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The best Sobolev trace constant as limit of the usual Sobolev constant for small strips near the boundary

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In this paper we prove that the best constant in the Sobolev trace embedding $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ in a bounded smooth domain can be obtained as the limit as $\varepsilon \to 0$ of the best constant of the usual Sobolev embedding $H^1(\Omega) \hookrightarrow L^q(\omega_\varepsilon, \mathrm{d}x/\varepsilon)$, where $\omega_\varepsilon = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$ is a small neighbourhood of the boundary. We also analyse symmetry properties of extremals of the latter embedding when Ω is a ball.

1. Introduction

The main goal of this paper is to obtain the best Sobolev trace constant for a given domain as the limit of the usual Sobolev constant in small strips near the boundary of the domain when the width of the strip tends to zero.

We consider a bounded smooth (C^2 is sufficient for our arguments) domain $\Omega \subset \mathbb{R}^N$ and we deal with the best constant of the Sobolev trace embedding $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$. For every critical or subcritical exponent, $1 \leqslant q \leqslant 2_* = 2(N-1)/(N-2)$, we have the Sobolev trace inequality: there exists a constant C such that

$$C\left(\int_{\partial\Omega} |v|^q \,\mathrm{d}S\right)^{2/q} \leqslant \int_\Omega (|\nabla v|^2 + v^2)\,\mathrm{d}x$$

for all $v \in H^1(\Omega)$. The best Sobolev trace constant is the largest C such that the above inequality holds, that is,

$$T_q = \inf_{v \in H^1(\Omega)\setminus H^1_0(\Omega)} \left(\int_\Omega |\nabla v|^2 + v^2 \,\mathrm{d}x\right)\Bigg/\left(\int_{\partial\Omega} |v|^q \,\mathrm{d}S\right)^{2/q}. \qquad (1.1)$$

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For subcritical exponents, $1 \leqslant q < 2_*$, the embedding is compact, so we have the existence of extremals, i.e. functions where the infimum is attained. These extremals can be taken to be strictly positive in $\bar{\Omega}$ and smooth up to the boundary. If we normalize the extremals with

$$\int_{\partial\Omega} |u|^q \, \mathrm{d}S = 1, \tag{1.2}$$

it follows that they are weak solutions of the problem

$$\left.\begin{aligned} -\Delta u + u = 0 \qquad & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = T_q |u|^{q-2} u \quad & \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.3}$$

where ν is the unit outward normal vector. In the special case $q = 2$, (1.3) is a linear eigenvalue problem of Steklov type [14]. In the rest of this paper we will assume that the extremals are normalized according to (1.2).

As we have mentioned, we want to see how the best trace constant, T_q, can be obtained as the limit of the usual Sobolev constant for some subdomains. To this end, let us consider the subset of Ω,

$$\omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon\}.$$

Notice that this set has measure $|\omega_\varepsilon| \sim \varepsilon |\partial\Omega|$ for small values of ε. For sufficiently small $\sigma \geqslant 0$ we can define the 'parallel' interior boundary $\Gamma_\sigma = \{y - \sigma\nu(y),\ y \in \partial\Omega\}$, where $\nu(y)$ denotes the outward unitary normal at $y \in \partial\Omega$. Note that $\Gamma_0 = \partial\Omega$. Then, we can also look at the set ω_ε as the neighbourhood of Γ_0 defined by

$$\omega_\varepsilon = \{x = y - \sigma\nu(y),\ y \in \partial\Omega,\ \sigma \in (0, \varepsilon)\} = \bigcup_{0<\sigma<\varepsilon} \Gamma_\sigma$$

for sufficiently small ε, say $0 < \varepsilon < \varepsilon_0$. We also define $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$ and for δ small we have that $\partial\Omega_\delta = \Gamma_\delta$.

Let us consider the usual Sobolev embedding associated to the set ω_ε, that is,

$$H^1(\Omega) \hookrightarrow L^q\left(\omega_\varepsilon, \frac{\mathrm{d}x}{\varepsilon}\right).$$

We have normalized the size of ω_ε by taking $\mathrm{d}x/\varepsilon$ as measure in ω_ε. In this case the embedding is continuous for exponents q such that $1 \leqslant q \leqslant 2^* = 2N/(N-2)$. Note that $2^* = 2N/(N-2)$ is larger than $2_* = 2(N-1)/(N-2)$. The best constant associated to this embedding is given by

$$S_q(\varepsilon) = \inf_{v \in H^1(\Omega)} \left(\int_\Omega |\nabla v|^2 + v^2 \, \mathrm{d}x \right) \Big/ \left(\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |v|^q \, \mathrm{d}x \right)^{2/q}. \tag{1.4}$$

For $q < 2^*$, by compactness, the infimum is attained. The extremals, normalized by

$$\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |u|^q \, \mathrm{d}x = 1 \tag{1.5}$$

are weak solutions of

$$\left.\begin{aligned} -\Delta u + u &= \frac{S_q(\varepsilon)}{\varepsilon}\chi_{\omega_\varepsilon}(x)|u|^{q-2}u && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.6}$$

where $\chi_{\omega_\varepsilon}$ denotes the characteristic function.

