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PAPER

Languages of higher-dimensional automata

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Abstract

We introduce languages of higher-dimensional automata (HDAs) and develop some of their properties. To this end, we define a new category of precubical sets, uniquely naturally isomorphic to the standard one, and introduce a notion of event consistency. HDAs are then finite, labeled, event-consistent precubical sets with distinguished subsets of initial and accepting cells. Their languages are sets of interval orders closed under subsumption; as a major technical step, we expose a bijection between interval orders and a subclass of HDAs. We show that any finite subsumption-closed set of interval orders is the language of an HDA, that languages of HDAs are closed under binary unions and parallel composition, and that bisimilarity implies language equivalence.

Keywords: Higher-dimensional automaton; concurrency theory; pomset; directed topology

1. Introduction

Higher-dimensional automata (HDAs) are a formalism for modeling and reasoning about behaviors of concurrent systems, introduced by Pratt (1991) and van Glabbeek (1991). Like Petri nets (Petri, 1962), event structures (Nielsen et al., 1981), configuration structures (van Glabbeek and Plotkin, 1995, 2009), asynchronous transition systems (Bednarczyk, 1987; Shields, 1985), and similar approaches (Johansen, 2015; Pratt, 1995, 2003; van Glabbeek and Goltz, 2001), they form a model of non-interleaving concurrency as they differentiate between interleaving and "truly" concurrent computations, that is, $a\|b \neq a.b + b.a$ (using CCS notation (Milner, 1989)). van Glabbeek (2006a) has shown that HDAs generalize "the main models of concurrency proposed in the literature," including those mentioned above.

HDAs extend finite automata with additional structure that distinguishes interleavings from concurrency. As an example, Figure 1 shows Petri net and HDA models for a system with two events, labeled a and b. The Petri net and HDA on the left model the (mutually exclusive) interleaving of a and b as either $a.b$ or $b.a$; those on the right model concurrent execution of a and b. In the HDA, this independence is indicated by a filled-in square.

HDAs thus have states and transitions like finite automata, but may also contain squares, cubes, and higher-dimensional cubical structures. A square stands for the concurrent execution of two events, a cube for the concurrent execution of three events, and so on.

This paper is concerned with *languages* of HDAs. Like languages related to other formalisms for concurrency, these need to account for both the sequential and the concurrent nature of computations. Their elements will therefore be finite *pomsets* or *partial words* (Winkowski, 1977). As an example, Figure 2 displays an HDA consisting of two squares, with three events labeled a, c,

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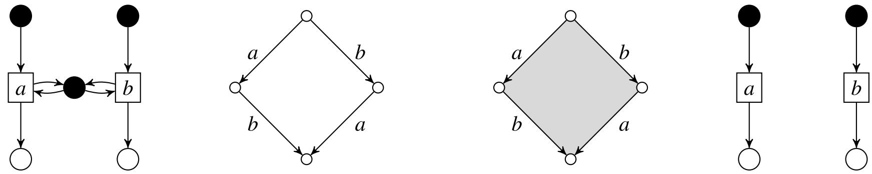

Figure 1. Petri net and HDA models distinguishing interleaving (left) from non-interleaving (right) concurrency. Left: Petri net and HDA models for $a.b + b.a$; right: HDA and Petri net models for $a \parallel b$.

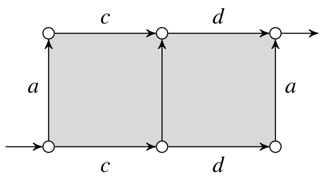

Figure 2. HDA which executes a in parallel with $c.d$. Initial and accepting cells marked with incoming and outgoing arrows.

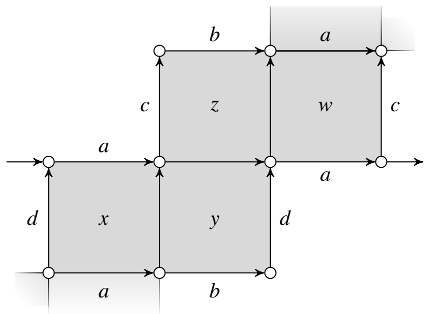

Figure 3. HDA which generates infinite set of pomsets (bottom left and top right edges identified).

and d. Here the a-labeled event is executed concurrently to the sequence $c.d$, so that the language of this HDA will contain the pomset

$$\left(\begin{smallmatrix} & a & \\ c & \longrightarrow & d \end{smallmatrix} \right). \tag{1}$$

(It will contain other elements; but in a sense to be made precise below, they are all generated by this one pomset.)

Partial words and pomsets have been introduced by Winkowski (1977) and have a long history as semantics for concurrent systems (Pratt, 1986; Vogler, 1992). The subclass of *interval orders*, introduced by Fishburn (1970), has seen abundant attention in concurrency theory and distributed systems (Fahrenberg et al., 2020; Herlihy and Wing, 1990; Janicki and Koutny, 1993; Janicki and Yin, 2017; Lamport, 1986b, 1986a; Vogler, 1991, 1992). A pomset is an interval order precisely if it is *2+2-free*, that is, does not contain an induced subpomset of the form

$$\text{2+2} = \left(\begin{smallmatrix} \bullet & \longrightarrow & \bullet \\ \bullet & \longrightarrow & \bullet \end{smallmatrix} \right).$$

We will show that languages of HDAs are sets of interval orders, and that any interval order may be generated by an HDA. For another example, the HDA in Figure 3 has a two-dimensional

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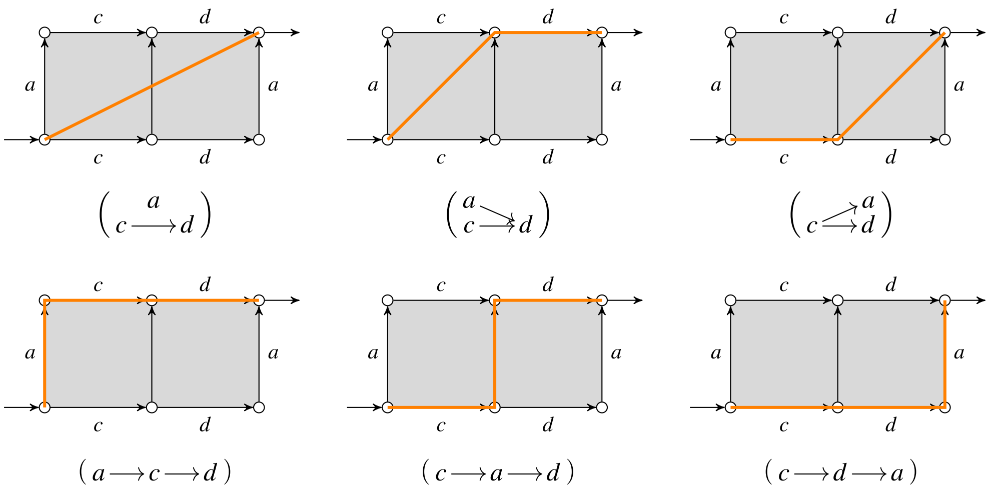

Figure 4. Directed paths in HDA of Figure 2 together with corresponding pomsets.

loop created by identifying the horizontal edges in the bottom-left and top-right of the automaton (together with their corresponding faces). Its language includes the infinite set

$$\Big\{ (a \longrightarrow b \longrightarrow a), \Big(\begin{smallmatrix} a \longrightarrow b \longrightarrow a \longrightarrow b \longrightarrow a \\ c \longrightarrow d \end{smallmatrix} \Big), \dots \Big\}, \tag{2}$$

where the second pomset is obtained by traversing the squares z, w, x, and y in that order. We will only be concerned with *finite* HDAs in this paper, yet as the above example shows, languages of finite HDAs may well be infinite.

A precursor to this work is van Glabbeek's (2006a), which introduces *tracks* in HDAs (there called paths) and then defines their observable content in terms of *ST-traces*. We have shown in Fahrenberg et al. (2020) that there is a bijective correspondence between ST-traces and interval orders. Another precursor is Fajstrup *et al.*'s (2006), where the authors define computations as *directed paths* through geometric cubical complexes. We introduce languages based on van Glabbeek's tracks and languages based on Fajstrup *et al.*'s directed paths and show that they define the same objects.

Grabowski (1981) has introduced a notion of *smoothing* for pomsets which is nowadays mostly called *subsumption* (Fanchon and Morin, 2002; Gischer, 1988): a pomset P subsumes a pomset Q if Q is at least as ordered as P. Sets of pomsets closed under subsumption are generally called *weak* (Fanchon and Morin, 2002; Grabowski, 1981). We show that languages of HDAs are weak sets of interval orders.

Figure 4 exhibits six directed paths through the HDA A of Figure 2 together with the corresponding pomsets. The language of A consists precisely of these six pomsets; it is also the weak closure of the pomset in (1) corresponding to the first directed path displayed. The language of the HDA in Figure 3 is the weak closure of the infinite set in (2).

We finish the paper by showing that languages of HDAs are closed under binary union and parallel composition, and further that *bisimilarity* of HDAs (Fahrenberg, 2005c; van Glabbeek, 2006a) implies language equivalence. A comprehensive treatment of regular operations on HDAs and their languages is left for future work.

We start this paper with an overview section which introduces the main concepts and results without going into too much technical detail. In order to properly define and develop languages of HDAs, we first introduce a new base category for precubical sets, identify a new subclass of *event consistent* precubical sets, and make clear the relationship between tracks and interval orders. This is also why we define HDAs only on page 587.

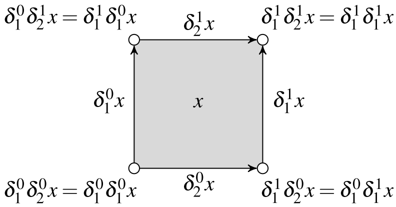

Figure 5. A square x with its four elementary faces δ_1^0x, δ_1^1x, δ_2^0x, δ_2^1x and four corners.

We detail the main technical contributions of this paper at the end of the overview Section 2. Afterwards, we introduce precubical sets, event consistency, and HDAs in Section 3. Section 4 is concerned with pomsets with interfaces, their gluing composition, and representations of interval orders. The connection between interval orders and tracks in precubical sets is made in Section 5, and directed paths are introduced in Section 6. Section 7 concludes the paper by defining languages of HDAs and developing some basic properties.

2. Overview

HDAs are built on *precubical sets* (Grandis, 2009; Serre, 1951), a generalization of directed graphs to higher dimensions. To be precise, a precubical set consists of a graded set $X = \bigcup_{n\geq 0} X_n$ of n-cells together with *elementary face maps* $\delta_{i,n}^{\nu} : X_n \to X_{n-1}$, $i \in \{1, \ldots, n\}$, $\nu \in \{0, 1\}$ that specify boundaries of n-cells. These are required to satisfy the *precubical identities*

$$\delta_{i,n-1}^{\nu}\delta_{j,n}^{\mu} = \delta_{j-1,n-1}^{\mu}\delta_{i,n}^{\nu},$$

for every $i < j \leq n$, which identify common elementary faces of elementary faces. Figure 5 shows an example of a 2-cell with all its faces; we will generally omit parentheses for elementary faces and the subscript n and thus, for example, write δ_1^0x instead of $\delta_{1,n}^0(x)$.

A precubical set X with $X_1 = \emptyset$, hence $X_i = \emptyset$ for all $i \geq 1$, is simply a set (of 0-cells or *points*). A one-dimensional precubical set X, with $X_2 = \emptyset$, is a directed graph. 1-cells are generally called *edges*, 2-cells, *squares*, and 3-cells, *cubes*. Modifying the standard setting (Grandis, 2009), we introduce precubical sets as presheaves over a category of linearly ordered sets with suitable morphisms (Definition 1). From a technical point of view, this does not matter, as our "large" category of precubical sets is uniquely isomorphic to the standard one (Proposition 9); yet it clarifies the relation between ordered sets, presimplicial sets, and precubical sets. This simplifies later developments.

An HDA is a tuple (X, I, F, λ) with X a precubical set, $I, F \subseteq X$ subsets of initial and accepting cells, and λ a labeling on X. This labeling is generated by a function $\lambda_1 : X_1 \to \Sigma$, into an alphabet Σ, which satisfies $\lambda_1(\delta_1^0x) = \lambda_1(\delta_1^1x)$ and $\lambda_1(\delta_2^0x) = \lambda_1(\delta_2^1x)$ for every $x \in X_2$, but we will extend it to a precubical morphism $\lambda : X \to !\Sigma$ into a special labeling object $!\Sigma$ (Definition 13).

One-dimensional HDAs are equivalent to ordinary finite automata, with 0-cells as states and 1-cells as transitions. Two-dimensional HDAs are equivalent to asynchronous transition systems, with the 2-cells denoting independence of events.

Most formalisms for non-interleaving concurrency have a notion of *events*: unique occurrences of actions in space and time. HDAs, on the other hand, do not have a well-defined notion of event (Fahrenberg et al., 2021; Pratt, 2000). Going back to the example in Figure 1, we see that the Petri nets on each side of the figure have two events each, induced by their transitions and labeled a and b, respectively. In the two HDAs on the other hand, every label appears twice, and there is no immediate conception of events. For the HDA on the right, we may deduce from the presence of the square, which indicates *two* events running concurrently, that there are indeed precisely two events in the system; but on the left, there might as well be four.

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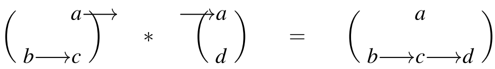

Figure 6. Two ipomsets and their gluing composition (interfaces marked with incoming and outgoing arrows).

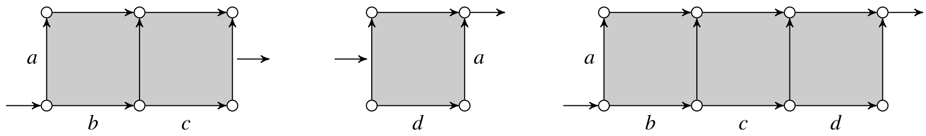

Figure 7. Three HDAs corresponding to ipomsets of Figure 6.

We make the notion of event identification precise in Definition 13 and identify a subclass of *event consistent* precubical sets: precubical sets X that admit an equivalence relation $\sim$ on X_1 such that for all $x \in X_2$, $\delta_1^0 x \sim \delta_1^1 x$, $\delta_2^0 x \sim \delta_2^1 x$, and $\delta_1^0 x \not\sim \delta_2^0 x$ (Lemma 18). The equivalence classes of the smallest such equivalence are called the *universal events* of X: the largest possible identification of events which is consistent with the structure of the HDA.

In the example in Figure 1, the HDA on the left has four universal events, whereas the one on the right has two. (An example of a precubical set which is not event consistent is shown in Figure 10 on page 585.)

Any labeling factors uniquely through the universal events (Proposition 20), so that we could have written this paper only with *unlabeled* (but event consistent) HDAs in mind and then added labels as an afterthought, much in the spirit of Winskel and Nielsen (1995). For sake of readability, we have refrained from doing so.

In Section 4, we recapitulate the notion of *pomset with interfaces* (*ipomset*) from Fahrenberg et al. (2020). An ipomset $(P, <_P, \lambda_P, S_P, T_P)$ consists of a labeled partial order $(P, <_P, \lambda_P)$ together with subsets $S_P, T_P \subseteq P$ of minimal and maximal elements which designate starting and terminating interfaces. Ipomsets may be *glued* along their interfaces: if $(Q, <_Q, \lambda_Q, S_Q, T_Q)$ is another ipomset such that $P \cap Q = T_P = S_Q$, then $P * Q$ is the ipomset

$$(P \cup Q, <_P \cup <_Q \cup (P \setminus T_P) \times (Q \setminus S_Q), \lambda_P \cup \lambda_Q, S_P, T_Q),$$

with the order defined by those of P and Q together with imposing that every event not in Q precedes every event not in P. Hence, events in the overlap $P \cap Q$ are continued across the gluing composition; Figure 6 shows an example.

We extend ipomsets with an *event order*: a second strict order, denoted $\dashrightarrow_P$, which is required to be linear on $<_P$-antichains. This allows us to assign which interfaces are identified in gluing compositions and also establishes a close relation between interval-ordered ipomsets and a subclass of HDAs, see Definition 60. As an example, Figure 7 shows the three HDAs corresponding to the ipomsets of Figure 6; we show in Lemma 65 that gluing compositions of interval ipomsets correspond to pushouts of their induced HDAs.

Most papers in concurrency theory define pomsets as isomorphism classes of labeled partial orders. We find it convenient to work directly with labeled partial orders instead and consider properties up to isomorphism. As any isomorphic ipomsets are uniquely isomorphic (Lemma 34), the difference is without significance.

One central mathematical insight on which this paper is built is that both precubical sets and interval ipomsets can be obtained by gluing linear orders, that is, precubical sets glued as presheaves, and interval ipomsets as gluing compositions of discrete ipomsets. An ipomset $(P, <_P, \dashrightarrow_P, \lambda_P, S_P, T_P)$ is *discrete* if $<_P$ is trivial, thus $\dashrightarrow_P$ is a linear order. We always think of $<_P$ as a *precedence* order; hence all events are concurrent in a discrete ipomset, and the event order $\dashrightarrow_P$ is used as a book-keeping device. Seen as linear $\dashrightarrow_P$-ordered sets, discrete ipomsets form

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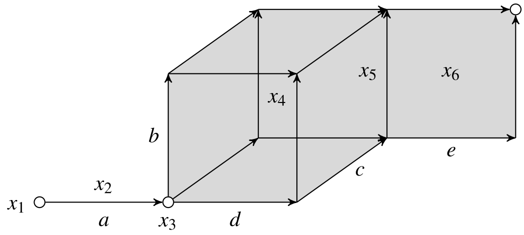

Figure 8. A track in a precubical set.

our base category for precubical sets; as trivial $<_P$-ordered sets, we may glue them into interval orders.

The subclass of precubical sets that correspond to interval ipomsets under this identification is comprised of *tracks*: sequences of cells connected at intermediate faces. This notion generalizes paths in finite automata to higher dimensions. Figure 8 shows an example, a track consisting of six cells $x_1, \ldots, x_6$ (x_4 is the central cube in the figure) with face relations

$$x_1 \lhd x_2 \rhd x_3 \lhd^3 x_4 \rhd^2 x_5 \lhd x_6,$$

where $x_1 \lhd x_2$ denotes that x_1 is a lower face of x_2, and $x_4 \rhd^2 x_5$ that x_5 is an upper face of an upper face of x_4 (in anticipation of notation introduced later on). We show in Section 5 how tracks give rise to interval ipomsets, but also how interval ipomsets can be converted into tracks.

In Section 6, we give a *geometric* interpretation of executions in HDAs, following (Fajstrup et al., 2006) and subsequent related work (Fajstrup, 2016; Ziemiański, 2017, 2020). Precubial sets may be realized geometrically as *directed spaces* (Grandis, 2009), and executions of HDAs may then be seen as *directed paths* through their geometric realization. We introduce the *interval arrangement* of a directed path, which tracks the events that are active during different phases of the execution and use this to define labels of directed paths.

