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Computability and totality in domains

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We survey the main results on computability and totality in Scott–Eršov-domains as well as their applications to the theory of functionals of higher types and the semantics of functional programming languages. A new density theorem is proved and applied to show the equivalence of the hereditarily computable total continuous functionals with the hereditarily effective operations over a large class of base types.

1. Introduction

This paper studies the following different concepts of computability on partial and total continuous functionals of finite types:

(1) Definability by a program in a functional language,
(2) Effective continuity, and
(3) Computability *via* a recursive transformation of codes (effective operations).

For (3) to make sense the hereditarily effective versions of the hierarchies have to be considered. Although apparently fundamentally different in nature, these concepts turn out to be equivalent provided in (1) an appropriate language is chosen. Plotkin has shown the equivalence of (1) and (2) on the partial continuous functionals with respect to the language PCF augmented by the parallel conditional and the parallel $\exists$ (Plotkin 1977). Normann proved the corresponding equivalence for the total continuous functionals and pure PCF (without parallel facilities) (Normann 1998a). The equivalence of (2) and (3) is due to Ersov for the partial and the total case (Eršov 1977), and for the partial case a similar result was given in Constable and Egli (1976).

Following Eršov (1977), we define the partial and total continuous functionals in the framework of effective Scott–Eršov-domains. Eršov showed that his total continuous functionals are isomorphic to those defined in Kleene (1959) and Kreisel (1959), and that domain-theoretic computability (that is, concept (2)) corresponds to recursive countability (that is, having a recursive associate). The domain-theoretic approach not only allows for a very elegant definition of the continuous functionals, but also gives deeper insights into the phenomena being studied. In fact, most of the results mentioned so far rest on quite general theorems on computability and totality in domains. For instance, the density of the total continuous functionals used in Normann's proof is an instance of a general domain-theoretic theorem (Berger 1993), and Eršov's results are instances of domain-theoretic generalizations of the well-known Myhill–Shepherdson Theorem (GMS) and Kreisel–Lacombe–Shoenfield Theorem (GKLS) (Eršov 1977; Berger 1993). The main new result of this paper follows this line. We prove a density theorem for abstract domains with totality generalizing the corresponding theorem in Berger (1993) and, also, a recent result of Normann's showing density for the total continuous functionals over the reals (Normann 1998b). In fact, our proof is inspired by Normann's proof. Using GMS and GKLS, we conclude that (2) and (3) are also equivalent for the partial and total continuous functionals over the reals. The equivalence of (1) and (2) for these functionals has been shown in Escardó (1996b) using a version of PCF, called Real PCF, extended by the parallel $\exists$. The corresponding problem for the total continuous functionals over the reals and Real PCF (without $\exists$) seems to be open.

We will also briefly discuss extensions of the partial and total continuous functionals to dependent and transfinite types (Palmgren and Stoltenberg-Hansen 1990; Normann 1993; Berger 1999). The equivalence of (2) and (3) for these types has been shown in Normann and Waagbø (1997) using results from Berger (1999).

2. Computability in Scott–Eršov-domains

In this section we prove the equivalence of the concepts (1), (2), and (3) discussed in the introduction for the partial continuous functionals.

In order to fix notation we recall some basic definitions concerning domains, mainly following Griffor *et al.* (1993). By a *Scott–Eršov-domain* we mean a partially orderd set $(D, \sqsubseteq)$ that is:

- *Directed complete*, that is, every directed set $A \subseteq D$ has a least upper bound $\bigsqcup A \in D$ (the set A is *directed* if $A \neq \emptyset$ and $\forall x, y \in A \exists z \in A(x, y \sqsubseteq z)$).
- *Algebraic*, that is, for every $x \in D$ the set $\{x_0 \in D : x_0 \text{ compact and } x_0 \sqsubseteq x\}$ is directed and has x as its least upper bound ($x_0 \in D$ is *compact* if for every directed set $A \subseteq D$ such that $x \sqsubseteq \bigsqcup A$ we have $x_0 \sqsubseteq y$ for some $y \in A$).
- *Countably based*, that is, the set of compact elements is countable.
- *Bounded complete*, that is, every non-empty bounded subset of D has a least upper bound in D.
- Equipped with a *least element*, usually denoted $\perp$.

We will assume in addition that all Scott–Eršov-domains in consideration are *coherent*, which means that a non-empty subset is bounded whenever all of its two element subsets are bounded. Although all results presented in this paper also hold without this assumption, many notions have an easier definition and some proofs become less clumsy when coherence is assumed. The set of compact elements of a Scott–Eršov-domain D is denoted by D_0. The *Scott-topology* on D is generated by the basic open sets $\{x \in D : x \sqsupseteq x_0\}$, where $x_0 \in D_0$. The elements $x, y \in D$ are called *consistent*, written $x \uparrow y$, if $\{x, y\}$ is bounded in D. Clearly, this is the case if, and only if, x and y cannot be separated by disjoint neighbourhoods.

