (CS.4), this functor is also full and faithful. □

We will say that a character sheaf $\mathcal{L} \in \mathcal{CS}(G)$ *descends to* G if it is isomorphic to some $B_G(\mathcal{E}, \mu_{\mathcal{E}})$.

Remark 1.9. In fact, it is not difficult to recognize character sheaves that descend to G: they are exactly those character sheaves $\mathcal{L} = (\bar{\mathcal{L}}, \mu, \phi)$ for which the action of W on $\bar{\mathcal{L}}$ given by ϕ extends to a continuous action of $\mathrm{Gal}(\bar{k}/k)$ on $\bar{\mathcal{L}}$; see [17, Exposé XIII, Rappel 1.1.3], for example.

1.5. Discrete isogenies

Here, we consider character sheaves on G that are defined by discrete isogenies onto G (§ 1.5); these will play a role in § 3.1.

A finite, étale, surjective morphism $H \to G$ of smooth group schemes over k for which the action of $\mathrm{Gal}(\bar{k}/k)$ on the kernel is trivial is called a *discrete isogeny*, inspired by [26, § 2.2].

Proposition 1.10. *Let $f : H \to G$ be a discrete isogeny, and let A be the kernel of f. Let V be a one-dimensional representation of A equipped with an isomorphism $V \to V \otimes V$. Let $\psi : A \to \bar{\mathbb{Q}}_\ell^\times$ be the character of V. Then $(f_! V_H)_\psi$ (the ψ-isotypic component of $f_! V_H$) is an object of $\mathcal{CS}_0(G)$.*

Proof. Let f, A, V, and ψ be as above, and set $\mathcal{E} = (f_! V_H)_\psi$. Since A is abelian, $\mathcal{E}$ is an ℓ-adic local system on G of rank one. We must show that $\mathcal{E}$ comes equipped with an isomorphism $\mu_{\mathcal{E}} : m^*\mathcal{E} \to \mathcal{E} \boxtimes \mathcal{E}$. To do this we use étale descent to see that pullback along

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f gives an equivalence between ℓ-adic local systems on G and A-equivariant local systems on H; see [4, Proposition 8.1.1]. In particular, $f^*\mathcal{E}$ is the A-equivariant constant sheaf V on H with character ψ. Since f is a morphism of group schemes, the functor f^* defines $\mu_{\mathcal{E}} : m^*\mathcal{E} \to \mathcal{E} \boxtimes \mathcal{E}$ from the isomorphism $m^*\psi \cong \psi \boxtimes \psi$ determined by $V \to V \otimes V$. □

Remark 1.11. Since V is one dimensional, the choice of $V \to V \otimes V$ is exactly the choice of an isomorphism $V \cong \bar{\mathbb{Q}}_\ell$.

Remark 1.12. A descent argument similar to the one employed in the proof of Lemma 1.10 is used in [8, Lemma 1.10], though in the more restrictive case of connected algebraic groups.

1.6. Recollections on character sheaves for connected algebraic groups

If the smooth commutative group scheme G is of finite type, then every character sheaf descends to G; we will see that this feature does not necessarily hold when G is not of finite type.

Lemma 1.13. *If G is a connected commutative algebraic group over k, then*

$$B_G : \mathcal{CS}_0(G) \to \mathcal{CS}(G)$$

is an equivalence of categories.

Using this equivalence of categories, we may give a good description of $\mathcal{CS}(G)$ when G is connected and finite type.

Proposition 1.14. *If G is a connected, commutative algebraic group over k, then the following hold.*

(1) $\mathrm{Tr}_G : \mathcal{CS}(G)_{/iso} \to G(k)^*$ *is an isomorphism of groups.*

(2) *Every character sheaf on G is isomorphic to one defined by a discrete isogeny.*

(3) $\mathrm{Aut}(\mathcal{L}) = 1$, *for all character sheaves $\mathcal{L}$ on G.*

Proof. By Lemma 1.13, we know that every character sheaf $\mathcal{L}$ on $\bar{G}$ descends to G; let $\mathcal{E}$ be an object of $\mathcal{CS}_0(G)$ for which $B_G(\mathcal{E}) \cong \mathcal{L}$. Since the functor $B_G : \mathcal{CS}_0(G) \to \mathcal{CS}(G)$ is full and faithful, $\mathrm{Aut}(\mathcal{L}) = \mathrm{Aut}(\mathcal{E})$. From here, Deligne's function-sheaf dictionary for connected commutative algebraic groups over finite fields, as in [14, Sommes trig.] or [28, 1.1.3], gives us all we need for points (1) and (2), as we briefly recall.

As in the proof of Proposition 1.10, use étale descent to see that pullback by the Lang isogeny $\mathsf{Lang} : G \to G$ defines an equivalence of categories between local systems on G and $G(k)$-equivariant local systems on G. Under this equivalence, local systems $\mathcal{E}$ on G arising from objects in $\mathcal{CS}_0(G)$ are matched with $G(k)$-equivariant constant local systems of rank one on G, and therefore with one-dimensional representations of $G(k)$. In the same way, pullback along the isogeny $\mathsf{Lang} \times \mathsf{Lang} : G \times G \to G \times G$ matches the extra structure $\mu_{\mathcal{E}} : m^*\mathcal{E} \to \mathcal{E} \boxtimes \mathcal{E}$ with an isomorphism $m^*V \to V \boxtimes V$ of one-dimensional representations of $G(k) \times G(k)$, which is exactly an isomorphism

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$V \to V \otimes V$ of one-dimensional representations, which is exactly the choice of an isomorphism $V \cong \bar{\mathbb{Q}}_\ell$. We see that $\mathcal{CS}_0(G)$ is equivalent to the category of characters of $G(k)$. Let $\bar{g}$ be a geometric point above $g \in G(k)$. If $\mathcal{E}$ matches $\psi : G(k) \to \bar{\mathbb{Q}}_\ell^\times$ under this equivalence, a simple calculation on stalks reveals that the action of Frobenius on $\mathcal{E}_{\bar{g}}$ is multiplication by $\psi(g)^{-1}$. In other words, for every $\mathcal{E}$ in $\mathcal{CS}_0(G)$, the trace of $\mathrm{Lang}^* \mathcal{E}$ is $t_{\mathcal{E}}^{-1}$ as a representation of $G(k)$, proving parts (1) and (2).

For part (3), suppose that $\mathrm{Lang}^* \mathcal{E} = V$ with isomorphism $V \to V \otimes V$. Observe that the equivalence above establishes a bijection between $\mathrm{Aut}(\mathcal{E})$ and the group of automorphisms of $\rho : V \to V$ for which

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \\ \downarrow & & \downarrow \\ V \otimes V & \xrightarrow{\rho \otimes \rho} & V \otimes V \end{array}$$

commutes. Since the only such isomorphism ρ is id_V, it follows that $\mathrm{Aut}(\mathcal{E}) = 1$, completing the proof. □

We have just seen that, for a connected commutative algebraic group G over k, the category of character sheaves on G is equivalent to the category of one-dimensional representations V of $G(k)$ equipped with an isomorphism $V \cong \bar{\mathbb{Q}}_\ell$, and therefore equivalent to the category of characters ψ of $G(k)$. We have also just seen that if the character of $\mathrm{Lang}^* \mathcal{E}$ is ψ then the canonical isomorphism $m^*\psi \cong \psi \boxtimes \psi$ determines the isomorphism $\mu_{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \boxtimes \mathcal{E}$. This fact leads (back) to a perspective on the function-sheaf dictionary common in the literature in which one considers one-dimensional local systems $\mathcal{E}$ on G for which *there exists* an isomorphism $m^*\mathcal{E} \cong \mathcal{E} \boxtimes \mathcal{E}$ [26, Introduction]. As a slight variation, one may also consider one-dimensional local systems $\bar{\mathcal{L}}$ on $\bar{G}$ for which *there exist* an isomorphism $\mathrm{Fr}_G^* \bar{\mathcal{L}} \cong \bar{\mathcal{L}}$ and an isomorphism $\bar{m}^* \bar{\mathcal{L}} \cong \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}$.

Although the category $\mathcal{CS}(G)$ of character sheaves on G specializes to $\mathcal{CS}_0(G)$ when G is of finite type (§ 3.6), this description is *not* sufficient when extending the dictionary to smooth commutative group schemes, as we will see already in § 2. In particular, for a given $\bar{\mathcal{L}}$ and ϕ there may be many μ that make $(\bar{\mathcal{L}}, \mu, \phi)$ a character sheaf. For étale G, Proposition 2.7 shows that $\mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}}$ measures the possibilities for μ. We will see in § 3 that $\mathrm{H}^2(\pi_0(\bar{G}), \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}}$ plays an analogous role for general smooth commutative group schemes G.

2. Character sheaves on étale commutative group schemes over finite fields

In this section we give a complete characterization of the category of character sheaves on étale commutative group schemes over finite fields.

2.1. Stalks of character sheaves

The equivalence $G \mapsto G(\bar{k})$, from the category of étale commutative group schemes over k to the category of commutative groups equipped with a continuous action of $\mathrm{Gal}(\bar{k}/k)$, provides the following simple description of character sheaves. A character sheaf $\mathcal{L}$ on an étale commutative group scheme G over k is

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(cs.1) an indexed set of one-dimensional $\bar{\mathbb{Q}}_\ell$-vector spaces $\bar{\mathcal{L}}_x$, as x runs over $G(\bar{k})$;

(cs.2) an indexed set of isomorphisms $\mu_{x,y} : \bar{\mathcal{L}}_{x+y} \xrightarrow{\cong} \bar{\mathcal{L}}_x \otimes \bar{\mathcal{L}}_y$, for all $x, y \in G(\bar{k})$, such that

$$\begin{array}{ccc} \bar{\mathcal{L}}_{x+y+z} & \xrightarrow{\mu_{x+y,z}} & \bar{\mathcal{L}}_{x+y} \otimes \bar{\mathcal{L}}_z \\ {\scriptstyle \mu_{x,y+z}} \downarrow & & \downarrow {\scriptstyle \mu_{x,y} \otimes \mathrm{id}} \\ \bar{\mathcal{L}}_x \otimes \bar{\mathcal{L}}_{y+z} & \xrightarrow{\mathrm{id} \otimes \mu_{y,z}} & \bar{\mathcal{L}}_x \otimes \bar{\mathcal{L}}_y \otimes \bar{\mathcal{L}}_z \end{array}$$

commutes, for all $x, y, z \in G(\bar{k})$; and

(cs.3) an indexed set of isomorphisms $\phi_x : \bar{\mathcal{L}}_{\mathrm{Fr}(x)} \to \bar{\mathcal{L}}_x$ such that

$$\begin{array}{ccc} \bar{\mathcal{L}}_{\mathrm{Fr}(x)+\mathrm{Fr}(y)} & \xrightarrow{\mu_{\mathrm{Fr}(x),\mathrm{Fr}(y)}} & \bar{\mathcal{L}}_{\mathrm{Fr}(x)} \otimes \bar{\mathcal{L}}_{\mathrm{Fr}(y)} \\ {\scriptstyle \phi_{x+y}} \downarrow & & \downarrow {\scriptstyle \phi_x \otimes \phi_y} \\ \bar{\mathcal{L}}_{x+y} & \xrightarrow{\mu_{x,y}} & \bar{\mathcal{L}}_x \otimes \bar{\mathcal{L}}_y \end{array}$$

commutes, for all $x, y \in G(\bar{k})$.

Under this equivalence, a morphism $\rho : \mathcal{L} \to \mathcal{L}'$ of character sheaves on G is given by

(cs.4) an indexed set $\bar{\rho}_x : \bar{\mathcal{L}}_x \to \bar{\mathcal{L}}'_x$ of linear transformations such that

$$\begin{array}{ccc} \bar{\mathcal{L}}_{\mathrm{Fr}(x)} & \xrightarrow{\bar{\rho}_{\mathrm{Fr}(x)}} & \bar{\mathcal{L}}'_{\mathrm{Fr}(x)} \\ {\scriptstyle \phi_x} \downarrow & & \downarrow {\scriptstyle \phi'_x} \\ \bar{\mathcal{L}}_x & \xrightarrow{\bar{\rho}_x} & \bar{\mathcal{L}}'_x \end{array} \quad \text{and} \quad \begin{array}{ccc} \bar{\mathcal{L}}_{x+y} & \xrightarrow{\bar{\rho}_{x+y}} & \bar{\mathcal{L}}'_{x+y} \\ {\scriptstyle \mu_{x,y}} \downarrow & & \downarrow {\scriptstyle \mu'_{x,y}} \\ \bar{\mathcal{L}}_x \otimes \bar{\mathcal{L}}_y & \xrightarrow{\bar{\rho}_x \otimes \bar{\rho}_y} & \bar{\mathcal{L}}'_x \otimes \bar{\mathcal{L}}'_y \end{array}$$

both commute, for all $x, y \in G(\bar{k})$.

We will see that $\mathrm{Tr}_G : \mathcal{CS}(G)_{/\mathrm{iso}} \to G(k)^*$ may not provide complete information about isomorphism classes of character sheaves on G when G is not a connected algebraic group. Our main tool for understanding this phenomenon is a group homomorphism $S_G : \mathcal{CS}(G)_{/\mathrm{iso}} \to \mathrm{H}^2(E_G^\bullet)$ defined in § 2.3, for which the next two sections are preparation.