In Section 7, we show that languages of HDAs defined by directed paths are the same as languages defined by tracks. We also see that languages of HDAs are weak sets of interval orders, and that any *finite* weak set of interval orders may be generated by an HDA.

In summary, the main contributions of this paper are as follows:

- New Definitions 1 and 10 of precubical sets as presheaves over a category $\boxdot$. This has linearly ordered sets as objects, and the morphisms are pairs (f, ε) of a poset map f and a function ε which partitions elements not in the image of f into two classes. This is similar to constructions in Awodey (2018); Bezem et al. (2013); the standard base category $\square$ Grandis (2009) of precubical sets is uniquely isomorphic to the skeleton of $\boxdot$.
- The identification of a new subclass of *event consistent* precubical sets in Definitions 13 and 17 and the introduction of universal events for such precubical sets.
- The exposition of a bijection, in Definitions 55 and 60, between interval-ordered ipomsets and HDA tracks. The first of these definitions introduces the *label* of a track in an HDA X, which forms the basis on which we define track-based languages of HDAs; the second defines the *track object* $\square^P$ pertaining to an interval ipomset P. These notions unite in the important Proposition 89: P is contained in the language of X precisely if there is an HDA morphism from $\square^P$ into X.
- The notion of *interval arrangement* of a directed path through the geometric realization of an HDA and the subsequent Definition 77 of labels of directed paths.
- Definition 86 of the language of an HDA, the closure properties (under binary union and parallel composition) in Theorems 100 and 108, and Theorems 110 and 111 that bisimilarity implies language equivalence. We expect that together with Proposition 89 this may form the basis of a theory of regular pomset languages, but leave this for future work.

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3. Precubical Sets and HDAs

In this section, we introduce precubical sets and HDAs, but we start with order-theoretic definitions, mainly to fix notation. We will return to posets and posets with interfaces in the next section.

A *poset* is a pair $(P, <)$ consisting of a set P and a *strict* partial order $<$ on P. We henceforth assume tacitly that the set P is finite. For any *alphabet*, that is, finite set Σ, a *pomset* is a triple $(P, <, \lambda)$ with $(P, <)$ a poset and $\lambda : P \to \Sigma$ the *labeling* of P. If the order is linear, that is, a total relation in which $x = y$, $x < y$, or $y < x$ for all $x, y \in P$, then we will speak of *linear posets* and *linear pomsets* (and generally denote linear po(m)sets by S, T, U instead of P, Q, R).

Elements $x, y \in P$ of a po(m)set P are *comparable* if $x = y$, $x < y$, or $y < x$; otherwise they are *incomparable*, denoted $x \parallel y$. An element $x \in P$ is *minimal* if there exists no $y \in P$ with $y < x$; and *maximal* if there is no $y \in P$ with $x < y$.

A subset $Q \subseteq P$ of a po(m)set P is an *antichain* if its elements are pairwise incomparable. A *maximal antichain* is one which is not a proper subset of any other antichain. The sets of minimal, respectively maximal elements of P are both maximal antichains.

A function $f : P \to Q$ between posets P, Q is a *poset map* (we use *map* and *morphism* interchangeably for the arrows in a category) if $x <_P y$ implies $f(x) <_Q f(y)$ for all $x, y \in P$. By irreflexivity, f is *injective* on comparable elements: if $x <_P y$ or $y <_P x$, then $f(x) \neq f(y)$. If P is linearly ordered, then f must be injective.

A function $f : P \to Q$ between pomsets P, Q is a *pomset map* if it is a poset map that preserves the labeling, that is, $\lambda_Q \circ f = \lambda_P$. Posets and poset maps form the category Pos, and pomsets and pomset maps form the category Poms. Isomorphism in these and all subsequent categories will be denoted $\cong$.

3.1 Precube categories

Precubical sets are usually defined as presheaves over a small skeletal category $\square$, see Definition 2 below. We find it more convenient to work with a large version of $\square$, denoted $\boxdot$ and defined below, which as objects has all linear posets.

Definition 1. *The* large precube category $\boxdot$ *consists of the following data:*

- *objects are linear posets* $(S, \dashrightarrow)$*;*
- *morphisms* $S \to T$ *in* $\boxdot(S, T)$ *are pairs* (f, ε)*, where* $f : S \to T$ *is a poset map and* $\varepsilon : T \to \{0, ⅃, 1\}$ *a function such that* $f(S) = \varepsilon^{-1}(⅃)$*;*
- *the composition of morphisms* $(f, \varepsilon) : S \to T$ *and* $(g, \zeta) : T \to U$ *is* $(g \circ f, \eta)$*, where*

$$\eta(u) = \begin{cases} \varepsilon(g^{-1}(u)) & \textit{for } u \in g(T), \\ \zeta(u) & \textit{otherwise.} \end{cases}$$

The function ε distinguishes events that have not yet started (labeled by 0) from those that have finished (labeled by 1) and those that are executing (labeled by ⅃). This notation is inspired by Chu spaces (Pratt, 1995); see also Fahrenberg et al. (2021) for the relation between HDAs and Chu spaces. For every morphism (f, ε), the isomorphism $f : S \to \varepsilon^{-1}(⅃) \subseteq T$ is unique; the map f is therefore determined by ε.

In an inclusion $(f, \varepsilon) : S \to T$, the events in $f(S) = \varepsilon^{-1}(⅃)$ are executing, whereas the events in $T \setminus f(S)$ are either not started or terminated. In the composition $(g \circ f, \eta)$, η is defined such that events in $U \setminus g(T)$ retain their status from the inclusion of T in U, events properly in T preserve their status from the inclusion of S in T, and events coming from S are executing. See Figure 9 for an example.

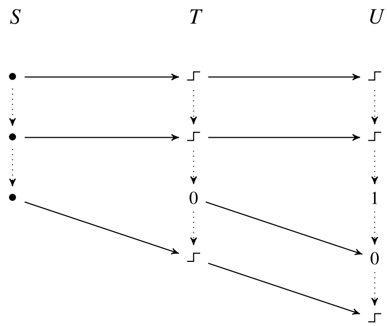

Figure 9. Composition of morphisms $S \to T \to U$ in $\square$

For each $n \geq 1$ denote by $[n]$ the linear poset

$$[n] = \{1 \dashrightarrow 2 \dashrightarrow \cdots \dashrightarrow n\},$$

together with $[0] = \emptyset$.

Definition 2. *The* precube category *is the full subcategory* $\square \subseteq \boxdot$ *on objects* $[n]$ *for all* $n \geq 0$.

Proposition 3. *The category* $\square$ *is skeletal, and the inclusion* $\square \subseteq \boxdot$ *is an equivalence of categories and admits a unique left inverse.*

Proof. It is clear that $\square$ contains no non-trivial isomorphisms, hence is skeletal. For every $S \in \boxdot$ of cardinality $|S| = n$, there is a unique isomorphism $\iota_S : S \to [n]$ in $\boxdot$. Hence there is a unique functor $\rho : \boxdot \to \square$, which is a left inverse of the inclusion $\square \subseteq \boxdot$; it is given by $\rho(S) = [\,|S|\,] = [n]$ on objects and for any $(f, \varepsilon) : S \to T$ by $\rho(f, \varepsilon) = (\iota_T \circ f \circ \iota_S^{-1}, \varepsilon \circ \iota_T^{-1})$ on morphisms. $\square$

Remark 4. The construction above mimics the situation for the base category of *presimplicial sets*. Let $\mathbb{\Delta}$ be the full subcategory of Pos spanned by the linear posets and $\Delta \subseteq \mathbb{\Delta}$ the full subcategory on objects $[n]$ for $n \geq 0$. Except for the maps being injective, Δ is the *augmented simplex category*, see Lane, (1998), VII.5 and presheaves on Δ, that is, functors from the opposite category Δ^{op} into Set, are presimplicial sets. The category Δ is skeletal, and the inclusion $\Delta \subseteq \mathbb{\Delta}$ is an equivalence of categories and admits a unique left inverse. Consequently, the presheaf categories $\mathsf{Set}^{\Delta^{\mathrm{op}}}$ and $\mathsf{Set}^{\mathbb{\Delta}^{\mathrm{op}}}$ are uniquely naturally isomorphic, and one may be used as drop-in replacement of the other. See nLab authors (2021) for more discussion on this subject.

Remark 5. A similar base category is introduced in Bezem et al. (2013) for cubical homotopy type theory, see also Awodey (2018); Bezem et al. (2019). Let $\mathscr{B}$ be the category with objects linear posets and morphisms in $\mathscr{B}(S, T)$ those functions $f : S \to T \sqcup \{0, 1\}$ (disjoint union) for which the restrictions $f_{|f^{-1}(T)}$ to elements which do not map to 0 or 1 are poset isomorphisms. Then $\mathscr{B} = \boxdot^{\mathrm{op}}$, as any $f \in \mathscr{B}(S, T)$ is uniquely determined by $\varepsilon : S \to \{0, \lrcorner, 1\}$ given by

$$\varepsilon(x) = \begin{cases} \lrcorner & \text{if } f(x) \in T, \\ f(x) & \text{if } f(x) \in \{0, 1\}. \end{cases}$$

The category defined in Bezem et al. (2013) uses unordered sets and also permits morphisms f for which $f_{|f^{-1}(T)}$ is merely injective. These two extensions are independent of each other; removing the order amounts to introducing *symmetries*, and removing surjectivity equips precubical sets with degeneracies (thus passing to *cubical* sets). See Grandis and Mauri (1990) for the presheaf categories of cubical and symmetric cubical sets.

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We proceed to show that the (reduced) precube category $\Box$ is isomorphic to the standard base category for precubical sets (Grandis, 2009; Grandis and Mauri, 1990). For any $n \geq 1$, $i \in [n]$, and $\nu \in \{0, 1\}$, define a $\Box$-map $d^{\nu}_{i,n} = (d_{i,n}, \varepsilon^{\nu}_{i}) : [n-1] \to [n]$ by

$$d_{i,n}(k) = \begin{cases} k & \text{for } 1 \leq k < i, \\ k+1 & \text{for } i \leq k \leq n-1 \end{cases} \qquad \text{and} \qquad \varepsilon^{\nu}_{i}(k) = \begin{cases} \nu & \text{for } k = i, \\ \text{\_}\!\ulcorner & \text{for } k \neq i. \end{cases}$$

Lemma 6. *Let* $(f, \varepsilon) \in \Box([m], [n])$, $m < n$, *and let* $s = \max\{i \in [n] \mid i \notin f([m])\}$. *Then* $(f, \varepsilon) = d^{\varepsilon(s)}_{s,n} \circ (g, \zeta)$, *where* $(g, \zeta) \in \Box([m], [n-1])$ *is given by*

$$g(i) = \begin{cases} f(i) & \textit{for } f(i) < s, \\ f(i-1) & \textit{for } f(i) > s \end{cases} \qquad \textit{and} \qquad \zeta(j) = \begin{cases} \varepsilon(j) & \textit{for } j < s, \\ \varepsilon(j+1) & \textit{for } j \geq s. \end{cases}$$

Proof. Elementary calculations. $\Box$

Lemma 7. *Let* $(f, \varepsilon) \in \Box([n-s], [n])$ *and denote* $[n] \setminus f([n-s]) = \{a_1 < \cdots < a_s\}$. *Then*

$$(f, \varepsilon) = d^{\varepsilon(a_s)}_{a_s,n} \circ d^{\varepsilon(a_{s-1})}_{a_{s-1},n-1} \circ \cdots \circ d^{\varepsilon(a_2)}_{a_2,n-s+2} \circ d^{\varepsilon(a_1)}_{a_1,n-s+1}.$$

Proof. From Lemma 6 by induction. $\Box$

Proposition 8. *The category* $\Box$ *is generated by morphisms* $d^{\nu}_{i,n}$ *and the co-precubical identities* $d^{\nu}_{j,n} \circ d^{\mu}_{i,n-1} = d^{\mu}_{i,n} \circ d^{\nu}_{j-1,n-1}$ *for* $1 \leq i < j < n$ *and* $\nu, \mu \in \{0, 1\}$. *Every* $\Box$*-map* $[n-s] \to [n]$ *can be written uniquely as a composition*

$$d^{\nu_1,\ldots,\nu_s}_{a_1,\ldots,a_s} = d^{\nu_s}_{a_s,n} \circ d^{\nu_{s-1}}_{a_{s-1},n-1} \circ \cdots \circ d^{\nu_2}_{a_2,n-s+2} \circ d^{\nu_1}_{a_1,n-s+1}$$

where $\nu_i \in \{0, 1\}$ *and* $1 \leq a_1 < \cdots < a_s \leq n$.

Proof. Lemma 7 implies that every morphism $(f, \varepsilon) \in \Box([m], [n])$ can be presented as a composition of elementary morphisms in which the sequence of lower indices is strictly decreasing. Such presentations are unique since f can be recovered from the sequences $a_s, \ldots, a_1$ and $\varepsilon(a_s), \ldots, \varepsilon(a_1)$. It remains to show that the co-precubical relations hold, which is elementary. $\Box$

3.2 Precubical sets

Precubical sets are usually defined as presheaves over $\Box$, that is, functors $\Box^{\mathsf{op}} \to \mathsf{Set}$ (Grandis, 2009). Using Proposition 3, we may instead use presheaves over $\boxdot$:

Proposition 9. *The presheaf categories* $\mathsf{Set}^{\Box^{\mathsf{op}}}$ *and* $\mathsf{Set}^{\boxdot^{\mathsf{op}}}$ *are uniquely naturally isomorphic.*

Proof. Each functor $F : \Box^{\mathsf{op}} \to \mathsf{Set}$ extends uniquely to a functor $\dot{F} = \rho \circ F : \boxdot^{\mathsf{op}} \to \mathsf{Set}$ by composition with the functor ρ from the proof of Proposition 3. The functor $\dot{F}$, in turn, restricts to F on $\Box^{\mathsf{op}}$. $\Box$

Definition 10. *The category of* precubical sets *is the presheaf category* $\mathsf{Set}^{\Box^{\mathsf{op}}}$ *or, equivalently,* $\mathsf{Set}^{\boxdot^{\mathsf{op}}}$. *That is, a precubical set is a functor* $\Box^{\mathsf{op}} \to \mathsf{Set}$ *or* $\boxdot^{\mathsf{op}} \to \mathsf{Set}$, *and a precubical map is a natural transformation of precubical sets.*

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We write X_n for $X([n])$, $\delta^{\nu_1,\dots,\nu_s}_{\{a_1,\dots,a_s\}}$ for $X(d^{\nu_1,\dots,\nu_s}_{a_1,\dots,a_s})$, and $\delta^{\nu}_{\{a_1,\dots,a_s\}}$ for $X(d^{\nu,\dots,\nu}_{a_1,\dots,a_s})$. The map $X(d^{\nu}_{i,n}) : X_n \to X_{n-1}$ is denoted by δ^{ν}_i. For any $x \in X_n$, n is called the *dimension* of x and indicated by $\dim x = n$. The maps δ^{ν}_i are called *elementary face maps* and the maps $\delta^{\nu_1,\dots,\nu_s}_{\{a_1,\dots,a_s\}}$, *face maps*.

Definition 11. *The* standard S-cube *on a linear poset* $(S, \dashrightarrow)$ *is the precubical set* $\square^S$, *where*

- $\square^S_k$ *is the set of functions* $x : S \to \{0, \lrcorner, 1\}$ *taking value* $\lrcorner$ *on exactly k elements;*
- δ^{ν}_i *converts the i-th occurrence of* $\lrcorner$ *into* $\nu \in \{0, 1\}$, *that is, if* $x^{-1}(\lrcorner) = \{p_1 \dashrightarrow \cdots \dashrightarrow p_k\}$, *then*

$$\delta^{\nu}_i x(p) = \begin{cases} \nu & \text{for } p = p_i, \\ x(p) & \text{otherwise.} \end{cases}$$

Every function $x : S \to \{0, \lrcorner, 1\}$ in $\square^S_k$ determines a unique poset map $f_x : [k] \to S$ by the isomorphism $f_x : [k] \to f_x([k]) = x^{-1}(\lrcorner)$. Denote the unique top-dimensional cell of $\square^S$ (the unique element of $\square^S(S)$) by $\mathbf{y}_S$. Then $\mathbf{y}_S(x) = \lrcorner$ for all $x \in S$. The order on S is necessary to define face maps: it determines which of the $\lrcorner$ values should be converted into ν.

For $n \geq 0$, the *standard n-cube* is $\square^n := \square^{[n]}$, and its unique n-cell is denoted by $\mathbf{y}_n$.

Regarded as a presheaf, $\square^S$ is the functor represented by S, that is, $\square^S(T) = \boxdot(T, S)$. The cell $\mathbf{y}_S$ corresponds to the identity morphism on S. The following is an immediate consequence of the Yoneda lemma.

Lemma 12. *Let X be a precubical set and* $x \in X_n$. *Then there exists a unique precubical map* $\mathbf{i}_x : \square^n \to X$ *such that* $\mathbf{i}_x(\mathbf{y}_n) = x$. □

3.3 Labelings and events

Definition 13. *Let A be a finite set. The* labeling object *on A is the precubical set* $!A$ *with* $!A_n = A^n$ *and* δ^{ν}_i *defined by*

$$\delta^{\nu}_i((a_1, \dots, a_n)) = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n).$$

The event object *on A is the precubical subset* $!!A \subseteq !A$ *given by*

$$!!A_n = \{(a_1, \dots, a_n) \mid a_i \neq a_j \text{ whenever } i \neq j\}.$$

Regarded as a presheaf, $!A(S) = \mathsf{Set}(S, A)$, hence $!A$ is representable in Set via the forgetful functor $\boxdot \to \mathsf{Set}$. In particular, $!A_n$ is exactly the set of isomorphism classes of linear posets over A with n elements. Similarly, $!!A(S) = \mathsf{Inj}(S, A)$, where Inj is the category of sets and *injective* maps, so that $!!A$ is representable in Inj via the forgetful functor $\boxdot \to \mathsf{Inj}$. Also note that $!A$ is infinite, whereas $!!A$ is finite: if A has m elements, then $!!A_m$ consists of all permutations of these elements and $!!A_n = \emptyset$ for $n > m$.

Every function $f \in \mathsf{Set}(A, B)$ induces a precubical map $!f : !A \to !B$, and every injective function $g \in \mathsf{Inj}(A, B)$ induces a precubical map $!!g : !!A \to !!B$, turning them into functors $! : \mathsf{Set} \to \mathsf{Set}^{\square^{\mathrm{op}}}$ and $!! : \mathsf{Inj} \to \mathsf{Inj}^{\square^{\mathrm{op}}}$. These are left adjoint to the functors $\mathsf{Set}^{\square^{\mathrm{op}}} \to \mathsf{Set}$ and $\mathsf{Inj}^{\square^{\mathrm{op}}} \to \mathsf{Inj}$ mapping X to X_1, hence $!A$ and $!!A$ are *free* in the following sense.