A Scott–Eršov-domain D is *effective* if it is endowed with a numbering $\nu_0 : \mathbf{N} \to D_0$, called *effectivation*, such that

1 The sets $\{(n, m) \mid \nu_0 n \sqsubseteq \nu_0 m\}$, and $\{(n, m) \mid \nu_0 n \uparrow \nu_0 m\}$ are decidable.

2 There is a recursive function $f:\mathbf{N}\times\mathbf{N}\to\mathbf{N}$ such that $v_0 f(n,m) = v_0 n \sqcup v_0 m$ whenever $v_0 n \uparrow v_0 m$.

An element $x \in D$ is called *computable* if the set $\{n \mid v_0 n \sqsubseteq x\}$ is recursively enumerable. We let D_{comp} denote the set of computable elements of D.

Convention: In the following, effective Scott–Eršov-domains will simply be called *domains*. We will let the letters D, E, F range over domains.

Basic examples of domains are the flat domains $\mathbf{N}_\perp := \mathbf{N} \cup \{\perp\}$ and $\mathbf{B}_\perp := \mathbf{B} \cup \{\perp\}$ ($\mathbf{B} = \{\mathrm{tt}, \mathrm{ff}\}$) of *partial integers* and *boolean values*. We will also consider the domain R of *partial reals*. R is the ideal completion of the partial order

$$I_{\mathbf{Q}} := \{[a,b] \mid a \in \{-\infty\} \cup \mathbf{Q},\ b \in \mathbf{Q} \cup \{+\infty\},\ a \leqslant b\},$$

where $\mathbf{Q}$ is the set of rational numbers. The ordering on $I_{\mathbf{Q}}$ corresponds to reverse inclusion of closed intervals, that is, $[a,b] \sqsubseteq [a',b']$ iff $a \leqslant a'$ and $b' \leqslant b$. The elements of R are downward closed directed subsets $A \subseteq I_{\mathbf{Q}}$ (ideals). The ordering on R is set inclusion. An ideal $A \in R$ that is 'converging' (that is, $\delta(A) := \inf\{b - a \mid [a,b] \in A\} = 0$) represents in a natural way the real number $r := \sup\{a \mid [a,b] \in A\} = \inf\{b \mid [a,b] \in A\}$.

It is well known that domains and continuous functions form a cartesian closed category. A continuous function between domains is called *effectively continuous* if it is a computable element of the function space.

The cartesian closed subcategory generated by the domains $\mathbf{N}_\perp$ and $\mathbf{B}_\perp$ is usually called the hierarchy of *partial continuous functionals of finite types*. The objects of this category are a family of domains D_ρ, where

$$D_\iota := \mathbf{N}_\perp, \quad D_o := \mathbf{B}_\perp, \quad D_{\rho\times\sigma} := D_\rho \times D_\sigma, \quad D_{\rho\to\sigma} := D_\rho \to D_\sigma.$$

From now on τ will refer to one of the base types ι and o. For brevity we will ignore product types.

The notion of an effective operation (that is, concept (3)) refers to a standard numbering $v:\mathbf{N}\to D_{\text{comp}}$ of the computable elements of a domain $(D, \sqsubseteq, v_0)$ called *principal constructivation*. It always exists and is characterized up to recursive equivalence by the conditions:

1 The set $\{(n,m) \mid v_0 n \sqsubseteq vm\}$ is recursively enumerable.
2 There is a recursive function $g:\mathbf{N}\to\mathbf{N}$ such that $v_0 n = vg(n)$ for all n.
3 For any other numbering $v':\mathbf{N}\to D_{\text{comp}}$ satisfying (1) and (2) there is a recursive function $h:\mathbf{N}\to\mathbf{N}$ such that $v'n = vh(n)$ for all n.

For example, a principal constructivation of the domain $\mathbf{N}_\perp$ is $v:\mathbf{N}\to\mathbf{N}_\perp$, defined by $v\langle n,m\rangle := \{n\}m$, where $\langle .,.\rangle : \mathbf{N}^2 \to \mathbf{N}$ is some primitive recursive pairing function and $\{.\}.$ is partial recursive function application.

An *effective operation between domains* is a function $F: D_{\text{comp}} \to E_{\text{comp}}$ that is tracked by some recursive function $f:\mathbf{N}\to\mathbf{N}$, that is, $F(vn) = \mu f(n)$ for all $n \in \mathbf{N}$, where v and μ are the principal constructivations of D and E. Note that in this definition F is not required to be continuous.

It is easy to see that a continuous function between domains is effectively continuous

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iff its restriction to computable arguments is an effective operation. Therefore, in order to prove the equivalence of the computability concepts (2) and (3) it remains to show that every effective operation is continuous. The latter follows from domain-theoretic generalizations of two theorems from elementary recursion theory. They establish a surprising connection between recursion theory and topology. Their generalization to domain theory is due to Eršov (1977).