2.2. A spectral sequence

Let G be a smooth commutative group scheme over k. The zeroth page of the Hochschild–Serre spectral sequence is a double complex $E^{\bullet,\bullet}$ defined by

$$E^{i,j} = C^i(\mathcal{W}, C^j(G(\bar{k}), \bar{\mathbb{Q}}_\ell^\times));$$

see [34, § 1.7], expanding on [36, Chapter 5 and § 7.5]. The standard derivative on cochains yields two derivatives,

$$\begin{aligned} d_G &: E^{i,j} \to E^{i,j+1} \quad \text{and} \\ d_{\mathcal{W}} &: E^{i,j} \to E^{i+1,j}; \end{aligned}$$

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we use the first as the derivative d_0 on the zeroth page, and the second to induce d_1. Combining them also yields a derivative $d = d_G + (-1)^j d_{\mathcal{W}}$ on the total complex

$$E_G^n = \bigoplus_{i+j=n} E^{i,j}.$$

The machinery of spectral sequences gives us a sequence of pages $E_r^{i,j}$, converging to a page $E_\infty^{i,j}$. We summarize the key properties of this spectral sequence in the following proposition.

Proposition 2.1. *In the spectral sequence defined above,*

(1) *the second page is given by* $E_2^{i,j} = \mathrm{H}^i(\mathcal{W}, \mathrm{H}^j(G(\bar{k}), \bar{\mathbb{Q}}_\ell^\times))$*;*

(2) *there is an isomorphism* $\mathrm{H}^n(\mathcal{W} \ltimes G(\bar{k}), \bar{\mathbb{Q}}_\ell^\times) \cong \mathrm{H}^n(E_G^\bullet)$*; and*

(3) *there is a filtration* $\mathrm{H}^n(\mathcal{W} \ltimes G(\bar{k}), \bar{\mathbb{Q}}_\ell^\times) = F_n \supset \cdots \supset F_{-1} = 0$, *with* $F_i/F_{i-1} \cong E_\infty^{i,n-i}$.

Moreover, since $\mathcal{W} \cong \mathbb{Z}$ has cohomological dimension 1, $E_2^{i,j} = 0$ for $i > 1$, and the sequence degenerates at the second page: $E_\infty^{i,j} = E_2^{i,j}$. We obtain the following corollary.

Corollary 2.2. *There is a short exact sequence*

$$0 \to \mathrm{H}^0(\mathcal{W}, \mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) \to \mathrm{H}^2(E_G^\bullet) \to \mathrm{H}^1(\mathcal{W}, \mathrm{H}^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) \to 0.$$

This sequence will play a key role in understanding the kernel of Tr_G, as described in the next few sections. For this application, we need a good understanding of these maps to and from the total complex.

Proposition 2.3. *Consider the short exact sequence in Corollary 2.2.*

(1) *Every class* $[\alpha \oplus \beta \oplus \gamma] \in \mathrm{H}^2(E_G^\bullet)$ *is cohomologous to one with* $\gamma = 0$.

(2) *The map* $\mathrm{H}^2(E_G^\bullet) \to \mathrm{H}^1(\mathcal{W}, \mathrm{H}^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times))$ *is given by* $[\alpha \oplus \beta \oplus 0] \mapsto [\beta]$.

(3) *Suppose that* $a \in Z^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)$ *represents a class in* $\mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)$ *fixed by Frobenius. The map* $\mathrm{H}^0(\mathcal{W}, \mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) \to \mathrm{H}^2(E_G^\bullet)$ *is given by* $[a] \mapsto [a \oplus 0 \oplus 0]$.

Proof. Since $\mathrm{H}^2(\mathcal{W}, C^0(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) = 0$, we may find a $\gamma_1 \in C^1(\mathcal{W}, C^0(\bar{G}, \bar{\mathbb{Q}}_\ell^\times))$ with $d_{\mathcal{W}}\gamma_1 = \gamma$. Subtracting $d\gamma_1$ from $\alpha \oplus \beta \oplus \gamma$, we may assume that $\gamma = 0$.

The latter two claims follow from tracing through the definition of latter pages in the spectral sequence. □

2.3. From character sheaves to the total complex

Let G be a smooth commutative group scheme over k. In this section we define a group homomorphism

$$S_G : \mathcal{CS}(G)_{/\mathrm{iso}} \to \mathrm{H}^2(E_G^\bullet).$$

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Let $\mathcal{L} = (\bar{\mathcal{L}}, \mu, \phi)$ be a character sheaf on G. For each geometric point $x \in \bar{G}$, choose a basis $\{v_x\}$ for $\bar{\mathcal{L}}_x$. Through this choice, $\mathcal{L}$ determines functions

$$a : \bar{G} \times \bar{G} \to \bar{\mathbb{Q}}_\ell^\times \qquad b : \bar{G} \to \bar{\mathbb{Q}}_\ell^\times$$
$$\mu_{x,y}(v_{x+y}) = a(x, y) v_x \otimes v_y \qquad \phi_x(v_{\mathrm{Fr}_G(x)}) = b(x) v_x.$$

Condition (CS.2) implies that

$$a(x + y, z) a(x, y) = a(x, y + z) a(y, z) \tag{3}$$

for all $x, y, z \in \bar{G}$, so $a \in Z^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)$. Similarly, condition (CS.3) gives

$$\frac{a(\mathrm{Fr}_G(x), \mathrm{Fr}_G(y))}{a(x, y)} = \frac{b(x + y)}{b(x) b(y)} \tag{4}$$

for $x, y \in \bar{G}$. Let $\alpha \in C^0(\mathcal{W}, C^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times))$ be the 0-cochain corresponding to a, and let $\beta \in C^1(\mathcal{W}, C^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times))$ be the cocycle such that $\beta(\mathrm{Fr})$ is b. We will write both α and β additively. Then

$$d_G \alpha = 0, \quad d_{\mathcal{W}} \alpha = d_G \beta, \quad d_{\mathcal{W}} \beta = 0;$$

in other words,

$$\alpha \oplus \beta \in Z^2(E_G^\bullet).$$

Although the cocycle $\alpha \oplus \beta$ is not well defined by $\mathcal{L}$, its class in $\mathrm{H}^2(E_G^\bullet)$ is. To see this, let $\{v'_x \in \bar{\mathcal{L}}_x^\times \mid x \in \bar{G}\}$ be another choice, and let $\alpha' \oplus \beta' \in Z^2(E_G^\bullet)$ be defined by $\mathcal{L}$ and this choice, as above. Now let $\delta \in C^0(\mathcal{W}, C^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times))$ correspond to the function $d : \bar{G} \to \bar{\mathbb{Q}}_\ell^\times$ defined by $v'_x = d(x) v_x$. Chasing through the diagrams in (CS.2) and (CS.3), we find that

$$\alpha' \oplus \beta' = \alpha \oplus \beta + d\delta,$$

so the class $[\alpha \oplus \beta]$ of $\alpha \oplus \beta$ in $\mathrm{H}^2(E_G^\bullet)$ is independent of the choice made above. It is also easy to see that $[\alpha \oplus \beta] = [\alpha_0 \oplus \beta_0]$ when $\mathcal{L} \cong \mathcal{L}_0$, which concludes the definition of the function

$$S_G : \mathcal{CS}(G)_{/\mathrm{iso}} \to \mathrm{H}^2(E_G^\bullet)$$
$$[\mathcal{L}] \mapsto [\alpha \oplus \beta].$$

It is also easy to see that $[\alpha_1 \oplus \beta_1] + [\alpha_2 \oplus \beta_2] = [\alpha_3 \oplus \beta_3]$ when $\mathcal{L}_3 = \mathcal{L}_1 \otimes \mathcal{L}_2$, so S_G is a group homomorphism.

Proposition 2.4. *If G is étale then $S_G : \mathcal{CS}(G)_{/iso} \to \mathrm{H}^2(E_G^\bullet)$ is an isomorphism.*

Proof. Suppose that $[\mathcal{L}] \in \mathcal{CS}(G)_{/\mathrm{iso}}$ with $S_G([\mathcal{L}]) = [\alpha \oplus \beta] = 0$, so that $\alpha \oplus \beta = d\sigma$ for some $\sigma \in C^0(\mathcal{W}, C^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) = C^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)$. For each $x \in \bar{G}$, define $\sigma_x : \bar{\mathcal{L}}_x \to \bar{\mathbb{Q}}_\ell$ by $\sigma_x : v_x \mapsto \sigma(x)$. Then the indexed set of isomorphisms $\{\sigma_x : \bar{\mathcal{L}}_x \to \bar{\mathbb{Q}}_\ell \mid x \in \bar{G}\}$ defines an isomorphism $\mathcal{L} \to (\bar{\mathbb{Q}}_\ell)_G$. Since $\mathcal{L} = 0 \in \mathcal{CS}(G)_{/\mathrm{iso}}$, S_G is injective.

To see that S_G is surjective, begin with $\alpha \oplus \beta \oplus 0 \in Z^2(E_G^\bullet)$. Since $d_{\mathcal{W}} \beta = 0$, we may define $a = \alpha \in C^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)$ and $b = \beta(\mathrm{Fr}) \in C^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)$, which are related to α and β as above. Set $\bar{\mathcal{L}}_x = \bar{\mathbb{Q}}_\ell$, and define $\mu_{x,y} : \bar{\mathcal{L}}_{x+y} \to \bar{\mathcal{L}}_x \otimes \bar{\mathcal{L}}_y$ by $\mu_{x,y}(1) = a(x, y)(1 \otimes 1)$ and $\phi_x : \bar{\mathcal{L}}_{\mathrm{Fr}_G(x)} \to \bar{\mathcal{L}}_x$ by $\phi_x(1) = b(x)$. Then (CS.1) holds since $d_G \alpha = 0$, and (CS.2) holds

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since $d_{\mathcal{W}}\alpha = d_G\beta$. Tracing the construction backward, we have defined a character sheaf $\mathcal{L}$ on G with $S_G(\mathcal{L}) = [\alpha \oplus \beta \oplus 0]$, showing that S_G is surjective. □

2.4. Objects in the étale case

In this section we fit the group homomorphisms Tr_G and S_G into a commutative diagram, determining the kernel and cokernel of Tr_G when G is an étale commutative group scheme over k. We begin with a simple general result relating duals, invariants, and coinvariants.

Lemma 2.5. *Let X be an abelian group equipped with an action of $\mathcal{W}$. Then*

$$(X^*)_{\mathcal{W}} \to (X^{\mathcal{W}})^*$$
$$[f] \mapsto f|_{X^{\mathcal{W}}}$$

is an isomorphism.

Proof. We can describe $X^{\mathcal{W}}$ as the kernel of the map $X \xrightarrow{\mathrm{Fr}-1} X$; let $Y = (\mathrm{Fr}-1)X$ be the augmentation ideal. Dualizing the sequence

$$0 \to X^{\mathcal{W}} \to X \to Y \to 0$$

yields

$$0 \to Y^* \to X^* \to (X^{\mathcal{W}})^* \to \mathrm{Ext}^1_{\mathbb{Z}}(Y, \bar{\mathbb{Q}}_\ell^\times).$$

Since $\mathrm{Ext}^1_{\mathbb{Z}}(-, \bar{\mathbb{Q}}_\ell^\times)$ vanishes, we get a natural isomorphism from the cokernel of $Y^* \xrightarrow{\mathrm{Fr}-1} X^*$ to $(X^{\mathcal{W}})^*$. □

Proposition 2.6. *If G is étale, then $\mathrm{Tr}_G : \mathcal{CS}(G)_{/iso} \to G(k)^*$ is surjective and split.*

Proof. Pick $\chi \in G(k)^*$. Let $[\beta] \in \mathrm{H}^1(\mathcal{W}, \bar{G}^*)$ be the class corresponding to χ under Lemma 2.5. Every representative cocycle $\beta \in Z^1(\mathcal{W}, \bar{G}^*)$ determines a homomorphism $\beta(\mathrm{Fr}) : G(\bar{k}) \to \bar{\mathbb{Q}}_\ell^\times$ such that $\beta(\mathrm{Fr})|_{G(k)} = \chi$. Set $\bar{\mathcal{L}}_x = \bar{\mathbb{Q}}_\ell$ for every $x \in G(\bar{k})$. Define $\mu_{x,y} : \bar{\mathcal{L}}_{x+y} \to \bar{\mathcal{L}}_x \otimes \bar{\mathcal{L}}_y$ by $\mu_{x,y}(1) = 1 \otimes 1$, and $\phi_x : \bar{\mathcal{L}}_{\mathrm{Fr}(x)} \to \bar{\mathcal{L}}_x$ by $\phi_x(1) = \beta(\mathrm{Fr})(x)$. Since $\beta(\mathrm{Fr}) : G(\bar{k}) \to \bar{\mathbb{Q}}_\ell^\times$ is a group homomorphism, condition (4) is satisfied with $a = 1$. So $\mathcal{L} = (\bar{\mathcal{L}}, \mu, \phi)$ is a character sheaf with $t_{\mathcal{L}} = \chi$. This shows that Tr_G is surjective.

Now let $\beta' \in Z^1(\mathcal{W}, \bar{G}^*)$ be another representative for $[\beta]$ so $\beta - \beta' = d_{\mathcal{W}}\delta$ for some $\delta \in C^0(\mathcal{W}, \bar{G}^*)$ defining $d \in \mathrm{Hom}(G(\bar{k}), \bar{\mathbb{Q}}_\ell^\times)$. Let $\mathcal{L}'$ be the character sheaf on G defined by β', as above. For each $x \in G(\bar{k})$, define $\bar{\rho}_x : \mathcal{L}_x \to \mathcal{L}'_x$ by $\bar{\rho}_x(1) = d(x)$. The collection of isomorphisms $\{\bar{\rho}_x \mid x \in G(\bar{k})\}$ satisfies condition (CS.4), so it defines a morphism $\rho : \mathcal{L} \to \mathcal{L}'$, which is clearly an isomorphism. We have now defined a section of Tr_G.