Lemma 14. *Let X be a precubical set and A a finite set.*

(1) *Any function* $\lambda_1 : X_1 \to A$ *for which* $\lambda_1(\delta^0_1 x) = \lambda_1(\delta^1_1 x)$ *and* $\lambda_1(\delta^0_2 x) = \lambda_1(\delta^1_2 x)$ *for all* $x \in X_2$ *extends uniquely to a precubical map* $\lambda : X \to !A$.
(2) *Any function* $\mathsf{ev}_1 : X_1 \to A$ *for which* $\mathsf{ev}_1(\delta^0_1 x) = \mathsf{ev}_1(\delta^1_1 x)$, $\mathsf{ev}_1(\delta^0_2 x) = \mathsf{ev}_1(\delta^1_2 x)$, *and* $\mathsf{ev}_1(\delta^0_1 x) \neq \mathsf{ev}_1(\delta^0_2 x)$ *for all* $x \in X_2$ *extends uniquely to a precubical map* $\mathsf{ev} : X \to !!A$.

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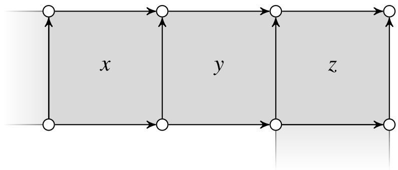

Figure 10. Example of a precubical set that is not event consistent (left and bottom right edges identified).

Proof. For the first claim, define functions $f_i : [1] \to [n]$, for all $n \geq 0$ and $i \in [n]$, by $f_i(1) = i$. Then define λ by $\lambda(x) = (\lambda_1((f_1, \varepsilon_1)^{\mathsf{op}}(x)), \ldots, \lambda_1((f_n, \varepsilon_n)^{\mathsf{op}}(x)))$ for $x \in X_n$. Because of $\lambda_1 \circ \delta_i^0 = \lambda_1 \circ \delta_i^1$, the choices of ε_i do not matter. It is clear that λ is the unique extension of λ_1.

For the second claim, we already know that ev_1 extends uniquely to $\mathsf{ev} : X \to !A$. We show that the image of ev lies in $!!A$. With a slight abuse of notation, write $\mathsf{ev}(x) = (\mathsf{ev}_1(x), \ldots, \mathsf{ev}_n(x))$ for $x \in X_n$, and suppose there exists an $x \in X_n$ with $\mathsf{ev}_i(x) = \mathsf{ev}_j(x)$ for $i < j$. Let $(f, \varepsilon) \in \square([2], [n])$ be the morphism that satisfies $f(1) = i$ and $f(2) = j$ (ε is again irrelevant), and let $y = (f, \varepsilon)^{\mathsf{op}}(x) \in X_2$. Then $\mathsf{ev}_1(\delta_1^0 y) = \mathsf{ev}_i(x) = \mathsf{ev}_j(x) = \mathsf{ev}_1(\delta_2^0 y)$, in contradiction to the second property of ev_1. □

Let henceforth Σ be a fixed finite set.

Definition 15. *Let X be a precubical set. A* labeling *of X is a precubical map $\lambda : X \to !\Sigma$. An* event identification *on X is a map $\mathsf{ev} : X \to !!\Sigma$.*

Every event identification on X is also a labeling on X, but the converse does not hold; in fact, all precubical sets admit labelings, but not necessarily event identifications:

Example 16. Figure 10 shows a precubical set with three 2-cells x, y, z and $\delta_1^0 x = \delta_2^0 z$. Any event identification $\mathsf{ev} : X \to !!\Sigma$ must fulfill

$$\mathsf{ev}(\delta_2^0 z) = \mathsf{ev}(\delta_1^0 x) = \mathsf{ev}(\delta_1^1 x) = \mathsf{ev}(\delta_1^0 y) = \mathsf{ev}(\delta_1^1 y) = \mathsf{ev}(\delta_1^0 z),$$

a contradiction.

Definition 17. *A precubical set X is* event consistent *if it admits an event identification $\mathsf{ev} : X \to !!\Sigma$.*

Lemma 18. *A precubical set X is event consistent iff there exists an equivalence relation $\sim$ on X_1 such that for all $x \in X_2$, $\delta_1^0 x \sim \delta_1^1 x$, $\delta_2^0 x \sim \delta_2^1 x$, and $\delta_1^0 x \not\sim \delta_2^0 x$.*

Proof. First suppose that X is event consistent. Let $\mathsf{ev} : X \to !!\Sigma$ and define the equivalence relation $\sim$ on X_1 by $x \sim y$ iff $\mathsf{ev}(x) = \mathsf{ev}(y)$. From the definition of $!\Sigma$, we have $\mathsf{ev}(\delta_i^0 x) = \delta_i^0 \mathsf{ev}(x) = \delta_i^1 \mathsf{ev}(x) = \mathsf{ev}(\delta_i^1 x)$ and therefore $\delta_i^0 x \sim \delta_i^1 x$ holds for all $x \in X_2$ and for $i \in \{1, 2\}$. From the definition of $!!\Sigma$, we have $\mathsf{ev}(\delta_1^0 x) = \delta_1^0 \mathsf{ev}(x) \neq \delta_2^0 \mathsf{ev}(x) = \mathsf{ev}(\delta_2^0 x)$ and therefore $\delta_1^0 x \not\sim \delta_2^0 x$ holds for all $x \in X_2$.

For the other direction, suppose there is a relation $\sim$ that satisfies the properties in the lemma and let $\Sigma = X_1/{\sim}$ be the set of equivalence classes. The quotient map $X_1 \to \Sigma$ then extends uniquely to $\mathsf{ev} : X \to !!\Sigma$ by Lemma 14, which yields the event identification needed. □

Any event consistent precubical set admits a *smallest* equivalence relation owing to Lemma 18, denoted $\sim_{\mathsf{ev}}$. It is given as the transitive closure of $\{(\delta_i^0 x, \delta_i^1 x) \mid x \in X_2, i \in \{1, 2\}\}$, we call its equivalence classes the *universal events* of X.

We will generally only concern ourselves with event consistent precubical sets in the rest of this work, but come back to the more general case at the end of Section 7.1.

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Example 19. The standard S-cube $\square^S$ from Definition 11 is event consistent for any linear poset S. Its universal event equivalence is given by $x \sim_{\mathsf{ev}} y$ iff $x^{-1}(\lrcorner\!\ulcorner) = y^{-1}(\lrcorner\!\ulcorner)$, induced by the event identification $\mathsf{ev} : \square^S \to !!S$ with $\mathsf{ev}(x) = x^{-1}(\lrcorner\!\ulcorner)$. If X is a precubical subset of $\square^S$, then X is also event consistent. Such precubical subsets of standard cubes are called *sculptures* in Fahrenberg et al. (2021), where it is shown that they correspond to Chu spaces over $\{0, \lrcorner\!\ulcorner, 1\}$ (Pratt, 1995, 2000).

The term "universal events" is justified by the following factorization property, which follows immediately from the definitions.

Proposition 20. *Every labeling $\lambda : X \to !\Sigma$ factors uniquely through E_X, that is, there is a unique factorization $\lambda_1 = \lambda_1^{\mathsf{ev}} \circ \mathsf{ev}_1 : X_1 \to E_X \to \Sigma$ that extends to a factorization $\lambda = \lambda^{\mathsf{ev}} \circ \mathsf{ev} : X \to !!E_X \to !\Sigma$.* □

Hence also any event identification factors uniquely through the universal events.

Proposition 21. *Let $\lambda : X \to !\Sigma$ be a labeling. If $x \in X_n$ and $(f, \varepsilon) \in \square([m], [n])$, then $\lambda((f, \varepsilon)^{\mathsf{op}}(x)) = \lambda(x) \circ f$. In particular,*

$$\lambda(\delta_i^\nu x) = (\lambda_1(x), \ldots, \lambda_{i-1}(x), \lambda_{i+1}(x), \ldots, \lambda_n(x)).$$

Proof. Straightforward from the definitions. □

Lemma 22. *Let X be an event consistent precubical set, $n \geq 0$, $x \in X_n$, $A, B \subseteq [n]$, and $\nu \in \{0, 1\}$. Then $\delta_A^\nu x = \delta_B^\nu x$ implies $A = B$.*

Proof. Applying Proposition 21 to $\mathsf{ev} : X \to !!E_X$ yields $\mathsf{ev}(\delta_A^\nu x) = (\mathsf{ev}_i(x))_{i \in [n] \setminus A}$ and $\mathsf{ev}(\delta_B^\nu x) = (\mathsf{ev}_i(x))_{i \in [n] \setminus B}$. Since events $\mathsf{ev}_i(x)$ are pairwise distinct, we obtain $A = B$. □

Remark 23. For $n = 2$, the above lemma reduces to the definition of event consistency. The lemma will show its importance once we consider *tracks*, that is, sequences of cells connected at faces, in Section 5. There is a related property of being *non-selfintersecting* which has been used for the same purpose, see, for example Fajstrup (2005): a precubical set X is non-selfintersecting if $\delta_A^\nu x = \delta_B^\mu x$ implies $A = B$ and $\nu = \mu$ for all $x \in X$. Example 16 shows that precubical sets may be non-selfintersecting, but not event consistent; similarly, event consistency does not imply the non-selfintersecting property, see Fahrenberg et al. (2021). Finally, also Section 4 of van Glabbeek (1996) contains some precursors to our notion of event consistency.

If $f : X \to Y$ is a precubical map, then $x \sim_{\mathsf{ev}} y$ implies $f(x) \sim_{\mathsf{ev}} f(y)$ for $x, y \in X_1$, and $f_1 : X_1 \to Y_1$ induces a map $E_f : E_X \to E_Y$. This defines a functor $E : \mathsf{Set}^{\square^{\mathsf{op}}} \to \mathsf{Set}$.

3.4 Higher-dimensional automata

Higher-dimensional automata are labeled precubical sets with initial and accepting cells.

Definition 24. *A* labeled precubical set *is an event consistent precubical set X together with a labeling $\lambda : X \to !\Sigma$. Maps $f : X \to Y$ of labeled precubical sets preserve labelings: $\lambda_X = \lambda_Y \circ f$.*

That is to say, the category of labeled precubical sets is the full subcategory of the slice category $\mathsf{Set}^{\square^{\mathsf{op}}}/!\Sigma$ on event consistent objects.

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Definition 25. *Let (X, λ) be a labeled precubical set. The* label *of a cell $x \in X_n$ is the linear pomset $\ell(x) = (\mathsf{ev}(x), \dashrightarrow, \lambda_{\mathsf{ev}(x)})$ with*

$$\mathsf{ev}(x) = (\mathsf{ev}_1(x) \dashrightarrow \mathsf{ev}_2(x) \dashrightarrow \cdots \dashrightarrow \mathsf{ev}_n(x))$$

and $\lambda_{\mathsf{ev}(x)}(\mathsf{ev}_i(x)) = \lambda_i(x)$.

Note that $\ell(x)$ is basically the tuple $\lambda(x)$, but using Proposition 20 we now regard it as a pomset. Event consistency is essential for this to make sense. Of the two following elementary lemmas, the first follows from Proposition 21 and Lemma 18; the second one is trivial.

Lemma 26. *If x is a face of y, then there is a pomset inclusion $\ell(x) \subseteq \ell(y)$.* □

Lemma 27. *Let $f : X \to Y$ be a map of labeled precubical sets and $x \in X_n$. Then $\ell(f(x)) = E_f(\ell(x)) \cong \ell(x)$.* □

Definition 28. *A higher-dimensional automaton (HDA) is a tuple (X, I, F, λ), where (X, λ) is a labeled precubical set, $I \subseteq X$ is a set of* initial cells, *and $F \subseteq X$ a set of* accepting cells. *If X and Y are HDAs, then a precubical map $f : X \to Y$ is an* HDA map *if it preserves labels and initial and accepting cells: $\lambda_X = \lambda_Y \circ f$, $f(I_X) \subseteq I_Y$, and $f(F_X) \subseteq F_Y$.*

4. Pomsets with Interfaces

We now return to posets and pomsets and introduce interfaces for them, building on our work in Fahrenberg et al. (2020) but enriched with event orders. Recall that Σ is a fixed finite set.

4.1 Ipomsets

Definition 29. *An* ipomset *is a tuple $(P, <_P, \dashrightarrow_P, \lambda_P, S_P, T_P)$, where P is a finite set;*

- *$<_P$ is a strict partial order on P called* precedence order;
- *$\dashrightarrow_P$ is a strict partial order on P called* event order;
- *$\lambda_P : P \to \Sigma$ is a function called* labeling;
- *S_P is a subset of the $<$-minimal elements of P called* source set;
- *T_P is a subset of the $<$-maximal elements of P called* target set.

We require that the relation $<_P \cup \dashrightarrow_P$ is total: if $x \neq y \in P$, then x and y are comparable by $<_P$ or by $\dashrightarrow_P$.

Note that $<_P \cup \dashrightarrow_P$ need not be a partial order, see Example 32 below. The linear pomset $(S_P, \dashrightarrow_P \cap (S_P \times S_P), \lambda_{P|S_P})$, where $\lambda_{P|S_P}$ denotes the domain restriction of $\lambda_P : P \to \Sigma$ to S_P, is called *source interface* of P; we often simply write $(S_P, \dashrightarrow_P, \lambda_P)$. Similarly, $(T_P, \dashrightarrow_P, \lambda_P)$ is the *target interface* of P. If $S = T = \emptyset$, then P is a pomset in the classical sense (Gischer, 1988; Pratt, 1986) (ignoring the event order). If $S = T = \emptyset$ and $<$ is linear, then P corresponds to a string.

Remark 30. In Fahrenberg et al. (2020), we defined ipomsets without an event order. Instead we picked out sources and targets using injections $s : [n] \to P$ and $t : [m] \to P$. This implicitly defines (linear) event orders on the subsets $S_P = s([n]) \subseteq P$ and $T_P = t([m]) \subseteq P$, so the only essential difference between (Fahrenberg et al., 2020) and our present setting is that the event order is extended to the whole of P. When $S_P \cap T_P = \emptyset$, such an extension is always possible, and we will see later that

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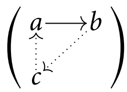
the ordered structures properly corresponding to HDAs are our present event-ordered ipomsets, see Definitions 55 and 60.

Remark 31. The ordered structure $(P, <_P, \dashrightarrow_P)$ underlying an ipomset is a *biposet* in the sense of Ésik and Németh (2004), but because of the requirement that $<_P \cup \dashrightarrow_P$ be total, not all biposets may be used. Ésik and Németh (2004) is concerned with n-posets, that is, finite sets with n partial orders, and then introduces a notion of higher-dimensional automata as recognizers of such structures. Except for the name, the higher-dimensional automata of Ésik and Németh (2004) have nothing to do with our HDAs.

Example 32. By definition, the maximal antichains of the precedence order are linearly ordered by the event order, but the event order may contain further arrows. As an example, consider the ipomset

$P =$ 

with precedence order $a < b$ and event order $b \dashrightarrow c \dashrightarrow a$. Both maximal antichains $a \parallel c$ and $b \parallel c$ are linearly $\dashrightarrow$-ordered, but by transitivity, also $b \dashrightarrow a$.

Let $Q \subseteq P$ be a subset of the ipomset $(P, <, \dashrightarrow, \lambda, S, T)$. Then the restriction

$$P_{|Q} := (Q, < \cap (Q \times Q), \dashrightarrow \cap (Q \times Q), \lambda_{|Q}, S \cap Q, T \cap Q)$$

is also an ipomset.

Definition 33. *Ipomsets P and Q are* isomorphic *if there exists a bijection $f : P \to Q$ (an* ipomset isomorphism*) that*

- *respects precedence: for all $x, y \in P$, $x <_P y$ iff $f(x) <_Q f(y)$;*
- *respects essential event ordering: for all $x, y \in P$ with $x \parallel_P y$, $x \dashrightarrow_P y$ iff $f(x) \dashrightarrow_Q f(y)$;*
- *respects labels and interfaces: $\lambda_Q \circ f = \lambda_P$, $f(S_P) = S_Q$, and $f(T_P) = T_Q$.*

By definition, f is only required to respect the part of the event ordering which orders events in antichains. In Section 4.4, we will introduce a notion of morphism between ipomsets for which the above form the isomorphisms.

Isomorphisms between ipomsets are unique:

Lemma 34. *There is at most one isomorphism between any two ipomsets.*

Proof. Using poset filtrations, we can combine the two orders on an ipomset into a linear order and then use the fact that isomorphisms between linearly ordered sets are unique:

Let P be an ipomset and P_0 its set of $<_P$-minimal elements. Let P_1 be the set of $<_P$-minimal elements of the sub-ipomset $P \setminus P_0$, P_2 the set of $<_p$-minimal elements of $P \setminus P_0 \setminus P_1$, and so on. The finite disjoint union $P = P_0 \sqcup P_1 \sqcup P_2 \sqcup \ldots$ is called *filtration* of P. (More precisely, one can set $P_{>-1} = P$ and then inductively for $i \geq 0$, until exhaustion, let P_i be the $<_P$-minimal elements of $P_{>i-1}$ and $P_{>i} = P_{>i-1} \setminus P_i$.)

Now all P_i are $<_P$-antichains and hence linearly ordered by $\dashrightarrow_P$. Let $\prec_P$ be the relation on P defined by $x \prec_P y$ if $x \in P_i$ and $y \in P_j$ for $i < j$, or $x, y \in P_i$ for some common i and $x \dashrightarrow_P y$. Then $\prec_P$ is a linear order on P. Further, if $f : P \to Q$ is an ipomset isomorphism, then $f : (P, \prec_P) \to (Q, \prec_Q)$ is an isomorphism of linear orders; hence f is unique. □

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4.2 Gluing and parallel compositions

The *gluing composition* $P * Q$ of two ipomsets is defined if the target interface of P is isomorphic to the source interface of Q, in which case it identifies the targets of P with their corresponding sources in Q and makes all non-interface elements in P precede all non-interface elements in Q. Below, $(\,\cdot\,)^+$ is used for transitive closure.

Definition 35. *Let P and Q be ipomsets such that $(T_P, \dashrightarrow_P, \lambda_P)$ is isomorphic to $(S_Q, \dashrightarrow_Q, \lambda_Q)$. The* gluing composition *of P and Q is $P * Q = (P \sqcup (Q \setminus S_Q), <, \dashrightarrow, \lambda, S_P, T_Q)$ with $<$, $\dashrightarrow$, and λ defined as follows:*

$$< \;=\; <_P \cup <_Q \cup (P \setminus T_P) \times (Q \setminus S_Q), \qquad \lambda(x) = \begin{cases} \lambda_P(x) & \text{if } x \in P, \\ \lambda_Q(x) & \text{if } x \in Q. \end{cases}$$
$$\dashrightarrow \;=\; (\dashrightarrow_P \cup \dashrightarrow_Q)^+,$$

It is clear that $P * Q$, if defined, is indeed again an ipomset: the only non-trivial property to check is irreflexivity of $\dashrightarrow$, which follows from the fact that the restrictions of $\dashrightarrow$ to P and Q are precisely $\dashrightarrow_P$ and $\dashrightarrow_Q$, respectively. It is also clear that gluing composition respects isomorphisms:

Lemma 36. *If $P \cong P'$ and $Q \cong Q'$, then $P * Q$ is defined iff $P' * Q'$ is, and in that case, $P * Q \cong P' * Q'$.* □

Gluing composition of *pomsets*, that is, ipomsets with empty interfaces $S_P = T_P = \emptyset$, is the same as the standard *serial composition* (Gischer, 1988; Grabowski, 1981). Gluing composition of *strings* is concatenation. We also introduce a parallel composition of ipomsets that generalizes the eponymous operation for pomsets.