Our main result is the following.

Theorem 1.1. *Let Ω be a bounded C^2 domain and let T_q and $S_q(\varepsilon)$ be the best Sobolev constants given by (1.1) and (1.4).*

(i) *For critical or subcritical q, $1 \leqslant q \leqslant 2_* = 2(N-1)/(N-2)$, we have*

$$\lim_{\varepsilon \to 0} S_q(\varepsilon) = T_q. \tag{1.7}$$

Moreover, for subcritical q, $1 \leqslant q < 2_* = 2(N-1)/(N-2)$, the extremals of $S_q(\varepsilon)$ normalized according to (1.5) converge strongly (along subsequences) in $H^1(\Omega)$ and in $C^\beta(\Omega)$, for some $\beta > 0$, to an extremal of (1.1),

$$\lim_{\varepsilon \to 0} u_\varepsilon = u_0 \quad \textit{strongly in } H^1(\Omega) \textit{ and in } C^\beta(\Omega).$$

In the critical case, $q = 2_* = 2(N-1)/(N-2)$, the extremals of $S_q(\varepsilon)$ converge weakly (along subsequences) in $H^1(\Omega)$ to a limit, u_0, that is a weak solution of (1.3). This convergence is strong in $H^1(\Omega)$ if and only if the limit verifies $\int_{\partial\Omega} u_0^q = 1$ and in this case u_0 is an extremal for T_{2_*}.

(ii) *For supercritical q, $2_* = 2(N-1)/(N-2) < q < 2^* = 2N/(N-2)$, we have*

$$\lim_{\varepsilon \to 0} S_q(\varepsilon) = 0. \tag{1.8}$$

A reference closely related to this work is [1], in which the authors consider concentrated reactions near the boundary in an elliptic problem. They prove that the solutions converge to a solution of a problem with a non-homogeneous flux condition at the boundary. Our results can be viewed as a complement of the results of [1], since here we deal with (nonlinear) eigenvalue problems when the reactions are concentrated near the boundary (see the right-hand side of (1.6)).

Next, we look at the symmetry for extremals of (1.4) in the special case when Ω is a ball, $\Omega = B(0, R)$. In this case we prove the following result.

Theorem 1.2. *Let $S_q(\varepsilon)$ be the best Sobolev constant given by (1.4) with $\Omega = B(0, R)$.*

(i) *For $1 \leqslant q \leqslant 2$ and for every $R, \varepsilon > 0$, the extremals of (1.4) in a ball are radial functions that do not change sign. In particular, there exists a unique non-negative extremal of (1.4) satisfying (1.5).*

(ii) *For $2 < q < 2_* = 2(N-1)/(N-2)$, there exist $0 < R_0 \leqslant R_1 < \infty$ such that:*

(a) *for $0 < R \leqslant R_0$ and ε small (possibly depending on R) the extremals of (1.4) are radial;*

(b) *for $R \geqslant R_1$ and ε small (possibly depending on R) the extremals of (1.4) are not radial.*

Remark 1.3. As a consequence of our results, we find that extremals for the Sobolev trace embedding in small balls are radial. For symmetry results of extremals of Sobolev inequalities see, for example, [6, 7] and references therein. Also, for references concerning Sobolev trace embeddings we refer the reader to [2, 5, 8–10] and references therein.

2. Proof of theorem 1.1

This section is devoted to the proof of theorem 1.1. First, we prove that the Sobolev trace constant is continuous as a function of the domain. We believe that this result is interesting for itself.

Lemma 2.1. *Let $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$. Then the function*

$$\delta \to T_q(\Omega_\delta)$$

is continuous at $\delta = 0$.

Proof. Consider a sufficiently small fixed $\varepsilon_0 > 0$. For all $0 < \delta < \varepsilon_0$, let us consider a smooth increasing function ψ_δ such that $\psi_\delta(0) = \delta$, $\psi_\delta(s) = s$ for all $s \geqslant \varepsilon_0$ and $\psi_\delta(s) \to s$ as $\delta \to 0$ in $C^1([0, \infty))$. Now we take the diffeomorphism

$$A_\delta : \Omega \to \Omega_\delta,$$

$$A_\delta(x) = \begin{cases} y - \psi_\delta(s)\nu(y) & \text{for } x = y - s\nu(y), \text{ with } y \in \partial\Omega, \ s \in (0, \varepsilon_0), \\ x & \text{for } x \in \Omega \setminus \bar{\omega}_{\varepsilon_0}. \end{cases}$$

Observe that if $y \in \partial\Omega$ and $0 < s < \varepsilon_0$, then $x = y - s\nu(y) \in \omega_{\varepsilon_0}$. Moreover, A_δ is also a diffeomorphism when restricted to the boundary,

$$A_\delta : \partial\Omega \to \partial\Omega_\delta.$$

This diffeomorphism has bounded derivatives and, furthermore,

$$\lim_{\delta \to 0} \|DA_\delta(x) - I\| = 0 \tag{2.1}$$

uniformly in $\bar{\Omega}$. Here $I \in \mathcal{M}_{n \times n}$ is the identity matrix.