Now suppose that $\chi_1, \chi_2 \in G(k)^*$. Pick cocycles $\beta_1, \beta_2 \in Z^1(\mathcal{W}, \bar{G}^*)$ and construct character sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ on G, as above. Since $\mathcal{L}_1 \otimes \mathcal{L}_2$ is exactly the character sheaf built from the cocycle $\beta_1 \cdot \beta_2$, and since $t_{\mathcal{L}_1 \otimes \mathcal{L}_2} = t_{\mathcal{L}_1} \cdot t_{\mathcal{L}_2}$, the section of Tr_G defined here is a homomorphism. □

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Proposition 2.7. *If G is étale then the map $S_G : \mathcal{CS}(G)_{/iso} \to \mathrm{H}^2(E^\bullet_G)$ induces an isomorphism of split short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \mathrm{Tr}_G & \longrightarrow & \mathcal{CS}(G)_{/iso} & \xrightarrow{\mathrm{Tr}_G} & G(k)^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle S_G} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^0(\mathcal{W}, \mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) & \longrightarrow & \mathrm{H}^2(E^\bullet_G) & \longrightarrow & \mathrm{H}^1(\mathcal{W}, \mathrm{H}^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) & \longrightarrow & 0 \end{array}$$

Proof. This result follows from Propositions 2.3, 2.4, and 2.6. □

Definition 2.8. We call a character sheaf $\mathcal{L}$ on G *invisible* if it is non-trivial and $\mathrm{Tr}_G(\mathcal{L}) = 1$.

The proposition gives a method for determining whether a given G admits invisible character sheaves.

Remark 2.9. Recall the Künneth formula in group cohomology [9, Proposition I.0.8]: if A and A' are groups and M and M' are abelian groups with M $\mathbb{Z}$-free, then

$$\begin{aligned} &\mathrm{H}^n(A \times A', M \otimes M') \\ &\cong \bigoplus_{i+j=n} \mathrm{H}^i(A, M) \otimes \mathrm{H}^j(A', M') \oplus \bigoplus_{i+j=n+1} \mathrm{Tor}_1^{\mathbb{Z}}\left(\mathrm{H}^i(A, M), \mathrm{H}^j(A', M')\right). \end{aligned}$$

Now suppose that $\bar{G} = \mathbb{Z}^r \times \prod_{i=1}^m \mathbb{Z}/N_i\mathbb{Z}$ is an arbitrary finitely generated abelian group, with $N_i \mid N_{i+1}$. Then the Künneth formula implies that

$$\mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times) \cong \left(\bar{\mathbb{Q}}_\ell^\times\right)^{r(r-1)/2} \times \prod_{i=1}^m (\mathbb{Z}/N_i\mathbb{Z})^{m+r-i}. \tag{5}$$

We see that $\mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)$ is trivial if and only if $\bar{G}$ is cyclic. Of course, $\mathrm{H}^0(\mathcal{W}, \mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times))$ may or may not be trivial, even when $\mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)$ is non-trivial.

Example 2.10. Consider the simplest non-trivial case, where $\bar{G} = \{1, i, j, k\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Using (5), we have $\mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times) \cong \mathbb{Z}/2\mathbb{Z}$, on which $\mathcal{W}$ must act trivially, regardless of its action on $\bar{G}$ itself. The non-trivial element corresponds to the extension

$$1 \to \bar{\mathbb{Q}}_\ell^\times \to Q \to \bar{G} \to 1, \tag{6}$$

where $Q = \{c + c_i i + c_j j + c_k k \mid c, c_i, c_j, c_k \in \bar{\mathbb{Q}}_\ell \text{ with exactly one non-zero}\}$ is a subgroup of the quaternion algebra over $\bar{\mathbb{Q}}_\ell$. Let a be a 2-cocycle corresponding to this extension, with values in $\{\pm 1\}$. When Fr_G acts trivially on $\bar{G}$, any homomorphism $b : \bar{G} \to \bar{\mathbb{Q}}_\ell^\times$ will satisfy (4), and the corresponding $\alpha \oplus \beta$ are non-cohomologous in $\mathrm{H}^2(E^\bullet_G)$. When Fr_G exchanges i and j, then we may take $b(1) = 1, b(i) = -1$ and $b(j) = b(k) = \pm 1$, up to coboundaries. Finally, when Fr_G cycles i, j, and k, any homomorphism $b : \bar{G} \to \bar{\mathbb{Q}}_\ell^\times$ will satisfy (4), but now the corresponding $\alpha \oplus \beta$ are all cohomologous in $\mathrm{H}^2(E^\bullet_G)$. In each case, we may produce an explicit character sheaf from the listed a and b.

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Note that these character sheaves arise from discrete isogenies, as in § 1.5. Let $\bar{H}$ be the quaternion group of order 8: the subgroup of Q with $c, c_i, c_j, c_k \in \{\pm 1\}$. The sequence (6) is the pushforward of

$$1 \to \{\pm 1\} \to \bar{H} \to \bar{G} \to 1$$

along the inclusion $\{\pm 1\} \hookrightarrow \bar{\mathbb{Q}}_\ell^\times$. Note that these character sheaves arise from a non-commutative cover of $\bar{G}$, justifying the inclusion of such covers in the definition of a discrete isogeny.

2.5. On the necessity of working with Weil sheaves

In this section we justify the appearance of Weil sheaves in Definition 1.1.

Proposition 2.11. *Let G be a commutative étale group scheme over k. Then the image of $\mathcal{CS}_0(G)$ under $\mathrm{Tr}_G : \mathcal{CS}(G) \to G(k)^*$ is $\mathrm{Hom}(G(k), \bar{\mathbb{Z}}_\ell^\times)$.*

Proof. Objects in $\mathcal{CS}_0(G)$ may be described by a small modification to the technique used in §§ 2.2 and 2.3. Set $F^{i,j} := C^i_{\mathrm{cts}}(\mathrm{Gal}(\bar{k}/k), C^j(G(\bar{k}), \bar{\mathbb{Q}}_\ell^\times))$. Then the results of § 2.2 adapt to give a short exact sequence in continuous Galois cohomology

$$0 \to \mathrm{H}^0(k, \mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) \to \mathrm{H}^2(F_G^\bullet) \to \mathrm{H}^1(k, \mathrm{H}^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) \to 0,$$

for which the maps are given by the analogues of Proposition 2.3. Moreover, using [17, Exposé XIII, Rappel 1.1.3], we see that Proposition 2.4 adapts to provide an isomorphism $\mathcal{CS}_0(G)_{/\mathrm{iso}} \to \mathrm{H}^2(F_G^\bullet)$ compatible with $\mathcal{CS}_0(G) \to \mathcal{CS}(G)$ and with the following map of exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^0(k, \mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) & \longrightarrow & \mathrm{H}^2(F_G^\bullet) & \longrightarrow & \mathrm{H}^1(k, \mathrm{H}^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^0(\mathcal{W}, \mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) & \longrightarrow & \mathrm{H}^2(E_G^\bullet) & \longrightarrow & \mathrm{H}^1(\mathcal{W}, \mathrm{H}^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) & \longrightarrow & 0 \end{array}$$

In this way, Proposition 2.11 is now reduced to the claim

$$\mathrm{H}^1(k, \mathrm{H}^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) = \mathrm{Hom}(G(k), \bar{\mathbb{Z}}_\ell^\times).$$

To see that, one may argue as follows. Pick $i \in \pi_0(G)$, and let $G^i \hookrightarrow G$ be the corresponding connected component. Pick a geometric point x on G^i, and observe that, since G^i is connected as a k-scheme, $G^i(\bar{k})$ is canonically identified with the $\mathrm{Gal}(\bar{k}/k)$-orbit of x. We remark that, while G^i is defined over k, the set $G^i(k)$ is non-empty only when $G^i(\bar{k}) = \{x\}$. Since $\mathrm{H}^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times) = \mathrm{Hom}(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)$, evaluation $\chi \mapsto \chi(x)$ defines $\mathrm{H}^1(k, \mathrm{H}^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) \to \mathrm{H}^1(k, \bar{\mathbb{Q}}_\ell^\times)$. By continuity, $\mathrm{H}^1(k, \bar{\mathbb{Q}}_\ell^\times) = \mathrm{H}^1(k, \bar{\mathbb{Z}}_\ell^\times)$. Letting i range over $\pi_0(G)$, we conclude that $\mathrm{H}^1(k, \mathrm{H}^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) = \mathrm{H}^1(k, \mathrm{H}^1(\bar{G}, \bar{\mathbb{Z}}_\ell^\times))$. When adapted to abelian groups with continuous action of $\mathrm{Gal}(\bar{k}/k)$, the strategy of the proof of Lemma 2.5 gives $\mathrm{H}^1(k, \mathrm{H}^1(\bar{G}, \bar{\mathbb{Z}}_\ell^\times)) = \mathrm{Hom}(G(k), \bar{\mathbb{Z}}_\ell^\times)$, concluding the proof. □

Proposition 2.11 reveals the necessity of working with Weil sheaves in Definition 1.1: one cannot geometrize all characters of $G(k)$ using local systems on G, for general smooth commutative groups schemes G. Proposition 2.11 is extended to all smooth commutative groups schemes in § 3.4.

Example 2.12. Consider the case when G is the étale group scheme $\mathbb{Z}$ over k with Fr_G trivial. If $\chi : \mathbb{Z} \to \bar{\mathbb{Q}}_\ell^\times$ is the character of $G(k)$ determined by $\chi(1) = \ell$, and if $\mathcal{L}$ is a character sheaf on G in the isomorphism class corresponding to χ under Proposition 2.6, then $\mathcal{L}$ does not descend to G, since the image of χ is not bounded. If $\chi' : \mathbb{Z} \to \bar{\mathbb{Q}}_\ell^\times$ is the character of $G(k)$ determined by $\chi'(1) = 1 + \ell$, and if $\mathcal{L}'$ corresponds to χ' under Proposition 2.6, then $\mathcal{L}'$ does descend to G, since the image of χ' is bounded. However, $\mathcal{L}'$ is not defined by a discrete isogeny (§ 1.5). If $\chi'' : \mathbb{Z} \to \bar{\mathbb{Q}}_\ell^\times$ is the character of $G(k)$ determined by $\chi''(1) = \zeta$, a root of unity in $\bar{\mathbb{Q}}_\ell^\times$, and if $\mathcal{L}''$ corresponds to χ'' under Proposition 2.6, then $\mathcal{L}''$ is defined by a discrete isogeny.

2.6. Morphisms in the étale case

A complete understanding of the morphisms in $\mathcal{CS}(G)$ also requires a description of the automorphisms of an arbitrary character sheaf $\mathcal{L}$.

Proposition 2.13. *Let G be an étale commutative group scheme over k. If $\mathcal{L}$ and $\mathcal{L}'$ are character sheaves on G then every $\rho \in \mathrm{Hom}(\mathcal{L}, \mathcal{L}')$ is either trivial or an isomorphism. Moreover, the trace map induces an isomorphism of groups*

$$\mathrm{Aut}(\mathcal{L}) \to \mathrm{Hom}(G(\bar{k})_{\mathcal{W}}, \bar{\mathbb{Q}}_\ell^\times).$$

Proof. We have already seen, in Lemma 1.3, that every $\rho \in \mathrm{Hom}(\mathcal{L}, \mathcal{L}')$ is either trivial or an isomorphism. Now suppose that $\rho \in \mathrm{Aut}(\mathcal{L})$. The second diagram in (cs.4) shows that the association $x \mapsto \bar{\rho}_x$ is a homomorphism from $G(\bar{k})$ to $\bar{\mathbb{Q}}_\ell^\times$, and the first diagram in (cs.4) shows that it factors through $G(\bar{k}) \to G(\bar{k})_{\mathcal{W}}$.

Conversely, if $\rho : G(\bar{k})_{\mathcal{W}} \to \bar{\mathbb{Q}}_\ell^\times$ is any homomorphism, then defining $\bar{\rho}_x$ as multiplication by $\rho(x)$ will define a morphism $\bar{\mathcal{L}} \to \bar{\mathcal{L}}'$ that will satisfy the two diagrams in (cs.4).

Composition of morphisms corresponds to pointwise multiplication in this correspondence, showing that the resulting bijection is actually a group isomorphism. □

3. Character sheaves on smooth commutative group schemes over finite fields

3.1. Restriction to the identity component

Consider the short exact sequence defining the component group scheme for G:

$$0 \longrightarrow G^0 \xrightarrow{\iota_0} G \xrightarrow{\pi_0} \pi_0(G) \longrightarrow 0 \tag{7}$$

Since $\pi_0(G)$ is an étale commutative group scheme (and thus smooth), Lemma 1.4 implies that (7) defines a sequence of functors

$$\mathcal{CS}(0) \longrightarrow \mathcal{CS}(\pi_0(G)) \xrightarrow{\pi_0^*} \mathcal{CS}(G) \xrightarrow{\iota_0^*} \mathcal{CS}(G^0) \longrightarrow \mathcal{CS}(0) \tag{8}$$

and therefore, after passing to isomorphism classes, a sequence of abelian groups

$$0 \longrightarrow \mathcal{CS}(\pi_0(G))_{/\mathrm{iso}} \xrightarrow{\pi_0^*} \mathcal{CS}(G)_{/\mathrm{iso}} \xrightarrow{\iota_0^*} \mathcal{CS}(G^0)_{/\mathrm{iso}} \longrightarrow 0 \tag{9}$$

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Note that we found the groups $\mathcal{CS}(\pi_0(G))_{/\mathrm{iso}}$ and $\mathcal{CS}(G^0)_{/\mathrm{iso}}$ in §§ 2.4 and 1.6, respectively. We will shortly see that (9) is exact.