Definition 37. *The* parallel composition *of ipomsets P and Q is $P \parallel Q = (P \sqcup Q, <, \dashrightarrow, \lambda, S_P \sqcup S_Q, T_P \sqcup T_Q)$ with $<$, $\dashrightarrow$, and λ defined as follows:*

$$< \;=\; <_P \cup <_Q, \qquad \lambda(x) = \begin{cases} \lambda_P(x) & \text{if } x \in P, \\ \lambda_Q(x) & \text{if } x \in Q. \end{cases}$$
$$\dashrightarrow \;=\; \dashrightarrow_P \cup \dashrightarrow_Q \cup P \times Q,$$

It is easy to see that $P \parallel Q$ is again an ipomset. Parallel composition of ipomsets is not commutative because of the event order. It is again clear that parallel composition respects isomorphisms:

Lemma 38. *If $P \cong P'$ and $Q \cong Q'$, then $P \parallel Q \cong P' \parallel Q'$.* □

4.3 Interval orders

An *interval order* is a poset P in which $x < z$ and $y < w$ imply $x < w$ or $y < z$ for all $x, y, z, w \in P$.

Lemma 39. Fishburn (1970, 1985); Janicki and Koutny (1993). *The following are equivalent for any poset P:*

(1) P is an interval order;

(2) P does not contain an induced subposet $2{+}2 = \left(\begin{smallmatrix} \bullet \longrightarrow \bullet \\ \bullet \longrightarrow \bullet \end{smallmatrix}\right)$;

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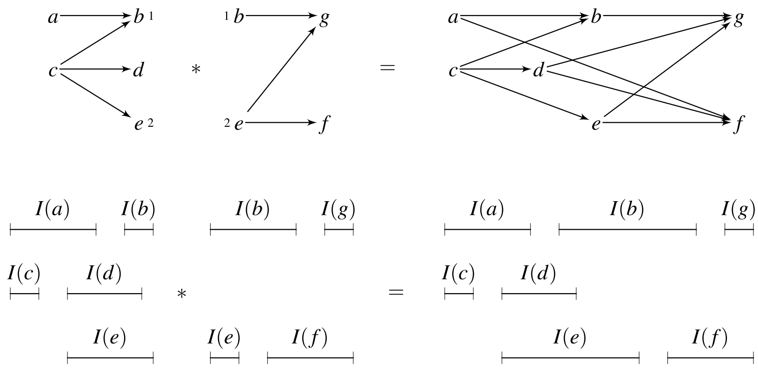

Figure 11. Two interval ipomsets and their gluing: above as ipomsets, below using interval representations (event order not shown)

(3) *P has an* interval representation*: a pair of functions $s, t : P \to Q$ into a linear poset $(Q, <_Q)$ such that for all $x, y \in P$, $s(x) <_Q t(x)$, and $x <_P y$ iff $t(x) <_Q s(y)$;*
(4) *the order $\prec$ on maximal antichains of P defined by $X \prec Y$ if $X \neq Y$ and $y \not<_P x$ for all $x \in X$, $y \in Y$ is linear.*

Definition 40. *An* interval ipomset *is an ipomset P for which the underlying precedence poset $(P, <_P)$ is an interval order.*

Restrictions of interval ipomsets are again interval. The following is shown in Fahrenberg et al. (2020) using interval representations, see Figure 11 for an example.

Lemma 41. *If P and Q are interval ipomsets and $P * Q$ exists, then $P * Q$ is an interval ipomset.* □

We develop a decomposition property for interval ipomsets which will be useful later.

Definition 42. *An ipomset P is* discrete *if $<_P$ is empty (thus, $\dashrightarrow_P$ is a linear order). In addition, P is a* starter *if $T_P = P$, and P is a* terminator *if $S_P = P$. A starter P is* elementary *if $P \setminus S_P$ is a singleton, and a terminator P is elementary if $P \setminus T_P$ is a singleton.*

Starters may be used to start events and terminators to terminate them. In compositions, they can switch off parts of starting or terminating interfaces, see Figure 12.

Every starter is a gluing of elementary starters, and every terminator a gluing of elementary terminators (both not necessarily unique). Identity ipomsets are both starters and terminators, and any discrete ipomset can be written as a gluing of a starter followed by a terminator.

We introduce special notation for discrete ipomsets: for subsets $S, T \subseteq U$ of a linear pomset $(U, \dashrightarrow, \lambda)$ we write

$$ {}_S U_T = (U, \emptyset, \dashrightarrow, \lambda, S, T). $$

The next lemma follows easily.

Lemma 43. *Let S, T, U be linear pomsets. If $S \subseteq T \subseteq U$, then ${}_S T_T * {}_T U_U \cong {}_S U_U$ and ${}_U U_T * {}_T T_S \cong {}_U U_S$. If $S, T \subseteq U$, then ${}_S U_U * {}_U U_T \cong {}_S U_T$.* □

Proposition 44. *For an ipomset P, the following are equivalent:*

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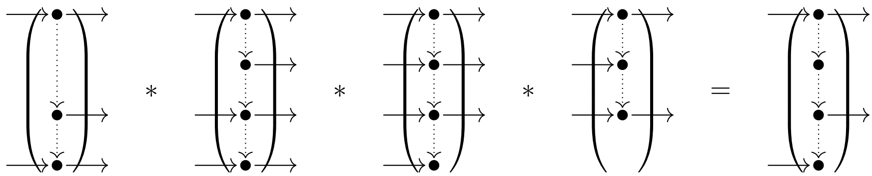

Figure 12. Decomposition of discrete ipomset into elementary starters and terminators.

(1) P is an interval ipomset;
(2) P is a finite gluing of discrete ipomsets;
(3) P is a finite gluing of elementary starters and terminators.

Proof. Equivalence of (2) and (3) is clear. Given that discrete ipomsets are interval, (2) implies (1) by Lemma 41.

To show that (1) implies (2), let P be an interval ipomset and $P_1 \prec \cdots \prec P_m$ the sequence of maximal antichains in P given by Lemma 39(4). Each P_i is linearly ordered by the restriction $\dashrightarrow_i$ of $\dashrightarrow_P$ to P_i. Let $S_1 = S_P$, $T_m = T_P$, and $T_i = S_{i+1} = P_i \cap P_{i+1}$ for $i \in [m-1]$, and define ipomsets $P_i = (P_i, \emptyset, \dashrightarrow_i, \lambda_{|P_i}, S_i, T_i)$. Then the gluing $Q = P_1 * \cdots * P_m$ is defined; we show that $Q = P$.

It is clear that the underlying sets of Q and P are equal and that the source and target interfaces agree. Further, $\dashrightarrow_Q = (\dashrightarrow_1 \cup \cdots \cup \dashrightarrow_m)^+ = \dashrightarrow_P$. To see that $<_Q = <_P$, we note that $x <_P y$ implies that $x \in P_i$ and $y \in P_j$ with $i < j$ and $x, y \notin P_i \cap P_j$, and vice versa. □

Figure 12 shows an example of a decomposition of a discrete ipomset into elementary starters and terminators. This proposition also gives an alternate proof of Lemma 41.

4.4 Subsumption

Pomsets may be *smoothened*, or made less concurrent, by strengthening precedence relations. The corresponding relation between pomsets has been introduced by Grabowski (1981) and is nowadays often called *subsumption* (Gischer, 1988). We adapt it to ipomsets.

Definition 45. *An ipomset Q* subsumes *an ipomset P if there exists a bijection $f : P \to Q$, called a* subsumption map, *such that*

- *for all $x, y \in P$, $f(x) <_Q f(y)$ implies $x <_P y$;*
- *for all $x, y \in P$ with $x \parallel_P y$, $x \dashrightarrow_P y$ implies $f(x) \dashrightarrow_Q f(y)$; and*
- *$\lambda_Q \circ f = \lambda_P$, $f(S_P) = S_Q$, and $f(T_P) = T_Q$.*

We write $P \sqsubseteq Q$ if Q subsumes P. That is, the points of P and Q are in bijection, but P may be more precedence ordered than Q, and Q may be more event ordered than P.

If P is discrete or Q is linear, then any subsumption map $f : P \to Q$ is an isomorphism and, in particular, unique. We extend gluing composition to subsumptions:

Definition 46. *Let $f : P \to P'$ and $g : Q \to Q'$ be subsumption maps and assume $P * Q$ and $P' * Q'$ to be defined. Define $h = f * g : P * Q \to P' * Q'$ by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in P, \\ g(x) & \text{if } x \in Q, \end{cases}$$

Lemma 47. *The map h from Definition 46 is well-defined and a subsumption map.*

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Proof. By Lemma 36, we may assume that $T_P = S_Q$ and $T_{P'} = S_{Q'}$, showing that h is well-defined and a bijection. The other properties follow easily. □

Lemma 48. *If $P \sqsubseteq P'$ and $Q \sqsubseteq Q'$, then $P * Q$ is defined iff $P' * Q'$ is, and in that case, $P * Q \sqsubseteq P' * Q'$.*

Proof. Let $f : P \to P'$ and $g : Q \to Q'$ be the subsumption maps. The first claim is clear as f and g respect interfaces and labels. The second claim follows from Definition 46. □

Using subsumption maps as 2-morphisms, ipomsets assemble as morphisms into a bicategory. Below, the *identity* on a linear pomset $(S, \dashrightarrow, \lambda)$ is the discrete ipomset $\mathsf{id}_S = {}_S S_S = (S, \emptyset, \dashrightarrow, \lambda, S, S)$, with trivial precedence order and all points in both interfaces.

Proposition 49. *Ipomsets form a (large) bicategory* iPoms *with objects linear pomsets $(S, \dashrightarrow, \lambda)$, ipomsets $(P, <, \dashrightarrow, \lambda, S, T)$ as morphisms from $(S, \dashrightarrow, \lambda)$ to $(T, \dashrightarrow, \lambda)$ with gluing as composition and identities* id_S*, subsumptions as 2-morphisms, and ipomset isomorphisms as 2-isomorphisms.*

Proof. It is clear that subsumption maps compose associatively and are invertible precisely when they are ipomset isomorphisms. Gluing composition is associative up-to 2-isomorphism, and the ipomsets id_S are on-the-nose units for $*$. The pentagon identity is trivially satisfied due to uniqueness of 2-isomorphisms. □

Remark 50. The bicategory iPoms is large as its objects and morphisms form a proper class. However, given that any ipomset is uniquely isomorphic to one on points $[k]$ and with interfaces $[n]$ and $[m]$, for some $k, n, m \geq 0$, iPoms is equivalent to its skeleton, which is a small 2-category; hence iPoms is *essentially small*.

By Lemma 41, interval ipomsets form a sub-bicategory of iPoms which we will denote iiPoms.

5. Tracks and Their Labels

We are now ready to introduce *tracks* in precubical sets, which are our model of computations, that is, sequences of cells connected at faces. We define labels of tracks as interval ipomsets and show, conversely, how interval ipomsets give rise to tracks.

5.1 Tracks

Let X be an event consistent precubical set. For $x, y \in X$, we say that x is an *elementary lower face* of y, denoted $x \lhd y$, if $x = \delta_i^0 y$ for some i; x is an *elementary upper face* of y, denoted $y \rhd x$, if $x = \delta_i^1 y$. The reflexive transitive closures of the relations $\lhd$ and $\rhd$ are denoted $\lhd^*$ and $\rhd^*$.

We say that x is a *lower*, resp. *upper face* of y if $x \lhd^* y$, resp. $y \rhd^* x$. This is equivalent to the condition that $x = \delta_A^{0,\dots,0} y$ for some (possibly empty) A, resp. $x = \delta_A^{1,\dots,1} y$. By Lemma 22, A is determined uniquely by x and y.

Definition 51. *A* track *in X is a non-empty sequence $\rho = (x_1, \dots, x_m)$, $m \geq 1$, of elements of X such that for all $i = 1, \dots, m-1$, $x_i \lhd^* x_{i+1}$ or $x_{i+1} \rhd^* x_i$. A track ρ as above is* from x_1 to x_m*, denoted $\rho : x_1 \rightsquigarrow x_m$.*

We allow repeated cells $x_i = x_{i+1}$ in tracks for notational convenience. A track $(x_1, \dots, x_m)$ is *full* if it does not contain such repeated cells and all face relations are elementary, that is, $x_i \lhd x_{i+1}$

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or $x_i \rhd x_{i+1}$ for all $i = 1, \dots, m-1$. Any track without repeated cells may be *filled* to a full track by inserting appropriate (not necessarily unique) cells. In van Glabbeek (2006a), full tracks are called *execution paths*.

Example 52. Figure 8 in the introduction displays the track $(x_1, x_2, x_3, x_4, x_5, x_6)$. As $x_3 \lhd^3 x_4$, this track is not full; it may be filled by inserting appropriate faces of x_4 and x_6, for example

$$(x_1, x_2, x_3, \delta_1^0\delta_1^0 x_4, \delta_1^0 x_4, x_4, \delta_1^1 x_4, x_5, x_6).$$

Definition 53. *Let $\rho = (x_1, \dots, x_m)$, $\tau = (y_1, \dots, y_k)$ be tracks in X. The* concatenation *$\rho * \tau$ of ρ and τ is defined if $x_m = y_1$, and in that case, $\rho * \tau = (x_1, \dots, x_m, y_2, \dots, y_k)$.*

The *unit* tracks are (x) for $x \in X$. Tracks containing exactly two cells are called *basic*. Concatenation is associative, and every non-unit track is a unique concatenation of basic tracks. The following is clear.

Lemma 54. *Tracks in X form a small category $\mathsf{Track}(X)$ with objects $x \in X$, tracks $\rho : x \rightsquigarrow y$ as morphisms, $*$ as composition, and identities $\mathsf{id}_x = (x)$.* □

5.2 Labels of tracks

Let (X, λ) be a labeled precubical set.

Definition 55. *The* label *of a track ρ in X is the ipomset $\ell(\rho)$ defined recursively as follows:*

- *If $\rho = (x)$ is a unit track, then $\ell(\rho) = \mathsf{id}_{\ell(x)}$: the identity ipomset on $\ell(x)$.*
- *If $\rho = (x, y)$ with $x \lhd^* y$, then $\ell(\rho) = {}_{\ell(x)}\ell(y)_{\ell(y)}$: a starter.*
- *If $\rho = (y, x)$ with $y \rhd^* x$, then $\ell(\rho) = {}_{\ell(y)}\ell(y)_{\ell(x)}$: a terminator.*
- *If $\rho = \tau * \rho'$ with τ a basic track, then $\ell(\rho) = \ell(\tau) * \ell(\rho')$.*

By Proposition 44, labels of tracks are interval ipomsets, and $\ell(\rho_1 * \rho_2) \cong \ell(\rho_1) * \ell(\rho_2)$ for all tracks ρ_1, ρ_2. The following is therefore clear.

Proposition 56. *Labeling defines a functor $\ell : \mathsf{Track}(X) \to \mathsf{iiPoms}$.*

Next we see that filling a track with extra cells does not change its label. Let $\sim$ be the equivalence on sets of tracks generated by $(x_1, \dots, x_m) \sim (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m)$ for $x_{i-1} \lhd^* x_i \lhd^* x_{i+1}$ or $x_{i-1} \rhd^* x_i \rhd^* x_{i+1}$.

Lemma 57. *If $\rho \sim \tau$, then $\ell(\rho) \cong \ell(\tau)$.*

Proof. If $x \lhd^* y \lhd^* z$, then

$$\ell((x, y, z)) = \ell((x, y)) * \ell((y, z)) = {}_{\ell(x)}\ell(y)_{\ell(y)} * {}_{\ell(y)}\ell(z)_{\ell(z)} = {}_{\ell(x)}\ell(z)_{\ell(z)} = \ell((x, z))$$

from Lemma 43. The computations for $x \rhd^* y \rhd^* z$ are similar, and the result then follows by induction. □

The next lemma shows that labels of tracks that consist of a cell and two of its faces on either side are discrete ipomsets. Its proof is a straightforward application of the definition.

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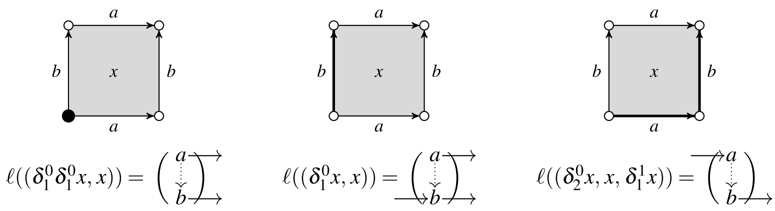


Figure 13. Some tracks and their labels in $X = \square^{\{a \dashrightarrow b\}}$ (solid cells indicate the respective faces).

Lemma 58. *If $x \lhd^* y \rhd^* z$, then $\ell((x, y, z)) = {}_{\ell(x)}\ell(y)_{\ell(z)}$.* □

Figure 13 shows some examples of simple tracks and their labels.

5.3 Interval ipomsets as tracks

We have seen how labels of tracks in HDAs can be computed as interval ipomsets. Now we show the inverse: how interval ipomsets may be converted into HDAs consisting essentially of a single track. To this end, first introduce a relation $\prec$ on the set $\{0, \text{⨼}, 1\}$ by

$$\prec \,= \{(0, 0), (\text{⨼}, 0), (1, 0), (1, \text{⨼}), (1, 1)\}.$$

The relation $\prec$ (which is *not* a partial order, given that it is neither reflexive nor irreflexive) corresponds to the meaning that we associate to the elements of $\{0, \text{⨼}, 1\}$: 0 meaning the event has not yet started; ⨼ for an executing event; and 1 if the event has terminated. The intuition is that when x and y are events so that $x < y$ in the precedence order, then either y has not yet started, in which case x may be in any state, hence the first three pairs $(0, 0)$, $(\text{⨼}, 0)$, $(1, 0)$; or x has terminated and y may be in any state, hence the last three pairs $(1, 0)$, $(1, \text{⨼})$, $(1, 1)$. In particular, it is impossible that both are active, so that $(\text{⨼}, \text{⨼}) \notin \prec$.

Remark 59. In Chu spaces for concurrency Pratt (1995), $K_3 = \{0 <_C \text{⨼} <_C 1\}$ is the structure that defines the possible execution forms that an event can take. The intuition of the order $<_C$ is that 0 (the event has not yet started) can happen only before ⨼ (the event is executing), which can happen only before 1 (the event has terminated). This order can be extended to sets of (execution values of) events, which in Chu terminology is called a state. For states, the order $<_C$ expresses admissible sequences of executions of the system. For two independent events x, y, all execution forms (e_x, e_y) with $e_x, e_y \in \{0, \text{⨼}, 1\}$ would thus be admissible; but if there is a precedence order $x < y$, then the allowed tuples are precisely the ones in the relation $\prec$ above.