Therefore, we can change variables with

$$u(x) = v(A_\delta(x))$$

for $x \in \Omega$ or $x \in \partial\Omega$. This induces a map denoted, similarly, by

$$A_\delta : H^1(\Omega) \mapsto H^1(\Omega_\delta),$$

which is a diffeomorphism. Moreover, we see that the following diagram is commutative:

$$\begin{array}{ccc} H^1(\Omega) & \longrightarrow & L^q(\partial\Omega) \\ {\scriptstyle A_\delta}\downarrow & & \downarrow{\scriptstyle A_\delta} \\ H^1(\Omega_\delta) & \longrightarrow & L^q(\partial\Omega_\delta). \end{array} \tag{2.2}$$

Therefore, from (2.1), we obtain

$$C_1(\delta)\int_{\Omega}|\nabla u|^2+u^2\,\mathrm{d}x \leqslant \int_{\Omega_\delta}|\nabla v|^2+v^2\,\mathrm{d}x \leqslant C_2(\delta)\int_{\Omega}|\nabla u|^2+u^2\,\mathrm{d}x,$$

where $C_i(\delta)\to 1$ as $\delta\to 0$.

In a similar way, we get

$$C_1(\delta)\int_{\partial\Omega}|u|^q\,\mathrm{d}S \leqslant \int_{\partial\Omega_\delta}|v|^q\,\mathrm{d}S \leqslant C_2(\delta)\int_{\partial\Omega}|u|^q\,\mathrm{d}S, \tag{2.3}$$

with $C_i(\delta)\to 1$ as $\delta\to 0$.

From the previous inequalities we obtain that there exist two constants K_1, K_2 such that $K_i(\delta)\to 1$ as $\delta\to 0$ and

$$K_1(\delta)T_q(\Omega)\leqslant T_q(\Omega_\delta)\leqslant K_2(\delta)T_q(\Omega).$$

The desired continuity is proved. □

The next result shows that the traces on $\partial\Omega_\delta$ also behave continuously as $\delta\to 0$. In order to do this, we first figure out a device that allows us to compare traces taken on different surfaces close to the boundary of Ω. For this, observe that, for any $q\leqslant 2_*$, we can define the mapping

$$\gamma_\delta : H^1(\Omega)\to L^q(\partial\Omega_\delta)\longleftrightarrow L^q(\partial\Omega).$$

Here the first arrow denotes traces and the second denotes the diffeomorphism induced by A_δ^{-1} as in (2.2).

Then, we have the following result which, in particular, complements some results in [1].

Lemma 2.2. *Denoting by γ the trace operator on $\partial\Omega$, we have*

$$\lim_{\delta\to 0}\gamma_\delta=\gamma \quad \text{in } L^q(\partial\Omega)$$

uniformly on compact sets of $H^1(\Omega)$ if $q=2_*$ or in $\mathcal{L}(H^1(\Omega),L^q(\partial\Omega))$ if $q<2_*$.

In particular, for $q\leqslant 2_*$, if u_ε is a bounded sequence in $H^1(\Omega)$, then

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|u_\varepsilon|^q$$

is also bounded.

Moreover, if $u_\varepsilon\to u_0$ strongly in $H^1(\Omega)$ and $q\leqslant 2_*$, then

$$\int_{\partial\Omega_{\delta(\varepsilon)}}|u_\varepsilon|^q\,\mathrm{d}S\to\int_{\partial\Omega}|u_0|^q\,\mathrm{d}S \tag{2.4}$$

as $\delta(\varepsilon)\to 0$ and

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|u_\varepsilon|^q\,\mathrm{d}x\to\int_{\partial\Omega}|u_0|^q\,\mathrm{d}S \tag{2.5}$$

as $\varepsilon\to 0$.

Proof. If $q \leqslant 2_*$ and u_ε is a bounded sequence in $H^1(\Omega)$, using the co-area formula and the fact that the gradient of the distance to the boundary has length 1, we write

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u_\varepsilon|^q \,\mathrm{d}x &= \frac{1}{\varepsilon}\int_0^\varepsilon \int_{\partial\Omega_\delta} |u_\varepsilon|^q \,\mathrm{d}S\,\mathrm{d}\delta \\ &\leqslant \frac{1}{\varepsilon}\int_0^\varepsilon T_q(\Omega_\delta)^{-q/2}\|u_\varepsilon\|^q_{H^1(\Omega_\delta)}\,\mathrm{d}\delta \\ &\leqslant \sup_{\delta\in[0,\varepsilon]}[T_q(\Omega_\delta)^{-q/2}]\|u_\varepsilon\|^q_{H^1(\Omega)},\end{aligned}$$

which is bounded using lemma 2.1 and the fact that the sequence u_ε is bounded in $H^1(\Omega)$.

Note that if $q < 2_*$, there exists some $0 < s < 1$, such that

$$\gamma_\delta : H^1(\Omega) \hookrightarrow H^s(\Omega) \to L^q(\partial\Omega_\delta) \longleftrightarrow L^q(\partial\Omega).$$

In a similar fashion, if $q = 2_*$, we take $s = 1$.