Lemma 3.1. *Every discrete isogeny to G^0 extends to a discrete isogeny to G inducing an isomorphism on component groups.*

Proof. Let $\pi : B \to G^0$ be a discrete isogeny, and set $A := \ker \pi$. We will find a discrete isogeny $f : H \to G$ such that $H^0 = B$, $f^0 = \pi$, and $\pi_0(f) : \pi_0(H) \to \pi_0(G)$ is an isomorphism of component groups. Namely, we will fit π into the following diagram.

$$\begin{array}{ccccc} A & \longrightarrow & A & & \\ \downarrow & & \downarrow & & \\ B & \longrightarrow & H & \longrightarrow & \pi_0(H) \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle\pi_0(f)}\downarrow\wr \\ G^0 & \longrightarrow & G & \longrightarrow & \pi_0(G) \end{array} \tag{10}$$

Here, all rows and columns are exact and all maps are defined over k. We will do so by passing back and forth between group schemes over k and their $\bar{k}$-points.

Extensions of $G^0(\bar{k})$ by $A(\bar{k})$ with $\mathcal{W}$-equivariant maps, such as $B(\bar{k})$, correspond to classes in $\mathrm{Ext}^1_{\mathbb{Z}[\mathcal{W}]}(G^0(\bar{k}), A(\bar{k}))$. Similarly, extensions of $G(\bar{k})$ by $A(\bar{k})$ with $\mathcal{W}$-equivariant maps correspond to classes in $\mathrm{Ext}^1_{\mathbb{Z}[\mathcal{W}]}(G(\bar{k}), A(\bar{k}))$. The map $G^0(\bar{k}) \to G(\bar{k})$ induces a homomorphism

$$\mathrm{Ext}^1_{\mathbb{Z}[\mathcal{W}]}(G(\bar{k}), A(\bar{k})) \to \mathrm{Ext}^1_{\mathbb{Z}[\mathcal{W}]}(G^0(\bar{k}), A(\bar{k}))$$

fitting into the long exact sequence

$$\mathrm{Ext}^1_{\mathbb{Z}[\mathcal{W}]}(G(\bar{k}), A(\bar{k})) \to \mathrm{Ext}^1_{\mathbb{Z}[\mathcal{W}]}(G^0(\bar{k}), A(\bar{k})) \to \mathrm{Ext}^2_{\mathbb{Z}[\mathcal{W}]}(\pi_0(G)(\bar{k}), A(\bar{k}))$$

derived from applying the functor $\mathrm{Hom}(—, A(\bar{k}))$ to $G^0(\bar{k}) \to G(\bar{k}) \to \pi_0(G)(\bar{k})$. Since $\mathcal{W} \cong \mathbb{Z}$ has cohomological dimension 1 [9, Example 4.3], $\mathrm{Ext}^2_{\mathbb{Z}[\mathcal{W}]}(\pi_0(G)(\bar{k}), A(\bar{k}))$ vanishes [10, Theorem 2.6].

We therefore have the existence of diagram (10) at the level of $\bar{k}$-points. This expresses $H(\bar{k})$ as a disjoint union of translates of $B(\bar{k})$; by transport of structure we may take H to be a group scheme over $\bar{k}$. Similarly, the restriction of f to each component of H is a morphism of schemes, and thus f is as well. Finally, the whole diagram descends to a diagram of k-schemes since the $\bar{k}$-points of the objects come equipped with continuous $\mathrm{Gal}(\bar{k}/k)$-actions and the morphisms are $\mathrm{Gal}(\bar{k}/k)$-equivariant. □

We now wish to apply the results of § 1.6 to the identity component of G, for which we must confirm that the identity component of G is actually an algebraic group over k.

Lemma 3.2. *If G is a commutative smooth group scheme over k, then its identity component, G^0, is a connected algebraic group over k.*

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Proof. Since G is a smooth group scheme over k, its identity component G^0 is a connected smooth group scheme of finite type over k, reduced over some finite extension of k [20, 3.17]. Since k is a finite field and hence perfect, G^0 is actually reduced over k [23, Proposition 6.4.1]. Since every group scheme over a field is separated [20, 3.12], it follows that G^0 is a connected algebraic group. □

Proposition 3.3. *The restriction functor* $\iota_0^* : \mathcal{CS}(G) \to \mathcal{CS}(G^0)$ *is essentially surjective.*

Proof. By Lemma 3.2 and Proposition 1.14, every character sheaf on G^0 is isomorphic to $(\pi_! \bar{\mathbb{Q}}_\ell)_\psi$ for some discrete isogeny $\pi : B \to G^0$ and character $\psi : \ker \pi \to \bar{\mathbb{Q}}_\ell^\times$. So to prove the proposition it suffices to show that $(\pi_! \bar{\mathbb{Q}}_\ell)_\psi$ extends to a character sheaf on G. By Lemma 3.1, there is an extension of the discrete isogeny $\pi : B \to G^0$ to a discrete isogeny $f : H \to G$ such that $\pi_0(f) : \pi_0(H) \to \pi_0(G)$ is an isomorphism. Then $(f_! \bar{\mathbb{Q}}_\ell)_\psi$ is a character sheaf on G, and $(f_! \bar{\mathbb{Q}}_\ell)_\psi|_{G^0} \cong (\pi_! \bar{\mathbb{Q}}_\ell)_\psi$. □

3.2. The component group sequence

Lemma 3.4. *The group homomorphism* $\pi_0^* : \mathcal{CS}(\pi_0(G))_{/iso} \to \mathcal{CS}(G)_{/iso}$ *is injective.*

Proof. Let $\mathcal{L}$ be a character sheaf on $\pi_0(G)$, and let $\rho : \pi_0^* \mathcal{L} \to (\bar{\mathbb{Q}}_\ell)_G$ be an isomorphism in $\mathcal{CS}(G)$. For each $x \in \pi_0(\bar{G})$, set $\bar{G}^x := \pi_0^{-1}(x)$. The restriction $\pi_0^* \bar{\mathcal{L}}|_{\bar{G}^x}$ is the constant sheaf $(\bar{\mathcal{L}}_x)_{\bar{G}^x}$, so the isomorphism $\bar{\rho}|_{\bar{G}^x} : (\bar{\mathcal{L}}_x)_{\bar{G}^x} \to (\bar{\mathbb{Q}}_\ell)_{\bar{G}^x}$ determines an isomorphism $\bar{\rho}_x : \bar{\mathcal{L}}_x \to (\bar{\mathbb{Q}}_\ell)_x$. The collection $\{\bar{\rho}_x \mid x \in \pi_0(\bar{G})\}$ determines an isomorphism $\mathcal{L} \to (\bar{\mathbb{Q}}_\ell)_{\pi_0(G)}$ in $\mathcal{CS}(\pi_0(G))$. □

Proposition 3.5. *The sequence*

$$0 \longrightarrow \mathcal{CS}(\pi_0(G))_{/iso} \xrightarrow{\pi_0^*} \mathcal{CS}(G)_{/iso} \xrightarrow{\iota_0^*} \mathcal{CS}(G^0)_{/iso} \longrightarrow 0$$

is exact.

Proof. Exactness at $\mathcal{CS}(G^0)_{/\text{iso}}$ follows from Proposition 3.3, and exactness at $\mathcal{CS}(\pi_0(G))_{/\text{iso}}$ from Lemma 3.4. Here we show that it is also exact at $\mathcal{CS}(G)_{/\text{iso}}$. First note that $\iota_0^* \circ \pi_0^*$ is trivial by Lemma 1.4. So it suffices to show that, if $\mathcal{L} = (\bar{\mathcal{L}}, \mu, \phi)$ is a character sheaf on G with $\mathcal{L}|_{G^0} = (\bar{\mathbb{Q}}_\ell)_{G^0}$, then $\mathcal{L}$ is in the essential image of π_0^*.

As above, set $\bar{G}^x := \pi_0^{-1}(x)$ for $x \in \pi_0(\bar{G})$. Let g, g' be geometric points in the same geometric connected component $\bar{G}^x$. Set $a = g^{-1} g'$, and note that a is a geometric point in $\bar{G}^0$. Let $\mu_{g,a} : \bar{\mathcal{L}}_{ga} \to \bar{\mathcal{L}}_g \otimes \bar{\mathcal{L}}_a$ be the isomorphism of vector spaces obtained by restriction of $\mu : m^* \bar{\mathcal{L}} \to \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}$ to the geometric point (g, a) on $\bar{G}^x \times \bar{G}^0$. Since $\mathcal{L}|_{G^0} = (\bar{\mathbb{Q}}_\ell)_{G^0}$, the stalk of $\bar{\mathcal{L}}$ at a is $\bar{\mathbb{Q}}_\ell$. In this way the pair of geometric points $g, g' \in \bar{G}^x$ determines an isomorphism $\varphi_{g,g'} := \mu_{g,a}^{-1}$ from $\bar{\mathcal{L}}_g$ to $\bar{\mathcal{L}}_{g'}$. The isomorphisms $\varphi_{g,g'} : \bar{\mathcal{L}}_g \to \bar{\mathcal{L}}_{g'}$ are canonical in the following sense: if $g, g' \in \bar{G}^x$ and $h, h' \in \bar{G}^y$, then it

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follows from (CS.2) and (CS.3) that

$$\begin{array}{ccc} \bar{\mathcal{L}}_{gh} & \xrightarrow{\varphi_{gh,g'h'}} & \bar{\mathcal{L}}_{g'h'} \\ \downarrow{\scriptstyle \mu_{g,h}} & & \downarrow{\scriptstyle \mu_{g',h'}} \\ \bar{\mathcal{L}}_g \otimes \bar{\mathcal{L}}_h & \xrightarrow{\varphi_{g,g'}\otimes\varphi_{h,h'}} & \bar{\mathcal{L}}_{g'} \otimes \bar{\mathcal{L}}_{h'} \end{array} \quad \text{and} \quad \begin{array}{ccc} \bar{\mathcal{L}}_{\mathrm{Fr}(g)} & \xrightarrow{\varphi_{\mathrm{Fr}(g),\mathrm{Fr}(g')}} & \bar{\mathcal{L}}_{\mathrm{Fr}(g')} \\ \downarrow{\scriptstyle \phi_g} & & \downarrow{\scriptstyle \phi_{g'}} \\ \bar{\mathcal{L}}_g & \xrightarrow{\varphi_{g,g'}} & \bar{\mathcal{L}}_{g'} \end{array} \tag{11}$$

both commute.

For each $x \in \pi_0(\bar{G})$, pick $g(x) \in \bar{G}^x$, and set $\bar{\mathcal{E}}_x := \bar{\mathcal{L}}_{g(x)}$. Let $\phi_x : \bar{\mathcal{E}}_{\mathrm{Fr}(x)} \to \bar{\mathcal{E}}_x$ be the isomorphism of $\bar{\mathbb{Q}}_\ell$-vector spaces obtained by composing $\varphi_{g(\mathrm{Fr}(x)),\mathrm{Fr}(g(x))} : \bar{\mathcal{L}}_{g(\mathrm{Fr}(x))} \to \bar{\mathcal{L}}_{\mathrm{Fr}(g(x))}$ with $\phi_{g(x)} : \bar{\mathcal{L}}_{\mathrm{Fr}(g(x))} \to \bar{\mathcal{L}}_{g(x)}$. For each pair $x, y \in \pi_0(\bar{G})$ let $\mu_{x,y} : \bar{\mathcal{E}}_{x+y} \to \bar{\mathcal{E}}_x \otimes \bar{\mathcal{E}}_y$ be the isomorphism of $\bar{\mathbb{Q}}_\ell$-vector spaces obtained by composing $\varphi_{g(x+y),g(x)g(y)} : \bar{\mathcal{L}}_{g(x+y)} \to \bar{\mathcal{L}}_{g(x)g(y)}$ with $\mu_{g(x),g(y)} : \bar{\mathcal{L}}_{g(x)g(y)} \to \bar{\mathcal{L}}_{g(x)} \otimes \bar{\mathcal{L}}_{g(y)}$. Using (11), it follows that (CS.1), (CS.2) and (CS.3) are satisfied for $\mathcal{E} := (\bar{\mathcal{E}}_x, \mu_{x,y}, \phi_x)$, thus defining a character sheaf on $\pi_0(G)$.

The pullback $\pi_0^*(\mathcal{E})$ of $\mathcal{E}$ along $\pi_0 : G \to \pi_0(G)$ is constant on geometric connected components, with stalks given by $(\pi_0^*\mathcal{E})_g = \mathcal{E}_x$ for all $g \in \bar{G}^x$. Thus both $\pi_0^*\mathcal{E}$ and $\mathcal{L}$ are constant on geometric connected components of G. The choices above define isomorphisms $\bar{\mathcal{L}}|_{\bar{G}^x} \to (\bar{\mathcal{E}}_x)_{\bar{G}^x}$ for each $x \in \pi_0(\bar{G})$. The resulting isomorphism $\bar{\mathcal{L}} \to \pi_0^*\bar{\mathcal{E}}$ satisfies (CS.4), thus defining an isomorphism $\mathcal{L} \to \pi_0^*\mathcal{E}$ in $\mathcal{CS}(G)$. □

3.3. The dictionary

We saw in Proposition 1.6 that $\mathrm{Tr}_G : \mathcal{CS}(G)_{/\mathrm{iso}} \to G(k)^*$ is a functorial group homomorphism. In this section we find the image and kernel of Tr_G.