The following generalizes Definition 11 of the standard S-cube $\square^S$ on a linear poset S to arbitrary ipomsets.

Definition 60. *For an ipomset P, define the HDA $(\square^P, I_{\square^P}, F_{\square^P}, \lambda_{\square^P})$ as follows:*

- *$\square^P_k$ is the set of all relation-preserving functions $x : (P, <_P) \to (\{0, \text{⨼}, 1\}, \prec)$ taking value ⨼ on exactly k elements;*
- *for $x \in \square^P_k$ and $x^{-1}(\text{⨼}) = \{p_1 \dashrightarrow_P \cdots \dashrightarrow_P p_k\}$,*

$$\delta_i^\nu(x)(p) = \begin{cases} \nu & \text{for } p = p_i, \\ x(p) & \text{for } p \neq p_i; \end{cases}$$

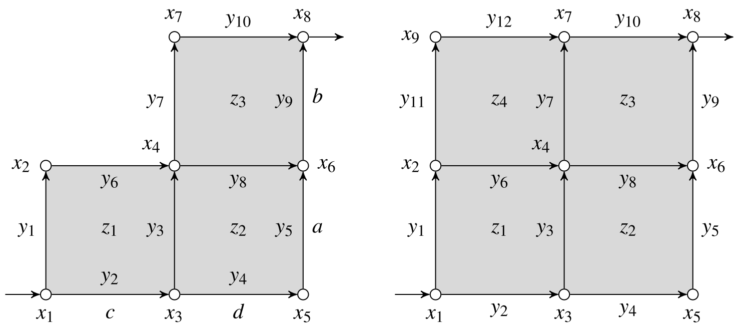


Figure 14. The HDAs $\square^P$ (left) and $\square^Q$ (right) from Examples 61 and 62.

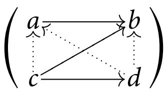
- *$I_{\square^P} = \{i_{\square^P}\}$ and $F_{\square^P} = \{f_{\square^P}\}$ are given by*

$$i_{\square^P}(p) = \begin{cases} \lrcorner & \text{if } p \in S_P, \\ 0 & \text{if } p \notin S_P, \end{cases} \qquad f_{\square^P}(p) = \begin{cases} \lrcorner & \text{if } p \in T_P, \\ 1 & \text{if } p \notin T_P; \end{cases}$$

- *For $x \in \square^P_k$ with $x^{-1}(\lrcorner) = \{p_1 \dashrightarrow_P \cdots \dashrightarrow_P p_k\}$, $\lambda_{\square^P}(x) = (\lambda_P(p_1), \ldots, \lambda_P(p_k))$.*

Above, $x^{-1}(\lrcorner)$ is indeed an antichain in the precedence order and hence linearly ordered by $\dashrightarrow_P$.

Example 61. Let P be the ipomset



The cells of $\square^P$ are as follows, in the increasing order of dimension (and with the event order omitted):

$$x_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad x_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad x_4 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$x_5 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad x_6 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad x_7 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad x_8 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$y_1 = \begin{pmatrix} \lrcorner & 0 \\ 0 & 0 \end{pmatrix} \quad y_2 = \begin{pmatrix} 0 & 0 \\ \lrcorner & 0 \end{pmatrix} \quad y_3 = \begin{pmatrix} \lrcorner & 0 \\ 1 & 0 \end{pmatrix} \quad y_4 = \begin{pmatrix} 0 & 0 \\ 1 & \lrcorner \end{pmatrix}$$

$$y_5 = \begin{pmatrix} \lrcorner & 0 \\ 1 & 1 \end{pmatrix} \quad y_6 = \begin{pmatrix} 1 & 0 \\ \lrcorner & 0 \end{pmatrix} \quad y_7 = \begin{pmatrix} 1 & \lrcorner \\ 1 & 0 \end{pmatrix} \quad y_8 = \begin{pmatrix} 1 & 0 \\ 1 & \lrcorner \end{pmatrix}$$

$$y_9 = \begin{pmatrix} 1 & \lrcorner \\ 1 & 1 \end{pmatrix} \quad y_{10} = \begin{pmatrix} 1 & 1 \\ 1 & \lrcorner \end{pmatrix}$$

$$z_1 = \begin{pmatrix} \lrcorner & 0 \\ \lrcorner & 0 \end{pmatrix} \quad z_2 = \begin{pmatrix} \lrcorner & 0 \\ 1 & \lrcorner \end{pmatrix} \quad z_3 = \begin{pmatrix} 1 & \lrcorner \\ 1 & \lrcorner \end{pmatrix}$$

The cells y_1, y_3, and y_5 are labeled by a, y_7 and y_9 by b, y_2 and y_6 by c, y_4, y_8, y_{10} by d. The labels of z_1, z_2, and z_3 are (c, a), (d, a), and (d, b), respectively. The order of letters in these pairs is determined by the event order on P. Geometrically, these are arranged as shown in Figure 14 (left).

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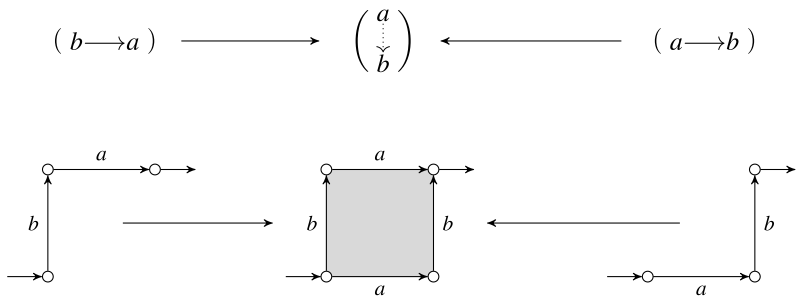
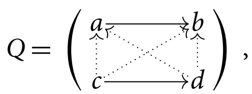


Figure 15. Subsumptions (top) give rise to HDA inclusions (bottom)

We will later apply Definition 60 to *interval* ipomsets to conclude in Proposition 92 that the language of $\square^P$ is generated by P. Our definition applies to general ipomsets, but as we will see, Proposition 92 fails for ipomsets which are not interval. It is an interesting open problem to characterize those HDA which are isomorphic to some $\square^P$.

Example 62. If we instead of the ipomset P of Example 61 take Q to be the 2+2-ipomset



then $\square^Q$ contains $\square^P$ and the following extra cells:

$$x_9 = \begin{pmatrix} 1 \longrightarrow 1 \\ 0 \longrightarrow 0 \end{pmatrix} \qquad y_{11} = \begin{pmatrix} 1 \longrightarrow \ulcorner \\ 0 \longrightarrow 0 \end{pmatrix} \qquad y_{12} = \begin{pmatrix} 1 \longrightarrow 1 \\ \ulcorner \longrightarrow 0 \end{pmatrix} \qquad z_4 = \begin{pmatrix} 1 \longrightarrow \ulcorner \\ \ulcorner \longrightarrow 0 \end{pmatrix}$$

Geometrically, this amounts to adding the top-left square to $\square^P$, see Figure 14 (right).

The following can be shown by easy calculations; Figure 15 shows some simple examples.

Lemma 63. *If $f : P \to Q$ is a subsumption map, then the function $\square^f : \square^P \to \square^Q$ given by $\square^f(x)(p) = x(f(p))$ is an injective HDA map.* □

For the next lemma, recall the notions of event consistency and universal events from Section 3.3.

Lemma 64. *Let P be an ipomset. Then $\square^P$ is event consistent, and $E_{\square^P} \cong P$ as labeled sets. For every $x \in \square^P$, $\mathsf{ev}(x) = x^{-1}(\ulcorner)$ as linear posets.*

Proof. To show that $\square^P$ is event consistent, let $f : P \to Q$ be a subsumption map into a discrete ipomset Q; Q may be obtained from any linearization of $\dashrightarrow_P$. Then the precubical set underlying $\square^Q$ is a standard cube, and by Lemma 63, $\square^f : \square^P \to \square^Q$ is an embedding. By Example 19, $\square^P$ is event consistent.

There is a P-labeling of $\square^P$, that is, a precubical map $\square^P \to {!P}$ that sends $x \in \square^P$ to $x^{-1}(\ulcorner) = (p_1 \dashrightarrow_P \cdots \dashrightarrow_P p_n)$. This induces a function $\pi : E_{\square^P} \to P$ (Proposition 20). For every $p \in P$, define $x_p \in \square_1^P$ by

$$x_p(q) = \begin{cases} 1 & \text{if } q <_P p, \\ \ulcorner & \text{if } q = p, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\pi(x_p) = p$, π is surjective.

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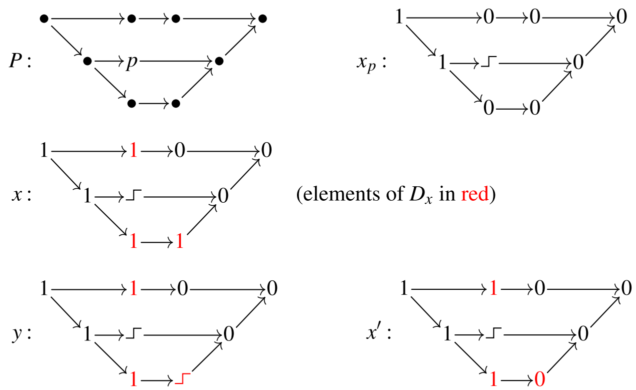


Figure 16. Pomsets and cells in the proof of Lemma 64.

It remains to show that π is injective. Let $x \in \square_1^P$ satisfy $\pi(x) = p$; we will show that $x \sim_{\mathrm{ev}} x_p$. Note that $x(q) = \lrcorner$ iff $q = p$. We proceed by induction on the number of elements in the set $D_x = \{q \in P \mid x(q) \neq x_p(q)\}$, see Figure 16 for an illustration.

All elements of D_x are parallel with p, since monotonicity of x and x_p implies that $x(q) = x_p(q) = 1$ if $q <_P p$ and $x(q) = x_p(q) = 0$ if $p <_P q$. Thus, $x_p(q) = 0$ and $x(q) = 1$ for all $q \in D_x$.

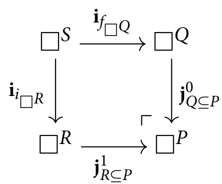
Let $q \in D_x$ be a $<_P$-maximal element. Let $y : P \to \{0, \lrcorner, 1\}$ be given by $y(q) = \lrcorner$ and $y(r) = x(r)$ for $r \neq q$. We show that y is monotone and hence a 2-cell in $\square^P$.

We have $y(r) = x(r) = 1$, hence $y(r) \prec y(q)$, for all $r <_P q$, since x preserves $<_P$. For $q <_P r$, on the other hand, maximality of q in D_x implies that $r \notin D_x$, so that $y(r) = x(r) = x_p(r) = 0$, hence $y(q) \prec y(r)$. Given that $y(r) = x(r)$ for $r \neq q$, we have shown that y is monotone.

Now $y(p) = y(q) = \lrcorner$, so y is a 2-cell in $\square^P$. Let $i = 1$ if $q \dashrightarrow p$ and $i = 2$ if $p \dashrightarrow q$, then $\delta_i^1(y) = x$. Let $x' = \delta_i^0(y)$, then also $\pi(x') = p$, and $D_{x'} = D_x \setminus \{q\}$. The inductive hypothesis asserts that $x = \delta_i^1(y) \sim_{\mathrm{ev}} \delta_i^0(y) = x' \sim_{\mathrm{ev}} x_p$. $\square$

Next we see that gluings of ipomsets correspond to pushouts of their HDA objects. Recall the Yoneda inclusions $\mathbf{i}_x$ from Lemma 12.

Lemma 65. *Let Q and R be composable ipomsets with $T_Q \cong S \cong S_R$ and $P = Q * R$. There is a pushout*



where

$$\mathbf{j}^0_{Q\subseteq P}(x)(p) = \begin{cases} x(p) & \text{for } p \in Q, \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbf{j}^1_{R\subseteq P}(x)(p) = \begin{cases} x(p) & \text{for } p \in R, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. It is clear that all the maps in the diagram are injective. Fix $x \in \square^P$.

- If there exists $q \in Q \setminus T_Q$ with $x(q) \in \{0, \lrcorner\}$, then obviously $x \notin \mathbf{j}^1_{R\subseteq P}(\square^R)$. But for all $r \in R \setminus S_R$ we have $q < r$ and then $x(r) = 0$. It is easy to verify that the restriction $x_{|Q} \in \square^Q$ and then $x = \mathbf{j}^0_{Q\subseteq P}(x_{|Q}) \in \mathbf{j}^0_{Q\subseteq P}(\square^Q)$.

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- Similarly, if $x(r) \in \{\lrcorner, 1\}$ for some $r \in R \setminus S_R$, then $x \in \mathbf{j}^1_{R\subseteq P}(\square^R)$.
- We have

$$\mathbf{j}^0_{Q\subseteq P} \circ \mathbf{i}_{f_{\square Q}}(y)(p) = \mathbf{j}^1_{R\subseteq P} \circ \mathbf{i}_{i_{\square R}}(y)(p) = \begin{cases} 1 & \text{for } p \in Q \setminus T_Q, \\ \lrcorner & \text{for } p \in S \cong T_Q \cong S_R, \\ 0 & \text{for } p \in R \setminus S_R. \end{cases}$$

Thus, the diagram commutes. Denote $j = \mathbf{j}^0_{Q\subseteq P} \circ \mathbf{i}_{f_{\square Q}} = \mathbf{j}^1_{R\subseteq P} \circ \mathbf{i}_{i_{\square R}}$. The condition $x(q) = 1$ for all $q \in Q \setminus T_Q$ and $x(r) = 0$ for all $r \in R \setminus S_R$ is equivalent to both $x \in \mathbf{j}^0_{Q\subseteq P}(\square^Q) \cup \mathbf{j}^1_{R\subseteq P}(\square^R)$ and $x \in j(\square^P)$.

As a consequence, $\square^P = \mathbf{j}^0_{Q\subseteq P}(\square^Q) \cup \mathbf{j}^1_{R\subseteq P}(\square^R)$ and $j(\square^S) = \mathbf{j}^0_{Q\subseteq P}(\square^Q) \cap \mathbf{j}^1_{R\subseteq P}(\square^R)$. $\square$

Lemma 66. *Let X be a labeled precubical set, $\rho : x \rightsquigarrow y \in \mathsf{Track}(X)$, and $P = \ell(\rho)$. There is a map of labeled precubical sets $g : \square^P \to X$ such that $g(i_{\square^P}) = x$ and $g(f_{\square^P}) = y$.*

Proof. Induction on the number of cells in ρ.

- If $\rho = (x)$, then $P = \mathrm{id}_{\ell(x)}$ and $\square^P = \square^{\mathrm{ev}(x)}$. The Yoneda map $\mathbf{i}_x : \square^{\mathrm{ev}(x)} \to X$ satisfies the required condition.
- If $\rho = (x, y)$ with $x \lhd^* y$, then $P = {}_{\ell(x)}\ell(y)_{\ell(y)}$. Again, we may take $g = \mathbf{i}_y$.
- The case $\rho = (y, x)$ with $y \rhd^* x$ is similar.
- In case $\rho = \sigma * \tau$, where both $\sigma : x \rightsquigarrow z$ and $\tau : z \rightsquigarrow y$ are shorter than ρ, let $Q = \ell(\sigma)$, $R = \ell(\tau)$. By the inductive hypothesis, there are labeled precubical maps $g_Q : \square^Q \to X$ and $g_R : \square^R \to X$ such that $g_Q(i_{\square^Q}) = x$, $g_R(f_{\square^R}) = y$, and $g_Q(f_{\square^Q}) = g_R(i_{\square^R}) = z$. The last equality, together with Lemma 65, guarantees that g_Q and g_R glue to a map $g : \square^P \to X$. It is clear that $g(i_{\square^P}) = g_Q(i_{\square^Q}) = x$ and $g(f_{\square^P}) = g_R(f_{\square^R}) = y$. $\square$

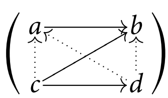
Proposition 67. *For any interval ipomset P, there exists a track $\rho : i_{\square^P} \rightsquigarrow f_{\square^P}$ such that $\ell(\rho) \cong P$.*

Proof. If $P = {}_SU_T$ is discrete, then $\ell((i_{\square^P}, \mathbf{y}_U, f_{\square^P})) \cong P$. If P is not discrete, then there is a presentation $P = Q * R$ (Proposition 44). If $\sigma : i_{\square^Q} \rightsquigarrow f_{\square^Q}$ is a track with $\ell(\sigma) \cong Q$ and $\tau : i_{\square^R} \rightsquigarrow f_{\square^R}$ a track with $\ell(\tau) \cong R$, then

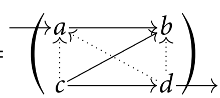
$$\ell(\mathbf{j}^0_{Q\subseteq P}(\sigma) * \mathbf{j}^1_{R\subseteq P}(\tau)) \cong Q * R = P$$

by Lemma 65. $\square$

Example 68. We follow up on Example 61. For

$P =$  ,

we have $i_{\square^P} = x_1$ and $f_{\square^P} = x_8$ (see also Figure 14), and the track ρ of the proposition is given by $\rho = (x_1, z_1, y_3, z_2, y_8, z_3, x_8)$. If we add interfaces to P, for example

$Q =$  ,

then $i_{\square^Q} = y_1, f_{\square^Q} = y_{10}$, and $\rho = (y_1, z_1, y_3, z_2, y_8, z_3, y_{10})$.

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6. The Geometric View

Precubical sets may be realized as directed topological spaces, and then directed paths through these spaces give an intuitive model of computations. In this section, we first recap the geometric realization and then introduce labels of directed paths in HDAs. We will see that for every directed path there exists a track with the same label, and vice versa, so that HDA languages defined using tracks and using directed paths are the same.

6.1 Geometric realization

Recall that the *concatenation* $\alpha * \beta$ of two paths $\alpha, \beta : I = [0, 1] \to \mathscr{X}$ in a topological space $\mathscr{X}$ is defined, if $\alpha(1) = \beta(0)$, as

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & \text{for } t \leq \frac{1}{2}, \\ \beta(2t-1) & \text{for } t \geq \frac{1}{2}. \end{cases}$$

A *directed topological space*, or *d-space* (Grandis, 2009) is a pair $(\mathscr{X}, \vec{P}\mathscr{X})$ consisting of a topological space $\mathscr{X}$ and a set $\vec{P}\mathscr{X} \subseteq \mathscr{X}^I$ of paths in $\mathscr{X}$ such that $\vec{P}\mathscr{X}$

- contains all constant paths;
- is closed under concatenation: if $\alpha, \beta \in \vec{P}\mathscr{X}$ and $\alpha(1) = \beta(0)$, then $\alpha * \beta \in \vec{P}\mathscr{X}$;
- is closed under reparametrization and subpath: for any $\alpha \in \vec{P}\mathscr{X}$ and $h : I \to I$ continuous and (weakly) increasing, also $\alpha \circ h \in \vec{P}\mathscr{X}$.