Theorem 1 (Generalized Rice–Shapiro Theorem). Let $U \subseteq D_{\text{comp}}$ be such that the set $\{n \mid \nu n \in U\}$ is recursively enumerable. Then U is an open subset of D_{comp} (with respect to the relativized Scott-topology).

The proof of this theorem uses Markov's principle.

Theorem 2 (Generalized Myhill–Shepherdson Theorem). Every effective operation between domains is continuous (with respect to the relativized Scott-topologies).

This can be used to show that the hierarchy of effectively continuous partial functionals coincides with the hierarchy of effective operations based on a principal constructivation of $\mathbf{N}^{\perp}$ (Berger 1990).

Theorem 3. The effective partial continuous functionals and the partial effective operations over $\mathbf{N}$ are effectively isomorphic.

Now we turn our attention to the concepts (1) and (2). It is well known that the partial continuous functionals form a model for the functional programming language PCF (Plotkin 1977) and also for Kleene's schemes (S1–S9).

The following theorem is due to Platek (1966) and Plotkin (1977).

Theorem 4. On the partial continuous functionals PCF-definability and (S1–S9) computability coincide, but are weaker than domain theoretic computability.

Hence equivalence of (1) and (2) does not hold for PCF and (S1–S9). The reason for this is that PCF is unable to define such simple functions as the parallel or, $\text{POR} : \mathbf{B}_\perp \to \mathbf{B}_\perp \to \mathbf{B}_\perp$ returning $\mathtt{tt}$ as soon as one of its argument is $\mathtt{tt}$ (whereas the other may be undefined, that is, $\perp$) (Plotkin 1977). This can be remedied by either restricting the continuous functionals to some 'sequential' fragment (Milner 1977), or extending PCF. Plotkin showed how to do the latter (Plotkin 1977). He proved that besides POR only one further functional $\exists : (\mathbf{N} \to \mathbf{B}_\perp) \to \mathbf{B}_\perp$ is needed, defined by $\exists(f) = \mathtt{tt}$ if $f(n) = \mathtt{tt}$ for some $n \in \mathbf{N}$, and $\exists(f) = \mathtt{ff}$ if $f(\perp) = \mathtt{ff}$.

Theorem 5. The partial continuous functionals definable in PCF+POR+$\exists$ are precisely the computable ones in the domain-theoretic sense.

Escardó introduced *Real PCF* (Escardó 1996a),which is PCF extended by a base type for the partial real numbers. He interpreted it in a corresponding extension of the partial continuous functionals by the domain R of partial reals. In Escardó (1996a) he extended Plotkin's result to this situation.

Theorem 6. In Real PCF+$\exists$ every effectively continuous functional over the partial reals is definable.

To be precise, Escardó works with continuous domains, but, since these are retracts of domains, it is fairly obvious how to translate his results to our framework. The corresponding problem for the total continuous functionals over the reals and Real PCF (without $\exists$) seems to be open.

3. Totality

The total continuous functionals are defined in Eršov (1977) as follows. For every type ρ define $\overline{D}_\rho \subseteq D_\rho$ by

$$\overline{D}_\iota := \mathbf{N}, \quad \overline{D}_o := \mathbf{B}, \quad \overline{D}_{\rho\to\sigma} := \{f \in D_{\rho\to\sigma} | f[\overline{D}_\rho] \subseteq \overline{D}_\sigma\},$$

and define equivalence relations $=_\rho$ on $\overline{D}_\rho$ by

$$x =_\tau y :\Leftrightarrow x = y, \quad f =_{\rho\to\sigma} g :\Leftrightarrow \forall x \in \overline{D}_\rho f(x) =_\sigma g(x).$$

Then the total continuous functionals are the equivalence classes of the $\overline{D}_\rho$. From the fact that $\overline{D}_\rho$ is dense in D_ρ (see below) it follows that for $x, y \in \overline{D}_\rho$

$$x =_\rho y \Leftrightarrow x \uparrow y.$$

Hence, $=_\rho$ is preserved by application, that is, application on the quotient structure is well-defined. Instead of the types $\mathbf{N}_\perp$ and $\mathbf{B}_\perp$ with total elements $\mathbf{N}$ and $\mathbf{B}$, other domains D and selected subsets $\overline{D}$ could also be used as base types. The resulting hierarchy $\overline{D}_\rho$ will then be called *total continuous functionals over* $\overline{D}$. Furthermore, the function space can be hereditarily restricted to computable elements:

$$\overline{D}^{\text{comp}}_{\rho\to\sigma} := \{f \in D_{\rho\to\sigma} \mid f \text{ computable and } f[\overline{D}^{\text{comp}}_\rho] \subseteq \overline{D}^{\text{comp}}_\sigma\}.$$

We will call this the *hereditarily computable total continuous functionals over* $\overline{D}$

In this section we establish general conditions on $(D, \overline{D})$ under which the construction of the (hereditarily computable) total continuous functionals over $(D, \overline{D})$ is possible. In particular, we will be interested in proving the crucial density property in all types.