Theorem 3.6. *If G is a smooth commutative group scheme over k then $\mathrm{Tr}_G : \mathcal{CS}(G)_{/iso} \to G(k)^*$ is surjective and has kernel canonically isomorphic to $\mathrm{H}^2(\pi_0(\bar{G}), \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}}$, so*

$$0 \longrightarrow \mathrm{H}^2(\pi_0(\bar{G}), \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}} \longrightarrow \mathcal{CS}(G)_{/iso} \xrightarrow{\mathrm{Tr}_G} G(k)^* \longrightarrow 0$$

is an exact sequence.

Proof. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{CS}(\pi_0(G))_{/\mathrm{iso}} & \longrightarrow & \mathcal{CS}(G)_{/\mathrm{iso}} & \longrightarrow & \mathcal{CS}(G^0)_{/\mathrm{iso}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Tr}_{\pi_0(G)}} & & \downarrow{\scriptstyle \mathrm{Tr}_G} & & \downarrow{\scriptstyle \mathrm{Tr}_{G^0}} & & \\ 0 & \longrightarrow & \pi_0(G)(k)^* & \longrightarrow & G(k)^* & \longrightarrow & G^0(k)^* & \longrightarrow & 0 \end{array} \tag{12}$$

be the commutative diagram of abelian groups obtained by applying Lemma 1.4 to (7). The sequence of abelian groups

$$1 \longrightarrow G^0(k) \longrightarrow G(k) \longrightarrow \pi_0(G)(k) \longrightarrow 0$$

is exact since $\mathrm{H}^1(k, G^0) = 0$ by Lemma 3.2 and Lang's theorem on connected algebraic groups over finite fields [27]. Since $\bar{\mathbb{Q}}_\ell^\times$ is divisible, $\mathrm{Hom}(- , \bar{\mathbb{Q}}_\ell^\times)$ is exact, and thus

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the dual sequence of character groups in (12) is exact. The upper row in (12) is exact by Proposition 3.5. Now Lemma 3.2 and Proposition 1.14 imply that $\ker \mathrm{Tr}_{G^0} = 0$ and $\operatorname{coker} \mathrm{Tr}_{G^0} = 0$, while Proposition 2.7 gives $\ker \mathrm{Tr}_{\pi_0(G)} \cong \mathrm{H}^0(\mathcal{W}, \mathrm{H}^2(\pi_0(\bar{G}), \bar{\mathbb{Q}}_\ell^\times))$ and $\operatorname{coker} \mathrm{Tr}_{\pi_0(G)} = 0$. It now follows from the snake lemma

$$\begin{array}{ccccccccc}
0 & \dashrightarrow & \ker \mathrm{Tr}_{\pi_0(G)} & \dashrightarrow & \ker \mathrm{Tr}_G & \dashrightarrow & \ker \mathrm{Tr}_{G^0} = 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{CS}(\pi_0(G))_{/\mathrm{iso}} & \longrightarrow & \mathcal{CS}(G)_{/\mathrm{iso}} & \longrightarrow & \mathcal{CS}(G^0)_{/\mathrm{iso}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Tr}_{\pi_0(G)}} & & \downarrow{\scriptstyle \mathrm{Tr}_G} & & \downarrow{\scriptstyle \mathrm{Tr}_{G^0}} & & \\
0 & \longrightarrow & \pi_0(G)(k)^* & \longrightarrow & G(k)^* & \longrightarrow & G^0(k)^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{coker} \mathrm{Tr}_{\pi_0(G)} = 0 & \dashrightarrow & \operatorname{coker} \mathrm{Tr}_G & \dashrightarrow & \operatorname{coker} \mathrm{Tr}_{G^0} = 0 & &
\end{array} \tag{13}$$

that $\operatorname{coker} \mathrm{Tr}_G = 0$ and $\ker \mathrm{Tr}_{\pi_0(G)} \to \ker \mathrm{Tr}_G$ is an isomorphism. This gives the promised short exact sequence

$$0 \longrightarrow \mathrm{H}^2(\pi_0(\bar{G}), \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}} \longrightarrow \mathcal{CS}(G)_{/\mathrm{iso}} \xrightarrow{\mathrm{Tr}_G} G(k)^* \longrightarrow 0 \qquad \square$$

Remark 3.7. Although $\mathrm{Tr}_{\pi_0(G)}$ is split and Tr_{G^0} is an isomorphism, we do not know if Tr_G is split, in general. Surjectively of Tr_G shows that every ℓ-adic character of $G(k)$ admits a geometrization, but without a splitting for Tr_G we do not know how to make this geometrization canonical.

3.4. Descent, revisited

We now extend Proposition 2.11 to all smooth commutative group schemes over k.

Proposition 3.8. *Let G be a smooth commutative group scheme over k. Then $\mathcal{L} \in \mathcal{CS}(G)$ descends to G if and only if $t_\mathcal{L} : G(k) \to \bar{\mathbb{Q}}_\ell^\times$ has bounded image.*

Proof. By Lemma 3.2, the identity component G^0 is a connected algebraic group over k. It follows from Proposition 1.14 that the restriction of $\mathcal{L}$ to G^0 descends to G. Also, since $G^0(k)$ is finite, the image of $t_\mathcal{L} : G(k) \to \bar{\mathbb{Q}}_\ell^\times$ is a finite subgroup, and therefore has bounded image. If $\chi \in G(k)^*$ then there is some finite-image character χ_0 with the same restriction to $G^0(k)$ since $G^0(k)$ lies inside the torsion part of the finitely generated abelian group $G(k)$. Therefore χ is bounded if and only if $\chi \cdot \chi_0^{-1}$ is bounded. But $\chi \cdot \chi_0^{-1}$ descends to a character of $\pi_0(G)$. Thus, it is enough to prove Corollary 3.8 for étale group schemes G, which is done in Proposition 2.11. $\square$

Proposition 3.8 shows that the full subcategory $\mathcal{CS}_0(G) \subset \mathcal{CS}(G)$ is not an equivalence, for general smooth commutative group schemes G. Again we see the necessity of working with Weil sheaves in Definition 1.1.

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3.5. Morphisms of character sheaves

Theorem 3.9. *Let G be a smooth commutative group scheme over k. There is a canonical isomorphism*

$$\mathrm{Aut}(\mathcal{L}) \cong \mathrm{Hom}(\pi_0(\bar{G})_{\mathcal{W}}, \bar{\mathbb{Q}}_\ell^\times).$$

Proof. Fix $\mathcal{L} = (\bar{\mathcal{L}}, \mu, \phi)$, and consider the group homomorphism from $\mathrm{Aut}(\mathcal{L})$ to $\mathrm{Hom}(\bar{G}_{\mathcal{W}}, \bar{\mathbb{Q}}_\ell^\times)$ defined in the proof of Proposition 2.13. This homomorphism is injective because morphisms of sheaves are determined by the linear transformations induced on stalks. Homomorphisms in the image of $\mathrm{Aut}(\mathcal{L}) \to \mathrm{Hom}(\bar{G}_{\mathcal{W}}, \bar{\mathbb{Q}}_\ell^\times)$ are continuous when $\bar{G}$ is viewed as the base of the *espace étalé* attached to $\bar{\mathcal{L}}$. Since ℓ is invertible in k, it follows that the image of $\mathrm{Aut}(\mathcal{L}) \to \mathrm{Hom}(\bar{G}_{\mathcal{W}}, \bar{\mathbb{Q}}_\ell^\times)$ is contained in $\mathrm{Hom}(\pi_0(\bar{G}_{\mathcal{W}}), \bar{\mathbb{Q}}_\ell^\times)$. We also have $\pi_0(\bar{G}_{\mathcal{W}}) = \pi_0(\bar{G})_{\mathcal{W}}$. To see that $\mathrm{Aut}(\mathcal{L}) \to \mathrm{Hom}(\pi_0(\bar{G})_{\mathcal{W}}, \bar{\mathbb{Q}}_\ell^\times)$ is surjective, begin with $\theta \in \mathrm{Hom}(\pi_0(\bar{G})_{\mathcal{W}}, \bar{\mathbb{Q}}_\ell^\times)$, and, for each $[x] \in \pi_0(\bar{G})_{\mathcal{W}}$, define $\bar{\rho}^y : \bar{\mathcal{L}}^y \to \bar{\mathcal{L}}^y$ by scalar multiplication by $\theta([x]) \in \bar{\mathbb{Q}}_\ell^\times$ for each $y \in [x]$. This defines an isomorphism $\bar{\rho} : \bar{\mathcal{L}} \to \bar{\mathcal{L}}$ of local systems on $\bar{G}$ compatible with μ and ϕ, and thus an isomorphism $\rho : \mathcal{L} \to \mathcal{L}$ which maps to θ under $\mathrm{Aut}(\mathcal{L}) \to \mathrm{Hom}(\pi_0(\bar{G})_{\mathcal{W}}, \bar{\mathbb{Q}}_\ell^\times)$. □

3.6. The dictionary for commutative algebraic groups over finite fields

Having extended the function-sheaf dictionary from connected commutative algebraic groups over k to smooth commutative group schemes G over k, we look back briefly to the case when G is a commutative algebraic group. Although Weil sheaves are not necessary in that case, the dictionary is still not perfect, generally.

Corollary 3.10. *Let G be a commutative algebraic group over k. All character sheaves on G descend to G: $\mathcal{CS}(G)$ is equivalent to $\mathcal{CS}_0(G)$ (§ 2.5). The trace of Frobenius determines a short exact sequence*

$$0 \longrightarrow \mathrm{H}^2(\pi_0(\bar{G}), \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}} \longrightarrow \mathcal{CS}_0(G)_{/iso} \xrightarrow{\mathrm{Tr}_G} G(k)^* \longrightarrow 0$$

The group $\mathrm{H}^2(\pi_0(\bar{G}), \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}}$ need not be trivial.

Proof. Since $G(k)$ is finite, the first statement follows from Propositions 1.8 and 3.8. The second statement is then a consequence of Theorem 3.6. The third statement is justified by Example 2.10. □

Remark 3.11. Suppose G is a reductive algebraic group over k with cyclic component group scheme. Such groups are considered in [30]. When commutative, such groups are extensions of finite cyclic groups by algebraic tori. Every Frobenius-stable character sheaf on G, in the sense of [30], is a character sheaf on G, in our sense, and vice versa. Moreover, since $\mathrm{H}^2(\pi_0(\bar{G}), \bar{\mathbb{Q}}_\ell^\times) = 0$ by Remark 2.10, it follows that each Frobenius-stable character sheaf on G as in [30] is determined by its trace of Frobenius, up to isomorphism.

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3.7. Base change

When using character sheaves to study characters, it is useful to understand how character sheaves behave under change of fields. Let k' be a finite extension of k. Then $k \hookrightarrow k'$ induces a group homomorphism $i_{k'/k} : G(k) \hookrightarrow G(k')$, and thus a homomorphism

$$\begin{aligned} i^*_{k'/k} : G(k')^* &\to G(k)^* \\ \chi &\mapsto \chi \circ i_{k'/k}. \end{aligned}$$

We can interpret this operation on characters in terms of character sheaves.

Proposition 3.12. *Set* $G_{k'} := G \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$, *and let*

$$\mathcal{CS}(\mathrm{Res}_{k'/k}(G_{k'})) \xrightarrow{\iota^*} \mathcal{CS}(G)$$

be the functor obtained by pullback along the canonical closed immersion

$$\iota : G \hookrightarrow \mathrm{Res}_{k'/k}(G_{k'})$$

of k*-schemes. The following diagram commutes.*

$$\begin{array}{ccc} \mathcal{CS}(\mathrm{Res}_{k'/k}(G_{k'}))_{/iso} & \xrightarrow{\iota^*} & \mathcal{CS}(G)_{/iso} \\ \downarrow{\scriptstyle \mathrm{Tr}_{\mathrm{Res}_{k'/k}(G_{k'})}} & & \downarrow{\scriptstyle \mathrm{Tr}_G} \\ G(k')^* & \xrightarrow{i^*_{k'/k}} & G(k)^* \end{array}$$

Proof. The closed immersion $\iota : G \hookrightarrow \mathrm{Res}_{k'/k}(G_{k'})$ is given by [7, § 7.6]. Proposition 3.12 follows immediately from Lemma 1.4 together with the identifications

$$\mathrm{Res}_{k'/k}(G_{k'})(k) \cong G_{k'}(k') \cong G(k')$$

from the definitions of Weil restriction and base change. □

In the opposite direction, let $\mathrm{Nm} : G(k') \to G(k)$ be the norm map, and consider the group homomorphism

$$\begin{aligned} \mathrm{Nm}^* : G(k)^* &\to G(k')^* \\ \chi &\mapsto \chi \circ \mathrm{Nm}\,. \end{aligned}$$

We can also interpret this operation in terms of character sheaves.