The elements of $\vec{P}\mathscr{X}$ are called directed paths or *d-paths*.

Prominent examples of d-spaces are the directed interval $\vec{I} = [0, 1]$ with the natural ordering on the real numbers and the directed n-cubes $\vec{I}^n$ for $n \geq 0$. Similarly, there are directed Euclidean spaces $\vec{\mathbb{R}}^n$ for all $n \geq 0$. In each of these, the d-paths a precisely the paths which are (weakly) increasing in each coordinate, that is, $\alpha : I \to \vec{\mathbb{R}}^n$ is a d-path iff $t_1 \leq t_2$ implies $\alpha(t_1) \leq \alpha(t_2)$ in the usual ordering $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ iff $x_i \leq y_i$ for all i.

Morphisms $f : (\mathscr{X}, \vec{P}\mathscr{X}) \to (\mathscr{Y}, \vec{P}\mathscr{Y})$ of d-spaces are *d-maps*; they are those continuous functions that also preserve directedness, that is, $f \circ \alpha \in \vec{P}\mathscr{Y}$ for all $\alpha \in \vec{P}\mathscr{X}$. For any d-space $(\mathscr{X}, \vec{P}\mathscr{X})$, we have $\vec{P}\mathscr{X} = \mathscr{X}^{\vec{I}}$ as function spaces.

The so-defined category dTop of d-spaces is complete and cocomplete (Grandis, 2009). In particular, quotients of d-spaces are well-defined. If $\mathscr{X}$ is a d-space and $\sim$ an equivalence on $\mathscr{X}$, then d-paths in the quotient space $\mathscr{X}/\sim$ are of the form

$$\alpha = (\pi(\beta_1) * \cdots * \pi(\beta_m)) \circ h$$

where all β_i are d-paths in $\mathscr{X}$ such that $\beta_i(1) \sim \beta_{i+1}(0)$ and $h : \vec{I} \to \vec{I}$ is a surjective d-map.

Surjective d-maps $h : \vec{I} \to \vec{I}$ as above are called *reparametrizations* and will play a central role below.

Definition 69. *The* geometric realization *of a precubical set X is the d-space*

$$|X| = \bigsqcup_{n \geq 0} X_n \times \vec{I}^n / \sim,$$

where the equivalence relation $\sim$ is generated by

$$(\delta_i^\nu x, (t_1, \ldots, t_{n-1})) \sim (x, (t_1, \ldots, t_{i-1}, \nu, t_{i+1}, \ldots, t_{n-1})).$$

The geometric realization of a precubical map $f : X \to Y$ is the d-map $|f| : |X| \to |Y|$ given by $|f|([x, (t_1, \ldots, t_n)]) = [f(x), (t_1, \ldots, t_n)]$.

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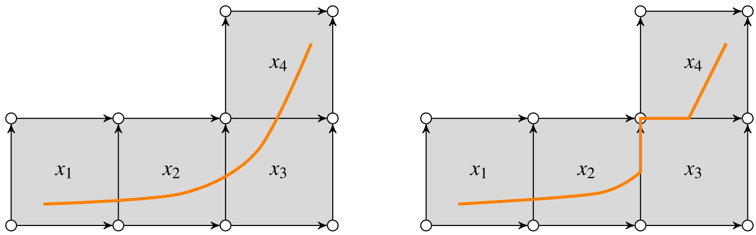

Figure 17. Left: d-path α with presentation $((|\mathbf{i}_{x_1}|\circ\beta_1)*(|\mathbf{i}_{x_2}|\circ\beta_2)*(|\mathbf{i}_{x_3}|\circ\beta_3)*(|\mathbf{i}_{x_4}|\circ\beta_4))\circ h$ (Lemma 73); right: counterexample in the proof of Lemma 73, with presentation $((|\mathbf{i}_{x_1}|\circ\beta_1)*(|\mathbf{i}_{x_2}|\circ\beta_2)*(|\mathbf{i}_{\delta_1^0 x_3}|\circ\gamma_1)*(|\mathbf{i}_{\delta_1^1 x_3}|\circ\gamma_2)*(|\mathbf{i}_{x_4}|\circ\beta_4))\circ h''$.

Above, $[x,(t_1,\dots,t_n)]$ is used to denote equivalence classes of $\sim$. Geometric realization is a functor from $\mathsf{Set}^{\square^{\mathrm{op}}}$ to dTop.

Example 70. The geometric realization of the n-cube $\square^n$ is the directed cube $\vec{I}^n$. The purpose of the equivalence relation $\sim$ in the definition is to embed faces as subspaces, for example, the elementary face $\delta_1^0\mathbf{y}_n$ of the top cell of $\square^n$ is the subset $\{(0,t_2,\dots,t_n)\mid 0\le t_i\le 1\}\subseteq I^n$.

The *interior image* $]x[\ \subseteq |X|$ of a cell $x\in X_n$ in a precubical set X is defined as

$$]x[\ =\{[x,(t_1,\dots,t_n)]\mid 0<t_i<1 \text{ for all } i\in[n]\}.$$

The set $]x[$ is open for $x\notin X_0$; for $x\in X_0$, $]x[\ =\{x\}$.

Definition 71. *The* carrier $\mathsf{carr}(p)$ *of a point* $p\in|X|$ *is the unique cell* $x\in X$ *such that* $p\in\ ]x[$.

For later use we record the following lemma, whose proof easily follows from the definition; see also Fahrenberg (2005b):

Lemma 72. *For a precubical map* $f:X\to Y$ *and* $p\in|X|$, $\mathsf{carr}(|f|(p))=f(\mathsf{carr}(p))$. □

We conclude with a description of d-paths on $|X|$. Recall the Yoneda inclusions $\mathbf{i}_x:\square^n\to X$ from Lemma 12. These induce d-maps $|\mathbf{i}_x|:\vec{I}^n\to|X|$.

Lemma 73. *Every d-path* $\alpha\in\vec{P}|X|$ *has a presentation*

$$\alpha=\big((|\mathbf{i}_{x_1}|\circ\beta_1)*(|\mathbf{i}_{x_2}|\circ\beta_2)*\cdots*(|\mathbf{i}_{x_m}|\circ\beta_m)\big)\circ h, \qquad (*)$$

where $x_1,\dots,x_m\in X$, $\beta_i\in\vec{P}(\vec{I}^{\dim x_i})$, $[x_i,\beta_i(1)]=[x_{i+1},\beta_{i+1}(0)]\in|X|$, *and* $h:\vec{I}\to\vec{I}$ *is a reparametrization. Moreover, we can assume that* $\mathsf{carr}(\beta_i(\frac{1}{2}))=\mathbf{y}_{\dim(x_i)}$.

Figure 17 shows an example: on the left, a d-path and a presentation; on the right, the counterexample used below in the proof.

Proof. Apart from the last statement, this follows immediately from the description of d-paths on quotient d-spaces and the definition of the geometric realization.

Let $\mathfrak{S}$ be the set of sequences $(d_0,d_1,\dots)$ of natural numbers which are eventually vanishing, that is, there exists $n\ge 0$ such that $d_j=0$ for all $j>n$. Equip $\mathfrak{S}$ with the reverse lexicographic order, that is, $(d_j)<(d'_j)$ if there exists n such that $d_n<d'_n$ and $d_j=d'_j$ for $j>n$. For every presentation ($*$) of α, we associate the sequence $(d_j)\in\mathfrak{S}$ such that d_j is the number of indices i such that $\dim(x_i)=j$. Choose a presentation ($*$) with a minimal associated sequence $(d_j)\in\mathfrak{S}$. Denote $n_i=\dim x_i$.

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Assume that for some i, $\beta_i(t) \notin]\mathbf{y}_{n_i}[$ for all t. But then $\beta_i \in \vec{P}|\square^{n_i}_{n_i-1}|$, the set of d-paths in the (n_i-1)-restriction of $\square^{n_i}$, and hence it has a presentation

$$\beta_i = \big((|\mathbf{i}_{y_1}| \circ \gamma_1) * \cdots * (|\mathbf{i}_{y_l}| \circ \gamma_l)\big) \circ h'.$$

Obviously dim $(y_k) < n_i$ for all k. Collecting these two presentations, we have

$$\begin{aligned}\alpha &= \big((|\mathbf{i}_{x_1}| \circ \beta_1) * \cdots * (|\mathbf{i}_{x_{i-1}}| \circ \beta_{i-1}) * (|\mathbf{i}_{x_i}| \circ ((|\mathbf{i}_{y_1}| \circ \gamma_1) * \cdots \\ &\qquad \cdots * (|\mathbf{i}_{y_l}| \circ \gamma_l)) \circ h') * (|\mathbf{i}_{x_{i+1}}| \circ \beta_{i+1}) * \cdots * (|\mathbf{i}_{x_m}| \circ \beta_m)\big) \circ h \\ &= \big((|\mathbf{i}_{x_1}| \circ \beta_1) * \cdots * (|\mathbf{i}_{x_{i-1}}| \circ \beta_{i-1}) * (|\mathbf{i}_{\mathbf{i}_{x_i}(y_1)}| \circ \gamma_1) * \cdots \\ &\qquad \cdots * (|\mathbf{i}_{\mathbf{i}_{x_i}(y_l)}| \circ \gamma_l) * (|\mathbf{i}_{x_{i+1}}| \circ \beta_{i+1}) * \cdots * (|\mathbf{i}_{x_m}| \circ \beta_m)\big) \circ h''\end{aligned}$$

for some reparametrization h'' obtained from h and h'. Let (d'_j) be the associated sequence of this presentation. Then $d'_j = d_j$ for $j > n_i$ and $d'_{n_i} = d_{n_i} - 1$, since x_i no longer appears in the presentation and cells $\mathbf{i}_{x_i}(y_k)$ have smaller dimensions: a contradiction to the minimality of (d_j).

As a consequence, for every i there exists t_i with $\mathsf{carr}(\beta_i(t_i)) = \mathbf{y}_{n_i}$. By reparametrizing β_i and adjusting h'' we can ensure that $t_i = \frac{1}{2}$. □

6.2 Labels of d-paths

For the rest of this section, (X, λ) is a labeled precubical set (which is, by definition, event consistent). We will associate to every d-path α in $|X|$ its *label* $\ell(\alpha)$ as an interval ipomset. In order to do so, we first need to find the (universal) events in X that are active during the execution α.

We say that an event $e \in E_X$ is *active* at point $p = [x, (t_1, \ldots, t_n)] \in |X|$, for $x \in X_n$, if there is $i \in [n]$ such that $\mathsf{ev}_i(x) = e$ and $0 < t_i < 1$. Otherwise, e is *inactive* at p. It is easy to verify that this does not depend on the choice of a presentation of p. Let $U^X_e \subseteq |X|$ be the set of points in which e is active. The following is clear.

Lemma 74. $U^X_e = \bigcup\{]x[\mid e \in \mathsf{ev}(x)\} = \{p \in |X| \mid e \in \mathsf{ev}(\mathsf{carr}(p))\}$. □

Note that all events are inactive at vertices, exactly one event is active along an edge, and so on: if dim $(\mathsf{carr}(p)) = n$, then exactly the n events in $\mathsf{ev}(\mathsf{carr}(p))$ are active at p. We will write U_e for U^X_e when X is clear.

Now fix a d-path $\alpha \in \vec{P}|X|$. For every event $e \in E_X$, let

$$J^\alpha_e = \alpha^{-1}(U_e) \subseteq [0, 1]$$

be the set of points in time when e is active. J^α_e is an open subset of $[0, 1]$, since U_e is open. Moreover, by Lemma 73 it has a finite number of connected components. Thus, there is a unique presentation

$$J^\alpha_e = I^\alpha_{e,1} \cup \cdots \cup I^\alpha_{e,n^\alpha_e} \tag{3}$$

as a union of connected components ordered increasingly. Each of these components is open in $[0, 1]$, though not necessarily in $\mathbb{R}$: possibly $I^\alpha_{e,1} = [0, t[$ or $T^\alpha_{e,n^\alpha_e} =]t, 1]$, or even $I^\alpha_{e,1} = [0, 1]$ for $n^\alpha_e = 1$. The collection of presentations (3) is called the *interval arrangement* of α.

Example 75. Figure 18 shows a d-path α through a labeled precubical set with a two-dimensional loop: α starts inside the bottom-left square with events a and c, continues until the upper face of the top-right square, which is identified with the lower face of the bottom-left square, and finishes in the right c-labeled edge. Assuming that α is parametrized so that $\alpha(\frac{i}{6}) = p_i$ for $i \in [5]$

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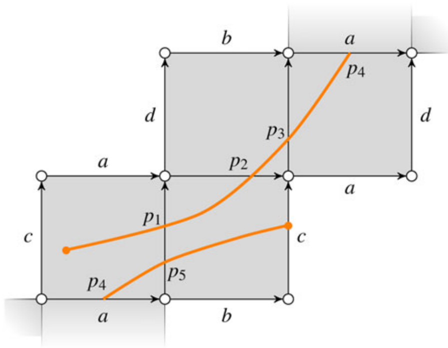

Figure 18. Directed path which wraps around a two-dimensional loop (bottom left and top right edges identified).

(the intersection points $p_1, \ldots, p_5$ of α with the edges are indicated in the figure), its interval arrangement is

$$J_a^\alpha = [0, \tfrac{1}{6}[\cup]\tfrac{1}{2}, \tfrac{5}{6}], \qquad J_b^\alpha =]\tfrac{1}{6}, \tfrac{1}{2}[\cup]\tfrac{5}{6}, 1[, \qquad J_c^\alpha = [0, \tfrac{1}{3}[\cup]\tfrac{2}{3}, 1], \qquad J_d^\alpha =]\tfrac{1}{3}, \tfrac{2}{3}[.$$

Now, for every $x \in X$, define a relation $\dashrightarrow_x$ on $\mathsf{ev}(x)$ by $e \dashrightarrow_x e'$ if $e = \mathsf{ev}_i(x)$ and $e' = \mathsf{ev}_j(x)$ for $i < j$. From Lemma 26, we immediately get

Lemma 76. *If $e, e' \in \delta_i^\nu x$ for some i and ν, then $e \dashrightarrow_{\delta_i^\nu x} e'$ iff $e \dashrightarrow_x e'$.* □

As a consequence, on every connected component $C \subseteq U_e \cap U_{e'}$ there is a well-defined relation $\dashrightarrow_C$ between e and e' (although it may differ between different components). We write $\dashrightarrow_p = \dashrightarrow_C$ for any point $p \in C$.

Definition 77. *The* label *of α is the ipomset $\ell(\alpha) = (P, <_P, \dashrightarrow_P, \lambda_P, S_P, T_P)$ given as follows:*

- $P = \{(e, i) \mid e \in E_X, 1 \le i \le n_e^\alpha\}$;
- $(e, i) <_P (e', i')$ *if* $I_{e,i}^\alpha < I_{e',i'}^\alpha$;
- $\dashrightarrow_P$ *is the transitive closure of the relations* $(e, i) \dashrightarrow_{\alpha(t)} (e', i')$ *for* $t \in I_{e,i}^\alpha \cap I_{e',i'}^\alpha$ *(this does not depend on the choice of t since $I_{e,i}^\alpha \cap I_{e',i'}^\alpha$ is connected);*
- $\lambda_P((e, i)) = \lambda^{\mathsf{ev}}(e)$, $S_P = \{(e, i) \in P \mid 0 \in I_{e,i}^\alpha\}$, *and* $T_P = \{(e, i) \in P \mid 1 \in I_{e,i}^\alpha\}$.

Hence all elements of S_P are of the form $(e, 1)$ and all elements of T_P are of the form (e, n_e^α). Further, $S_P \cong \mathsf{carr}(\alpha(0))$ and $T_P \cong \mathsf{carr}(\alpha(1))$ as linear posets.

Proposition 78. *The label $\ell(\alpha)$ is an interval ipomset.*

Proof. By definition, $<_P$ is an interval order, S_P contains only $<_P$-minimal elements, and T_P contains only $<_P$-maximal elements. Assume that (e, i) and (e', i') are $<_P$-incomparable, then $I_{e,i}^\alpha \cap I_{e',i'}^\alpha \neq \emptyset$. Let $t \in I_{e,i}^\alpha \cap I_{e',i'}^\alpha$, then $(e, i) \dashrightarrow_{\alpha(t)} (e', i')$ or $(e', i') \dashrightarrow_{\alpha(t)} (e, i)$, hence (e, i) and (e', i') are $\dashrightarrow_P$-comparable.

It remains to show that $\dashrightarrow_P$ is irreflexive. So let

$$(e_1, i_1) \dashrightarrow_{\alpha(t_1)} \cdots \dashrightarrow_{\alpha(t_{r-1})} (e_r, i_r) \dashrightarrow_{\alpha(t_r)} (e_1, i_1)$$

be a shortest loop of elementary relations and denote $H_k = I_{e_k,i_k}^\alpha =]a_k, b_k[$ (or $[a_k, b_k[$, $]a_k, b_k]$, or $[a_k, b_k]$, in case $a_k = 0$ or $b_k = 1$; this will not matter for our argument below).

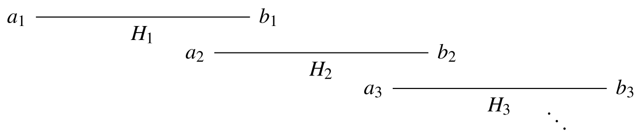

Figure 19. Progression of intervals in the proof of Proposition 78.

We have $H_k \cap H_{k+1} \neq \emptyset$ for $k \in [r-1]$, and also $H_r \cap H_1 \neq \emptyset$. On the other hand, $H_k \cap H_l = \emptyset$ for $k < l-1$ and $(k, l) \neq (1, r)$; otherwise we can construct a shorter loop. Further, $H_1 \cap \cdots \cap H_r = \emptyset$; otherwise, these elements would be linearly ordered by $\dashrightarrow_{\alpha(t)}$ for some $t \in \bigcap H_k$.

We show that for every k, H_{k+1} is either to the right or to the left of H_k. Let $k \in [r-2]$ and assume $H_{k+1} \subseteq H_k$. Then $H_k \cap H_{k+2} \supseteq H_{k+1} \cap H_{k+2} \neq \emptyset$, forcing $k = 1$ and $r = k+2 = 3$; but now also $H_1 \cap H_2 \cap H_3 \neq \emptyset$, a contradiction. A similar contradiction is obtained when assuming $H_k \subseteq H_{k+1}$, and also for $H_r \subseteq H_{r-1}$, $H_{r-1} \subseteq H_r$, $H_1 \subseteq H_r$, and $H_r \subseteq H_1$.

Now assume that H_2 is to the *right* of H_1 (the argument for the other case is similar), then $a_1 < a_2 < b_1 < b_2$, see Figure 19 for an illustration.