Abstracting from Eršov's approach to the total continuous functionals, Normann proposed the notion of a *domain with totality* (Normann 1997). For our purposes we will slightly modify his definitions. A pair $(D, \overline{D})$ where D is a domain and $\overline{D}$ is a subset of D is called a *domain with totality*. The set $\overline{D}$ is called the totality on D and the elements in $\overline{D}$ are called *total*. Sometimes we will refer to $(D, \overline{D})$ simply as $\overline{D}$ as long as this does not cause ambiguities. The totality $\overline{D}$ is called *strong* if the consistency relation $\uparrow$ on $\overline{D}$ is an equivalence relation. If $\overline{D}$ is strong, then for every total x we let $\mathbf{x} := \{y \in \overline{D} \mid x \uparrow y\}$ be the equivalence class of x. Furthermore, we let $\mathbf{D} := \overline{D}/\uparrow = \{\mathbf{x} \mid x \in \overline{D}\}$ denote the quotient structure endowed with the quotient topology. An equivalence class $\mathbf{x} \in \mathbf{D}$ is called *computable* if it contains a computable element. More generally, if P is a property of elements of $\overline{D}$, then $\mathbf{x} \in \mathbf{D}$ is said to have property P if some element of $\mathbf{x}$ has property P.

For every domain the set of its maximal elements is a strong totality. Further examples of domains with strong totality are (as we will see) the continuous functionals over **N** and **B**, and their hereditarily computable versions. As shown in Eršov (1975), their quotients define the Kleene–Kreisel total continuous functionals. The computable elements of $\mathbf{D}_\rho$ correspond to the recursively countable functionals. The domain R of partial reals becomes a domain with strong totality by setting $\overline{R} := \{A \in R \mid \delta(A) = 0\}$. The quotient space **R** is homeomorphic to the reals. See Blanck (1996) and Blanck *et al.* (1998) for many further examples of interesting topological spaces represented in the form **D**.

If $(D, \overline{D})$ and $(E, \overline{E})$ are domains with totality, the total elements of $D \to E$ are defined by

$$\overline{D \to E} := \overline{D} \to \overline{E} := \{f \in D \to E : f[\overline{D}] \subseteq \overline{E}\}.$$

The elements of $\overline{D} \to \overline{E}$ are called *total functions*. However, in general, $\overline{D} \to \overline{E}$ will not be strong even if $\overline{D}$ and $\overline{E}$ both are. Moreover, it is natural to consider $f, g \in \overline{D} \to \overline{E}$ as equivalent if $f(x) \uparrow g(x)$ for all $x \in \overline{D}$, but this notion of equivalence will in general not coincide with $\uparrow$ in $\overline{D} \to \overline{E}$.

This can be remedied by requiring the total elements to be dense. Note that $\overline{D} \subseteq D$ is dense iff

$$\forall x_0 \in D_0 \, \exists x \in \overline{D} \; x_0 \sqsubseteq x.$$

One can immediately check that if D, E are domains with totality such that $\overline{D}$ is dense and $\overline{E}$ is strong, then $\overline{D} \to \overline{E}$ is strong. Moreover, for $f, g \in \overline{D} \to \overline{E}$ we have $f \uparrow g$ iff $f(x) \uparrow g(x)$ for all $x \in \overline{D}$. The latter amounts to a principle of *extensionality*: two total functions are identified if they are extensionally equal on total arguments, that is, for $f, g \in \overline{D} \to \overline{E}$ we have $\mathbf{f} = \mathbf{g}$ iff $\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{x})$ for all $x \in \overline{D}$, where, of course, $\mathbf{f}(\mathbf{x})$ is the equivalence class of $f(x)$.

We are still not satisfied, since, even if $\overline{D}$ and $\overline{E}$ are both strong and dense, $\overline{D} \to \overline{E}$ need not be dense. Consider, for example, $D := \mathbf{N}_\perp$ with $\overline{D} := D$, and $E := \mathbf{N}_\perp$ with $\overline{E} := \mathbf{N}$. Both are strong and dense totalities, but $\overline{D} \to \overline{E}$ contains only constant functions and hence is not dense. What is wrong here is the fact that we declared $\perp \in D$ to be total. To exclude this we need a property forcing the elements of $\overline{D}$ to be in some sense 'large'. In Berger (1993), I introduced the notion of *codensity*, which, together with density, was preserved under function spaces. This solved the density problem for functionals over discrete base types like **N** or **B**, but, unfortunately, it excluded base types like the reals, since codensity of $\overline{D}$ implies that the quotient space **D** is strongly disconnected in the sense that any two different points can be separated by clopen sets.

In the following we introduce generalizations of the notions of density and codensity and prove a density theorem generalizing the results in Berger (1993) and also Normann's density theorem for the total continuous functionals over the reals (Normann 1998b). In fact, our proof has been obtained by an analysis of Normann's proof.