If $\mathcal{L} := (\bar{\mathcal{L}}, \mu, \phi)$ is a character sheaf on G, we define $\mathcal{L}' := (\bar{\mathcal{L}}, \mu, \phi_{k'})$ on the base change $G_{k'}$ of G to k' by setting

$$\phi_{k'} := \phi \circ \mathrm{Fr}^*_G(\phi) \circ \cdots \circ (\mathrm{Fr}^{n-1}_G)^*(\phi).$$

The commutativity of diagram (CS.3) for $\phi_{k'}$ follows from the fact that $\mathrm{Fr}_{G_{k'}} = \mathrm{Fr}^n_G$. Note that we may also think about the construction of $\phi_{k'}$ from ϕ as restricting the action φ of $\mathcal{W}_k$ on $\bar{\mathcal{L}}$, defined in § 1.2, to $\mathcal{W}_{k'}$.

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Proposition 3.13. *With the notation above, the rule* $\nu_{k'/k} : (\bar{\mathcal{L}}, \mu, \phi) \mapsto (\bar{\mathcal{L}}, \mu, \phi_{k'})$ *defines a monoidal functor* $\mathcal{CS}(G) \to \mathcal{CS}(G_{k'})$ *such that the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{CS}(G)_{/iso} & \xrightarrow{\nu_{k'/k}} & \mathcal{CS}(G_{k'})_{/iso} \\ \downarrow \mathrm{Tr}_G & & \downarrow \mathrm{Tr}_{G_{k'}} \\ G(k)^* & \xrightarrow{\mathrm{Nm}^*} & G(k')^* \end{array}$$

Proof. Let $\mathcal{L} := (\bar{\mathcal{L}}, \mu, \phi) \in \mathcal{CS}(G)$, and write F for Fr_G. For any $x \in G(k')$, we may compute the value of $\mathrm{Tr}_{G_{k'}}(\nu_{k'/k}\mathcal{L})(x) = t_{\nu_{k'/k}\mathcal{L}}(x)$ as the trace of $\phi_{k'}$ on $\bar{\mathcal{L}}_x$, and the value of $\mathrm{Nm}^*(\mathrm{Tr}_G(\mathcal{L}))(x)$ as the trace of ϕ on $\bar{\mathcal{L}}_{\mathrm{Nm}(x)}$. Applying (CS.3) to the stalk of $\bar{\mathcal{L}}^{\boxtimes n}$ at the point $(x, \mathrm{Fr}(x), \dots, \mathrm{Fr}^{n-1}(x))$ yields the following diagram.

$$\begin{array}{ccc} \bar{\mathcal{L}}_{\mathrm{Nm}(x)} & \longrightarrow & \bar{\mathcal{L}}_{F(x)} \otimes \bar{\mathcal{L}}_{F^2(x)} \otimes \cdots \otimes \bar{\mathcal{L}}_x \\ \downarrow \phi_{\mathrm{Nm}(x)} & & \downarrow \phi_x \otimes (F^*\phi)_x \otimes \cdots \otimes ((F^{n-1})^*\phi)_x \\ \bar{\mathcal{L}}_{\mathrm{Nm}(x)} & \longrightarrow & \bar{\mathcal{L}}_x \otimes \bar{\mathcal{L}}_{F(x)} \otimes \cdots \otimes \bar{\mathcal{L}}_{F^{n-1}(x)} \end{array}$$

Choose a basis vector v for $\bar{\mathcal{L}}_{\mathrm{Nm}(x)}$, and write $v_0 \otimes v_1 \otimes \cdots \otimes v_{n-1}$ for the image of v under the bottom map, for $v_i \in \bar{\mathcal{L}}_{\mathrm{Fr}^i(x)}$. By (CS.2), v maps to $v_1 \otimes v_2 \otimes \cdots \otimes v_0$ along the top of the diagram. Let $\alpha_i \in \bar{\mathbb{Q}}_\ell^\times$ represent $((F^i)^*\phi)_x$ with respect to these bases, and let α be the trace of $\phi_{\mathrm{Nm}(x)}$. We may now equate the trace α of ϕ on $\bar{\mathcal{L}}_{\mathrm{Nm}(x)}$ with the product $\alpha_0 \cdots \alpha_{n-1}$, which is the trace of $\phi_{k'}$ on $\bar{\mathcal{L}}_x$. □

Finally, let G' be a smooth commutative group scheme over k'. We explain how to geometrize the canonical isomorphism between characters of $G'(k')$ and of $(\mathrm{Res}_{k'/k} G')(k)$. We may decompose the base change $(\mathrm{Res}_{k'/k} G')_{k'}$ of $\mathrm{Res}_{k'/k} G'$ to k' into a product of copies of G', indexed by elements of $\mathrm{Gal}(k'/k)$:

$$(\mathrm{Res}_{k'/k} G')_{k'} \cong \prod_{\mathrm{Gal}(k'/k)} G'.$$

Since products and coproducts agree for group schemes we have a natural inclusion of k'-schemes

$$G' \hookrightarrow (\mathrm{Res}_{k'/k} G')_{k'},$$

mapping G' into the summand corresponding to $1 \in \mathrm{Gal}(k'/k)$. Composing $\nu_{k'/k}$ from Proposition 3.13 with pullback along this map yields a functor

$$\rho : \mathcal{CS}(\mathrm{Res}_{k'/k} G') \to \mathcal{CS}(G').$$

Proposition 3.14. *Let* k'/k *be a finite extension and let* G' *be a smooth commutative group scheme over* k'. *Then the functor*

$$\rho : \mathcal{CS}(\mathrm{Res}_{k'/k} G') \to \mathcal{CS}(G'),$$

defined above, induces

$$\begin{array}{ccc} \mathcal{CS}(\operatorname{Res}_{k'/k} G')_{/iso} & \xrightarrow{\rho} & \mathcal{CS}(G')_{/iso} \\ \downarrow {\scriptstyle \operatorname{Tr}_{\operatorname{Res}_{k'/k} G'}} & & \downarrow {\scriptstyle \operatorname{Tr}_{G'}} \\ G'(k')^* & \longrightarrow & G'(k')^* \end{array}$$

where the bottom map is the identity.

Proof. By Lemma 1.4, the pullback part of the definition of ρ corresponds to the map

$$(\operatorname{Res}_{k'/k} G')(k')^* \to G'(k')^*$$

induced by $g \mapsto (g, 1, \dots, 1)$. Since the action of $\operatorname{Gal}(k'/k)$ on

$$(\operatorname{Res}_{k'/k} G')_{k'} \cong \prod_{\operatorname{Gal}(k'/k)} G'$$

is given by permuting coordinates, composition with the norm map yields the identity on $G'(k')$. □

4. Quasicharacter sheaves for p-adic tori

Let K be a local field with ring of integers R and finite residue field k; in this section we denote the group $\mathcal{W}$ by $\mathcal{W}_k$. We continue to assume that ℓ is invertible in k.

4.1. Néron models

We will consider connected commutative algebraic groups over K that admit a Néron model, by which we mean a locally finite type Néron model. By [7, § 10.2, Theorem 2], these are precisely the connected commutative algebraic groups over K that contain no subgroup isomorphic to $\mathbb{G}_a$. Write $\mathcal{N}_K$ for the full subcategory of the category of algebraic groups consisting of such objects. This category is additive, and includes all algebraic tori over K, abelian varieties over K, and unipotent K-wound groups. We write $\mathcal{N}$ for the category of Néron models that arise in this way; in particular, $\mathcal{N}$ is a full subcategory of the category of smooth commutative group schemes over R.

Example 4.1. If $T_K = \mathbb{G}_{\mathbf{m},K}$, then a Néron model can be obtained by gluing copies of $\mathbb{G}_{\mathbf{m},R}$ (one for each $n \in \mathbb{Z}$) along their generic fibres, via the gluing morphisms $\operatorname{Spec}(\mathbb{Z}[t, t^{-1}]) \to \operatorname{Spec}(\mathbb{Z}[t, t^{-1}])$ defined by $t \mapsto \pi^n t$ [7, § 10.1, Example 5].

Suppose that K'/K is a quadratic extension, and that $T_K = U_1(K'/K)$ is the unitary group. When K'/K is unramified, the Néron model of T_K is a form of the Néron model for $\mathbb{G}_{\mathbf{m},K}$, with the non-trivial automorphism $\sigma \in \operatorname{Gal}(K'/K)$ mapping (x, n) to $(\sigma(x), -n)$ for $x \in R'^{\times}$ and $n \in \mathbb{Z}$ specifying the copy of $\mathbb{G}_{\mathbf{m},R'}$. This example illustrates the compatibility between Néron models and unramified base change [7, § 10.1, Proposition 3].

On the other hand, if $K' = K(\sqrt{\pi})$ is totally ramified over K, then the Néron model of $U_1(K'/K)$ is affine, namely $\operatorname{Spec}(R[x, y]/(x^2 - \pi y^2 - 1))$. The special fibre is the disjoint union of two affine lines.

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Finally, we remark that, if K'/K is any finite extension of local fields and X' is a Néron model for $X_{K'}$, then $\mathrm{Res}_{R'/R}(X')$ is a Néron model for $\mathrm{Res}_{K'/K}(X_{K'})$ [7, § 7.6, Proposition 6].

4.2. Quasicharacters

Write $\mathfrak{m}$ for the maximal ideal of R, and set $R_n = R/\mathfrak{m}^{n+1}$ for every non-negative integer n. Suppose that $X \in \mathcal{N}$. Note that $X(K) = X(R)$. A *quasicharacter of* $X(K)$ is a group homomorphism $X(K) \to \bar{\mathbb{Q}}_\ell^\times$ that factors through $X(R) \to X(R_n)$ for some non-negative integer n. We note that this definition is compatible with [11, Chapter XV, § 2.3]. The group of quasicharacters of $X(K)$ will be denoted by $\mathrm{Hom}(X(K), \bar{\mathbb{Q}}_\ell^\times)$, and the subgroup of those that factor through $X(R_n)$ will be denoted by $\mathrm{Hom}_n(X(K), \bar{\mathbb{Q}}_\ell^\times)$. In this section we will see how to geometrize and categorify quasicharacters of $X(K)$ using character sheaves.

4.3. Review of the Greenberg transform

Let K, R, and R_n be as above. For each $n \in \mathbb{N}$, the Greenberg functor maps schemes over R_n to schemes over k. See [5] for the definition and fundamental properties of the Greenberg functor as we use it; other useful references include [21], [22], [18, V, § 4, no. 1], [7, Chapter 9, § 6], and [31, § 2.2]. For any non-negative integer n, we will write

$$\mathrm{Gr}_n^R : \mathsf{Sch}_{/R} \to \mathsf{Sch}_{/k}$$

for the functor produced by composing pullback along $\mathrm{Spec}(R_n) \to \mathrm{Spec}(R)$ with the Greenberg functor. This functor respects open immersions, closed immersions, étale morphisms, smooth morphisms, and geometric components. Moreover, there is a canonical isomorphism

$$\mathrm{Gr}_n^R(X)(k) \cong X(R_n)$$

for any scheme X over R.

For any $n \leqslant m$, the surjective ring homomorphism $R_m \to R_n$ determines a natural transformation

$$\varrho^R_{n\leqslant m} : \mathrm{Gr}_m^R \to \mathrm{Gr}_n^R$$

between additive functors. Crucially, $\varrho^R_{n\leqslant m}(X) : \mathrm{Gr}_m^R(X) \to \mathrm{Gr}_n^R(X)$ is an affine morphism of k-schemes, for every R-scheme X and every $n \leqslant m$ [5, Proposition 4.3]. This observation is key to the proof that, for any scheme X over R, the projective limit

$$\mathrm{Gr}_R(X) := \varprojlim_{n\in\mathbb{N}} \mathrm{Gr}_n^R(X),$$

taken with respect to the surjective morphisms $\varrho^R_{n\leqslant m}(X) : \mathrm{Gr}_m^R(X) \to \mathrm{Gr}_n^R(X)$, exists in the category of group schemes over k; see [24, § 8.2]. This leads to the definition of the *Greenberg transform*:

$$\mathrm{Gr}_R : \mathsf{Sch}_{/R} \to \mathsf{Sch}_{/k}.$$

By construction, the k-scheme $\mathrm{Gr}_R(X)$ comes equipped with morphisms

$$\varrho_n^R(X) : \mathrm{Gr}^R(X) \to \mathrm{Gr}_n^R(X), \quad \forall n \in \mathbb{N}.$$

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4.4. Quasicharacter sheaves

Set $S = \mathrm{Spec}(R)$ and $S_n = \mathrm{Spec}(R_n)$; note that $S_0 = \mathrm{Spec}(k)$ is the special fibre of S. Let X be a smooth commutative group scheme over S. For every non-negative integer n, the Greenberg transform $\mathrm{Gr}_n^R(X)$ is a *smooth* commutative group scheme over S_0. The Greenberg transform $\mathrm{Gr}_R(X)$ of X is a commutative group scheme over k with k-rational points $X(R)$. The morphism of k-schemes $\varrho_n^R(X) : \mathrm{Gr}_R(X) \to \mathrm{Gr}_n^R(X)$ induces a functor

$$\varrho_n^R(X)^* : \mathcal{CS}(\mathrm{Gr}_n^R(X)) \to \mathcal{CS}(\mathrm{Gr}_R(X)),$$

as in Lemma 1.4.