We proceed by induction. Let $k \in [r-2]$ and assume H_{k+1} is to the right of H_k, then $a_k < a_{k+1} < b_k < b_{k+1}$. We show that also H_{k+2} is to the right of H_{k+1}. Assume otherwise, then $a_{k+2} < a_{k+1} < b_{k+2}$, hence $a_{k+2} < b_k$ and $a_k < b_{k+2}$, which implies $H_k \cap H_{k+2} \neq \emptyset$, again forcing $k = 1$ and $r = k+2 = 3$ and then a contradiction.

Hence if H_2 is to the right of H_1, then the sequence of intervals $H_1, \ldots, H_r$ proceeds to the right; but the same argument as above then also shows that H_1 is to the right of H_r which is impossible. Similarly, if H_2 is to the left of H_1, then the sequence proceeds to the left, and H_1 then has the impossible task of being to the left of H_r. Overwhelmed by contradictions, we are forced to accept that $\dashrightarrow_P$ is irreflexive. □

6.3 Properties of d-path labels

The main goal of this section is to prove that for every d-path α in $|X|$ there is a track ρ in X with the same labeling and vice versa. First, we show several properties of labels of d-paths.

Lemma 79. *Let $\alpha \in \vec{P}|X|$ and $h : \vec{I} \to \vec{I}$ a (surjective) reparametrization. Then $\ell(\alpha) \cong \ell(\alpha \circ h)$.*

Proof. If $J_e^\alpha = I_{e,1}^\alpha \cup \cdots \cup I_{e,n_e^\alpha}^\alpha$, then

$$J_e^{\alpha\circ h} = h^{-1}(J_e^\alpha) = h^{-1}(I_{e,1}^\alpha) \cup \cdots \cup h^{-1}(I_{e,n_e^\alpha}^\alpha)$$

is a presentation as a union of connected components, so that $n_e^{\alpha\circ h} = n_e^\alpha$ and $I_{e,i}^{\alpha\circ h} = h^{-1}(I_{e,i}^\alpha)$. The result follows from the definition of d-path label. □

Lemma 80. *Let $f : X \to Y$ be a map of labeled precubical sets and $e \in E_Y$. Then*

$$|f|^{-1}(U_e^Y) = \bigsqcup_{e' \in E_f^{-1}(e)} U_{e'}^X$$

as a disjoint union.

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Proof. For $p \in |X|$ we have

$$
\begin{aligned}
p \in |f|^{-1}(U_e^Y) &\iff |f|(p) \in U_e^Y \\
&\iff e \in \mathsf{ev}(\mathsf{carr}(|f|(p))) && (74) \\
&\iff e \in \mathsf{ev}(f(\mathsf{carr}(p))) && (72) \\
&\iff e \in E_f(\mathsf{ev}(\mathsf{carr}(p))) && (27) \\
&\iff \exists e' \in E_f^{-1}(e) : e' \in \mathsf{ev}(\mathsf{carr}(p)) \\
&\iff \exists e' \in E_f^{-1}(e) : p \in U_{e'}^X. && (74)
\end{aligned}
$$

If $p \in U_{e'}^X \cap U_{e''}^X$ for $e' \neq e'' \in E_X$, then $e', e'' \in \mathsf{ev}(\mathsf{carr}(p))$. By Lemma 27, $E_f(e'), E_f(e'') \in \mathsf{ev}(\mathsf{carr}(|f|(p)))$, so $E_f(e') \neq E_f(e'')$. Consequently, e' and e'' cannot both belong to $E_f^{-1}(e)$. □

Lemma 81. *For any map of labeled precubical sets $f : X \to Y$ and $\alpha \in \vec{P}|X|$, $\ell(\alpha) \cong \ell(|f| \circ \alpha)$.*

Proof. By Lemma 80, there is a bijection between connected components of $J_e^{|f|\circ\alpha}$ and $\bigcup_{e' \in E_f^{-1}(e)} J_{e'}^\alpha$ for every $e \in E_Y$. These induce a bijection between the ipomsets $\ell(|f| \circ \alpha)$ and $\ell(\alpha)$. It is easy to check that this is an ipomset isomorphism. □

Proposition 82. *Let $\alpha, \beta \in \vec{P}|X|$ be such that $\alpha(1) = \beta(0)$. Then $\ell(\alpha * \beta) = \ell(\alpha) * \ell(\beta)$.*

Proof. Let $p = \alpha(1) = \beta(0)$ and $l, r : [0, 1] \to [0, 1]$, $l(t) = \frac{t}{2}$, $r(t) = \frac{t+1}{2}$. Then, for each $e \in E_X$,

$$J_e^{\alpha*\beta} = \tfrac{1}{2} J_e^\alpha \cup (\tfrac{1}{2}(J_e^\beta + \tfrac{1}{2})) = l(J_e^\alpha) \cup r(J_e^\beta).$$

If $e \notin \mathsf{carr}(p)$, then $1 \notin J_e^\alpha$ and $0 \notin J_e^\beta$. Thus, $l(J_e^\alpha)$ and $r(J_e^\beta)$ are disjoint, $n_e^{\alpha*\beta} = n_e^\alpha + n_e^\beta$ and

$$I_{e,i}^{\alpha*\beta} = \begin{cases} l(I_{e,i}^\alpha) & \text{for } 1 \le i \le n_e^\alpha, \\ r(I_{e,i-n_e^\alpha}^\beta) & \text{for } n_e^\alpha < i \le n_e^{\alpha*\beta}. \end{cases}$$

If $e \in \mathsf{carr}(p)$, then $1 \in J_e^\alpha$ and $0 \in J_e^\beta$. Therefore, $l(J_e^\alpha)$ and $r(J_e^\beta)$ are glued along $\frac{1}{2}$ and consequently $n_e^{\alpha*\beta} = n_e^\alpha + n_e^\beta - 1$ and

$$I_{e,i}^{\alpha*\beta} = \begin{cases} l(I_{e,i}^\alpha) & \text{for } 1 \le i < n_e^\alpha, \\ l(I_{e,n_e^\alpha}^\alpha) \cup r(I_{e,1}^\beta) & \text{for } i = n_e^\alpha, \\ r(I_{e,i-n_e^\alpha}^\beta) & \text{for } n_e^\alpha < i < n_e^{\alpha*\beta}. \end{cases}$$

It follows that the maps $i_e^\alpha : \ell(\alpha) \ni (e, i) \mapsto (e, i) \in \ell(\alpha * \beta)$ and

$$i_e^\beta : \ell(\beta) \ni (e, i) \mapsto \begin{cases} (e, i + n_e^\alpha) & \text{if } e \notin \mathsf{carr}(p) \\ (e, i + n_e^\alpha - 1) & \text{if } e \in \mathsf{carr}(p) \end{cases}$$

glue to the bijection $i : \ell(\alpha) * \ell(\beta) \to \ell(\alpha * \beta)$. It is elementary to check that i is an ipomset isomorphism. □

We record the following easy fact for use in the next proof.

Lemma 83. *Let S be a linear pomset and $\alpha \in \vec{P}|\square^S|$ a path such that $\mathsf{carr}(\alpha(t)) = \mathbf{y}_S$ for some $t \in [0, 1]$. Let $x = \mathsf{carr}(\alpha(0))$ and $y = \mathsf{carr}(\alpha(1))$, then $\ell(\alpha) = {}_{\ell(x)}S_{\ell(y)}$.* □

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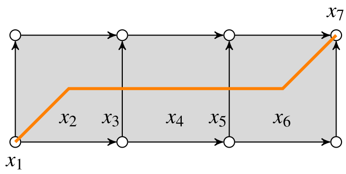

Figure 20. Track $\rho = (x_1, \dots, x_7)$ together with d-path α through center points of ρ.

Proposition 84. *For every d-path $\alpha \in \vec{P}|X|$, there exists a track $\rho : \mathsf{carr}(\alpha(0)) \rightsquigarrow \mathsf{carr}(\alpha(1))$ in X such that $\ell(\alpha) \cong \ell(\rho)$.*

Proof. By Lemma 73, there exists a presentation

$$\alpha = \big((|\mathbf{i}_{x_1}| \circ \beta_1) * (|\mathbf{i}_{x_2}| \circ \beta_2) * \cdots * (|\mathbf{i}_{x_m}| \circ \beta_m)\big) \circ h$$

such that $x_i \in X_{n_i}$, $\beta_i \in \vec{P}|\square^{\ell(x_i)}|$ and $\mathbf{i}_{x_i} : \square^{\ell(x_i)} \to X$ is the unique HDA map sending $\mathbf{y}_{\ell(x_i)}$ into x_i (we replace $\square^{n_i}$ with $\square^{\ell(x_i)}$ to obtain compatible labelings and make $\mathbf{i}_{x_i}$ HDA maps). We have

$$\begin{aligned}
\ell(\alpha) &= \ell(((|\mathbf{i}_{x_1}| \circ \beta_1) * \cdots * (|\mathbf{i}_{x_m}| \circ \beta_m)) \circ h) \\
&= \ell((|\mathbf{i}_{x_1}| \circ \beta_1) * \cdots * (|\mathbf{i}_{x_m}| \circ \beta_m)) && (79) \\
&= \ell(|\mathbf{i}_{x_1}| \circ \beta_1) * \cdots * \ell(|\mathbf{i}_{x_m}| \circ \beta_m) && (82) \\
&= \ell(\beta_1) * \cdots * \ell(\beta_m)) && (81) \\
&= {}_{\ell(\mathsf{carr}(\beta_1(0)))}\ell(x_1)_{\ell(\mathsf{carr}(\beta_1(1)))} * \cdots * {}_{\ell(\mathsf{carr}(\beta_m(0)))}\ell(x_m)_{\ell(\mathsf{carr}(\beta_m(1)))} && (83) \\
&= \ell(\mathsf{carr}(\beta_1(0)), x_1, \mathsf{carr}(\beta_1(1))) * \cdots * \ell(\mathsf{carr}(\beta_m(0)), x_m, \mathsf{carr}(\beta_m(1))) && (58) \\
&= \ell(\mathsf{carr}(\alpha(0)), x_1, \mathsf{carr}(\beta_1(1)), \dots, \mathsf{carr}(\beta_m(0)), x_m, \mathsf{carr}(\alpha(1))),
\end{aligned}$$

hence we can set $\rho = (\mathsf{carr}(\alpha(0)), x_1, \mathsf{carr}(\beta_1(1)), \dots, \mathsf{carr}(\beta_m(0)), x_m, \mathsf{carr}(\alpha(1)))$. $\square$

For the converse result, we construct a d-path through the center points of a given track, see also Fajstrup (2005) and Figure 20 for an example.

Proposition 85. *For every track $\rho : x \rightsquigarrow y$ in X, there is a d-path α with $\mathsf{carr}(\alpha(0)) = x$, $\mathsf{carr}(\alpha(1)) = y$, and $\ell(\alpha) = \ell(\rho)$.*

Proof. If $\rho = (x)$ is a unit track, we can let $\beta \in \vec{P}(\vec{I}^{\dim x})$ be the constant d-path $\beta(t) = (\frac{1}{2}, \dots, \frac{1}{2})$ and $\alpha = |\mathbf{i}_x| \circ \beta$. Otherwise, write $\rho = (x_1, \dots, x_m)$ with $m \geq 2$ and let $n_i = \dim x_i$ for $i \in [m]$. We construct α as a concatenation of d-paths $\alpha_1 * \cdots * \alpha_{m-1}$. Let $i \in [m-1]$.

- If $x_i \lhd^* x_{i+1}$, then $x_i = \delta_A^{0,\dots,0} x_{i+1}$ for a unique set $A \subseteq [n_{i+1}]$. Let $\beta_i \in \vec{P}(\vec{I}^{n_{i+1}})$ be the d-path

$$\beta_i(t) = (t_1, \dots, t_{n_{i+1}}), \qquad t_j = \begin{cases} \frac{1}{2}t & \text{if } j \in A, \\ \frac{1}{2} & \text{if } j \notin A \end{cases}$$

  and $\alpha_i = |\mathbf{i}_{x_{i+1}}| \circ \beta_i$. Then $\mathsf{carr}(\alpha_i(0)) = x_i$ and $\mathsf{carr}(\alpha_i(t)) = x_{i+1}$ for $0 < t \leq 1$.
- If $x_i \rhd^* x_{i+1}$, then $x_{i+1} = \delta_A^{1,\dots,1} x_i$ for a unique set $A \subseteq [n_i]$. Let $\beta_i \in \vec{P}(\vec{I}^{n_i})$ be the d-path

$$\beta_i(t) = (t_1, \dots, t_{n_i}), \qquad t_j = \begin{cases} \frac{1}{2} + \frac{1}{2}t & \text{if } j \in A, \\ \frac{1}{2} & \text{if } j \notin A \end{cases}$$

  and $\alpha_i = |\mathbf{i}_{x_i}| \circ \beta_i$. Then $\mathsf{carr}(\alpha_i(t)) = x_i$ for $0 \leq t < 1$ and $\mathsf{carr}(\alpha_i(1)) = x_{i+1}$.

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By construction, $\alpha_i(1) = \alpha_{i+1}(0)$ for all $i \in [m-1]$, so the concatenation $\alpha = \alpha_1 * \cdots * \alpha_{m-1}$ exists. Further, this is a representation as in Lemma 73, hence $\ell(\alpha) = \ell(\rho)$ by Proposition 84. □

7. Languages of Higher-Dimensional Automata

We define languages of HDAs and discuss some of their properties.

7.1 Languages

Using the work in Sections 5 and 6, we can define languages of HDAs in two different ways. The first one is a straight application of van Glabbeek's track-based approach from van Glabbeek (2006a), and the second one uses d-paths through geometric realizations in the spirit of Fajstrup et al. (2006).

Definition 86. *A track $\rho : x \rightsquigarrow y$ in an HDA (X, I, F, λ) is* accepting *if $x \in I$ and $y \in F$. The* track language *of X is $L_t(X) = \{\lambda(\rho) \in \mathsf{iiPoms} \mid \rho$ accepting track in $X\}$.*

A d-path $\alpha \in \vec{P}|X|$ is accepting *if $\mathsf{carr}(\alpha(0)) \in I$ and $\mathsf{carr}(\alpha(1)) \in F$. The* path language *of X is $L_p(X) = \{\lambda(\alpha) \in \mathsf{iiPoms} \mid \alpha$ accepting d-path in $|X|\}$.*

Theorem 87. *For every HDA X, $L_t(X) = L_p(X)$.*

Proof. Immediate from Propositions 84 and 85. □

From now on we write $L(X) = L_t(X) = L_p(X)$ and call this set simply the *language* of X. It follows immediately from Proposition 56 that languages of HDAs are sets of interval ipomsets:

Proposition 88. *For any HDA X, $L(X) \subseteq \mathsf{iiPoms}$.* □

The following property allows us to reason about languages using maps from objects $\square^P$.

Proposition 89. *For any HDA X and any interval ipomset P, $P \in L(X)$ iff there is an HDA map $\square^P \to X$.*

Proof. For the forward direction, assume $P \in L(X)$, then there exists a track $\rho : x \rightsquigarrow y$ with $x \in I_X$, $y \in F_X$, and $\lambda(\rho) = P$. The conclusion follows from Lemma 66.

For the reverse direction, let $g : \square^P \to X$. Then, by Proposition 67, there exists a track $\rho : i_{\square^P} \rightsquigarrow f_{\square^P}$ such that $\lambda(\rho) = P$, $g(i_{\square^P}) \in I_X$, and $g(f_{\square^P}) \in F_X$. Now Proposition 56 implies that $\lambda(f(\rho)) = P$. □

Remark 90. Thanks to Proposition 89, the language of an HDA X may alternatively be defined as the set of interval ipomsets P that admit an HDA map $\square^P \to X$. This definition remains valid even if we do *not* assume event consistency, hence it may be used to introduce languages also of HDA which are not event consistent. We will expand on this in future work.

We finish this section with some properties of languages of HDAs generated by interval ipomsets. The following is immediate from Proposition 67.

Lemma 91. *$P \in L(\square^P)$ for every interval ipomset P.* □

Proposition 92. *For all interval ipomsets P and Q, $Q \in L(\square^P)$ iff $Q \sqsubseteq P$.*

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Proof. The backwards direction is immediate from Lemma 63 and Proposition 89: a subsumption map $f : Q \to P$ gives rise to $\square^f : \square^Q \to \square^P$, thus $Q \in L(\square^P)$. For the forward direction, let $\rho : i_{\square^P} \rightsquigarrow f_{\square^P}$ be an accepting track in $\square^P$. We show that $\lambda(\rho) \sqsubseteq P$ by induction on the length of ρ.

If $\rho = (x)$, then $i_{\square^P} = x = f_{\square^P}$, which implies that $P = {}_P P_P = \ell(\rho)$. Otherwise, there is a presentation $\rho = (x, y) * \tau$. Note that $\ell(x) \cong S_P$. There are two cases to consider:

- $x \vartriangleleft^* y$. Then $y(p) = ⌐$ for $p \in \mathsf{ev}(y)$ and $y(p) = 0$ otherwise. Let Q be an interval ipomset with the same elements as P, $<_Q = <_P$, $\dashrightarrow_Q = \dashrightarrow_P$, $\lambda_Q = \lambda_P$, $T_Q = T_P$; the only difference is that $S_Q = \mathsf{ev}(y)$. Then $\square^P$ and $\square^Q$ are naturally isomorphic as labeled precubical sets and τ can be regarded as an accepting track in $\square^Q$. Moreover, $P = {}_{\ell(x)}\ell(y)_{\ell(y)} * Q$. By induction, $\ell(\tau) \sqsubseteq Q$; using Lemma 48, $\ell(\rho) = \ell(x, y) * \ell(\tau) \sqsubseteq P$.
- $x \vartriangleright^* y$. Then

$$y(p) = \begin{cases} 0 & \text{for } p \in P \setminus \mathsf{ev}(x), \\ ⌐ & \text{for } p \in \mathsf{ev}(y), \\ 1 & \text{for } p \in \mathsf{ev}(x) \setminus \mathsf{ev}(y). \end{cases}$$

  Let Q be the restriction of P to $P \setminus (\mathsf{ev}(x) \setminus \mathsf{ev}(y))$, then the precubical map $\mathbf{j}^1_{Q \subseteq P} : \square^Q \to \square^P$ is an injection onto $\{x \mid x(p) = 1 \text{ for } p \in \mathsf{ev}(x) \setminus \mathsf{ev}(y)\}$. Furthermore, τ is a track from $\mathbf{j}^1_{Q \subseteq P}(i_{\square^Q})$ to $\mathbf{j}^1_{Q \subseteq P}(f_{\square^Q})$ lying in $\mathbf{j}^1_{Q \subseteq P}(\square^Q)$. Thus, τ lifts uniquely to an accepting track τ' on $\square^Q$. By induction hypothesis, $\ell(\tau) \sqsubseteq Q$, and then with Lemma 48, $\ell(\rho) = \ell(x, y) * \ell(\tau) \sqsubseteq {}_{\ell(x)}\ell(x)_{\ell(y)} * Q = P$. □

Remark 93. Example 62 shows that the above proposition fails if P is not an interval ipomset: for $P = 2{+}2$, $P \notin L(\square^P)$. In general, Proposition 88 implies that if $P \notin \mathsf{iiPoms}$, then $P \notin L(\square^P)$. We will get back to this issue in Example 107 below.