In order to motivate the definitions below we briefly recall the definition of codensity in Berger (1993). A totality $\overline{D}$ on a domain D is *codense* if for any two inconsistent compacts $x_0, y_0 \in D_0$ (that is, $x_0 \not\uparrow y_0$) there is a continuous 'test' $t : D \to \mathbf{B}_\perp$ that separates x_0 and y_0, that is, $t(x_0) = \mathtt{tt}$ and $t(y_0) = \mathtt{ff}$, and is total, that is, $t[\overline{D}] \subseteq \mathbf{B}$ (therefore an element $x \in \overline{D}$ is 'rather large' because $t(x)$ is defined for 'many' tests t). Using

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an effectivation $v_0 : \mathbf{N} \to D_0$, we can join the separating and total tests above into one continuous function $f : D \to \mathbf{B}_\bot^\omega$ ($\mathbf{B}_\bot^\omega$ is the countably infinite product of $\mathbf{B}_\bot$ defined below), by setting $f(x)(\langle n, m\rangle) := t(x)$, where t is the (selected) total test separating $v_0 n$ and $v_0 m$ if $v_0 n \not\uparrow v_0 m$, otherwise $f(x)(\langle n, m\rangle) := \mathtt{tt}$. Clearly, f is total and sepaparating, that is, if $x \not\uparrow y$, then $f(x) \not\uparrow f(y)$. Conversely, if there is a continuous total and separating $f : D \to \mathbf{B}_\bot^\omega$, then clearly $\overline{D}$ is codense. The idea now is to replace in the situation above the domain $\mathbf{B}_\bot^\omega$ with total subset $\mathbf{B}^\omega$ by an arbitrary domain with totality $(A, \overline{A})$.

For any domain D we let D^ω denote the set of functions $s : \mathbf{N} \to D$. Ordered pointwise this is again a domain. Any totality $\overline{D}$ on D gives rise to the totality $\overline{D}^\omega := \{s \in D^\omega \mid s[\mathbf{N}] \subseteq \overline{D}\}$ on D^ω. $\overline{D}^\omega$ will be strong if $\overline{D}$ is. Note that because domains have a countable base, the totality $\overline{D}$ is dense in D iff there is an $s \in \overline{D}^\omega$ such that $s[\mathbf{N}]$ is dense in D. We call $\overline{D}$ *effectively dense* if there is a computable such s. Since we will only be interested in effective density, when we say 'dense' in the following, we will always mean 'effectively dense'.

A continuous function $f : D \to E$ is called *separating* if it preserves inconsistencies, that is, $\forall x, y \in D\,(x \not\uparrow y \Rightarrow f(x) \not\uparrow f(y))$.

Let $(A, \overline{A})$ be a domain with totality. A totality $\overline{D}$ on a domain D is called $\overline{A}$-*dense* if $\overline{A} \to \overline{D}$ is dense in $A \to D$. $\overline{D}$ is called $\overline{A}$-*codense* if $\overline{D} \to \overline{A}$ contains a computable separating element.

The following facts, whose simple proofs we omit, continue the motivating discussion above and are intended to shed some light on the notions of $\overline{A}$-density and $\overline{A}$-codensity.

(1) $\overline{D}$ is dense iff it is $\mathbf{B}^\omega$-dense.
(2) $\overline{D}$ is codense (in the sense of Berger (1993)) iff it is $\mathbf{B}^\omega$-codense.
(3) If $\overline{D}$ is $\overline{A}$-dense and $\overline{A}$ is non-empty, then $\overline{D}$ is dense.
(4) If $\overline{D}$ is $\overline{A}$-codense and $\overline{A}$ is strong, then $\overline{D}$ is strong.
(5) If $\overline{D}$ is non-empty, then $\overline{D}$ is $\overline{A}^\omega$-dense iff $\overline{A}^n \to \overline{D}$ is dense uniformly for all $n \in \mathbf{N}$.
(6) $\overline{D}$ and $(\overline{D}^\omega)^\omega$ are both $\overline{D}^\omega$-codense for every domain with totality $(D, \overline{D})$.

For example, to prove in (6) that $(\overline{D}^\omega)^\omega$ is $\overline{D}^\omega$-codense, consider the continuous function $\mathrm{melt} : (D^\omega)^\omega \to D^\omega$, defined by $\mathrm{melt}(s)(\langle m, n\rangle) := s(m)(n)$. Clearly melt is total and bijective, in particular, separating.

Theorem 7. Let $(A, \overline{A})$, $(D, \overline{D})$, and $(E, \overline{E})$ be domains with totality.

(1) If $\overline{D}$ is $\overline{A}$-codense and $\overline{E}$ is $\overline{A}$-dense, then $\overline{D} \to \overline{E}$ is $\overline{A}$-dense.
(2) If $\overline{D}$ is dense and $\overline{E}$ and $\overline{A}^\omega$ are both $\overline{A}$-codense, then $\overline{D} \to \overline{E}$ is $\overline{A}$-codense.