For any fixed $u \in H^s(\Omega)$, from (2.3), we have that these operators converge to the usual trace on $\partial\Omega$, that is

$$\lim_{\delta\to 0}\gamma_\delta(u) = \gamma(u).$$

Moreover, we have

$$\|\gamma_\delta\|_{\mathcal{L}(H^s(\Omega),L^q(\partial\Omega))} \leqslant C,$$

uniformly on δ. Hence, from the Banach–Alouglu–Bourbaki lemma, we get

$$\lim_{\delta\to 0}\gamma_\delta = \gamma$$

uniformly on compact sets of $H^s(\Omega)$.

In addition, if $u_\varepsilon \to u_0$ strongly in $H^1(\Omega)$,

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega} |\gamma_\varepsilon(u_\varepsilon)|^q \,\mathrm{d}S = \int_{\partial\Omega} |u_0|^q \,\mathrm{d}S,$$

which, combined with (2.3), gives (2.4).

On the other hand, to obtain (2.5) we write

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u_\varepsilon|^q \,\mathrm{d}x = \frac{1}{\varepsilon}\int_0^\varepsilon \int_{\partial\Omega_\delta} |u_\varepsilon|^q \,\mathrm{d}S\,\mathrm{d}\delta.$$

Since for every $\delta < \varepsilon$, $\int_{\partial\Omega_\delta} |u_\varepsilon|^q$ and $\int_{\partial\Omega} |u_0|^q$ are uniformly close, we obtain (2.5). □

Remark 2.3. The only property that we have actually used in the proof of the previous results is (2.1). Therefore, both lemmas above remain true for any family of domains Ω_δ such that there exists a diffeomorphism $A_\delta : \Omega \mapsto \Omega_\delta$ with $A_\delta : \partial\Omega \mapsto \partial\Omega_\delta$ such that (2.1) holds. Also note that in lemma 2.2 the conclusions remain true for $q < 2_*$ under the weaker assumption of convergence in $H^s(\Omega)$ for $s < 1$ but sufficiently close to 1.

Proof of theorem 1.1. We first prove (1.7) for critical or subcritical exponents, i.e. $1 \leqslant q \leqslant 2_* = 2(N-1)/(N-2)$. Given $k > 0$, let us take a regular function u_k such that

$$T_q + \frac{1}{k} \geqslant \biggl(\int_\Omega |\nabla u_k|^2 + u_k^2 \,\mathrm{d}x \biggr) \bigg/ \biggl(\int_{\partial\Omega} u_k^q \,\mathrm{d}S \biggr)^{2/q}.$$

By the regularity of u_k, from lemma 2.2 (see also [1]), we have, for a fixed k,

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\omega_\varepsilon} u_k^q \,\mathrm{d}x = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^\varepsilon \int_{\Gamma_s} u_k^q \,\mathrm{d}S \,\mathrm{d}s = \int_{\partial\Omega} u_k^q \,\mathrm{d}S.$$

Therefore, using u_k as test in (1.4) and taking limits we get

$$\limsup_{\varepsilon\to 0} S_q(\varepsilon) \leqslant T_q + \frac{1}{k}.$$

Letting $k \to \infty$ we obtain

$$\limsup_{\varepsilon\to 0} S_q(\varepsilon) \leqslant T_q. \tag{2.6}$$

Now let us prove that for $q \leqslant 2_*$ we have

$$\liminf_{\varepsilon\to 0} S_q(\varepsilon) \geqslant T_q. \tag{2.7}$$

For this, note that, for $u \in H^1(\Omega)$, using the restriction to Ω_δ, we obtain

$$\biggl(\int_{\partial\Omega_\delta} |u|^q \,\mathrm{d}S \biggr)^{2/q} \leqslant \frac{1}{T_q(\Omega_\delta)} \|u\|^2_{H^1(\Omega_\delta)} \leqslant \frac{1}{T_q(\Omega_\delta)} \|u\|^2_{H^1(\Omega)}.$$

Integrating for $\delta \in (0, \varepsilon)$, we obtain

$$\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |u|^q \,\mathrm{d}x = \frac{1}{\varepsilon} \int_0^\varepsilon \int_{\partial\Omega_\delta} |u|^q \,\mathrm{d}S \,\mathrm{d}\delta \leqslant \biggl(\frac{1}{\varepsilon} \int_0^\varepsilon \frac{\mathrm{d}\delta}{(T_q(\Omega_\delta))^{q/2}} \biggr) \|u\|^q_{H^1(\Omega)}.$$

Thus, we have obtained

$$\biggl(\frac{1}{\varepsilon} \int_0^\varepsilon \frac{\mathrm{d}\delta}{(T_q(\Omega_\delta))^{q/2}} \biggr)^{-2/q} \leqslant S_q(\varepsilon). \tag{2.8}$$

This fact, together with the continuity of the map

$$\delta \to T_q(\Omega_\delta),$$

proved in lemma 2.1, gives (2.7).