7.2 Languages are subsumption-closed

Because of Proposition 88, we henceforth restrict ourselves to *interval* ipomsets.

Definition 94. *The* weak closure *of a set* $\mathscr{S} \subseteq \mathsf{iiPoms}$ *is* $\mathscr{S}\downarrow = \{Q \in \mathsf{iiPoms} \mid \exists P \in \mathscr{S} : Q \sqsubseteq P\}$.

That is, $\mathscr{S}\downarrow$ is the smallest subsumption-closed superset of $\mathscr{S}$. The set $\mathscr{S}$ is called *weak* if $\mathscr{S}\downarrow = \mathscr{S}$.

Theorem 95. *For every HDA X, $L(X) \subseteq \mathsf{iiPoms}$ is weak.*

Proof. This follows from subsumption closedness of $L(\square^P)$, Proposition 92: Choose interval ipomsets $Q \sqsubseteq P$ with $P \in L(X)$. Proposition 89 gives a map $f : \square^P \to X$ and Proposition 92 gives a map $g : \square^Q \to \square^P$. The composition $f \circ g$ with Proposition 89 again gives the conclusion. □

As a partial converse, we will see in Theorem 101 below that any *finite* subsumption-closed set of interval ipomsets can be generated by an HDA.

Lemma 96. *If $f : X \to Y$ is an HDA map, then $L(X) \subseteq L(Y)$.*

Proof. Let $P \in L(X)$, then Proposition 89 gives a map $\square^P \to X$. Composition with f yields a map $\square^P \to Y$, hence $P \in L(Y)$. □

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For HDAs generated by interval pomsets, Proposition 92 implies the following.

Lemma 97. $L(\square^P) = \{P\}\downarrow$. □

7.3 Languages are closed under union

We now show that languages of HDAs are closed under union (that is, they form filters). To this end, we introduce *coproducts* of HDAs. First, the coproduct of precubical sets X and Y is $Z = X \sqcup Y$ given by

$$Z_n = X_n \sqcup Y_n, \qquad \delta_i^\nu(z) = \begin{cases} (\delta_X)_i^\nu(z) & \text{if } z \in X, \\ (\delta_Y)_i^\nu(z) & \text{if } z \in Y. \end{cases}$$

Definition 98. *The* coproduct *of HDAs* (X, I_X, F_X, λ_X) *and* (Y, I_Y, F_Y, λ_Y) *is the HDA* $X \sqcup Y = (X \sqcup Y, I_X \cup I_Y, F_X \cup F_Y, \lambda)$ *with* $\lambda(z) = \lambda_X(z)$ *if* $z \in X$ *and* $\lambda(z) = \lambda_Y(z)$ *if* $z \in Y$.

It can easily be shown that these are in fact the categorical coproducts in the categories of precubical sets and HDAs, respectively. Next we note that subsumption closure of sets of interval ipomsets distributes over union (Grabowski, 1981):

Lemma 99. *For any subsets* $\mathscr{S}_1, \mathscr{S}_2 \subseteq \mathsf{iiPoms}$, $(\mathscr{S}_1 \cup \mathscr{S}_2)\downarrow = \mathscr{S}_1\downarrow \cup \mathscr{S}_2\downarrow$. □

Theorem 100. *For HDAs* X *and* Y, $L(X \sqcup Y) = L(X) \cup L(Y)$.

Proof. By construction of $X \sqcup Y$, any accepting track in $X \sqcup Y$ is an accepting track in X or in Y, and vice versa. The result follows with Lemma 99. □

Theorem 101. *Let* $\mathscr{S} \subseteq \mathsf{iiPoms}$ *be weak and finite. There is an HDA* X *with* $L(X) = \mathscr{S}$.

Proof. Write $\mathscr{S} = \{P_1, \dots, P_n\}\downarrow$ and let $X = \square^{P_1} \sqcup \dots \sqcup \square^{P_n}$. By Lemma 97, $L(\square^{P_i}) = \{P_i\}\downarrow$ for all $i = 1, \dots, n$, so using Theorem 100, $L(X) = \{P_1\}\downarrow \cup \dots \cup \{P_n\}\downarrow = (P_1 \cup \dots P_n)\downarrow$ by Lemma 99. □

7.4 Languages are closed under parallel composition

We show below that parallel compositions of HDA languages are languages of *tensor products* of HDAs. First, the tensor product of precubical sets X and Y is $Z = X \otimes Y$ given by

$$Z_n = \bigsqcup_{k+l=n} X_k \times Y_l, \qquad \delta_i^\nu((x,y)) = \begin{cases} (\delta_X)_i^\nu(x) & \text{if } i \le \dim x, \\ (\delta_Y)_{i-\dim x}^\nu(y) & \text{if } i > \dim x. \end{cases}$$

We will below use the important fact that geometric realizations of tensor products are products of geometric realizations (Grandis, 2009):

Lemma 102. *For precubical sets* X *and* Y, $|X \otimes Y| = |X| \times |Y|$. □

Definition 103. *The* tensor product *of HDAs* (X, I_X, F_X, λ_X) *and* (Y, I_Y, F_Y, λ_Y) *is* $X \otimes Y = (X \otimes Y, I, F, \lambda)$ *with* $I = \{(x,y) \mid x \in I_X, y \in I_Y\}$, $F = \{(x,y) \mid x \in F_X, y \in F_Y\}$, *and* $\lambda((x,y)) = \lambda_X(x) * \lambda_Y(y)$.

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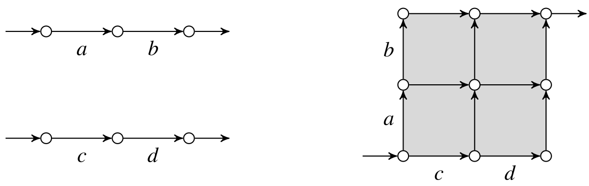

Figure 21. HDAs $X = \square^{(a<b)}$ and $Y = \square^{(c<d)}$ (left) and their tensor product $X \otimes Y$.

Above, $\lambda((x, y)) = \lambda_X(x) * \lambda_Y(y)$ denotes the concatenation of $\lambda_X(x)$ and $\lambda_Y(y)$ as sequences in $!\Sigma$. More formally, one can easily show that $!\Sigma \otimes !\Sigma = !\Sigma$, so that λ is the tensor product of the maps λ_X and λ_Y.

Remark 104. If X and Y are one-dimensional HDAs, that is, $X_2 = Y_2 = \emptyset$, then $Z = X \otimes Y$ is two-dimensional, with $Z_2 = X_1 \times Y_1$ and $Z_1 = X_1 \times Y_0 \sqcup X_0 \times Y_1$. The labels of 2-cells $(x, y) \in Z_2$ are $\lambda((x, y)) = (\lambda_X(x), \lambda_Y(y))$, and the labels of 1-cells $(x, y) \in Z_1$ are $\lambda(x, y) = \lambda_X(x)$ for $x \in X_1$ and $\lambda(x, y) = \lambda_Y(y)$ for $y \in Y_1$. Hence $X \otimes Y$ can be seen as the *synchronized product* (Winskel and Nielsen, 1995, Sec. 2.2.3) of the finite automata X and Y.

Lemma 105. *For ipomsets P and Q, $\square^{P\|Q} \cong \square^P \otimes \square^Q$.*

Proof. Let $X = \square^{P\|Q}$. As the underlying set of $P \parallel Q$ is the disjoint union $P \sqcup Q$ and $<_{P\|Q} = <_P \sqcup <_Q$, any poset map $x : (P \parallel Q, <_{P\|Q}) \to \{0, ⎍, 1\}$ has a unique decomposition $x = x_P \sqcup x_Q$ into poset maps $x_P : (P, <_P) \to \{0, ⎍, 1\}$ and $x_Q : (Q, <_Q) \to \{0, ⎍, 1\}$; and any two such maps give rise to a poset map x. Hence $X_n \cong \bigsqcup_{k+l=n} \square^P_k \times \square^Q_l$ as sets. It is easy to see that the face maps agree on both sides, and the same holds for the labeling. For the initial cell, we have $i_{\square^{P\|Q}}(p) = ⎍$ iff $p \in S_{P\|Q}$ (and 0 otherwise), iff $p \in S_P$ or $p \in S_Q$, hence $i_{\square^{P\|Q}}$ maps to $i_{\square^P} \sqcup i_{\square^Q}$ under the isomorphism; similarly for the accepting cell. $\square$

Definition 106. *The* parallel composition *of subsumption-closed subsets $\mathscr{S}_1, \mathscr{S}_2 \subseteq \mathsf{iiPoms}$ is $\mathscr{S}_1 \parallel \mathscr{S}_2 = \{P\|Q \mid P \in \mathscr{S}_1, Q \in \mathscr{S}_2\}\downarrow \cap\, \mathsf{iiPoms}$.*

We need to take the intersection with iiPoms above because parallel compositions of interval ipomsets may not be interval.

Example 107. Let P and Q be the ipomsets $P = (a \longrightarrow b)$, $Q = (c \longrightarrow d)$. Figure 21 shows the one-dimensional HDAs $X = \square^P$ and $Y = \square^Q$ as well as their tensor product $X \otimes Y = \square^{P\|Q}$ (*cf.* Example 62 and Figure 14). Now $L(X) = \{P\}\downarrow$ and $L(Y) = \{Q\}\downarrow$, but as $P\|Q$ is not an interval ipomset, $L(X \otimes Y) \neq \{P \parallel Q\}\downarrow$. Instead,

$$L(X \otimes Y) = \{P \parallel Q\}\downarrow \cap\, \mathsf{iiPoms} = \left\{ \left(\begin{smallmatrix} a \to b \\ \nearrow \\ c \to d \end{smallmatrix} \right), \left(\begin{smallmatrix} a \to b \\ \searrow \\ c \to d \end{smallmatrix} \right) \right\}\Big\downarrow.$$

Theorem 108. *For HDAs X and Y, $L(X \otimes Y) = L(X) \parallel L(Y)$.*

Proof. To show $L(X) \parallel L(Y) \subseteq L(X \otimes Y)$, let $R \in L(X) \parallel L(Y)$, then there are $P \in L(X)$ and $Q \in L(Y)$ such that $R \sqsubseteq P \parallel Q$. Let $f : \square^P \to X$ and $g : \square^Q \to Y$ be the maps given by Proposition 89. There is a composition

$$\square^R \xrightarrow{(63)} \square^{P\|Q} \xrightarrow{(105)} \square^P \otimes \square^Q \xrightarrow{f \otimes g} X \otimes Y,$$

thus $R \in L(X \otimes Y)$.

For showing $L(X \otimes Y) \subseteq L(X) \parallel L(Y)$, we have to do more work. Let $R \in L(X \otimes Y)$, then there is a d-path $\gamma \in \vec{P}|X \otimes Y|$ with $\lambda(\gamma) = R$, $\mathsf{carr}(\gamma(0)) \in I_{X \otimes Y}$, and $\mathsf{carr}(\gamma(1)) \in F_{X \otimes Y}$. Now $|X \otimes Y| = |X| \times |Y|$, so let α and β be the projections of γ to $|X|$ and $|Y|$, respectively. Let $P = \lambda(\alpha)$ and $Q = \lambda(\beta)$. We have $\mathsf{carr}(\alpha(0)) \in I_X$, $\mathsf{carr}(\alpha(1)) \in F_X$, $\mathsf{carr}(\beta(0)) \in I_Y$, and $\mathsf{carr}(\beta(1)) \in F_Y$, so that $P \in L(X)$ and $Q \in L(Y)$.

We show that $R \sqsubseteq P \parallel Q$. We have $E_{X \otimes Y} = E_X \sqcup E_Y$, so for every $e \in E_{X \otimes Y}$, $J_e^\alpha = \emptyset$ or $J_e^\beta = \emptyset$. Further, $J_e^\gamma = J_e^\alpha \cup J_e^\beta$ for every $e \in E_{X \otimes Y}$, so that the presentation $J_e^\gamma = I_{e,1}^\gamma \cup \cdots \cup I_{e,n_e^\gamma}^\gamma$ is the same as the one for J_e^α or J_e^β. Hence the underlying sets $R = P \sqcup Q$, and $x <_P y$ or $x <_Q y$ imply $x <_R y$.

Regarding the event orders, we work directly with the elementary relations $\dashrightarrow_{\gamma(t)}$. Assume $x \dashrightarrow_{\gamma(t)} y$, then $t \in I_x^\gamma \cap I_y^\gamma$. Now, writing $\mathscr{J}^\alpha = \{J_e^\alpha \mid e \in E_X\}$ and $\mathscr{J}^\beta = \{J_e^\beta \mid e \in E_Y\}$,

- if $I_x^\gamma, I_y^\gamma \in \mathscr{J}^\alpha$, then $x \dashrightarrow_{\alpha(t)} y$;
- if $I_x^\gamma, I_y^\gamma \in \mathscr{J}^\beta$, then $x \dashrightarrow_{\beta(t)} y$;
- if $I_x^\gamma \in \mathscr{J}^\alpha$ and $I_y^\gamma \in \mathscr{J}^\beta$, then $x \in P$ and $y \in Q$, hence $x \dashrightarrow_{P\parallel Q} y$; and
- the case $I_x^\gamma \in \mathscr{J}^\beta$ and $I_y^\gamma \in \mathscr{J}^\alpha$ cannot occur: this would imply $x \in Q$ and $y \in P$ and hence $y \dashrightarrow_{\gamma(t)} x$ instead of $x \dashrightarrow_{\gamma(t)} y$.

We have shown that $x \dashrightarrow_{\gamma(t)} y$ implies $x \dashrightarrow_{P\parallel Q} y$, so this also holds for the transitive closure $\dashrightarrow_R$. □

7.5 Language equivalence is implied by bisimulation

As a final sanity check of our notion of language, we show that bisimilarity of HDAs implies their language equivalence. Fahrenberg (2005c) has introduced a notion of *hd-bisimilarity* for HDAs which in our setting can be stated as follows. An *hd-bisimulation* between HDAs X and Y is a graded set $R = \bigcup R_n$ with $R_n \subseteq X_n \times Y_n$ such that

(1) R is closed under face maps: for all $(x, y) \in R_n$, $i \in [n]$ and $\nu \in \{0, 1\}$, $(\delta_i^\nu x, \delta_i^\nu y) \in R_{n-1}$;
(2) R respects labels: for all $(x, y) \in R$, $\lambda_X(x) = \lambda_Y(y)$;
(3) the restrictions $R \cap I_X \times I_Y$ and $R \cap F_X \times F_Y$ are bijections;
(4) for all $(x, y) \in R$ and any $x' \in X$ and $k \in [\dim x']$ such that $x = \delta_k^0 x'$, there exists $y' \in Y$ such that $y = \delta_k^0 y'$ and $(x', y') \in R$;
(5) for all $(x, y) \in R$ and any $y' \in Y$ and $k \in [\dim y']$ such that $y = \delta_k^0 y'$, there exists $x' \in X$ such that $x = \delta_k^0 x'$ and $(x', y') \in R$.

Hence initial and accepting cells are related bijectively (3), and (4) whenever a cell in X can be extended, then a related extension is available in Y, and vice versa (5). Finally, X and Y are *hd-bisimilar* if there exists an hd-bisimulation $R \subseteq X \times Y$: this is an equivalence relation.

As in Fahrenberg (2005c), we can express hd-bisimilarity using *open maps* (Joyal et al., 1996). We say that an HDA map $f : X \to Y$ is *open* if f is bijective on initial and accepting cells and the following *zig-zag property* holds for every $x \in X$: if $y' \in Y$ and $k \in [\dim y']$ are such that $f(x) = \delta_k^0 y'$, then there exists $x' \in X$ with $x = \delta_k^0 x'$ and $y' = f(x')$. The following is shown in Fahrenberg (2005c).

Lemma 109. *HDAs X and Y are hd-bisimilar iff there exists an HDA Z and a span of open maps $X \leftarrow Z \rightarrow Y$.* □

Theorem 110. *If HDAs X and Y are hd-bisimilar, then $L(X) = L(Y)$.*

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Proof. It suffices to assume an open HDA map $f : X \to Y$; the inclusion $L(X) \subseteq L(Y)$ is then clear by Lemma 96. For the reverse inclusion, let $\sigma = (y_1, \dots, y_m)$ be an accepting track in Y. By bijectivity of f on initial cells there is $x_1 \in I_X$ such that $f(x_1) = y_1$, and then inductive application of the zig-zag property yields a track $\rho = (x_1, \dots, x_m)$ in X with $f(x_i) = y_i$ for all i and $\lambda_X(\rho) = \lambda_Y(\sigma)$, with $x_m \in F_X$ because f is bijective on accepting cells. □

In van Glabbeek (2006a), van Glabbeek introduces a notion of *ST-bisimilarity* for HDAs which in our notation is given as follows. An *ST-bisimulation* between HDAs X and Y is a relation R between tracks in X and Y such that

(1) R is a bijection between initial unit tracks $\{(x) \mid x \in I_X\}$ and $\{(y) \mid y \in I_Y\}$;
(2) R respects accepting cells: for all $(\rho, \sigma) \in R$ such that $\rho : x \rightsquigarrow x'$ and $\sigma : y \rightsquigarrow y'$, $x' \in F_X$ iff $y' \in F_Y$;
(3) R respects labels: for all $(\rho, \sigma) \in R$, $\ell_X(\rho) = \ell_Y(\sigma)$;
(4) for all $(\rho, \sigma) \in R$ and track ρ' in X such that ρ and ρ' may be concatenated, there exists a track σ' in Y such that $(\rho * \rho', \sigma * \sigma') \in R$;
(5) for all $(\rho, \sigma) \in R$ and track σ' in Y such that σ and σ' may be concatenated, there exists a track ρ' in X such that $(\rho * \rho', \sigma * \sigma') \in R$.

That is, whenever a track in X can be extended, then a related extension is available in Y and vice versa. Finally, X and Y are *ST-bisimilar* if there exists an ST-bisimulation R between them; this is an equivalence relation.

Theorem 111. *If HDAs X and Y are ST-bisimilar, then $L(X) = L(Y)$.*

Proof. By symmetry, it suffices to show the inclusion $L(X) \subseteq L(Y)$. Let $P \in L(X)$, then there is a track $\rho : x \rightsquigarrow x'$ in X with $x \in I_X$, $x' \in F_X$ and $\ell(\rho) = P$. By (1), there is $y \in I_Y$ such that the unit tracks $((x), (y)) \in R$. Now $\rho = (x) * \rho$, so using (4) there exists a track $\sigma : y \rightsquigarrow y'$ in Y such that $(\rho, \sigma) \in R$, but then by (2), $y' \in F_Y$. Hence σ is an accepting track in Y, and by (3), $\ell(\sigma) = \ell(\rho) = P$, so that $P \in L(Y)$. □

In van Glabbeek (2006a), other notions of *history-preserving* and *hereditary* history-preserving bisimilarity for HDAs are introduced; both imply ST-bisimilarity and, thus, language equivalence.

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