Definition 4.2. Let X be a smooth group scheme over R. A *quasicharacter sheaf* of X is a triple

$$\mathcal{F} := (n, \{\mathcal{F}_i\}_{n\leqslant i}, \{\alpha_{i\leqslant j}\}_{n\leqslant i\leqslant j}),$$

where n is a non-negative integer, each $\mathcal{F}_i$ is a character sheaf on $\mathrm{Gr}_i^R(X)$, and each

$$\alpha_{i\leqslant j} : \mathcal{L}_j \to \varrho_{i\leqslant j}^R(X)^*\mathcal{L}_i$$

is an isomorphism; here, $\alpha_{i\leqslant i}$ is the identity, and the $\alpha_{i\leqslant j}$ are compatible with each other. If $\mathcal{F} := (n, \{\mathcal{F}_i\}, \{\alpha_{i\leqslant j}\})$ and $\mathcal{F}' := (m, \{\mathcal{F}_i'\}, \{\alpha_{i\leqslant j}'\})$ are objects, then $\mathrm{Hom}(\mathcal{F}, \mathcal{F}')$ is the set of equivalence classes of pairs $(k, \{\beta_i\}_{k\leqslant i})$, where $n, m \leqslant k$ and the $\beta_i : \mathcal{F}_i \to \mathcal{F}_i'$ are morphisms of character sheaves such that

$$\begin{array}{ccc} \mathcal{F}_j & \xrightarrow{\alpha_j} & \varrho_{i\leqslant j}^R(X)^*\mathcal{F}_i \\ \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle f_{i\leqslant j}^*\beta_j} \\ \mathcal{F}_j' & \xrightarrow{\alpha_j} & \varrho_{i\leqslant j}^R(X)^*\mathcal{L}_i' \end{array}$$

commutes for all $k \leqslant i \leqslant j$; we identify two such pairs $(k, \{\beta_i\})$ and $(l, \{\gamma_i\})$ if $\beta_i = \gamma_i$ for sufficiently large i. Identities and composites are defined in the natural way. Let $\mathcal{QCS}(X)$ denote the category of quasicharacter sheaves for X.

Remark 4.3. If $\varrho_n^R(X)^* : \mathcal{CS}(\mathrm{Gr}_n^R(X)) \to \mathcal{CS}(\mathrm{Gr}_R(X))$ is full, then the construction above can be improved by forming $\mathcal{QCS}(X)$ from the essential images of the functors $\varrho_n^R(X)^*$; however, we do not know if $\varrho_n^R(X)^*$ is full.

Remark 4.4. We offer the following alternate construction of $\mathcal{QCS}(X)$. As above, let X be a smooth group scheme over R. Although $\mathrm{Gr}_R(X)$ is not locally of finite type and therefore not smooth, let us consider the rigid monoidal category $\mathcal{CS}(\mathrm{Gr}_R(X))$ as defined in § 1.2, though without insisting that the commutative group k-scheme G is smooth. A quasicharacter sheaf for X is an object of the following rigid monoidal subcategory of $\mathcal{CS}(\mathrm{Gr}_R(X))$, denoted by $\mathcal{QCS}(X)$:

(1) objects in $\mathcal{QCS}(X)$ are the ℓ-adic sheaves $\varrho_n^R(X)^*\mathcal{L}$, for $n \in \mathbb{N}$ and $\mathcal{L} \in \mathcal{CS}(\mathrm{Gr}_n^R(X))$;

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(2) morphisms $\varrho_n^R(X)^*\mathcal{L} \to \varrho_m^R(X)^*\mathcal{L}'$ in $\mathcal{QCS}(X)$ are those morphisms in $\mathcal{CS}(\mathrm{Gr}_R(X))$ which take the form $\varrho_m^R(X)^*\rho$ for $\rho \in \mathrm{Hom}(\varrho_{n\leqslant m}^R(X)^*\mathcal{L}, \mathcal{L}')$ when $n \leqslant m$, and $\varrho_n^R(X)^*\rho$ for $\rho \in \mathrm{Hom}(\mathcal{L}, \varrho_{m\leqslant n}^R(X)^*\mathcal{L}')$ when $m \leqslant n$.

Theorem 4.5. *Let K be a local field with residue field k, in which ℓ is invertible; let R be the ring of integers of K.*

(0) *The trace of Frobenius provides a natural transformation between the additive functors*

$$X \mapsto \mathcal{QCS}(X)_{/iso} \quad \text{and} \quad X \mapsto \mathrm{Hom}(X(K), \bar{\mathbb{Q}}_\ell^\times)$$

as functors from $\mathcal{N}$ to the category of commutative groups.

Regarding this natural transformation, for every $X \in \mathcal{N}$,

(1) *there is a canonical short exact sequence of commutative groups*

$$0 \to \mathrm{H}^2(\pi_0(X)_{\bar{k}}, \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}_k} \to \mathcal{QCS}(X)_{/iso} \to \mathrm{Hom}(X(K), \bar{\mathbb{Q}}_\ell^\times) \to 0;$$

(2) *for all quasicharacter sheaves $\mathcal{F}$, $\mathcal{F}'$ on $\mathrm{Gr}_R(X)$, and for every $\rho \in \mathrm{Hom}(\mathcal{F}, \mathcal{F}')$, either ρ is trivial or ρ is an isomorphism;*

(3) *for all quasicharacter sheaves $\mathcal{F}$ for X, there is a canonical isomorphism*

$$\mathrm{Aut}(\mathcal{F}) \cong \mathrm{Hom}((\pi_0(X)_{\bar{k}})_{\mathcal{W}_k}, \bar{\mathbb{Q}}_\ell^\times).$$

Proof. To prove (0), use Proposition 1.6 with $G = \mathrm{Gr}_n^R(X)$, the fact that Néron models are unique up to isomorphism, the fact that every $\mathcal{CS}(\mathrm{Gr}_n^R(X))$ is a full subcategory of $\mathcal{QCS}(X)$, and the observation that every object in $\mathcal{QCS}(X)$ is in the essential image of $\mathcal{CS}(\mathrm{Gr}_n^R(X))$ for some n. To prove (1), use Theorem 3.6 with $G = \mathrm{Gr}_n^R(X)$, and then argue as in part (0). To prove (2), argue as in the proof of Lemma 1.3. To prove (3), use the fact that the component group of $\mathrm{Gr}_n^R(X)$ is independent of n, Theorem 3.9 with $G = \mathrm{Gr}_n^R(X)$, in which case $\pi_0(G) = \pi_0(X \times_S S_0)$ and $\pi_0(\bar{G}) = \pi_0(X)_{\bar{k}}$, and then argue as in part (0). □

Remark 4.6. In § 4.5 we see that the étale site on $\mathrm{Gr}_R(X)$ is rich enough to geometrize all quasicharacters of $X(K)$ as ℓ-adic local systems on $\mathrm{Gr}_R(X)$, where X is a Néron model for an algebraic torus or an abelian variety over a local field K. It is natural to ask if the étale site on the generic fibre X_K would have sufficed. This seems unlikely, since the geometric étale fundamental group of $\mathbb{G}_{\mathbf{m},K}$ is $\hat{\mathbb{Z}}$; however, limited results in this direction were established in [13] when $K = \mathbb{Q}_p$.

4.5. Quasicharacter sheaves for p-adic tori

As we explained in the Introduction, our original motivation for this paper was to find a geometrization of quasicharacters of p-adic tori. This is now provided by the following adaptation of Theorem 4.5 in the case when $T \in \mathcal{N}$ is a Néron model for an algebraic torus over K.

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Corollary 4.7. *Let T be a Néron model for an algebraic torus over K. The following is a commutative diagram of exact sequences.*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{H}^2(X_*(T)_{\mathcal{I}_K}, \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}_k} & & \mathrm{H}^2(X_*(T)_{\mathcal{I}_K}, \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}_k} & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{QCS}(\pi_0(T))_{/iso} & \xrightarrow{\pi_0^*} & \mathcal{QCS}(T)_{/iso} & \xrightarrow{\iota_0^*} & \mathcal{QCS}(T^0)_{/iso} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Tr}_{\pi_0(T)}} & & \downarrow{\scriptstyle \mathrm{Tr}_{\mathrm{Gr}_R(T)}} & & \downarrow{\scriptstyle \mathrm{Tr}_{\mathrm{Gr}_R(T)^0}} & & \\
0 & \longrightarrow & \mathrm{Hom}(\pi_0(T)(k), \bar{\mathbb{Q}}_\ell^\times) & \xrightarrow{inf} & \mathrm{Hom}(T(K), \bar{\mathbb{Q}}_\ell^\times) & \xrightarrow{res} & \mathrm{Hom}(T^0(R), \bar{\mathbb{Q}}_\ell^\times) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Proof. The horizontal sequence of groups coming from categories of quasicharacter sheaves is exact by Proposition 3.5, together with the observation that the functors π_0^* and ι^* preserve limits. It is elementary that the horizontal sequence of quasicharacters is exact. Accordingly, by Theorem 4.5, the kernel of $\mathrm{Tr}_{\mathrm{Gr}_R(T)}$ is $\mathrm{H}^2(\pi_0(T)_{\bar{k}}, \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}}$. By [6, Equation (3.1)], the special fibre of the component group scheme for T is given by

$$\pi_0(T)_{\bar{k}} = X_*(T)_{\mathcal{I}_K},$$

where $X_*(T)$ is the cocharacter lattice of T_K and $\mathcal{I}_K$ is the inertia group for K. Thus,

$$\mathrm{H}^2(\pi_0(T)_{\bar{k}}, \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}_k} = \mathrm{H}^2(X_*(T)_{\mathcal{I}_K}, \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}_k}.$$

Thus, the middle vertical sequence is exact. Since T^0 and $\pi_0(T)$ do not lie in $\mathcal{N}$, we cannot use Theorem 4.5 to determine the image and trace of Frobenius for these schemes. Instead, we observe that T^0 and $\pi_0(T)$ are smooth commutative group schemes over R, so Definition 4.2 gives meaning to categories $\mathcal{QCS}(T^0)$ and $\mathcal{QCS}(\pi_0(T))$, and, moreover, that $\pi_0(\mathrm{Gr}_n^R(T^0)) = \pi_0(T^0)_k = 1$ and $\pi_0(\mathrm{Gr}_n^R(\pi_0(T))) = \pi_0(T)_k$ are both independent of n. It follows that the vertical sequences through $\mathcal{QCS}(T^0)_{/\mathrm{iso}}$ and $\mathcal{QCS}(\pi_0(T))_{/\mathrm{iso}}$ are exact by Theorem 3.6 and Definition 4.2. The diagram commutes by Lemma 1.4. □

Example 4.8. When $T_K = \mathbb{G}_{\mathbf{m},K}$ or $T_K = U_1(K'/K)$, the geometric component group $\pi_0(T)_{\bar{k}}$ is cyclic, so $\mathrm{Tr}_{\mathrm{Gr}_R(T)}$ is an isomorphism. Conversely, when $T_K = \mathbb{G}_{\mathbf{m},K}^2$, then

$$\mathrm{H}^0(\mathcal{W}_k, \mathrm{H}^2(\pi_0(T), \bar{\mathbb{Q}}_\ell^\times)) = \bar{\mathbb{Q}}_\ell^\times,$$

and there are uncountably many invisible quasicharacter sheaves for T.

We may also give examples of tori whose Néron models have component groups appearing in Example 2.10. Let $L = K'(\sqrt{\pi})$ be a quadratic ramified extension of K'. When $K = K'$ and $T_K = U_1(L/K) \times U_1(L/K)$, the component group $\pi_0(T)_{\bar{k}}$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with trivial Frobenius action. When K'/K is an unramified quadratic extension

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and $T_K = \mathrm{Res}_{K'/K}\, U_1(L/K')$, then $\pi_0(T)_{\bar{k}}$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with Frobenius exchanging the direct factors. Finally, let K'/K be a cubic unramified extension and $S_K = \mathrm{Res}_{K'/K}\, U_1(L/K')$. If T_K is the subtorus with character lattice $X^*(S_K)/\langle(1,1,1)\rangle$, then $\pi_0(T)_{\bar{k}}$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with Frobenius of order 3. Each of these tori will have one invisible quasicharacter sheaf.

We may also extract information about the automorphism groups of quasicharacter sheaves from Theorem 4.5.

Corollary 4.9. *Let T be a Néron model for an algebraic torus over K. For $\mathcal{E} \in \mathcal{QCS}(\pi_0(T))$, $\mathcal{F} \in \mathcal{QCS}(T)$, and $\mathcal{F}^0 \in \mathcal{QCS}(T^0)$, there are canonical isomorphisms*

$$\mathrm{Aut}(\mathcal{E}) \cong (\check{T}_\ell)^{\mathcal{W}_K}, \quad \mathrm{Aut}(\mathcal{F}) \cong (\check{T}_\ell)^{\mathcal{W}_K}, \quad \mathrm{Aut}(\mathcal{F}^0) \cong 1,$$

where $\check{T}_\ell := \mathrm{Hom}(X_*(T), \bar{\mathbb{Q}}_\ell^\times)$, the ℓ-adic dual torus to T_K.