Proof. (1) By assumption, there is a computable separating and total $f \in D \to A$, and $\overline{A} \to \overline{E}$ is dense. We have to show that $\overline{A} \to \overline{D} \to \overline{E}$ is dense in $A \to D \to E$. Using isomorphism, it suffices to show that $\overline{A} \times \overline{D} \to \overline{E}$ is dense in $A \times D \to E$. Let h_0 be a compact element of $A \times D \to E$. We have to construct some total $h \in A \times D \to E$ extending h_0. Since h_0 is compact, there are finitely many compacts $a_i \in A_0$, $d_i \in D_0$, $e_i \in E_0$ ($i \in I$, I finite) such that

$$h_0(a, d) = \bigsqcup\{e_i \mid i \in I,\, a_i \sqsubseteq a,\, d_i \sqsubseteq d\}$$

for all $(a, d) \in A \times D$. Define a function $\mathrm{pair} \in A \times A \to A$ by $\mathrm{pair}(a, b) := \mathrm{melt}(\lambda n.\mathbf{if}\; n =$

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0 **then** a **else** b). Clearly, pair is computable, total and separating. Since f is also separating, we have for all $i, j \in I$

$$(a_i, d_i) \uparrow (a_j, d_j) \Leftrightarrow \text{pair}(a_i, f(d_i)) \uparrow \text{pair}(a_j, f(d_j)).$$

By algebraicity of A, there are compacts $b_i \sqsubseteq \text{pair}(a_i, f(d_i))$ $(i \in I)$ such that for all $i, j \in I$

$$(a_i, d_i) \uparrow (a_j, d_j) \Leftrightarrow b_i \uparrow b_j.$$

Hence the function $g : A \to E$

$$g_0(a) := \bigsqcup\{e_i \mid i \in I,\, b_i \sqsubseteq a\}$$

is a well-defined compact in $A \to E$. By assumption, there is a total $g \in A \to E$ extending g_0. Now define $h : A \times D \to E$ by

$$h(a, d) := g(\text{pair}(a, f(d))).$$

Obviously, h is continuous and total. For $i \in I$ we have

$$h(a_i, d_i) = g(\text{pair}(a_i, f(d_i))) \sqsupseteq g(b_i) \sqsupseteq g_0(b_i) \sqsupseteq e_i.$$

Hence h extends h_0.
(2) By assumption, $\overline{D}$ is dense, which means that there is a computable total $s \in D^{\omega}$ such that $s[\mathbf{N}]$ is dense in D. Furthermore, we have computable separating and total functions $f \in E \to A$ and $f' \in A^{\omega} \to A$. We define $g \in (D \to E) \to A$ by

$$g(h) := f'(f \circ h \circ s).$$

Clearly, g is computable and total. To see that g is separating, let $h_0 \not\uparrow h_1$ in $D \to E$. Since $s[\mathbf{N}]$ is dense in D, there is n such that $h_0(s(n)) \not\uparrow h_1(s(n))$. Hence $f(h_0(s(n))) \not\uparrow f(h_1(s(n)))$ in A because f is separating. It follows that $f \circ h_0 \circ s \not\uparrow f \circ h_1 \circ s$. Therefore $g(h_0) \not\uparrow g(h_1)$, since f' is separating. □

By Facts (1), (2) and (3) listed above, this theorem generalizes the density theorem in Berger (1993). Note also that the witnesses of density and the separating functions in the conclusions of the theorem are defined explicitly from the corresponding objects given by the assumptions using just case analysis on $n = 0$, $n > 0$. Hence we have the following corollary.

Theorem 8. Let $(D, \overline{D})$ be a domain with strong totality such that $\overline{D}$ is non-empty and $\overline{D}^{\omega} \to \overline{D}$ is dense. Then the total continuous functionals $\overline{D}_{\rho}$ and, also, the hereditarily computable total continuous functionals over $\overline{D}$ are domains with strong dense totalities.

Moreover, for every type ρ a dense and total sequence in D_{ρ} can be defined explicitly from a computable dense and total sequence in $D^{\omega} \to D$, case analysis on $n = 0$, $n > 0$, and a bijection from $\mathbf{N}^2$ to ω.

Proof. By Fact (6), we know that $\overline{D}$ and $(\overline{D}^{\omega})^{\omega}$ are both $\overline{D}^{\omega}$-codense. Hence, since $\overline{D}$ is assumed to be $\overline{D}^{\omega}$ dense, Theorem 7 and Fact (3) imply that $\overline{D}_{\rho}$ is $\overline{D}^{\omega}$-dense and -codense and, also, dense for every type ρ. □

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In Normann (1998b) it was shown for the domain $(R, \overline{R})$ of partial and total reals that $\overline{R}^n \to \overline{R}$ is dense uniformly for all n. Fact (5) implies that $\overline{R}$ is $\overline{R}^\omega$-dense. Hence we obtain Normann's density theorem in Normann (1998b).