From (2.6) and (2.7) we obtain (1.7), i.e.

$$\lim_{\varepsilon\to 0} S_q(\varepsilon) = T_q.$$

Now we turn our attention to the convergence of extremals in the subcritical case $q < 2_*$. Let us consider u_ε an extremal of $S_q(\varepsilon)$ normalized by

$$\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |u_\varepsilon|^q \,\mathrm{d}x = 1. \tag{2.9}$$

Hence, for ε small, using (2.6), we obtain

$$\|u_\varepsilon\|^2_{H^1(\Omega)} = S_q(\varepsilon) \leqslant T_q + 1.$$

Therefore, the sequence u_ε is bounded in $H^1(\Omega)$ and we can extract a subsequence (that we still denote by u_ε) such that

$$\left.\begin{array}{ll} u_\varepsilon \rightharpoonup u_0 & \text{weakly in } H^1(\Omega), \\ u_\varepsilon \to u_0 & \text{strongly in } L^2(\Omega), \\ u_\varepsilon \to u_0 & \text{strongly in } H^s(\Omega) \text{ for all } s < 1, \\ u_\varepsilon \to u_0 & \text{strongly in } L^q(\partial\Omega), \\ u_\varepsilon \to u_0 & \text{almost everywhere in } \Omega. \end{array}\right\} \tag{2.10}$$

Now we claim that

$$\int_{\partial\Omega} |u_0|^q \, \mathrm{d}S = 1. \tag{2.11}$$

To prove this note that, as we have

$$1 = \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u_\varepsilon|^q \, \mathrm{d}x = \frac{1}{\varepsilon}\int_0^\varepsilon \int_{\partial\Omega_\delta} |u_\varepsilon|^q \, \mathrm{d}S \, \mathrm{d}\delta$$

from the integral mean-value theorem, there exists $0 \leqslant \delta(\varepsilon) \leqslant \varepsilon$ such that

$$\int_{\partial\Omega_\delta} |u_\varepsilon|^q \, \mathrm{d}S = 1.$$

Now, from the convergence of u_ε to u_0 in $H^s(\Omega)$, valid for $0 < s < 1$, we conclude that

$$\int_{\partial\Omega} |u_0|^q \, \mathrm{d}S = 1$$

(see remark 2.3). This completes the proof of the claim.

Bearing this in mind, we have

$$\begin{aligned} T_q &\leqslant \left(\int_\Omega |\nabla u_0|^2 + u_0^2 \, \mathrm{d}x\right)\left(\int_{\partial\Omega} |u_0|^q \, \mathrm{d}S\right)^{q/2} \\ &\leqslant \|u_0\|^2_{H^1(\Omega)} \leqslant \liminf_{\varepsilon\to 0} \|u_\varepsilon\|^2_{H^1(\Omega)} \\ &\leqslant \limsup_{\varepsilon\to 0} \|u_\varepsilon\|^2_{H^1(\Omega)} = \limsup_{\varepsilon\to 0} S_q(\varepsilon) = T_q. \end{aligned}$$

Therefore,

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{H^1(\Omega)} = \|u_0\|_{H^1(\Omega)}.$$

In particular, the convergence of the norms implies that the extremals of $S_q(\varepsilon)$ normalized according to (2.9) converge strongly in $H^1(\Omega)$ to an extremal of (1.1),

$$\lim_{\varepsilon\to 0} u_\varepsilon = u_0 \quad \text{strongly in } H^1(\Omega),$$

which satisfies (2.11).

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Now, let us prove that we have convergence in $C^{\beta}(\Omega)$, for some $\beta > 0$. To this end we will use some results from [1] that describe the behaviour of solutions of linear elliptic equations with concentrated potentials.

Denote by $V_\varepsilon(x) = S_q(\varepsilon)u_\varepsilon^{q-2}$ so that u_ε is a solution of the problem

$$-\Delta u_\varepsilon + u_\varepsilon = \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} V_\varepsilon u_\varepsilon \quad \text{in } \Omega,$$
$$\frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

First, note that as q is subcritical we can choose $r > N-1$ such that

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon|^r \,\mathrm{d}x = \frac{S_q(\varepsilon)^r}{\varepsilon}\int_{\omega_\varepsilon} |u_\varepsilon|^{(q-2)r}\,\mathrm{d}x \leqslant C,$$

with C independent of ε. Indeed, as u_ε is uniformly bounded in $H^1(\Omega)$, from lemma 2.2 for any $\theta \leqslant 2(N-1)/(N-2)$, we have

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u_\varepsilon|^{\theta}\,\mathrm{d}x \leqslant C.$$

Now, we just write $\theta = (q-2)r$ and use the fact that $q < 2(N-1)/(N-2)$ (this implies that $(q-2) < 2/(N-2)$) to obtain $(q-2)r \leqslant 2(N-1)/(N-2)$ for some $r > N-1$.

Moreover, since $S_q(\varepsilon) \to T_q$, $u_\varepsilon \to u_0$ in $H^1(\Omega)$ and q is subcritical, we have that

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon \phi \,\mathrm{d}x \to \int_{\partial\Omega} V_0 \phi \,\mathrm{d}S$$

for any smooth function ϕ, where $V_0(x) = T_q u_0^{q-2}(x)$. Hence, u_0 satisfies

$$-\Delta u_0 + u_0 = 0 \qquad \text{in } \Omega,$$
$$\frac{\partial u_0}{\partial \nu} = V_0 u_0 \quad \text{on } \partial\Omega.$$

With all this at hand, we can apply [1, theorem 3.1 and corollary 3.2], which guarantee the convergence in the Hölder norm $C^{\beta}(\Omega)$, for some $\beta > 0$.