Proof. We already know that $\mathrm{Aut}(\mathcal{E}) = 1$ from Proposition 1.14, part (3). By Theorem 4.5,

$$\mathrm{Aut}(\mathcal{F}) \cong \mathrm{Hom}((\pi_0(T)_{\bar{k}})_{\mathcal{W}_k}, \bar{\mathbb{Q}}_\ell^\times).$$

By [6, Equation (3.1)] again,

$$\mathrm{Hom}((\pi_0(T)_{\bar{k}})_{\mathcal{W}_k}, \bar{\mathbb{Q}}_\ell^\times) \cong \mathrm{Hom}(X_*(T)_{\mathcal{W}_K}, \bar{\mathbb{Q}}_\ell^\times).$$

But $\mathrm{Hom}(X_*(T)_{\mathcal{W}_K}, \bar{\mathbb{Q}}_\ell^\times) \cong \mathrm{Hom}(X_*(T), \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}_K}$. So, for any quasicharacter sheaf $\mathcal{F}$ for T,

$$\mathrm{Aut}(\mathcal{F}) \cong (\check{T}_\ell)^{\mathcal{W}_K},$$

canonically. The case when $X = \pi_0(T)$ is handled by the same argument, replacing Theorem 4.5 with Theorem 3.6 and Definition 4.2, as in the proof of Corollary 4.7, after observing that $\pi_0(\pi_0(T)_{\bar{k}}) = \pi_0(T)_{\bar{k}}$. □

Remark 4.10. Since $\pi_0(T)_{\bar{k}} = X_*(T)_{\mathcal{I}_K}$ by [6, Equation (3.1)], we have

$$\mathrm{Hom}(\pi_0(T)(k), \bar{\mathbb{Q}}_\ell^\times) = \mathrm{Hom}((X_*(T)_{\mathcal{I}_K})^{\mathcal{W}_k}, \bar{\mathbb{Q}}_\ell^\times) = \mathrm{Hom}(X_*(T)_{\mathcal{I}_K}, \bar{\mathbb{Q}}_\ell^\times)_{\mathcal{W}_k} = \mathrm{H}^1(\mathcal{W}_k, \check{T}_\ell^{\mathcal{I}_K}).$$

By the Langlands correspondence for p-adic tori [37],

$$\mathrm{Hom}(T(K), \bar{\mathbb{Q}}_\ell^\times) \cong \mathrm{H}^1(\mathcal{W}_K, \check{T}_\ell),$$

where we refer to continuous cohomology, since $\mathrm{Hom}(T(K), \bar{\mathbb{Q}}_\ell^\times)$ refers to continuous group homomorphisms $T(K) \to \bar{\mathbb{Q}}_\ell^\times$. It now follows from the inflation-restriction exact sequence that the following diagram commutes.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(\pi_0(T)(k), \bar{\mathbb{Q}}_\ell^\times) & \xrightarrow{\text{inf}} & \mathrm{Hom}(T(K), \bar{\mathbb{Q}}_\ell^\times) & \xrightarrow{\text{res}} & \mathrm{Hom}(T^0(R), \bar{\mathbb{Q}}_\ell^\times) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \mathrm{H}^1(\mathcal{W}_k, \check{T}_\ell^{\mathcal{I}_K}) & \xrightarrow{\text{inf}} & \mathrm{H}^1(\mathcal{W}_K, \check{T}_\ell) & \xrightarrow{\text{res}} & \mathrm{H}^1(\mathcal{I}_K, \check{T}_\ell)^{\mathcal{W}_k} & \longrightarrow & 0
\end{array}$$

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Combining this with Corollary 4.7 produces the following commutative diagram of exact sequences.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{H}^2(X_*(T)_{\mathcal{I}_K}, \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}_k} & & \mathrm{H}^2(X_*(T)_{\mathcal{I}_K}, \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}_k} & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{QCS}(\pi_0(T))_{/\mathrm{iso}} & \xrightarrow{\pi_0^*} & \mathcal{QCS}(T)_{/\mathrm{iso}} & \xrightarrow{\iota_0^*} & \mathcal{QCS}(T^0)_{/\mathrm{iso}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^1(\mathcal{W}_k, \check{T}_\ell^{\mathcal{I}_K}) & \xrightarrow{\mathrm{inf}} & \mathrm{H}^1(\mathcal{W}_K, \check{T}_\ell) & \xrightarrow{\mathrm{res}} & \mathrm{H}^1(\mathcal{I}_K, \check{T}_\ell)^{\mathcal{W}_k} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\qquad (14)
$$

It is natural to ask if the vertical surjections can be defined directly, without making use of local class field theory, for which the results of [32, 33] may be helpful. The case when $T_K = \mathbb{G}_{\mathbf{m},K}$ is already very interesting, in which case (14) becomes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{QCS}(\mathbb{Z})_{/\mathrm{iso}} & \xrightarrow{\pi_0^*} & \mathcal{QCS}(\mathbb{G}_{\mathbf{m},K}^{\mathrm{Neron}})_{/\mathrm{iso}} & \xrightarrow{\iota_0^*} & \mathcal{QCS}(\mathbb{G}_{\mathbf{m},R})_{/\mathrm{iso}} & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \mathrm{Hom}(\mathcal{W}_k, \bar{\mathbb{Q}}_\ell^\times) & \xrightarrow{\mathrm{inf}} & \mathrm{Hom}(\mathcal{W}_K, \bar{\mathbb{Q}}_\ell^\times) & \xrightarrow{\mathrm{res}} & \mathrm{Hom}(\mathcal{I}_K, \bar{\mathbb{Q}}_\ell^\times) & \longrightarrow & 0
\end{array}
\qquad (15)
$$

We suspect that the general case of (14), where K is any local field and T_K is any torus over K, may be deduced from (15). In § 4.6 we develop a tool for further work in that direction.

4.6. Weil restriction and quasicharacter sheaves

Let K'/K be a finite Galois extension of local fields, and let k'/k be the corresponding extension of residue fields. Let R and R' be the rings of integers of K and K', respectively. Suppose that $X \in \mathcal{N}$, set $X_{K'} := X_K \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K')$, and let X' be a Néron model for $X_{K'}$.

Proposition 4.11. *The canonical closed immersion*

$$X_K \hookrightarrow \mathrm{Res}_{K'/K} X_{K'}$$

of K-group schemes induces a map of k-group schemes

$$f : \mathrm{Gr}_R(X) \to \mathrm{Res}_{k'/k} \mathrm{Gr}_{R'}(X')$$

which, through quasicharacter sheaves, induces

$$\mathrm{Hom}(X_K(K'), \bar{\mathbb{Q}}_\ell^\times) \xrightarrow{\chi \mapsto \chi|_{X(K)}} \mathrm{Hom}(X_K(K), \bar{\mathbb{Q}}_\ell^\times)$$

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Proof. By the Néron mapping property, the canonical closed immersion

$$X_K \hookrightarrow \mathrm{Res}_{K'/K}(X_{K'})$$

extends uniquely to a morphism

$$X \to \mathrm{Res}_{R'/R}(X') \tag{16}$$

of smooth R-group schemes. Applying the functor Gr_n^R to (16) and using [5, Theorem 1.1] defines the morphism of smooth group schemes

$$f_n : \mathrm{Gr}_{n-1}^R(X) \to \mathrm{Res}_{k'/k} \mathrm{Gr}_{en-1}^{R'}(X'), \tag{17}$$

where e is the ramification index of K'/K. Using Lemma 1.4, (17) induces a functor

$$f_n^* : \mathcal{CS}(\mathrm{Res}_{k'/k} \mathrm{Gr}_{en-1}^{R'}(X')) \to \mathcal{CS}(\mathrm{Gr}_{n-1}^R(X)). \tag{18}$$

Since

$$\left(\mathrm{Res}_{k'/k} \mathrm{Gr}_{en-1}^{R'}(X')\right)(k) = \left(\mathrm{Gr}_{en-1}^{R'}(X')\right)(k'),$$

it follows from Lemma 1.4 that the pullback functor (18) actually induces

$$\mathrm{Hom}_{en-1}(X'(R'), \bar{\mathbb{Q}}_\ell^\times) \to \mathrm{Hom}_{n-1}(X(R), \bar{\mathbb{Q}}_\ell^\times).$$

Since X' is a Néron model for $X_{K'}$ and X is a Néron model for X_K, this may be rewritten as

$$\mathrm{Hom}_{en-1}(X_{K'}(K'), \bar{\mathbb{Q}}_\ell^\times) = \mathrm{Hom}_{en-1}(X_K(K'), \bar{\mathbb{Q}}_\ell^\times) \to \mathrm{Hom}_{n-1}(X_K(K), \bar{\mathbb{Q}}_\ell^\times).$$

Passing to limits now defines

$$\mathrm{Hom}(X_K(K'), \bar{\mathbb{Q}}_\ell^\times) \to \mathrm{Hom}(X_K(K), \bar{\mathbb{Q}}_\ell^\times).$$

Argue as in Proposition 3.12 to see that this is indeed restriction of characters. □

4.7. Transfer of quasicharacters

Let K and L be local fields with rings of integers $\mathcal{O}_K$ and $\mathcal{O}_L$, respectively. Pick uniformizers ϖ_K and ϖ_L for $\mathcal{O}_K$ and $\mathcal{O}_L$, respectively; what follows will not depend on these choices. Suppose that ℓ is invertible in the residue fields of K and L.

We begin with $X_K \in \mathcal{N}_K$ with Néron model X and $Y_L \in \mathcal{N}_L$ with Néron model Y. Suppose that m is a positive integer such that

$$\mathcal{O}_K/\varpi_K^m \mathcal{O}_K \cong \mathcal{O}_L/\varpi_L^m \mathcal{O}_L.$$

Suppose also that

$$X \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(\mathcal{O}_K/\varpi_K^m \mathcal{O}_K) \cong Y \times_{\mathrm{Spec}(\mathcal{O}_L)} \mathrm{Spec}(\mathcal{O}_L/\varpi_L^m \mathcal{O}_L) \tag{19}$$

as smooth group schemes over $\mathcal{O}_K/\varpi_K^m \mathcal{O}_K$. Then

$$\mathrm{Gr}_{m-1}^{\mathcal{O}_K}(X) \cong \mathrm{Gr}_{m-1}^{\mathcal{O}_L}(Y)$$

as smooth group schemes over k. Accordingly, by Lemma 1.4, the isomorphism above determines an equivalence of categories

$$\mathcal{CS}(\mathrm{Gr}_{m-1}^{\mathcal{O}_K}(X)) \cong \mathcal{CS}(\mathrm{Gr}_{m-1}^{\mathcal{O}_L}(Y)) \tag{20}$$

which induces an isomorphism

$$\mathrm{Hom}_{m-1}(X(K), \bar{\mathbb{Q}}_\ell^\times) \cong \mathrm{Hom}_{m-1}(Y(L), \bar{\mathbb{Q}}_\ell^\times). \tag{21}$$

The isomorphism (21) is an instance of *transfer* of (certain) quasicharacters between $X(K)$ and $Y(L)$. We now recognize this transfer of quasicharacters as a consequence of the equivalence of categories of quasicharacter sheaves (20).

The isomorphism (19) can indeed exist between quasicharacters of algebraic tori over local fields, even when the characteristics of K and L differ. Suppose that T_K and M_L are tori over local fields K and L, splitting over K' and L', respectively. Then T_K and M_L are said to be n-*congruent* [12, § 2] if there are isomorphisms

$$\begin{aligned} \alpha &: \mathcal{O}_{K'}/\varpi_K^n\mathcal{O}_{K'} \to \mathcal{O}_{L'}/\varpi_L^n\mathcal{O}_{L'} \\ \beta &: \mathrm{Gal}(K'/K) \to \mathrm{Gal}(L'/L) \\ \phi &: X^*(T_K) \to X^*(M_L) \end{aligned}$$

satisfying the following conditions:

(1) α induces an isomorphism $\mathcal{O}_K/\varpi_K^n\mathcal{O}_K \to \mathcal{O}_L/\varpi_L^n\mathcal{O}_L$;

(2) α is $\mathrm{Gal}(K'/K)$-equivariant relative to β; and

(3) ϕ is $\mathrm{Gal}(K'/K)$-equivariant relative to β.

If T_K and M_L are n-congruent then α, β, and ϕ determine an isomorphism

$$\mathrm{Hom}_{n-1}(T_K(K), \bar{\mathbb{Q}}_\ell^\times) \cong \mathrm{Hom}_{n-1}(M_L(L), \bar{\mathbb{Q}}_\ell^\times). \tag{22}$$

Note that, if T_K and M_L are n-congruent, then they are m-congruent for every $m \leqslant n$.

Now let T be a Néron model for T_K, and let M be a Néron model for M_L. One of the main results of [12] gives an isomorphism of group schemes

$$T \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(\mathcal{O}_K/\varpi_K^m\mathcal{O}_K) \cong M \times_{\mathrm{Spec}(\mathcal{O}_L)} \mathrm{Spec}(\mathcal{O}_L/\varpi_L^m\mathcal{O}_L)$$

assuming that T_K and M_L are sufficiently congruent. In [12], the authors define a quantity h (the smallest integer such that ϖ^h lies in the Jacobian ideal associated to a natural embedding of T_K into an induced torus [12, § 8.1]) and show that, if $n > 3h$ and T_K and M_L are n-congruent, then there is a canonical isomorphism of smooth group schemes $\mathrm{Gr}_{n-3h-1}(T) \to \mathrm{Gr}_{n-3h-1}(M)$ determined by α, β, and ϕ [12, Theorem 8.5]. Combining this with the paragraph above gives the following instance of the geometrization of the transfer of quasicharacters.

Proposition 4.12. *With the notation above, suppose that T_K and M_L are n-congruent and that $n > 3h$. Set $m = n - 3h$. Then there is a canonical equivalence of categories*

$$\mathcal{CS}(\mathrm{Gr}_{m-1}^{\mathcal{O}_K}(T)) \cong \mathcal{CS}(\mathrm{Gr}_{m-1}^{\mathcal{O}_L}(M))$$

determined by α, β and ϕ inducing an isomorphism

$$\mathrm{Hom}_{m-1}(T(K), \bar{\mathbb{Q}}_\ell^\times) \cong \mathrm{Hom}_{m-1}(M(L), \bar{\mathbb{Q}}_\ell^\times)$$

compatible with (22).

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