Theorem 9. The total continuous functionals over the reals, $\overline{R}_\rho$, are dense in R_ρ for all finite types ρ.

In Normann (1998b), Normann proved a more general theorem than Theorem 9 connecting the discrete (**N**) and the continuous (**R**) case by admitting certain partial equivalence relations different from the consistency relation $\uparrow$.

In the same spirit, D. Scott and his students A. Awodey, A. Bauer and L. Birkedal have recently developed (Scott *et al.* 1998) a rather general theory of topological spaces endowed with a partial equivalence relation (equilogical spaces), which might yield a good framework for putting the work presented here into a more general (categorical) context.

Closing this section we state a simple but important application of density (Kreisel 1959; Schwichtenberg 1996; Berger 1993).

Theorem 10 (Effective choice principle). Let $(D_\rho)_\rho$ be the hierarchy of partial continuous functional over the integers. For all types ρ and σ there is a PCF+POR definable total functional of type $(\rho \times \sigma \to o) \to (\rho \to \sigma)$ computing for every total functional f of type $\rho \times \sigma \to o$ such that

$$\forall x \in \overline{D}_\rho \exists y \in \overline{D}_\sigma \; f(x, y) = 0$$

a total functional g of type $\rho \to \sigma$ such that

$$\forall x \in \overline{D}_\rho \; f(x, g(x)) = 0.$$

4. Computability and totality

Now we use the results of the previous section to prove the equivalence of the computability concepts (1), (2) and (3) for total continuous functionals.

It was proved in Gandy and Hyland (1977) that the fan-functional computing a modulus of uniform continuity of a total type-2 functional restricted to a compact fan is definable by Kleene's schemata (S1–S9) (see also Normann (1980)), but in Berger (1993) it was shown that the fan-functional *is* (S1–S9) definable when Kleene's schemata are interpreted in the partial continuous functionals. It was then conjectured that *every* computable total continuous functional over the integers is (S1–S9)-computable, that is, PCF-definable. Again it was Normann who in 1998 proved this conjecture (Normann 1998a), thus showing the computation concepts (1) and (2). to be equivalent for the total continuous functionals.

Theorem 11. Every computable total continuous functional over the integers is PCF-definable.

Moreover, for every type ρ there is a PCF-computable functional of type $(o \to o) \to \rho$ computing from every enumeration of the compact approximations of total functional f of type ρ (where this enumeration is coded as a sequence of integers) a total functional $\hat{f} \sqsubseteq f$.

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Normann's proof uses the density theorem in an essential way.

In order to prove the equivalence of the computation principles (2) and (3) we have to look for a total analogue of the generalized Myhill–Shepherdson Theorem (Theorem 2). Several such theorems are proved in Berger (1993). They may be viewed as a generalization of the *Kreisel–Lacombe–Shoenfield Theorem* (Kreisel *et al.* 1959). Here, we only present one of them.

An element y of a domain is called *almost maximal* if it cannot be extended in two inconsistent ways, that is, $\forall y', y''(y \sqsubseteq y', y'' \Rightarrow y' \uparrow y'')$. Using the axiom of choice, this can be shown to be equivalent to the property that y has precisely one maximal extension (but we will not use this fact). For instance, the elements of a codense set are almost maximal. Also, all elements of $\overline{R}$ are almost maximal (although $\overline{R}$ is not a codense set).

Theorem 12. Let D, E be effective domains with totality. Assume that $\overline{D}$ is effectively dense and all elements of $\overline{E}$ are almost maximal. Then every effective operation $f : \overline{D} \to \overline{E}$ can be extended to an effective (and by the generalized Myhill–Shepherdson Theorem continuous) operation $f' : D \to E$, in the sense that $f(x) \sqsubseteq f'(x)$ for all $x \in \overline{D}$.

From this theorem we may deduce the equivalence of the computation concepts (2) and (3) for many instances of total functions. For instance, it immediately entails the well-known theorem of Ceitin and Moschovakis saying that every effective operator on the reals is continuous. We will see that it also implies the equivalence of the hereditarily computable total continuous functionals and hereditarily effective total operations of finite types over a large class of base types.