In the critical case $q = 2_*$ we also obtain a uniform bound in $H^1(\Omega)$ for the extremals u_ε of $S_q(\varepsilon)$. Therefore, we can extract a subsequence such that (2.10) holds. Passing to the limit in the weak form of (1.6), we find that the limit u_0 is a weak solution of (1.3). However, due to the lack of compactness, we cannot ensure that u_0 verifies $\int_{\partial\Omega} |u_0|^q = 1$ in this case.

To finish the proof of the theorem it remains to show (1.8) in the supercritical case $2_* = 2(N-1)/(N-2) < q < 2^* = 2N/(N-2)$. To see this, assume that $0 \in \partial\Omega$ and consider

$$u(x) = |x|^{-\lambda},$$

where we choose λ such that $u \in H^1(\Omega)$, i.e. $\lambda < (N-2)/2$. Now we choose $\lambda = \lambda(q)$ such that

$$\int_{\partial\Omega} |u|^q \,\mathrm{d}S = +\infty,$$

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that is, $\lambda \geqslant (N-1)/q$, which is possible since $q > 2_*$. We observe that with this choice we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\omega_\varepsilon} |u|^q \, \mathrm{d}x = +\infty.$$

The proof is complete. □

REMARK 2.4. Observe that, in the critical case, using a sequence of minimizers and subsequences if necessary, we have $u_\varepsilon \to u_0$ weakly in $H^1(\Omega)$ and $S_\varepsilon(q) \to T_q$. Also, we have

$$\|u_0\|^2_{H^1(\Omega)} \leqslant \liminf_{\varepsilon\to 0} \|u_\varepsilon\|^2_{H^1(\Omega)} \leqslant \limsup_{\varepsilon\to 0} \|u_\varepsilon\|^2_{H^1(\Omega)} = \limsup_{\varepsilon\to 0} S_q(\varepsilon) = T_q$$

and

$$T_q \leqslant \biggl(\int_\Omega |\nabla u_0|^2 + u_0^2 \, \mathrm{d}x\biggr)\biggl(\int_{\partial\Omega} |u_0|^q \, \mathrm{d}S\biggr)^{q/2}.$$

Hence, if u_0 is a minimizer, then $\int_{\partial\Omega} |u_0|^q \, \mathrm{d}S \leqslant 1$. Conversely, if $\int_{\partial\Omega} |u_0|^q \, \mathrm{d}S \geqslant 1$, then the argument above shows that this integral is actually equal to 1 and u_0 is a minimizer. Moreover, in such a case, we obtain the convergence of the $H^1(\Omega)$ norms and, hence, the strong convergence in this space.

Thus, u_0 is a minimizer if and only if $\int_{\partial\Omega} |u_0|^q \, \mathrm{d}S = 1$, which in turn is equivalent to the strong convergence.

Also, in the critical case it may then be the case that one obtains (1.5) and $\int_{\partial\Omega} |u_0|^q \, \mathrm{d}S < 1$.

3. Proof of theorem 1.2

We divide the proof of theorem 1.2 into several lemmas. Throughout this section we take $\Omega = B(0,R)$, except in the next result.

LEMMA 3.1. *Let Ω be arbitrary. Then for any $1 \leqslant q \leqslant 2$ and any $\varepsilon > 0$, every extremal is of constant sign. Moreover, there exists a unique positive extremal of (1.4), normalized according to (1.5).*

Proof. Note that non-negative extremals of (1.4) are indeed positive solutions of (1.6), i.e. when normalized as in (1.5), they satisfy

$$\left.\begin{aligned} -\Delta u = f(x,u) = a(x)u^\rho - u \quad &\text{in } \Omega,\\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,& \end{aligned}\right\} \tag{3.1}$$

where

$$a(x) = \frac{S_q(\varepsilon)}{\varepsilon}\chi_{\omega_\varepsilon}(x) \geqslant 0 \quad \text{and} \quad \rho = q-1.$$

Also note that, from (1.4), non-negative extremals exist, since the absolute value of an extremal is an extremal.

Now, we use an argument from [15] (see also [12, 13]). Note that if $q < 2$, then $\rho < 1$. Hence, if $x \in \Omega \setminus \bar{\omega}_\varepsilon$ we have $f(x,u) = -u \leqslant C(x)u + D(x)$ if we take $C(x) = -1$ and $D(x) = 0$.