To make this precise, we need the notion of a *partially numbered set*, introduced by Eršov, which is a pair (S, ν) where S is a set and ν is a surjection from a subset $\delta\nu \subseteq \mathbf{N}$ onto S (Eršov 1975). For example, if $\overline{D}$ is a strong totality on a domain D such that $\overline{D} \subseteq D_{\mathrm{comp}}$, then any principal constructivation ν of D induces a partial numbering $\nu : \nu^{-1}\overline{D} \to \mathbf{D}$ defined by $\nu n := [\nu n]$. An *effective operation* between two numbered sets (S, ν), (T, μ) is a mapping $F : S \to T$ that is tracked by some partial recursive function f, that is, f is defined on $\delta\nu$, $f[\delta\nu] \subseteq \delta\mu$ and $F(\nu n) = \mu f(n)$ for all $n \in \delta\nu$. We use $\mathbf{EO}(S, T)$ to denote the set of effective operations from T to S, partially numbered by Kleene indices of tracking functions. Obviously, this corresponds precisely to the exponential in the category PER of partial equivalence relations. Given a partially numbered set S as base type, we define the *effective operations of finite types over* S by

$$S_\tau := S, \quad S_{\rho\to\sigma} := \mathbf{EO}(S_\rho, S_\sigma).$$

Starting with the base type $\mathbf{N}$ numbered by the identity, we obtain the hereditarily effective operations (Troelstra 1973). If we start with $\mathbf{N}_\perp$ numbered by a principal constructivation, we obtain the hereditarily partial effective operations (see Theorem 3). From Theorems 8 and 12 one can now easily derive the following theorem.

Theorem 13. Let D be a domain and set

$$\overline{D} := \{x \in D \mid x \text{ computable and almost maximal}\}.$$

Assume that $\overline{D}^{\omega} \to \overline{D}$ is dense and that there is a total computable function select $\in D \to D$ such that $\forall x, y \in \overline{D}(x \uparrow y \Rightarrow x \uparrow \text{select}(x) = \text{select}(y))$.

Then the hereditarily effective operations of finite type over $\overline{D}$ are effectively isomorphic with the hereditarily computable total continuous functionals over $\overline{D}$.

The theorem applies, for instance, to $\overline{D} := \mathbf{N}$ and to $\overline{D} := \overline{R}$.

5. Dependent domains and universes

In this paper we have focused on the type constructor → (function space). However, some of the work described has been extended to dependent products, dependent sums and universe operators in the sense of Martin-Löf type theory: Palmgren and Stoltenberg-Hansen developed the notion of a *dependent domain* and gave a domain interpretation of a partial type theory (Palmgren and Stoltenberg-Hansen 1990); Kristiansen and Normann used a universe of dependent domains with dense totality to represent computations relative to certain non-continuous functionals like ${}^{3}E$ (Normann 1993; Kristiansen and Normann 1994); Waagbø modified Palmgren's and Stoltenberg-Hansen's work for interpreting (the usual) total type theory using dependent domains with totality (Waagbø1997); abstract density theorems for dependent types and universe operators were proved in Berger (1999); and this was used in Waagbø (1997) to prove the equivalence of the computation concepts (2) and (3) for functionals of dependent types over **N**.

Theorem 14. Normann's well-founded hierarchy of hereditarily computable total continuous functionals of dependent types and Beeson's model of total effective operations of dependent types (Beeson 1982) are effectively isomorphic.

6. Related work

There exists a substantial literature discussing different notions of computability for higher types: for example, Platek (1966), Moldestad (1977), Moschovakis (1977), Gandy and Hyland (1977), Feferman (1977), Hyland (1979), Normann (1980) and Cook (1990). Much of this work focuses on Kleene's schemata (S1–S9) (Kleene 1959), which, when interpreted on the *partial* continuous functionals, are equivalent to PCF (see Theorem 4). In combination with Normann's result mentioned above, this implies that for the total continuous functionals effective continuity and (S1–S9) interpreted on the partial continuous functionals coincide. This seems to contradict earlier results showing that (S1–S9) computability is strictly weaker than effective continuity (*cf.*, for example, Normann (1980)). In fact, however, it just shows that the two interpretations of (S1–S9) give rise to different notions of computability on the total continuous functionals. This answers a question posed in Cook (1990, p. 59).

Cook's paper also gives an introduction to feasibility for higher type functionals, a subject that is not touched on here, since our concepts (1), (2) and (3) are all Turing complete in the sense that they define exactly those (partial) number theoretic functions (represented by strict functions of type level one) that are (partial) recursive. There are versions of (2) that induce a notion of subrecursiveness or feasibility in higher type (see,

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for example, Cook and Kapron (1990) and Bellantoni *et al.* (2000)), but I do not know of any convincing approach defining subrecursiveness in higher types *via* restrictions of the concepts (1) or (3).

In Plotkin (1998) it is shown how the equivalence of (1) and (2) can be used to prove full abstraction results for functional languages with respect to the continuous denotational semantics. In Blanck (1996), Eršov's method of defining the total continuous functionals within domain theory is generalized and used to define effective domain representations for large classes of topological spaces. In Blanck *et al.* (1998), this theory is applied to an analysis of continuous stream transformers. The material in Stoltenberg-Hansen and Tucker (1999) compares effective domain representations of topological algebras with other approaches to computability on topological spaces. The equivalence of the concepts (2) and (3) for effective metric spaces, which was first proved in Ceĭtin (1962) and Moschovakis (1964), has been generalized to effective topological spaces in Spreen (1990).

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