On the other hand, if $x \in \omega_\varepsilon$, for sufficiently small δ, Young's inequality yields

$$f(x,u) \leqslant (\delta - 1)u + \beta \left[\frac{S_q(\varepsilon)}{\delta^\rho \varepsilon}\right]^{1/(1-\rho)}$$

for some constant $\beta > 0$ and we can take $C(x) = \delta - 1$ and

$$D(x) = \beta \left[\frac{S_q(\varepsilon)}{\delta^\rho \varepsilon}\right]^{1/(1-\rho)}.$$

In summary,

$$C(x) = \delta \chi_{\omega_\varepsilon}(x) - 1, \qquad D(x) = \beta \left[\frac{S_q(\varepsilon)}{\delta^\rho \varepsilon}\right]^{1/(1-\rho)} \chi_{\omega_\varepsilon}(x)$$

and, for $u > 0$ and $x \in \Omega$, we have

$$f(x,u) \leqslant C(x)u + D(x).$$

Note that, for sufficiently small δ, the semigroup generated by $\Delta + C(x)$ in Ω with Neumann boundary conditions decays exponentially. Then, since $D \in L^\infty(\Omega)$, we see from [12,15] that there exists a solution of (3.1) which is maximal in the sense of pointwise ordering. In particular, it is non-zero since it bounds above in a pointwise sense any normalized positive extremal.

Now, the proof concludes by showing that, in fact, (3.1) has a unique solution, which follows from the fact that

$$\frac{f(x,u)}{u} = \frac{a(x)}{u^{1-\rho}} - 1$$

is non-increasing for $u > 0$ and strictly decreasing on a set of positive measure. Indeed, let φ be the maximal positive solution of (3.1) and $0 < \psi \leqslant \varphi$ any other solution. Then, multiplying the equation satisfied by φ by ψ, and multiplying that for ψ by φ, subtracting and integrating by parts in Ω, we have

$$0 = \int_\Omega \frac{f(x,\varphi)}{\varphi}\varphi\psi - \int_\Omega \frac{f(x,\psi)}{\psi}\varphi\psi = \int_\Omega \left(\frac{f(x,\varphi)}{\varphi} - \frac{f(x,\psi)}{\psi}\right)\varphi\psi.$$

Now, since $\psi \leqslant \varphi$ we obtain that

$$\frac{f(x,\varphi)}{\varphi} - \frac{f(x,\psi)}{\psi} \leqslant 0$$

and is non-zero in a set of positive measure. Therefore, we must have $\psi \equiv 0$.

When $q = 2$ the conclusion of the lemma follows easily, since the first eigenvalue of the elliptic problem (1.6) is simple [11]. Therefore, there exists a unique positive eigenfunction such that (1.5) holds. □

With this, if $\Omega = B(0,R)$, we get the following result, which actually proves the first part of theorem 1.2.

COROLLARY 3.2. *For every $1 \leqslant q \leqslant 2$ and every $R, \varepsilon > 0$, every extremal of (1.4) is radial and does not change sign in Ω.*

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Proof. Note that in any case when $q < 2$ or $q = 2$ the absolute value of an extremal is also an extremal. Therefore, the absolute value is a non-negative extremal and must then coincide with the unique positive extremal. This, in turn, must be radial, since, by uniqueness, it must coincide with any of its rotations. □

The following lemma proves theorem 1.2(a).

LEMMA 3.3. *For $2 < q < 2_* = 2(N-1)/(N-2)$ there exists R_1 such that, for every $R > R_1$, there exists ε_0 such that the extremals (1.4) are not radial for $\varepsilon < \varepsilon_0$.*

Proof. The results of [3] imply that in this case the extremals of the best Sobolev trace constant $T_q(B(0,R))$ are not radial (since they develop a concentration phenomena). Since the extremals for $S_q(\varepsilon)$ converge to the extremals of $T_q(B(0,R))$ as $\varepsilon \to 0$, they cannot be radial for sufficiently small ε (possibly depending on R). □

Now we finish the proof of theorem 1.2.

LEMMA 3.4. *For $2 < q < 2_* = 2(N-1)/(N-2)$ there exists R_0 such that, for every $R \leqslant R_0$, there exists ε_0 such that there exists a radial extremal of (1.4) for $\varepsilon < \varepsilon_0$.*

Proof. First, let us choose R_0 in such a way that, for any $R < R_0$, the problem

$$\left.\begin{aligned} -\Delta u + R^2 u = 0 \quad &\text{in } B(0,1), \\ \frac{\partial u}{\partial \nu} = R^2 \frac{T_q(R)}{R^\beta} u^{q-1} \quad &\text{on } \partial B(0,1) \end{aligned}\right\} \tag{3.2}$$

has a unique positive solution close to $u_0 \equiv |\partial B(0,1)|^{-1/q}$ normalized with the usual constraint $\int_{\partial B(0,1)} u^q = 1$ (see [6]). Here

$$\beta = \frac{qN - 2N + 2}{q}.$$

Observe that the above problem is merely (1.3) (together with (1.2)) rescaled from the ball of radius R to the ball of radius 1. Also note that, from the results of [4], we have

$$\lim_{R\to 0} \frac{T_q(R)}{R^\beta} = \frac{|B(0,1)|}{|\partial B(0,1)|^{2/q}}.$$

Moreover, we can assume (taking R_0 smaller if necessary) that, for $R < R_0$, the linearization of (3.2) is invertible. This can be obtained since, for small R, there is a unique solution to (3.2) with

$$\int_{\partial B(0,1)} u^q = 1$$

and the linearized problem is invertible at $R = 0$, $u = 1/(|\partial B(0,1)|^{1/q})$ and then invertible at (R, u_R) for small R (see [6] for details).

Now we want to use the implicit function theorem in (1.4). To this end, let us rescale (1.6) to the unit ball defining $v(x) = R^\alpha u(Rx)$, where u is the solution of (1.6) satisfying (1.5). If $\alpha = (N-1)/q$, we have that v satisfies

$$\frac{1}{\varepsilon R^{-1}} \int_{\Delta_{\varepsilon,R}} |v|^q \, \mathrm{d}x = 1, \tag{3.3}$$

where $\Delta_{\varepsilon,R} = B(0,1) \setminus B(0, 1-\varepsilon R^{-1})$ and also

$$\begin{gathered} -\Delta v + R^2 v = R^2 \frac{S_q(\varepsilon)}{R^\beta \varepsilon R^{-1}} \chi_{\varepsilon,R}(x) v^{q-1} \quad \text{in } B(0,1), \\ \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial B(0,1), \end{gathered}$$

where $\chi_{\varepsilon,R}(x)$ is the characteristic function of $\Delta_{\varepsilon,R}$. Let

$$S = \left\{ v \in H^1(B(0,1));\ \int_{\partial B(0,1)} |v|^q \, \mathrm{d}S = 1 \right\}.$$

If we multiply v by an adequate constant μ in order to have $w = \mu v \in S$, we have

$$\mu = \left(\int_{\partial B(0,1)} v^q \right)^{-1/q}$$

and we are left with a solution of

$$\left.\begin{gathered} -\Delta w + R^2 w = R^2 \frac{\tilde{A}(\varepsilon)}{\varepsilon R^{-1} R^\beta} \chi_{\varepsilon,R}(x) w^{q-1} \quad \text{in } B(0,1), \\ \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial B(0,1). \end{gathered}\right\} \tag{3.4}$$

Here

$$\tilde{A}(\varepsilon) = S_q(\varepsilon) \left(\int_{\partial B(0,1)} v^q \, \mathrm{d}S \right)^{1-2/q},$$

where the integral term also depends on ε through v. From (1.7) and the convergence of the extremals in theorem 1.1, using (3.3) and lemma 2.2, we find that

$$\tilde{A}(\varepsilon) \to T_q$$

as $\varepsilon \to 0$.

Let us consider the functional

$$F : S \times [0, \varepsilon_0] \mapsto (H^1(B(0,1)))^*,$$

given by

$$\begin{aligned} F(w,\varepsilon)(\phi) = \int_{B(0,1)} \nabla w \nabla \phi \, \mathrm{d}x + R^2 \int_{B(0,1)} w \phi \, \mathrm{d}x \\ - \frac{R^2 \tilde{A}(\varepsilon)}{\varepsilon R^{-1} R^\beta} \int_{B(0,1) \setminus B(0,1-\varepsilon R^{-1})} w^{q-1} \phi \, \mathrm{d}x. \end{aligned}$$

This functional is C^1 with respect to $w \in S$ (since $q > 2$).

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Recall that we are looking for pairs (w, ε) that are solutions of $F(w, \varepsilon) = 0$ (these are weak solutions of (3.4)).

To apply the implicit function theorem we need to compute

$$\frac{\partial F}{\partial w}(u, 0).$$

First, let us compute the derivative

$$\begin{aligned}\frac{\partial F}{\partial w}(w,\varepsilon)(\phi)(\chi) = \int_{B(0,1)} \nabla\chi\nabla\phi \,\mathrm{d}x + R^2 \int_{B(0,1)} \chi\phi \,\mathrm{d}x \\ - \frac{R^2 \tilde{A}(\varepsilon)}{\varepsilon R^{-1} R^{\beta}} \int_{B(0,1)\setminus B(0,1-\varepsilon R^{-1})} (q-1) w^{q-2} \phi\chi \,\mathrm{d}x.\end{aligned}$$

Taking the limit as $\varepsilon \to 0$ and evaluating at $w = u$, we obtain (by [1] or by the results of the previous section)

$$\begin{aligned}\frac{\partial F}{\partial w}(u,0)(\phi)(\chi) = \int_{B(0,1)} \nabla\chi\nabla\phi \,\mathrm{d}x + R^2 \int_{B(0,1)} \chi\phi \,\mathrm{d}x \\ - R^2 \frac{T_q}{R^{\beta}} \int_{\partial B(0,1)} (q-1) u^{q-2} \phi\chi \,\mathrm{d}x.\end{aligned}$$

This problem corresponds exactly with the linearization of (3.2) that is invertible by our choice $R < R_0$.

Therefore, by the implicit function theorem, we find that there exists ε_0 such that for any $\varepsilon < \varepsilon_0$ there exists a unique solution $w_\varepsilon \in S$ of

$$F(w_\varepsilon, \varepsilon) = 0$$

close to u, that is, a unique weak solution of (3.4), with

$$\lim_{\varepsilon \to 0} w_\varepsilon = u.$$

Since we have proved that every extremal of (1.4) tends to u as $\varepsilon \to 0$ and we have uniqueness of solutions of (3.4) in a neighbourhood of u, the extremals must be radial. □

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